MATH 151, 5.4 Indefinite Integrals and Net Change Theorem

An antiderivative of $f(x)$ and is called an

$$\int f(x)\,dx = F(x) + C \quad \text{means} \quad F'(x) = f(x)$$

Table of Indefinite Integrals (table of antiderivative formula of Section 4.9)

$\int cf(x)\,dx = c\int f(x)\,dx$

$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C \quad (n \neq -1)$

$\int \frac{1}{x}\,dx = \ln|x| + C$

$\int e^x\,dx = e^x + C$

$\int a^x\,dx = \frac{a^x}{\ln a} + C$

$\int \sin x\,dx = -\cos x + C$

$\int \cos x\,dx = +\sin x + C$

u-sub in Sect. 5.5

$\int \tan x\,dx = -\ln|\cos x| + C$

$\int \sec^2 x\,dx = \tan x + C$

$\int \sec x \tan x\,dx = \sec x + C$

$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$

$\int \frac{-1}{\sqrt{1-x^2}}\,dx = \arccos x + C$

$\int \frac{1}{1+x^2}\,dx = \arctan x$

Definite vs. Indefinite Integrals

- A **definite integral** $\int_a^b f(x)\,dx =$ Number
- An **indefinite integral** $\int f(x)\,dx =$ a function.

The Fundamental Theorem of Calculus, Part II (FTC II) explains the relationship between definite and indefinite integrals:

$$\int_a^b f(x)\,dx = \int f(x)\Big]_{x=a}^{x=b} = F(b) - F(a) = \cdots$$

Example 1 Find the indefinite integral (* the most general antiderivative)

(1) $\int \left(2\sqrt{t} + t + t\sqrt{t}\right)\,dt = \int 2t^{\frac{1}{2}} + t^1 + t^{3/2}\,dt = 2\cdot\frac{2}{3}t^{\frac{3}{2}} + \frac{1}{2}t^2 + \frac{2}{5}t^{\frac{5}{2}} + C.$

(2) $\int \frac{x^5 - x^3 + 6x + 5\sqrt{x}}{x^4}\, dx = \int x^1 - x^{-1} + 6x^{-3} + 5x^{-\frac{7}{2}}\, dx$

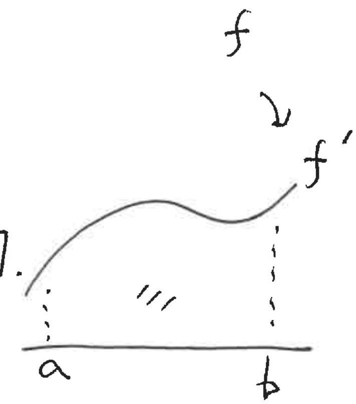
$\frac{1}{2}x^2 - \ln|x| + \frac{6}{-2}x^{-2} + 5\cdot\left(-\frac{2}{5}\right)x^{-\frac{5}{2}} + C$

(3) $\int \left(8\sqrt{x^3} + \sqrt[3]{x^2}\right) dx = \int 8x^{\frac{3}{2}} + x^{\frac{2}{3}}\, dx = 8\cdot\frac{2}{5}x^{\frac{5}{2}} + \frac{3}{5}x^{\frac{5}{3}} + C.$

Net Change Theorem

Consider a differentiable function $f(x)$ defined on $[a, b]$. FTC II implies that

$$\int_a^b f'(x)\, dx = \text{Net change of } f(x) \text{ on } [a,b].$$
$$= f(b) - f(a)$$



Net Change Theorem: The definite integral of the rate of change over $[a, b]$ gives the total change of f on $[a,b]$.

Examples of Net Change Calculations

Chemistry $\int_{t_1}^{t_2} \frac{d}{dt}[C]\, dt = C(t_2) - C(t_1)$ $\qquad \frac{d}{dt}[C]$ = concentration

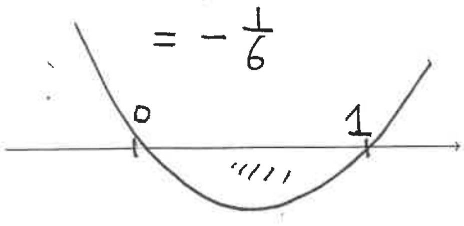
Mechanics $\int_a^b \rho(x)\, dx = m(b) - m(a).$ $\qquad \rho(x) =$ density $= m'(x)$

Compare the following integrals.

$\int_0^1 x^2 - x\, dx = \frac{1}{3}x^3 - \frac{1}{2}x^2\Big]_0^1$

$x(x-1)$ $\quad = \left(\frac{1}{3} - \frac{1}{2}\right) - (0 - 0)$

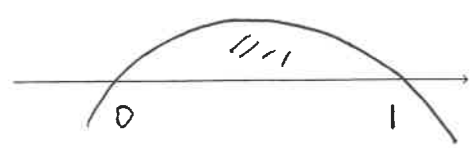
$= -\frac{1}{6}$



$\int_0^1 -(x^2 - x)\, dx = \int_0^1 -x^2 + x\, dx$

$-\frac{1}{3}x^3 + \frac{1}{2}x^2\Big]_0^1 = -\frac{1}{3} + \frac{1}{2} = \frac{1}{6}$

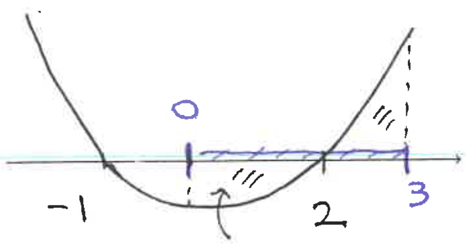
$-\left(-\frac{1}{6}\right) = \frac{1}{6}$

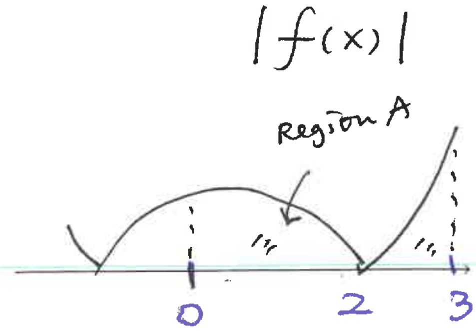


Example 2 Use subintervals to find the area of regions determined by $f(x)$.

(1) Given $f(x) = x^2 - x - 2$ on $[0, 3]$.

$(x-2)(x+1)$





Divide the interval $[0,3]$ into $[0,2]$ and $[2,3]$

$$\int_0^3 |f(x)|\,dx = \int_0^2 -f(x)\,dx + \int_2^3 f(x)\,dx$$

$$= \int_0^2 -(x^2-x-2)\,dx + \int_2^3 x^2-x-2\,dx = -\frac{1}{3}x^3+\frac{1}{2}x^2+2x\Big]_0^2$$

$$+ \frac{1}{3}x^3-\frac{1}{2}x^2-2x\Big]_2^3 = \left(-\frac{1}{3}2^3+\frac{1}{2}2^2+2\cdot 2\right)-0+\left(\frac{1}{3}3^3-\frac{1}{2}3^2-2\cdot 3\right)$$

$$-\left(\frac{1}{3}\cdot 2^3-\frac{1}{2}\cdot 2^2-2\cdot 2\right) = \underbrace{\left(-\frac{8}{3}+2+4\right)}_{\frac{10}{3}\ (A)} + \left(9-\frac{9}{2}-6\right)-\left(\frac{8}{3}-2-4\right)$$

$$-\frac{3}{2} + \frac{10}{3}\ (B)$$

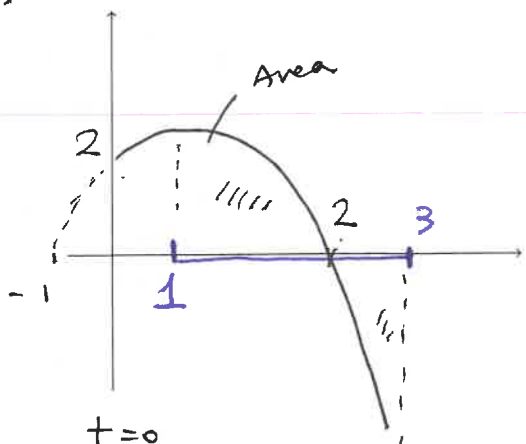
(2)

$$\frac{20-9}{6} = \frac{11}{6}$$

Example 3 The velocity function is given by $v(t) = -t^2 + t + 2$, $1 \le t \le 3$. Find the displacement and distance traveled by the particle over the given time interval.

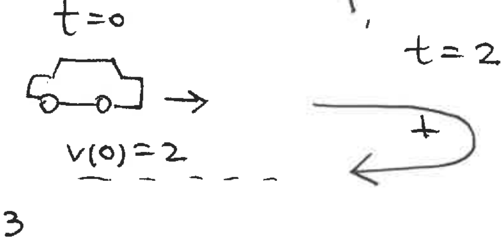
$$v(t) = -(t^2 - t - 2) = -(t-2)(t+1)$$

displacement is



$$\int_1^3 v(t)\,dt = \int_1^3 -t^2+t+2\,dt$$

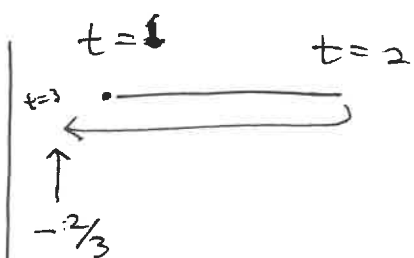
$$= -\frac{1}{3}t^3 + \frac{1}{2}t^2 + 2t \Big]_1^3$$



$$= \left(-\frac{1}{3}\cdot 3^3 + \frac{1}{2}\cdot 3^2 + 2\cdot 3\right) - \left(-\frac{1}{3}\cdot 1 + \frac{1}{2}\cdot 1 + 2\cdot 1\right)$$

$$= -9 + \frac{9}{2} + 6 + \frac{1}{3} - \frac{1}{2} - 2 = -9 + 4 + 6 - 2 + \frac{1}{3} = -1 + \frac{1}{3} = -\frac{2}{3}$$

For distance,



$$\int_1^3 |v(t)|\,dt = \int_1^2 v(t)\,dt + \int_2^3 -v(t)\,dt$$

$$= -\frac{1}{3}t^3 + \frac{1}{2}t^2 + 2t \Big]_1^2 + \frac{1}{3}t^3 - \frac{1}{2}t^2 - 2t \Big]_2^3 \qquad 9$$

$$\left(-\frac{1}{3}2^3 + \cancel{\frac{1}{2}\cdot 2^2}^{\,2} + 2\cdot 2\right) - \left(-\frac{1}{3}\cdot 1 + \frac{1}{2}\cdot 1 + 2\cdot 1\right) + \left(\cancel{\frac{1}{3}3^3} - \frac{1}{2}3^2 - 2\cdot 3\right)$$

$$-\left(\frac{1}{3}2^3 - \cancel{\frac{1}{2}\cdot 2^2}^{\,-6} - 2\cdot 2\right) = \left(-\frac{8}{3} + 6\right) - \left(2 + \frac{1}{6}\right) + \left(3 - \frac{9}{2}\right)$$

$$-\left(-6 + \frac{8}{3}\right) = \cdots = (6 - 2 + 3 + 6) + \left(-\frac{8}{3}\cdot 2 - \frac{1}{6} - \frac{9}{2}\right)$$

$$= 13 + \frac{-32-1-27}{6} = 13 - 10 = 3.$$