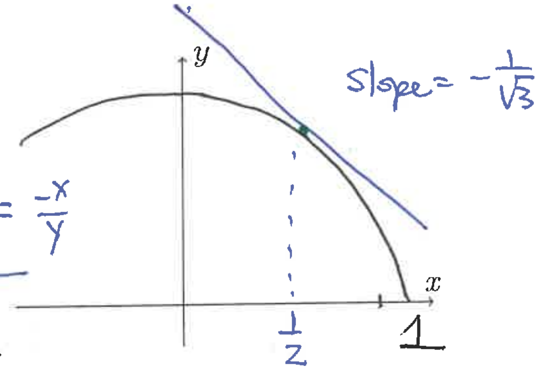
MATH 151 Section 3.5 Implicit Differentiation ($\Leftarrow$ Chain Rule)

Why do we need to differentiate Implicitly? Sometimes, functions are given implicitly.

The equation for the unit circle is $x^2+y^2=1$



So $y = \sqrt{1-x^2} = (1-x^2)^{\frac{1}{2}}$

$y' = \frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x) = \frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2}} = -\frac{x}{\sqrt{1-x^2}} = \frac{-x}{y}$

If $x = \frac{1}{2}$ then $y' \doteq -\frac{\frac{1}{2}}{\sqrt{1-\frac{1}{4}}} = -\frac{\frac{1}{\cancel{2}}}{\frac{\sqrt{3}}{\cancel{2}}} = -\frac{1}{\sqrt{3}}$

On the other hand, we use **the Chain Rule** to find y' as follows.

Step 1. The Chain Rule implies $\frac{d}{dx}(f(x))^2 = 2f(x)\cdot f'(x)$ or $\frac{d}{dx}y^2 = 2y\cdot y'$

We're taking derivative **with respect to** x.

We treat y as a function of x, i.e., $y = y(x)$ and then take derivative of each term w. r. t. x

$x^2+y^2=1 \Rightarrow 2x+2y\cdot y'=0 \Rightarrow 2y\cdot y' = -2x$

Step 2. Simplify or Solve the above for y' to get $y' = -\frac{\cancel{2}x}{\cancel{2}y} = -\frac{x}{y}$

Compare this y' with the above. They are the same

The above process is called Implicit Differentiation

We treat y as $y = y(x)$ then apply Chain Rule.

Example 1 Find $\frac{dy}{dx}$ or y' for the following equations.

(1) $x^2+4y^2=4$

$2x+4\cdot 2y\cdot y' = 0 \Rightarrow y' = \frac{-2x}{8y} = -\frac{1}{4}\frac{x}{y}$

(2) $\sqrt{x}+\sqrt{y}=1 \Rightarrow x^{\frac{1}{2}}+y^{\frac{1}{2}} = 1$

$\frac{1}{2}x^{-\frac{1}{2}}+\frac{1}{2}y^{-\frac{1}{2}}y' = 0 \Rightarrow y' = \frac{-x^{-\frac{1}{2}}}{y^{-\frac{1}{2}}} = -\frac{\sqrt{y}}{\sqrt{x}}$

(3) $y^3 = 2x^2 + y^4$

$$3y^2 y' = 4x + 4y^3 y' \Rightarrow (3y^2 - 4y^3)\, y' = 4x$$

$$y' = \frac{4x}{3y^2 - 4y^3}$$

(4) $e^y \cos x = 5 + \sin(xy)$

$$e^y y' \cos x + e^y(-\sin x) = 0 + \cos(xy) \cdot (xy)'$$

$$= 0 + \cos(xy)(1 \cdot y + x y')$$

$$e^y \cdot y' \cos x - x\cos(xy) \cdot y' = y\cos(xy) + e^y \sin x$$

$$y' = \frac{y\cos(xy) + e^y \sin x}{e^y \cos x - x\cos(xy)}$$

(5) $\sin(xy^2) + e^{y-x} = 9$ Exam II Fall 2022

$$\cos(xy^2)(xy^2)' + e^{y-x}(y-x)' = 0$$

$$\cos(xy^2)(1 \cdot y^2 + x \cdot 2y y') + e^{y-x}(y' - 1) = 0$$

$$\cos(xy^2) \cdot x \cdot 2y \cdot y' + e^{y-x} \cdot y' = e^{y-x} - \cos(xy^2)\, y^2$$

$$y' = \frac{e^{y-x} - \cos(xy^2) \cdot y^2}{\cos(xy^2) \cdot x2y + e^{y-x}}$$

Example 2 Find the tangent line to the curve $xe^y = x^2$ at the point ($1, 0$)

$$x = 1 \Rightarrow 1 \cdot e^y = 1 \Rightarrow e^y = 1 \Rightarrow y = 0$$

$$1 \cdot e^y + x \cdot e^y \cdot y' = 2x$$

$$\begin{cases} x = 1 \\ y = 0 \end{cases} \Rightarrow 1 \cdot e^0 + 1 \cdot e^0 \cdot y' = 2 \Rightarrow y' = 2 - 1 = 1$$

Eq'n $y - 0 = 1 \cdot (x - 1)$