MATH 151 Section 3.4 The Chain Rule

Find derivatives of the following compositions of functions   Power Rule

(1) $h(x) = (2x-1)^2 = (2x-1)(2x-1) = 4x^2 - 4x + 1$ | $((2x-1)^2)' = 2(2x-1)^1$

So $h'(x) = 8x - 4 = 2(4x-2) = 2(2x-1)\cdot 2$

(2) $h(x) = (3x+1)^2 = (3x+1)(3x+1) = 9x^2 + 6x + 1$ | $((3x+1)^2)' = 2(3x+1)$

So $h'(x) = 18x + 6 = 2(9x+3) = 2(3x+1)\cdot 3$

(3) $h(x) = (x^2+1)^2 = (x^2+1)(x^2+1) = x^4 + 2x^2 + 1$ | $((x^2+1)^2)' = 2(x^2+1)$

So $h'(x) = 4x^3 + 4x = 2(2x^3+2x) = 2(x^2+1)\cdot 2x$

(4) $h(x) = \sqrt{5x+1} = (5x+1)^{\frac{1}{2}}$

$h'(x) = \frac{1}{2}(5x+1)^{-\frac{1}{2}}\ 5$

$h(x) = \dfrac{3}{4x-1} = 3(4x-1)^{-1}$

$h'(x) = \boxed{-3(4x-1)^{-2}}\ 4$

Note that each function h is indeed compositions of functions, $h = f \circ g = f(g(x))$

For the derivative of $f(g(x))$, you need $f'(g(x))$ as well as an **extra term**, namely $g'(x)$

The Chain Rule

Suppose the derivatives $g'(x)$ and $f'(g(x))$ both exist. Then the composition $F = f \circ g = f(g(x))$ has the derivative

$$F'(x) = f'(g(x))\ g'(x)$$

$$\begin{tikzcd} x \arrow[r, "g"] \arrow[rr, bend right, "f\circ g"'] & g(x) \arrow[r, "f"] & f(g(x)) \end{tikzcd}$$

In Leibniz notation, if $y = f(u)$ and $u = u(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{df}{du}\,\frac{du}{dx}$$

$$\begin{tikzcd} x \arrow[r, "u"] \arrow[rr, bend right] & u = u(x) \arrow[r, "f"] & f(u) = y \end{tikzcd}$$

Is it really canceling? $\dfrac{df}{\cancel{du}}\,\dfrac{\cancel{du}}{dx} = \dfrac{df}{dx} = \dfrac{dy}{dx}$

NO and YES in Sect 3.10

for now

The Power Rule combined with the Chain Rule (Generalized Power Rule)

If n is a real number then

$$\frac{d}{dx}\left(g(x)\right)^n = n\left(g(x)\right)^{n-1} g'(x) \qquad \text{or} \qquad \frac{d}{dx}(\,u\,)^n = n\,u^{n-1}\frac{du}{dx}$$

Example 1 Differentiate the following functions.

(1) $y = (1+2x)^{10} = (1+2x)(1+2x)\cdots(1+2x)$

$$y' = 10(1+2x)^9\,2$$

(2) $y = \dfrac{1}{(1+2x)^{2024}} = (1+2x)^{-2024}$

$$y' = -2024(1+2x)^{-2025}\cdot 2$$

(3) $F(x) = \sqrt[3]{x^2+2x-3} = (x^2+2x-3)^{\frac{1}{3}}$

$$F'(x) = \frac{1}{3}(x^2+2x-3)^{-\frac{2}{3}}(2x+2)$$

(4) $f(x) = \dfrac{1}{\sqrt[4]{x^2+x+1}} = (x^2+x+1)^{-\frac{1}{4}}$

$$f'(x) = -\frac{1}{4}(x^2+x+1)^{-\frac{5}{4}}(2x+1)$$

(5) $y = (2x+1)^5(x^3-x+1)^4$

$$y' = \left((2x+1)^5\right)'\cdot(x^3-x+1)^4 + (2x+1)^5\left((x^3-x+1)^4\right)'$$

$$= 5(2x+1)^4\cdot 2\,(x^3-x+1)^4 + (2x+1)^5\,4\,(x^3-x+1)^3(3x^2-1)$$

(6) $g(x) = \sin 2x$

$$x \longrightarrow 2x \xrightarrow{\sin} \sin(2x)$$

$$g'(x) = \cos(2x)\cdot 2$$

(7) $f(x) = \sin\dfrac{1}{x}$

$$x \longrightarrow \underset{x^{-1}}{\frac{1}{x}} \xrightarrow{\sin} \sin\frac{1}{x}$$

$$f' = \cos\left(\frac{1}{x}\right)\cdot\left(-x^{-2}\right)$$

(8) $g(x) = \sin x^2 + \cos^2 x = \sin(\underline{x^2}) + (\underline{\cos x})^2$

$$g'(x) = \cos x^2 \cdot 2x + 2\cos x \cdot (-\sin x)$$

(9) $g(x) = \tan(\underline{\cos x})$

$$g'(x) = \sec^2(\cos x)(\cos x)'$$
$$= \sec^2(\cos x)(-\sin x)$$

(10) $g(x) = \tan x \cos x$

$$g'(x) = (\tan x)'\cos x + \tan x(\cos x)'$$
$$= \sec^2 x \cos x + \tan x(-\sin x)$$

(11) $G(t) = e^{7t\sin 2t}$

$$\underline{t \longrightarrow 7t \cdot \sin 2t} \xrightarrow{\exp} e^{7t\sin 2t}$$

$$(e^x)' = e^x$$

$$G'(t) = e^{7t\sin 2t} \cdot (7t \cdot \sin 2t)'$$
$$= e^{7t\sin 2t}(7 \cdot \sin 2t + 7t \cdot \cos 2t \cdot 2)$$

(12) $h(t) = \left(\frac{t^4+5}{t^2+5}\right)^3$

$$h'(t) = 3\left(\frac{t^4+5}{t^2+5}\right)^2 \left(\frac{t^4+5}{t^2+5}\right)'$$
$$= 3\left(\frac{t^4+5}{t^2+5}\right)^2 \left(\frac{4t^3(t^2+5) - (t^4+5)2t}{(t^2+5)^2}\right)$$

(13) $y = \sin^2(\cos 3x) = (\sin(\cos 3x))^2$

$$x \longrightarrow 3x \xrightarrow{\cos} \cos 3x \xrightarrow{\sin} \sin(\cos 3x) \xrightarrow{\text{square}} (\sin(\cos 3x))^2$$

$$y' = 2\sin(\cos 3x)(\sin(\cos 3x))'$$
$$= 2\sin(\cos 3x)\cos(\cos 3x)(\cos 3x)'$$
$$= 2\sin(\cos 3x)\cos(\cos 3x)(-\sin 3x)(3x)'$$
$$= 2 \text{ ——— '' ——— } (-\sin 3x) \cdot 3$$

Example 2 Find $h'(4)$ for $h(x) = \sqrt{4+5f(x)}$ where $f(4) = 9$ and $f'(4) = 7$

$$= (4+5f(x))^{\frac{1}{2}}$$

$$h'(x) = \frac{1}{2}\left(4+5f(x)\right)^{-\frac{1}{2}} \, 5f'(x)$$

$$h'(4) = \frac{1}{2}\left(4+5f(4)\right)^{-\frac{1}{2}} \, 5f'(4) = \frac{1}{2}(4+45)^{-\frac{1}{2}} \cdot 5 \cdot 7$$

$$= \frac{1}{2} \, \frac{1}{\cancel{7}} \cdot 5 \cdot \cancel{7} = \frac{5}{2}.$$

Derivative of $y = a^x$

For any $a > 0$, $e^{\ln a} = a$ (why?)

$$a \overset{\ln}{\longmapsto} \ln a \xrightarrow{\exp} e^{\ln a} = a$$

$\ln x$ is the inverse of e^x

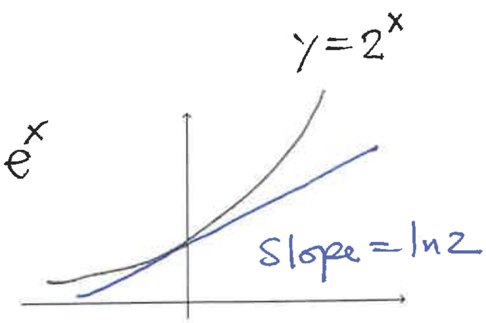
By the chain rule,

$$\frac{d}{dx} a^x = \frac{d}{dx}\left(e^{\ln a}\right)^x = \frac{d}{dx}\left(e^{\ln a x}\right)$$

$$= e^{(\ln a)x} \cdot \ln a = a^x \cdot \ln a$$

Theorem The derivative of exponential function $y = a^x$ is

$$(a^x)' = \ln a \; a^x, \quad (e^x)' = \ln e \cdot e^x = e^x$$



For example, the tangent line of $y = 2^x$ at $x = 0$ has the slope

$$(2^x)' = \ln 2 \; 2^x \Rightarrow \ln 2 \cdot 2^0 = \ln 2$$

Example 3 (1) $f(x) = 4^{\cos \pi x}$

$$x \longmapsto \cos \pi x \overset{4^x}{\longmapsto} 4^{\cos \pi x}$$

$$f'(x) = \ln 4 \cdot 4^{\cos \pi x} (\cos \pi x)' = \ln 4 \cdot 4^{\cos \pi x} (-\sin \pi x)\pi$$

(2) $f(x) = x^6 \, 4^{(x^6+2)}$

$$f' = 6x^5 \, 4^{(x^6+2)} + x^6 \ln 4 \cdot 4^{x^6+2} (6x^5)$$

What about the derivative of (x^x)? $(x^x)' = \ln x \cdot x^x$? No

Logarithmic Differentiation

Example 4 A table of values for f, g, f', and g' is given.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	9	4
2	1	8	8	-2
3	7	2	3	-7

(1) If $h(x) = f(g(x))$, find $h'(1)$

$$h'(x) = f'(g(x)) \cdot g'(x) \Rightarrow h'(1) = f'(g(1)) \cdot g'(1)$$
$$= f'(2) \cdot 4 = 8 \cdot 4 = 32.$$

(2) If $H(x) = g(f(x))$, find $H'(2)$.

$$H'(x) = g'(f(x))\, f'(x) \Rightarrow H'(2) = g'(f(2))\, f'(2)$$
$$= g'(1) \cdot 8 = 4 \cdot 8 = 32$$

Consider the cycle $(\cos x)' \to (\cos x)'' \to (\cos x)''' \to (\cos x)''''$

$-\sin x \quad -\cos x \quad \sin x \quad \cos x$

Quiz 4

Example 5 Find the 2023^{th} derivative of $y = \cos(2x)$.

$2020 + 3 = 2023$

$2^{2023} \sin 2x$

$$\left(e^{-x} f(x)\right)' = e^{-x}\left(-f(x) + f'(x)\right)$$

Example 6 Find the 2024^{th} derivative of $f(x) = xe^{-x}$.

$f'(x) = e^{-x}(-x+1)$, $f''(x) = e^{-x}(x-1-1)$

$f'''(x) = e^{-x}(-x+2+1)$ $f^{(4)}(x) = e^{-x}(x-3-1)$

$f^{(2023)}(x) = e^{-x}(-x+2023)$ $f^{(2024)}(x) = e^{-x}(x-2024)$

Example 7 Find $\dfrac{d^{2025}}{dx^{2025}}\left(\dfrac{1}{x}\right)$.

$f(x) = x^{-1}$ $f'(x) = -x^{-2}$, $f''(x) = 2 \cdot x^{-3}$

$f'''(x) = -3 \cdot 2 \cdot 1\, x^{-4}$, $f^{(4)} = 4 \cdot 3 \cdot 2 \cdot 1\, x^{-5}$

$f^{(2025)}(x) = -2025!\, x^{-2026}$

$1 \cdot 2 \cdot 3 \cdot \cdots \cdot 2025$