Finding Optimal Triadic Transformational Spaces with Dijkstra's Shortest Path Algorithm

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Abstract. This paper presents a computational approach to a particular theory in the work of Julian Hook—Uniform Triadic Transformations (UTTs). A UTT defines a function for transforming one chord into another, and is useful for explaining triadic transitions that circumvent traditional harmonic theory. By combining two UTTs and extrapolating, it is possible to create a two-dimensional chord graph. Meanwhile, graph theory has long been studied in the field of Computer Science. This work describes a software tool which can compute the shortest path between two points in a two-dimensional transformational chord space. Utilizing computational techniques, it is then possible to find the optimal chord space for a given musical piece. The musical work of Michael Nyman is analyzed computationally, and the implications of a weighted chord graph are explored.

Keywords: Neo-Riemannian Theory, Shortest Path Algorithm, Transformational Spaces, Computational Musicology

1 Introduction

Innovation has always been the impetus for further innovation; the creation of new concepts and new ideas immediately demands new methods of analysis, and new tools for that analysis. Contemporary Art Music is no exception. Indeed, innovative compositions of the late 20^{th} century have thwarted the methods of analysis on which many theorists rely. In the late 1980s, a new sub-discipline of music theory arose out of the need to describe triadic and intervallic motions irrespective of a common key tonic. This sub-discipline has been labelled as transformational theory [7], and has inspired a handful of different theories that are concerned with triadic and intervallic relationships outside of the more common diatonic framework—those that relate pitch-based units directly by transforming one into the other.

The somewhat narrower field of neo-Riemannian theory is similarly focused on the transformation of musical objects, specifically triads. Neo-Riemannian theory encapsulates the work of Richard Cohn [2], and more recently Julian

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Hook [6]. All of these theorists are concerned with the relationships between musical objects independent of a centralized tonic reference. Together, the theories have been expanded to analyze composers in different eras, for example Milton Babbitt, Béla Bartók and György Ligeti. [8,5,1].

For the purposes of this research, the definition of an optimal chord space is the transformational space which most efficiently represents the transitions of the triadic sequence of an entire piece. Since there are many possible harmonic spaces, this paper seeks to find the optimal space for a given piece. It is a logical extension, then, to employ one of the most effective tools in computational analysis to analyse the possible harmonic spaces of new music: the computer. Specifically, this essay will outline the application of UTTs to the first three pieces of Michael Nyman's *Six Celan Songs*, and a computational approach to discovering the most appropriate space.

1.1 Uniform Triadic Transformations

In Lewin's Generalized Musical Intervals and Transformations [7], he defines sets of pitches (labelled "spaces"), as well as corresponding groups of transformations. Based on Lewin's foundational work, Julian Hook similarly uses transformational spaces to define movement between triads [6]. Hook developed the concept of Uniform Triadic Transformations (UTT), which is based on Riemann's fundamental idea of major and minor triads' being mirror images of one another. The format for defining a UTT is <+/−, m, n>. The plus / minus sign defines whether or not the resulting triad will be the same (+) or the opposite (-) type of triad, and the m and n variables define the movement of the triad's root for the major and minor subsets, respectively [3].

1.2 Song Harmonies

Nyman's Six Celan songs are very triadic in their harmonic structure, but also don't follow standard tonal syntax. Thus, they are a good starting point for the utilization of digital UTT spaces. Most of the harmonic content in Nyman's songs have relatively regular durations, which is ideal since the UTT space does not consider durations of the triads. One drawback of the UTT space is that only major or minor triads can be considered. Therefore, if certain parts of the piece contain more complicated harmonies than simple triads, they must be simplified to the closest approximation of a major or minor triad. This is an unfortunate drawback, however most of the content in the pieces are simple triads, so the majority of the harmonies will be represented correctly.

The reason this piece is so relevant is that there are chord movements which cannot be ascribed to traditional forms of functional harmony. For example, the opening progression is Dm→ Fm→Am. The harmony rests on Am for more than 1 measure, which suggests that the tonic is Am. However, this movement could only be described as a iv→vi→i movement in harmonic minor, which is not a very common nor functional progression. We select the first three songs from

Nyman's Six Celan songs as a proof of concept for the new optimal chord space method.

2 Finding the Optimal Harmonic Space of a Song

Hook describes certain groups of UTTs that have a unique transformation for every triadic transition in the group. These are called *simply transitive* groups, and Hook explains that "Simply transitive groups also bring a measure of clarity to situations involving apparently redundant transformations"[6]. Since our approach is statistical—that is, we are looking for the most repeated transformations of a particular piece—a simply transitive group would be a good starting point. We consider the simply transitive group K(1,1) as defined by Hook, shown in Figure 1. Here, "T" and "E" represent the integers 10 and 11.

This subgroup was also the focus of analysis in [3]. In that research, Cook assembles his own toroidal space by creating a 2-dimensional graph for which each axis represents the repeated application of a particular UTT in K(1,1). Cook measures distance on that 2-dimensional space as the total number of transformations required to traverse the minimum path from one triad to another on that triadic graph. The approach presented here will be much the same.

Fig. 1: The UTTs which comprise the K(1,1) simply transitive subgroup.

Mode Preserving:			Mode Reversing:		
<+, 0, 0>	<+, 4, 4>	<+, 8, 8>	<–, 0, 1>	<–, 4, 5>	<–, 8, 9>
<+, 1, 1>	<+, 5, 5>	<+, 9, 9>	<–, 1, 2>	<–, 5, 6>	<–, 9, T>
<+, 2, 2>	<+, 6, 6>	<+, T, T>	<–, 2, 3>	<–, 6, 7>	<–, T, E>
<+, 3, 3>	<+, 7, 7>	<+, E, E>	<–, 3, 4>	<–, 7, 8>	<–, E, 0>

To select a pair of UTTs from the K(1,1) we look to the shortest path algorithm. Conceived in 1956 by Edsger Dijkstra, the shortest path algorithm will find the shortest path between two vertices in a graph, as long as a path exists between the two vertices [4]. Intuitively, the pair of UTTs that best represent a sequence of triads should have the shortest cumulative distance between each pair of triads in the sequence. This is a purely computational challenge. We have a list of UTTs provided. For every UTT, we must do the following procedure[1]:

1. Loop through every UTT that is not identical to the current UTT.
2. Create the 2-dimensional UTT space that represents all the triads that can possibly be visited given the two current transformations.
   a. Ensure that the created space contains every triad in the set of all major and minor triads.

[1] The code for the generation of UTT spaces can be found at Ryan Groves' github landing page at http://github.com/bigpianist/UTTSpaces.

3. Step through the triads in the given piece, and compute the shortest distance between each pair of triads, sequentially.
4. Add up the total number of transformations that were required for every triadic transition in the piece, based on its path in the 2-D space.

Every possible space created from pairs in the K(1,1) subgroup was computed for each piece, and the distances between each triadic transition computed. The most efficient spaces for the three analyzed Nyman songs are:

Song Title	Optimal UTT Pair	Dist.
Chanson einer Dame im Schatten	(<-, 4, 5>, 1, A), (<+, 2, 2>, 1, B)	246
Es war Erde in ihnen	(<-, 1, 2>, 1, A), (<-, 2, 3>, 1, B)	172
Psalm	(<-, 1, 2>, 1, A), (<+, 2, 2>, 1, B)	107

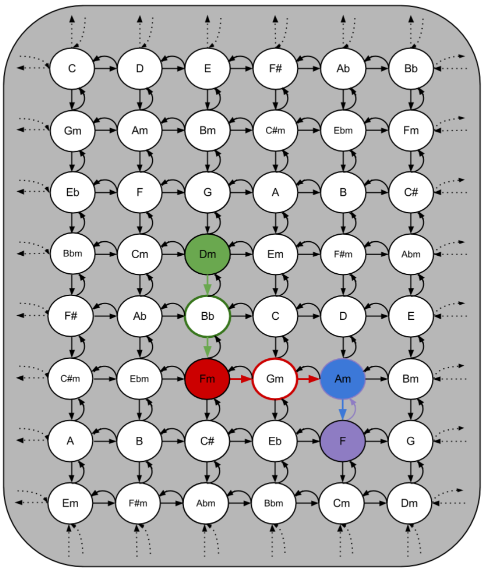
3 Application and Investigation

The efficient spaces discovered for each song can then be used to explore the transformational motion going on in each song. Take the first song, *Chanson einer Dame im Schatten*, for example. First, it is useful to also visualize the space.

In Figure 2, the opening phrase of *Chanson einer Dame im Schatten* is shown with its transformations in the UTT space of [(<-, 4, 5>, 1, A), (<+, 2, 2>, 1, B)].

It is difficult to tell how effective this new space is at first glance. However, with deeper inspection, one can start to understand the effectiveness of representing the harmony in such a way. Take, for example, the ending of most phrases. Often, each chord phrase ends in two successive chords, often on the last two beats of the last bar of the phrase. The opening phrase defines this clearly with its F→Am movement. Given the quickness of the transition between these two chords, it seems that an accurate harmonic space would likely find these chords to be

Fig. 2: UTT space of [(<-, 4, 5>, 1, A), (<+, 2, 2>, 1, B)], with the transformations of the opening chord sequence [Dm, Fm, Am, F, Am] shown in alternating colors.



close in harmonic distance. Grouping each of these quick transitions with their corresponding transformation(s) show that this is, indeed, the case (Table 1).

Two things are interesting about the transformations that the phrase-ending chord transitions create. For one, they are mostly a single transformation. The transitions with a single transformation also represent single transformations in *both* axes of the UTT space (i.e., there exists both a single ‘A’ transformation, as well as a single ‘B’ transformation), which implies that both UTTs are effective. Secondly, these transformations occur from different starting points, showing that it is not just the same chord sequence occurring repeatedly in the song.

The UTT space for *Es war Erde in ihnen* is necessarily an approximation because of its large quantity of triad simplifications (roughly 20% were augmented or diminished, and had to be converted to major and minor triads, respectively). Thus, it is more useful to consider the space of *Psalm*, and to investigate the commonalities between the space of *Psalm* and *Chanson einer Dame im Schatten.* One of the most striking characteristics of the transformations of the chord transitions in *Psalm* is the combination of three identical transformations. For example, for a single chord transition in the song, the combination of three ‘B’ (<+, 2, 2>) transformations in sequence, or the inversely equivalent transformation of three ‘B^{-1}’ transformations in sequence occurs a total of 8 times in the song. Similarly, the total number of times the combination of three ‘A’ transformations (‘A|A|A’) or three ‘A^{-1}’ transformations (‘$A^{-1}|A^{-1}|A^{-1}$’) occurs is also 8. This is very convincing evidence that the UTT space is appropriate, since similar movements in both axes are equally likely. Furthermore, simple movements of ‘A’ and ‘A^{-1}’ occur a total sum of 13 times. The exception is the simple movement of a single ‘B’ or a single ‘B^{-1}’ transformation, both of which are completely missing from the song.

Table 1: Table of quick, phrase-ending chord transitions with their corresponding transformations

Start Chord	End Chord	Transformation
F	Am	A
Am	C	$B^{-1}\|A$
Eb	Gm	A
Gm	Eb	A^{-1}
Eb	F	B
Dm	F	$B^{-1}\|A$
Am	Em	A^{-1}
Bb	Dm	A

Still, there is a distinct motion of three consecutive, identical transformations in both the ‘A’ and ‘B’ axes for the UTT space [(<-, 1, 2>, 1, A), (<+, 2, 2>, 1, B)] in *Psalm.* Let us consider the UTT created when applying the UTT of <-, 1, 2> three times consecutively. From a major chord, the root movement will be 1, then 2 (since the mode was reversed), then 1 again, for a total of 4. Starting on a minor chord, the root movement will be 2, then 1, then 2, for a total of 5, with the mode changing three times (which is equivalent to one reversal). The resulting UTT is exactly that from *Chanson einer Dame im Schatten*, <-, 4, 5>!

4 Conclusion

These new methods for transforming triads and intervals provide a new perspective on the possible harmonic spaces in which songs may have been composed. Furthermore, computational models of harmonic spaces provide immense power when considering multiple spaces. Commonly, the approach for utilizing these new transformational spaces has been a top-down approach, where the musicologist would identify what she thought were the most important harmonic movements, and shape the harmonic space around those. The computational method, on the other hand, provides a bottom-up approach, where every transition is considered when deciding on an optimal space. However, one must not rely entirely on this new tool set; still, there is a need for the application of musical intuition. As was evident in the analysis of the first three of Nyman's *Six Celan Songs*, the shortest distance algorithm found the most efficient harmonic space for each song. In the end, it was still up to the analyst to investigate the possible consequences of the identified spaces.

There is no doubt that the computer can serve music theorists for representing complex and abstract concepts. The current state of computer technologies affords a wide range of tools for the application of a computational approach to even the most cutting-edge of musical theories. It is with these tools that one can gain a new perspective on the music, and lead music analysts to new theories and innovative insights.

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