1.0 MEASUREMENT

Physics is a science based upon exact measurement of physical quantities. Therefore it is essential that student first becomes familiar with the various methods of measurement and the units in which these measurements are expressed.

A **unit** is a value quantity or magnitude in terms of which other values, quantities or magnitudes are expressed.

1.1 FUNDAMENTAL QUANTITIES AND UNITS

A **fundamental quantity** also known as **base quantity** is a quantity which cannot be expressed in terms of any other physical quantity. The units in which the fundamental quantities are measured are called fundamental units. In mechanics (study of the effects of external forces on bodies at rest or in motion), the quantities **length**, **mass** and **time** are chosen as fundamental quantities.

Fundamental Quantity	Fundamental Unit	Unit Symbol
Length	Meter	m
Mass	Second	s
Time	Kilogram	kg

1.2 SYSTEM OF UNITS

The following systems of units have been in use –

(i) The French or C.G.S (Centimeter, Gramme, Second) System;
(ii) The British or F.P.S (Foot, Pound, Secons) System;
(iii) The M.K.S (Metre, Kilogram, Second) System; and
(iv) The S.I. (International System of Units).

1.3 THE INTERNATIONAL SYSTEM OF UNITS (S.I.)

The S.I. is the latest version of the system of units and only system likely to be used all over the world. This system consists of **seven** base or fundamental units from which we can derive other possible quantities of science. They are

S.No	**Physical Quantity**	**Unit**	**Unit Symbol**
1.	Length	Meter	M
2.	Mass	Kilogram	Kg
3.	Time	Second	S

4.	Electric Current	Ampere	A
5.	Temperature	Kelvin	K
6.	Amount of Substance	Mole	Mol
7.	Luminous Intensity	Candela	Cd

1.4 CONCEPT OF DIMENSION

Dimension of a physical quantity simply indicates the physical quantities which appear in that quantity and gives absolutely no idea about the magnitude of the quantity. In mechanics the length, mass and time are taken as the three base dimensions and are expressed by as letter [L], [M] and [T] respectively. Hence, a formula which indicates the relation between the derived unit and the fundamental units is called **dimensional formula**.

Example 1: Deduce the dimensional formula for the following physical quantities: (a) Velocity, (b) acceleration, (c) force, (d) pressure, (e) work, and (f) power.

Solution:

$$(a)\ Dimension\ of\ Velocity = \frac{Dimension\ of\ length}{Dimension\ of\ time} = \frac{[L]}{[T]} = [LT^{-1}]$$

$$Hence\ the\ dimnesional\ formula\ for\ velocity\ will\ be\ [M^0LT^{-1}]$$

$$(b)\ Dimension\ of\ acceleration = \frac{Dimension\ of\ velocity}{Dimension\ of\ time} = \frac{[LT^{-1}]}{[T]} = [LT^{-2}]$$
$$= [M^0LT^{-2}]$$

$$(c)\ Dimension\ of\ force = Dimension\ of\ mass\ \times Dimension\ of\ acceleration$$
$$= [M] \times [LT^{-2}] = [MLT^{-2}]$$

$$(d)\ Dimension\ of\ Pressure = \frac{Dimension\ of\ Force}{Dimension\ of\ Area} = \frac{[MLT^{-2}]}{[L^2]} = [ML^{-1}T^{-2}]$$

Students to attempt (e) and(f).

Example 2: Deduce the dimensional formula for (a) modulus of elasticity(Y), and (b) coefficient of viscosity(η).

Solution:

$$(a)\ Y = \frac{Stress}{Strain} = \frac{Force/Area}{Change\ in\ length/Original\ length}$$
$$= \frac{Dimension\ of\ force \times Dimension\ of\ length}{Dimension\ of\ Area\ \times Dimension\ of\ length} = \frac{[MLT^{-2}] \times [L]}{[L^2] \times [L]}$$
$$= [ML^{-1}T^{-2}]$$

(b) *the coefficient of viscosity* (η) *of a liquid is defined as tangential force required per unit area to maintain unit velocity gradient between two layers of the liquid unit distance apart.*

$$\eta = \frac{F}{A}\frac{1}{(dV/dx)}$$
$$Dimension\ of\ \eta = \frac{Dimension\ of\ Force\ \times Dimension\ of\ distance}{dimension\ of\ Area\ \times Dimension\ of\ Velocity}$$
$$= \frac{[MLT^{-2}] \times [L]}{[L^2] \times [LT^{-1}]} = \frac{[ML^2T^{-2}]}{[L^3T^{-1}]} = [ML^{-1}T^{-1}]$$

1.4 USES OF DIMENSIONAL EQUATIONS

(a) *To check the homogeneity of a derived physical equation (i.e. to check the correctness of a physical equation).*

Example 3:

Show that the following relation for the time period of a body executing simple harmonic motion is correct.

$$T = 2\pi\sqrt{\frac{l}{g}}$$

where l and g are the displacement and acceleration due to gravity respectively.

Solution:

For the relation to be correct, the dimension of L.H.S must be equal to dimension of R.H.S

$$Dimension\ of\ L.H.S = M^0L^0T^1\ or\ T$$
$$Dimension\ of\ R.H.S = \frac{[Dimension\ of\ l]^{1/2}}{[Dimension\ of\ g]^{1/2}} = \frac{[L]^{1/2}}{[LT^{-2}]^{1/2}} = \frac{[L]^{1/2}}{[L]^{1/2}T^{-1}}$$
$$= \frac{1}{T^{-1}} = T$$

Since $Dimension\ of\ L.H.S = Dimension\ of\ R.H.S,$ the relation is dimensionally correct.

(b) *To derive a relationship between different physical quantities.*

Example 4:
If the frequency f of a stretched string depends upon the length l of the string, the tension T in the string and the mass per unit length λ of the string. Establish a relation for the frequency using the concept of dimension.

Solution:

$$f \propto l^x T^y \lambda^z \qquad \ldots \qquad (1)$$
$$f = k l^x T^y \lambda^z \qquad \ldots \qquad (2)$$
$$[Dimension\ of\ f] = [Dimension\ of\ l]^x \times [Dimension\ of\ T]^y \times [Dimension\ of\ \lambda]^z$$
$$[M^0 L^0 T^{-1}] = [L]^x \times [MLT^{-2}]^y \times [ML^{-1}]^z$$
$$[M]^0[L]^0[T]^{-1} = [M]^{y+z}[L]^{x+y-z}[T]^{-2y}$$
$$Equating\ the\ powers$$
$$y + z = 0 \qquad \ldots \qquad (3)$$
$$x + y - z = 0 \qquad \ldots \qquad (4)$$
$$-2y = -1 \qquad \ldots \qquad (5)$$
$$from\ equation\ (5), \qquad y = \frac{1}{2} \qquad \ldots \qquad (6)$$
$$substituting\ (6) into\ (3), we\ have$$
$$\frac{1}{2} + z = 0$$
$$z = -\frac{1}{2} \qquad \ldots \qquad (7)$$
$$substituting\ (6)\ and\ (7)\ into\ (4)$$
$$x + \frac{1}{2} - \left(-\frac{1}{2}\right) = 0$$
$$x = -1 \qquad \ldots \qquad (8)$$
$$substituting\ (6), (7)\ and\ (8) into\ (2), we\ have$$
$$f = k l^{-1} T^{1/2} \lambda^{-1/2}$$
$$f = k \frac{1}{l} \sqrt{\frac{T}{\lambda}} \qquad \ldots (*)$$

Equation $(*)$ is the required relation.

(c) *To derive the unit of a Physical Quantity*

Example 5:
The viscous drag F between two layers of liquid with surface area of contact A in a region of velocity gradient dv/dx is given by

$$F = \eta A\, dv/dx$$

where η is the coefficient of viscosity of the liquid. Obtain the unit for η.

Solution:

$$F = \eta A\, dv/dx$$
$$Making\ \eta\ the\ subject\ of\ the\ formula$$
$$\eta = \frac{F}{A}\frac{1}{dv/dx}$$
$$Dimension\ of\ force\ = [MLT^{-2}]$$
$$Dimension\ of\ surface\ area = [L^2]$$
$$Dimension\ of\ velocity = [LT^{-1}]$$
$$Dimension\ of\ displacement = [L]$$
$$Dimension\ of\ \eta = \frac{[MLT^{-2}][L]}{[L^2][LT^{-1}]} = \frac{ML^2T^{-2}}{L^3T^{-1}} = ML^{-1}T^{-1}$$
$$\therefore Unit\ of\ \eta = kgm^{-1}s^{-1}$$

1.5 LIMITATIONS OF DIMENSIONAL ANALYSIS

(i) The method does not provide any information about the magnitude of dimensionless variables and dimensionless constants.
(ii) The method cannot be used if the quantities depend upon more than three-dimensional quantities: M,L and T.
(iii) The method is not applicable if the relationship involves trigonometric, exponential and logarithmic functions.

EXERCISE

1) Show on the basis of dimensional analysis that the following relations are correct:
   a. $v^2 - u^2 = 2aS$, where u is the initial velocity, v is final velocity, a is acceleration of the body and S is the distance moved.
   b. $\rho = 3g/4rG$ where ρ is the density of earth, G is the gravitational constant, r is the radius of the earth and g is acceleration due to gravity.
2) The speed of sound v in a medium depends on its wavelength λ, the young modulus E, and the density ρ, of the medium. Use the method of dimensional analysis to derive a formula for the speed of sound in a medium. (Unit for Young Modulus E: $kgm^{-1}s^{-2}$)

2.0 VECTORS

Physical quantities can generally be classified as (i) Scalars and (ii) Vectors.

Scalars are physical quantities which possess only magnitude and no direction in space. Examples are mass, time, temperature, volume, speed etc. On the other hand, **Vectors** are physical quantities which have both magnitude and direction in space. Examples are force, velocity, acceleration, etc.

2.1 RESOLUTION OF VECTORS

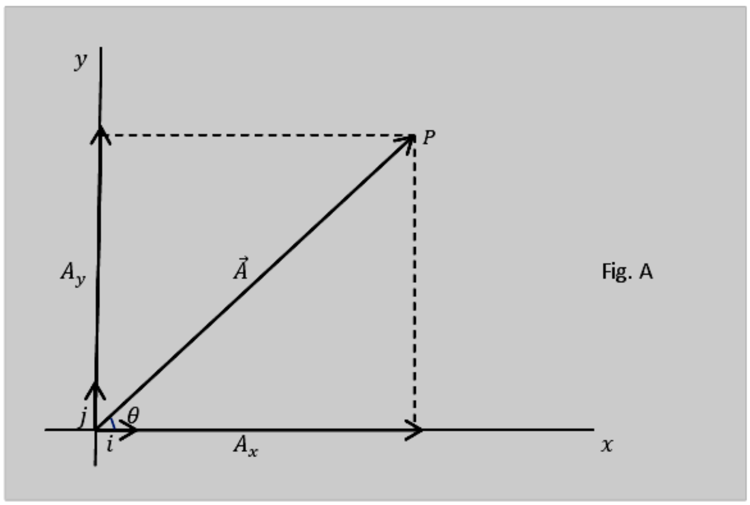


Fig. A

A two dimensional vector can be represented as the sum of two vectors. Consider figure A above, the vector $\vec{A}$ can be expressed as

$$\vec{A} = \vec{A}_x + \vec{A}_y \; or \; \vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} \quad \ldots \quad 2.1$$

Vectors $\vec{A}_x$ and $\vec{A}_y$ are called vector components of $\vec{A}$. $\hat{\imath}$ and $\hat{\jmath}$ are unit vectors along the x axis and y axis respectively. A **unit vector** is a vector that has a magnitude of exactly 1 and specify a particular direction.

Let θ be the angle which the vector $\vec{A}$ makes with the positive x-axis, then we have

$$A_x = A\cos\theta \; and \; A_y = A\sin\theta \quad \ldots \quad 2.2$$

$$Magnitude \; of \; vector \; A = |\vec{A}| = A = \sqrt{A_x^2 + A_y^2} \quad \ldots \quad 2.3$$

$$\tan\theta = \frac{A_y}{A_x} \quad \ldots \quad 2.4$$

In three dimensions, a vector $\vec{A}$ can be expressed as

$$\vec{A} = \vec{A}_x + \vec{A}_y + \vec{A}_z \ or\ \vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k} \qquad \dots \qquad 2.5$$

Here, the magnitude is expressed as

$$|\vec{A}| = A = \sqrt{A_x^2 + A_y^2 + A_z^2} \qquad \dots \qquad 2.6$$

2.2 ADDITION/SUBTRACTION OF VECTORS BY COMPONENTS

Example:

If $\vec{A} = 2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$ and $\vec{B} = \hat{\imath} + 4\hat{\jmath} + 5\hat{k}$. Find the magnitude of $(\vec{A} + \vec{B})$ and $(\vec{A} - \vec{B})$.

Solution:

$$(\vec{A} + \vec{B}) = (2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}) + (\hat{\imath} + 4\hat{\jmath} + 5\hat{k}) = 3\hat{\imath} + 7\hat{\jmath} + 9\hat{k}$$
$$(\vec{A} - \vec{B}) = (2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}) - (\hat{\imath} + 4\hat{\jmath} + 5\hat{k}) = \hat{\imath} - \hat{\jmath} - \hat{k}$$

Hence their magnitudes will be

$$|\vec{A} + \vec{B}| = \sqrt{(3)^2 + (7)^2 + (9)^2} = \sqrt{139}$$
$$|\vec{A} - \vec{B}| = \sqrt{(1)^2 + (-1)^2 + (-1)^2} = \sqrt{3}$$

2.3 MULTIPLICATION OF VECTORS

(i) **Dot Product or Scalar Product**

The scalar product of two vectors is defined as the product of the magnitude of two vectors and the cosine of the smaller angle between them.

$$\vec{A}.\vec{B} = |\vec{A}||\vec{B}|\cos\theta = AB\cos\theta = S$$

where S is a scalar quantity.

Example:

Find the angle between the two vectors $\vec{A} = 3\hat{\imath} + 4\hat{\jmath} + 5\hat{k}$ and $\vec{B} = 3\hat{\imath} + 4\hat{\jmath} - 5\hat{k}$.

Solution:

$$\vec{A}.\vec{B} = |\vec{A}||\vec{B}|\cos\theta$$
$$\vec{A}.\vec{B} = A_xB_x + A_yB_y + A_zB_z = 3\times 3 + 4\times 4 + 5(-5) = 0$$
$$|\vec{A}| = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50}$$
$$|\vec{B}| = \sqrt{3^2 + 4^2 + (-5)^2} = \sqrt{50}$$
$$\cos\theta = \frac{\vec{A}.\vec{B}}{|\vec{A}||\vec{B}|} = \frac{0}{\sqrt{50}\times\sqrt{50}} = 0$$
$$\theta = 90^o$$

(ii) **Cross Product or Vector product**

The vector product of two vectors is defined as a vector having a magnitude equal to the product of the magnitudes of the two vectors and the sine of the angle between them and is in the direction perpendicular to the plane containing the two vectors. Thus if $\vec{A}$ and $\vec{B}$ are two vectors, then their vector product (or cross product), written as $\vec{A} \times \vec{B}$, is a vector $\vec{C}$ defined as

$$\vec{C} = \vec{A} \times \vec{B} = |\vec{A}||\vec{B}| sin\theta n$$

where θ is the angle between the two vectors and n is the unit vector perpendicular to the plane of vectors $\vec{A}$ and $\vec{B}$.

3.0 KINEMATICS

Kinematics is the study of the motion of objects without referring to what causes the motion. Motion is a change in position in a time interval.

3.1 MOTION IN ONE DIMENSION/MOTION ALONG A STRAIGHT LINE

A straight line motion or one-dimensional motion could either be vertical (like that of a falling body), horizontal, or slanted, but it must be straight.

3.1.1 POSITION AND DISPLACEMENT

The **position** of an object in space is its location relative to some reference point, often the origin (or zero point). For example, a particle might be located at $x = 5m$, which means the position of the particle is $5m$ in positive direction from the origin. Meanwhile, the **displacement** is the change from one position x_1 to another position x_2. It is given as

$$\Delta x = x_2 - x_1 \quad ... \quad (3.1)$$

3.1.2 AVERAGE VELOCITY AND AVERAGE SPEED

The average velocity is the ratio of the displacement Δx tht occurs during a particular time interval Δt to that interval. It is a vector quantity and is expressed as:

$$v_{avg} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1} \quad .... \quad (3.2)$$

$$or$$

$$v_{avg} = \frac{\Delta y}{\Delta t} = \frac{y_2 - y_1}{t_2 - t_1} \quad .... \quad (3.2)$$

The average speed is the ratio of the total distance covered by a particle to the time. It is a scalar and is expressed as:

$$S_{avg} = \frac{total\ distance}{\Delta t} \quad \ldots \quad (3.3)$$

3.1.3 INSTANTANEOUS VELOCITY AND SPEED

The instantaneous velocity describes how fast a particle is moving at a given instant. It is expressed as:

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} \quad \ldots \quad (3.4)$$

Speed is the magnitude of instantaneous velocity; that is, speed is velocity that has no indication of direction either in words or via an algebraic sign. For example, a velocity of $+5m/s$ or $-5m/s$ is associated with a speed of $5m/s$.

Example 6:
The position of a particle moving on an x axis is given by $x = 7.8 + 9.2t - 2.1t^3$, with x in meters and t in seconds. What is its velocity at $t = 3.5s$? Is the velocity constant, or is it continuously changing?

Solution:

$$v = \frac{dx}{dt} = \frac{d(7.8 + 9.2t - 2.1t^3)}{dt} = 0 + 9.2 - (3)(2.1)t^2$$

$$v = 9.2 - 6.3t^2 \quad (*)$$

At $t = 3.5s$,

$$v = 9.2 - 6.3(3.5)^2 = -\frac{68m}{s}$$

$since\ eqn\ (*)\ involves\ t, the\ velocity\ v\ depends\ on\ time\ (t) and\ so\ is\ continuously\ changing.$

3.1.4 ACCELERATION

When a particle's velocity changes, the particle is said to undergo acceleration. The **average acceleration** a_{avg} over a time interval Δt is

$$a_{avg} = \frac{\Delta v}{\Delta t} = \frac{v_2 - v_1}{t_2 - t_1} \quad \ldots \quad 3.5$$

where the particle has velocity v_1 at time t_1 and then velocity v_2 at time t_2. The **instantaneous acceleration** (or simply acceleration) is the derivative of velocity with respect to time:

$$a = \frac{dv}{dt} \quad \ldots \quad 3.6$$

Equation 3.6 can also be written as

$$a = \frac{d}{dt}(v) = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} \quad \ldots \qquad 3.7$$

In words, the acceleration of a particle at any instant is the second derivative of its position $x(t)$ with respect to time.

Example 7:
A particle's position on the x-axis is given by $x = 4 - 27t + t^3$, with x in meters and t in seconds. (a) find the particle's velocity function $v(t)$ and acceleration $a(t)$. (b) Is there ever a time when $v = 0$?

Solution:

(a) $$v = \frac{dx}{dt} = \frac{d}{dt}(4 - 27t + t^3) = -27 + 3t^2$$

$$a = \frac{dv}{dt} = \frac{d}{dt}(-27 + 3t^2) = 6t$$

(b) $$0 = -27 + 3t^2$$

$$t^2 = {}^{27}\!/_{3} = 9$$

$$t = 3s$$

3.2 MOTION IN TWO AND THREE DIMENSIONS

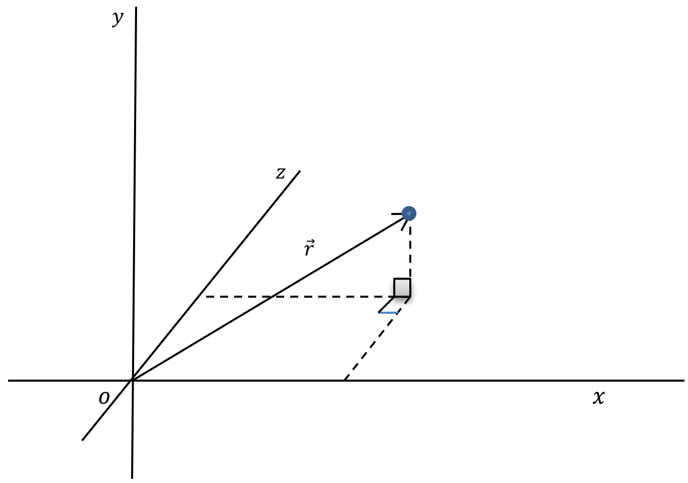
The concept of position, velocity and acceleration for motion along a straight line is similar to that two and three dimension, but more complex.

3.2.1 Position and Displacement

The **position vector** $\vec{r}$ is used to specify the position of a particle in space. It is generally expressed in the unit vector notation as

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \quad \ldots \qquad (3.8)$$

where $x\hat{\imath}, y\hat{\jmath}$ and $z\hat{k}$ are the vector components of $\vec{r}$, and the coefficients x, y, and z are its scalar components.



If a particle moves from point 1 to point 2 during a certain time interval, the position vector changes from $\vec{r}_1$ to $\vec{r}_2$, then the particle's **displacement** is

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1 \qquad \ldots \qquad (3.9)$$

Example 8:
The position vector of a particle is initially $\vec{r}_1 = (-3m)\hat{\imath} + (2m)\hat{\jmath} + (5m)\hat{k}$ and then later is $\vec{r}_2 = (9m)\hat{\imath} + (2m)\hat{\jmath} + (8m)\hat{k}$. What is the particle's displacement $\Delta\vec{r}$ from $\vec{r}_1$ to $\vec{r}_2$?

Solution:

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1 = [9-(-3)]\hat{\imath} + [2-2]\hat{\jmath} + [8-5]\hat{k}$$
$$\Delta\vec{r} = (12m)\hat{\imath} + (3m)\hat{k}$$

Example 9:
A rabbit runs across a parking lot. The coordinates of the rabbit as a function of time t are: $x = -0.3t^2 + 7.2t + 28$ and $y = 0.22t^2 - 9.1t + 30$ with t in seconds and x and y in meters. At $t = 15s$, what is the rabbit's position vector $\vec{r}$ in unit vector notation and as a magnitude and an angle?

Solution:

$$\vec{r} = x\hat{\imath} + y\hat{\jmath}$$

At $t = 15s$

$$x = -0.3(15)^2 + 7.2(15) + 28 = 68.5m$$
$$y = 0.22(15)^2 - 9.1(15) + 30 = -57m$$
$$\therefore \vec{r} = (68.5m)\hat{\imath} + (-57m)\hat{\jmath}$$

The magnitude of the position vector $\vec{r}$ is

$$r = \sqrt{x^2 + y^2} = \sqrt{(68.5)^2 + (-57)^2} = 89.11m$$

The angle of $\vec{r}$ is

$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\left(\frac{-57}{68.5}\right) = -39.76^{o}$$

3.2.1 Average Velocity and Instantaneous Velocity

If a particle moves through a displacement $\Delta\vec{r}$ in a time interval Δt, then its **average velocity** $\vec{v}_{avg}$ is

$$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\Delta x\hat{\imath} + \Delta y\hat{\jmath} + \Delta z\hat{k}}{\Delta t} = \frac{\Delta x}{\Delta t}\hat{\imath} + \frac{\Delta y}{\Delta t}\hat{\jmath} + \frac{\Delta z}{\Delta t}\hat{k} \qquad \dots \qquad 3.10$$

When we speak of the **velocity** of a particle, we usually mean the particle's **instantaneous velocity** $\vec{v}$. It is expressed as

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k} = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k} \qquad \dots \qquad 3.11$$

Example 10:
Find the velocity $\vec{v}$ at time $t = 15s$ of the rabbit in example 9 in unit vector notation and as a magnitude and angle.

Solution:
The components of the velocity are v_x and v_y

$$v_x = \frac{dx}{dt} = \frac{d}{dt}(x) = \frac{d}{dt}(-0.3t^2 + 7.2t + 28\,) = -0.6t + 7.2$$

$$At\ t = 15s, v_x = -0.6(15) + 7.2 = -1.8m/s$$

$$v_y = \frac{dy}{dt} = \frac{d}{dt}(y) = \frac{d}{dt}(0.22t^2 - 9.1t + 30) = 0.44t - 9.1$$

$$At\ t = 15s, v_y = 0.44(15) - 9.1 = -2.5m/s$$

$$\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath}$$

$$\vec{v} = (-1.8m/s)\hat{\imath} + (-2.5m/s)\hat{\jmath}$$

$$Magnitude\ of\ \vec{v} = v = \sqrt{v_x^2 + v_y^2} = \sqrt{(-1.8)^2 + (-2.5)^2} = 3.08m/s$$

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{-2.5}{-1.8}\right) = 54.25^{o}$$

3.2.2 Average Acceleration and Instantaneous Acceleration

The average acceleration is defined as the change in velocity from $\vec{v}_1$ to $\vec{v}_2$ in a time interval Δt. It is mathematically expressed as

$$\vec{a}_{avg} = \frac{\vec{v}_2 - \vec{v}_1}{\Delta t} = \frac{\Delta\vec{v}}{\Delta t} \qquad \dots \qquad 3.12$$

The instantaneous acceleration $\vec{a}$ (or acceleration) is the limit of average velocity as Δt approaches zero. It is expressed mathematically as

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} \qquad \ldots \qquad 3.13$$

We note that

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{v}) = \frac{d}{dt}\left(v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}\right)$$

$$\vec{a} = \frac{dv_x}{dt}\hat{\imath} + \frac{dv_y}{dt}\hat{\jmath} + \frac{dv_z}{dt}\hat{k}$$

$$\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$$

3.2.3 Constant Acceleration

In many types of motion, the acceleration is either constant or approximately so. The following equations describe the motion of a particle with constant acceleration:

s/n	Equation
1.	$v = v_o + at$
2.	$s = v_o t + \frac{1}{2}at^2$
3.	$v^2 = v_o^2 + 2as$
4.	$s = \frac{1}{2}(v_o + v)t$
5.	$s = vt - \frac{1}{2}at^2$
Note that these equations are only applicable if acceleration is constant.	

EXERCISE:
A driver spotted a police car and he braked from a speed of $100km/h$ to a speed of $80km/h$ during a displacement of $88m$, at a constant acceleration.
(a) *What is that acceleration*?
(b) *How much time is required for the given decrease in speed*.

Example 11:
Find the acceleration $\vec{a}$ at time $t = 15s$ of the rabbit in example 10 in unit vector notation and as a magnitude and angle.

Solution:

$$a_x = \frac{dv_x}{dt} = \frac{d}{dt}(v_x) = \frac{d}{dt}(-0.6t + 7.2) = -0.6ms^{-2}$$

$$a_y = \frac{dv_y}{dt} = \frac{d}{dt}(v_y) = \frac{d}{dt}(0.44t - 9.1) = 0.44ms^{-2}$$
$$\vec{a} = (-0.6ms^{-2})\hat{\imath} + (0.44ms^{-2})\hat{\jmath}$$
$$magnitude\ of\ \vec{a} = a = \sqrt{(-0.6)^2 + (0.44)^2} = 0.74ms^{-2}$$
$$\theta = \tan^{-1}\frac{a_y}{a_x} = \tan^{-1}\left(\frac{0.44}{-0.6}\right) = -36.25^o$$

EXERCISE

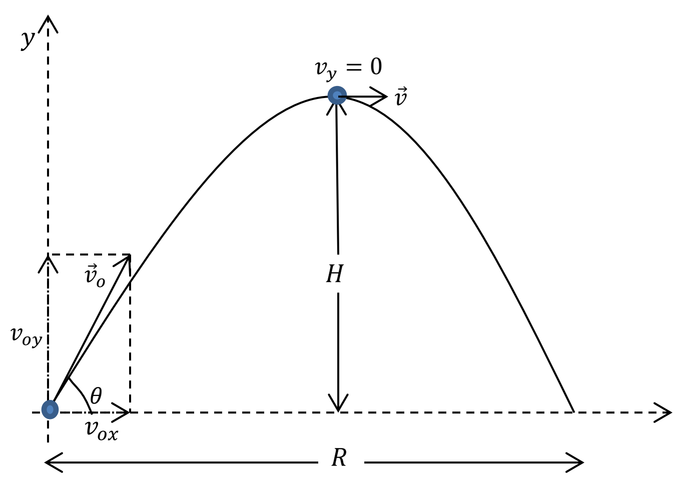
A particle with velocity $\vec{v}_o = -2\hat{\imath} + 4\hat{\jmath}$ in meters per second at $t = 0$ undergoes a constant acceleration $\vec{a}$ of magnitude $a = 3ms^{-2}$ at an angle $\theta = 130^o$ from the positive direction of the x axis. What is the particle's velocity $\vec{v}$ at $t = 5s$, in unit vector notation and as a magnitude and an angle?

$$Ans: \vec{v} = -12\hat{\imath} + 16\hat{\jmath}\ (m/s), \quad v = \frac{19m}{s}, \quad \theta = 127^o$$

3.3 PROJECTILE MOTION

A **projectile** is a particle which moves in a vertical plane with some initial velocity $\vec{v}_o$ but its acceleration is always the free fall acceleration $\vec{g}$ which is downward. Example of a projectile is a golf ball. The motion of a projectile is called **projectile motion**.

Consider the figure below.



When an object is object is projected at an angle to the horizontal, as shown in figure above, the following should be noted:

(i) the horizontal component of the velocity v_{ox} is constant;
(ii) the vertical component of the velocity v_{oy} changes;
(iii) the vertical component of the velocity is zero at the highest point;
(iv) the vertical acceleration $a_y = -g$ throughout the motion, while the horizontal acceleration a_x is zero.

In projectile motion, the horizontal motion and the vertical motion are independent of each other; that is, neither motion affects the other.

<u>Vertical motion</u>:
Initial velocity $v_{oy} = v_o \sin\theta$, acceleration $a_y = -g$
Let H be the **maximum height** reached. At maximum height $v_y = 0$.

$$Using \quad v_y^2 = v_{oy}^2 + 2a_yH$$
$$0 = (v_o \sin\theta)^2 + 2(-g)H$$
$$0 = v_o^2 sin^2\theta - 2gH$$
$$H = \frac{v_o^2 sin^2\theta}{2g} \quad \ldots \quad 3.14$$

Let the **time taken to reach the maximum height** be t.

$$Using \quad v_y = v_{oy} + a_yt$$
$$0 = v_o \sin\theta + (-g)t$$
$$0 = v_o \sin\theta - gt$$
$$t = \frac{v_o \sin\theta}{g} \quad \ldots \quad 3.15$$

When the object landed on the ground, the vertical displacement $H = 0$. Let T be the **total time of flight**.

$$Using \quad H_y = v_{oy}t + \frac{1}{2}a_yt^2$$
$$0 = v_o \sin\theta\, T + \frac{1}{2}(-g)T^2$$
$$Dividing\ through\ by\ t, we\ have$$
$$0 = v_o \sin\theta - \frac{1}{2}gT$$
$$\frac{gT}{2} = v_o \sin\theta$$
$$T = \frac{2v_o \sin\theta}{g} = 2t \quad \ldots \quad 3.16$$

<u>Horizontal Motion</u>:
Initial velocity $v_{ox} = v_o \cos\theta = constant, a_x = 0$
To obtain the **range** (horizontal displacement) of the projectile, we use

$$x - x_o = v_{ox}t + \frac{1}{2}a_xt^2$$

The range R of the projectile is

$$R = v_{ox}T + \frac{1}{2}a_xt^2$$

$Since\ a_x = 0, then$

$$R = v_{ox}T$$

$$R = (v_o \cos\theta)\left(\frac{2v_o \sin\theta}{g}\right)$$

$$R = \frac{v_o^2(2\sin\theta\cos\theta)}{g}$$

$$recall, 2\sin\theta\cos\theta = \sin 2\theta$$

$$R = \frac{v_o^2 \sin 2\theta}{g} \quad .. \quad 3.17$$

The maximum value of R is $\frac{v_o^2}{g}$ and it occurs when $\sin 2\theta = 1$, or $2\theta = 90^o, \therefore \ \theta = 45^o$. Therefore the maximum range is obtained if the object is projected at an angle 45^o to the horizontal.

Example 12:
A pirate ship is $560m$ from a military island base. A military cannon (large gun on wheels) located at sea level fires balls at initial speed $v_o = 82m/s$. (a) At what angle θ from the horizontal must a ball be fired to hit the ship? (b) How far should the pirate ship be from the cannon if it is to be beyond the maximum range of the cannonballs?

Solution:

(a)

$$R = \frac{v_o^2 \sin 2\theta}{g}$$

$$\sin 2\theta = \frac{gR}{v_o^2}$$

$$2\theta = \sin^{-1}\left(\frac{gR}{v_o^2}\right) = \sin^{-1}\left(\frac{9.8 \times 560}{82^2}\right) = \sin^{-1} 0.816 = 54.68^o$$

$$\theta = \frac{54.68^o}{2}$$

$$\theta = 27.34^o$$

(b)

$$R = \frac{v_o^2 \sin 2\theta}{g}$$

$$The\ range\ R\ is\ maximum\ at\ \theta = 45^o$$

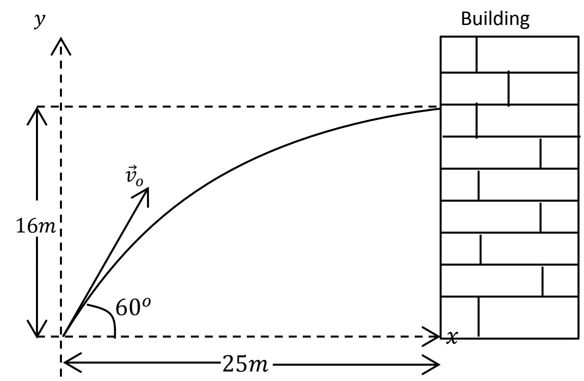
$$R = \frac{82^2 \times \sin(2 \times 45^o)}{9.8}$$

$$R = 686m$$

$$\therefore The\ ship\ should\ be\ beyond\ 686m\ (i.e. R > 686m) for\ the\ ship\ to\ be\ safe.$$

Example 13:
A projectile shot at an angle of 60^o above the horizontal strikes a building $25m$ away at a point $16m$ above the point of projection. Find the magnitude and direction of the velocity of the projectile as it strikes the building.

Solution:



Horizontal motion

$$Using\ x = v_{ox}t + \frac{1}{2}a_x t^2$$
$$recall\ a_x = 0, v_{ox} = v_o \cos\theta$$
$$x = v_o \cos\theta\, t$$
$$t = \frac{x}{v_o \cos\theta} = \frac{25}{v_o \times \cos 60} = \frac{50}{v_o}$$

Vertical motion

$$Using\ y = v_{oy}t + \frac{1}{2}a_y t^2$$
$$recall\ a_y = -g, v_{oy} = v_o \sin\theta$$
$$y = v_o \sin\theta\, t - \frac{1}{2}gt^2$$
$$16 = v_o \times \sin 60^o \times \left(\frac{50}{v_o}\right) - \frac{1}{2} \times 9.8 \times \frac{50^2}{v_o^2}$$
$$16 = 43.3 - \frac{12{,}250}{v_o^2}$$
$$v_o = 21.18m/s$$

$Now, t = \frac{50}{v_o} = \frac{50}{21.18} = 2.36s$

$v_{ox} = v_o \cos\theta = 21.18 \times \cos 60^o = 10.59m/s$

$v_{oy} = v_o \sin\theta = 21.18 \times \sin 60^o = 18.34m/s$

$$Using\ v = v_o + at$$
$$v_x = v_{ox} + a_x t = 10.59 + (0)t = 10.59m/s$$
$$v_y = v_{oy} + a_y t = 18.34 + (-9.8) \times 2.36 = -4.79m/s$$

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(10.59)^2 + (-4.79)^2} = 11.62 m/s$$

$$\theta = \tan^{-1}\left(\frac{-4.79}{10.59}\right) = -24.33^0$$

EXERCISE

1. A projectile leaves the ground at an angle of 60^o to the horizontal. Its kinetic energy is E. Neglecting air resistance, find in terms of E its kinetic energy at the highest point of the motion. (Answer: $\frac{1}{4}E$)

2. An object is released from an aeroplane which is diving at an angle of 30^o from the horizontal with a speed of $50 m/s$. If the plane is at a height of $4000 m$ from the ground when the object is released, find
   (a) the velocity of the object when it hits the ground.
   (b) the time taken for the object to reach the ground.

4.0 DYNAMICS

Dynamics is a branch of mechanics which deals with the forces that give rise to motion. Just as kinematics describes how objects move without describing the force that caused the motion, Newton's laws of motion are the foundation of dynamics which describes the motion and force responsible for the motion.

4.1 FORCE

Force is that which changes the velocity of an object. Force is a vector quantity. An external force is one which lies outside of the system being considered.

4.1.1 TYPES OF FORCES

Contact Force is that in which one object has to be in contact with another to exert a force on it. A push or pull on an object are examples of contact force.

Tension $\vec{T}$ is the force on a string or chain tending to stretch it.

Normal force $\vec{F}_N$ is the force which acts perpendicular to a surface which supports an object.

Weight $\overrightarrow{\boldsymbol{W}}$ of an object is the force with which gravity pulls downward upon it. It is given as $\vec{W} = m\vec{g}$. It is equal to the gravitational force on the body.

Frictional force $\vec{\boldsymbol{f}}$ is the force on a body when the body when the body slides or attempts to slide along a surface and is always parallel to the surface and directed so as to oppose the motion of the body.

Other important forces include **gravitational force** or simply gravity, **electromagnetic force**, **nuclear force**.

4.1.2 NET FORCE

The net force is the vector sum of all force vectors that act on an object. It is expressed as:

$$\vec{F}_{net} = \sum_{i=1}^{n} \vec{F}_i = \vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_n \qquad \ldots \qquad 4.1$$

In Cartesian components, the net force are given by

$$\begin{aligned} \vec{F}_{net,x} &= \sum_{i=1}^{n} F_{i,x} = \vec{F}_{1,x} + \vec{F}_{2,x} + \cdots + \vec{F}_{n,x} \\ \vec{F}_{net,y} &= \sum_{i=1}^{n} F_{i,y} = \vec{F}_{1,y} + \vec{F}_{2,y} + \cdots + \vec{F}_{n,y} \qquad \ldots \qquad 4.2 \\ \vec{F}_{net,z} &= \sum_{i=1}^{n} F_{i,z} = \vec{F}_{1,z} + \vec{F}_{2,z} + \cdots + \vec{F}_{n,z} \end{aligned}$$

4.2 NEWTON'S LAWS OF MOTION

4.2.1 Newton's First Law

If no net force acts on a body (i.e. $\vec{F}_{net} = 0$), then the body's velocity cannot change; that is, the body cannot accelerate. If it was moving, it will remain in motion in a straight line with the same constant velocity.

4.2.2 Newton's Second Law

If a net external force, $\vec{F}_{net}$ acts on an object with mass m, the force will cause an acceleration, $\vec{a}$, in the same direction as the force. Mathematically, the law is expressed as:

$$\vec{\boldsymbol{F}}_{net} = m\vec{\boldsymbol{a}} \qquad \ldots \qquad 4.3$$

Which may be written in Cartesian components as

$$\vec{F}_{net,x} = ma_x, \quad \vec{F}_{net,y} = ma_y, \quad \vec{F}_{net,z} = ma_z \qquad \dots \qquad 4.4$$

4.2.3 Newton's Third Law

The forces that two interacting objects exert on each other are always exactly equal in magnitude and opposite in direction:

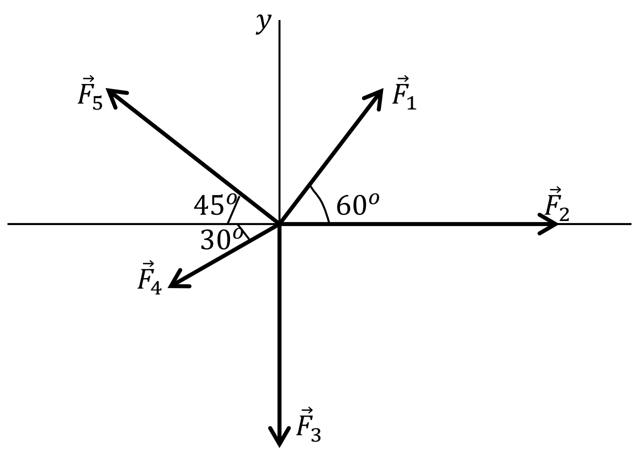
$$\vec{F}_{1\to 2} = -\vec{F}_{2\to 1} \qquad \dots \qquad 4.5$$

Example 14:
Three forces that act on a particle are given by $\vec{F}_1 = (20\hat{\imath} - 36\hat{\jmath} + 73\hat{k})N, \vec{F}_2 = (-17\hat{\imath} + 21\hat{\jmath} - 46\hat{k})N$ *and* $\vec{F}_3 = (-12\hat{k})N$. Find their resultant (net) vector. Also find the magnitude of the resultant to two significant figures.

<u>Solution</u>:

$$\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$$
$$\vec{F}_{net,x} = [20\hat{\imath} + (-17\hat{\imath})]N = 3\hat{\imath}N$$
$$\vec{F}_{net,y} = [-36\hat{\jmath} + 21\hat{\jmath}]N = -15\hat{\jmath}N$$
$$\vec{F}_{net,z} = [73\hat{k} - 46\hat{k} - 12\hat{k}]N = 15\hat{k}N$$
$$\vec{F}_{net} = [3\hat{\imath} - 15\hat{\jmath} + 15\hat{k}]N$$
$$F = |\vec{F}_{net}| = \sqrt{3^2 + (-15)^2 + 15^2} = \sqrt{459} = 21N$$

Example 15:
Five coplanar forces act on an object as shown in the figure below. Find the resultant of the forces. The magnitude of the forces are: $F_1 = 15N$, $F_2 = 19N$, $F_3 = 22N$, $F_4 = 11N, F_5 = 16N$.



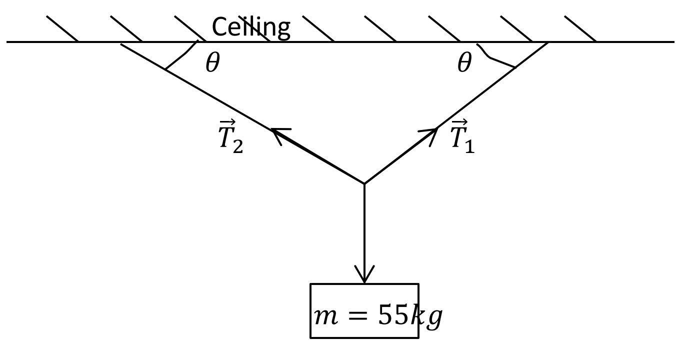
Solution:

$$\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 + \vec{F}_5$$
$$\vec{F}_1 = \vec{F}_{1,x} + \vec{F}_{1,y} = F_1 \cos 60^o\, \hat{\imath} + F_1 \sin 60^o\, \hat{\jmath}$$
$$\vec{F}_2 = \vec{F}_{2,x}\hat{\imath} = F_2\hat{\imath}$$
$$\vec{F}_3 = -\vec{F}_{3,y}\hat{\jmath} = -F_3\hat{\jmath}$$
$$\vec{F}_4 = -\vec{F}_{4,x}\hat{\imath} - \vec{F}_{4,y}\hat{\jmath} = -F_4 \cos 30^o\, \hat{\imath} - F_4 \sin 30^o\, \hat{\jmath}$$
$$\vec{F}_5 = -\vec{F}_{5,x} + \vec{F}_{5,y} = -F_5 \cos 45^o + F_5 \sin 45^o$$

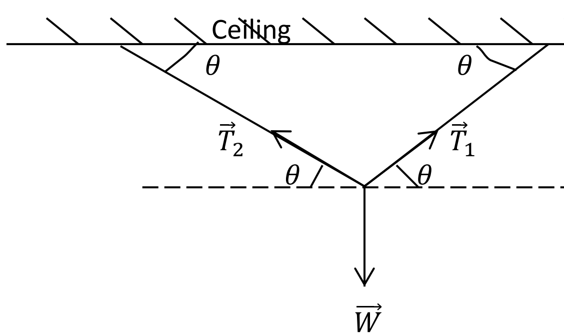
$$\vec{F}_1 = 15 \cos 60^o\, \hat{\imath} + 15 \sin 60^o\, \hat{\jmath} = 7.5\hat{\imath} + 12.99\hat{\jmath}$$
$$\vec{F}_2 = 19\hat{\imath}$$
$$\vec{F}_3 = -22\hat{\jmath}$$
$$\vec{F}_4 = -11 \cos 30^o\, \hat{\imath} - 11 \sin 30^o\, \hat{\jmath} = -9.53\hat{\imath} - 5.5\hat{\jmath}$$
$$\vec{F}_5 = -16 \cos 45^o + 16 \sin 45^o = -11.31\hat{\imath} + 11.31\hat{\jmath}$$

$$\vec{F}_{net,x} = (7.5 + 19 - 9.53 - 11.31)\hat{\imath}\ N = 5.66\hat{\imath}\ N$$
$$\vec{F}_{net,y} = (12.99 - 22 - 5.5 + 11.31)\hat{\jmath}\ N = -3.2\hat{\jmath}\ N$$
$$\vec{F}_{net} = \sqrt{(5.66)^2 + (-3.2)^2} = 6.5N$$

Example 16:
A mass of $55kg$ was suspended with two ropes as shown in the figure below. What is the tension in each rope if $\theta = 45^o$?



Solution:



$$\vec{F}_{net} = \sum \vec{F}_i = M(\vec{a}) = M(0) = 0$$

$$\sum \vec{F}_x = \vec{T}_{1x} + \vec{T}_{2x} = (T_1 \cos\theta - T_2 \cos\theta)\hat{\imath} = 0$$

$$\sum \vec{F}_y = \vec{T}_{1y} + \vec{T}_{2y} - \vec{W} = [T_1 \sin\theta + T_2 \sin\theta - 55(9.8)]\hat{\jmath} = 0$$

$$Note, both\ ropes\ support\ the\ load\ equally$$

$$\vec{T}_1 = \vec{T}_2 = \vec{T}$$

$$T\cos\theta - T\cos\theta = 0$$

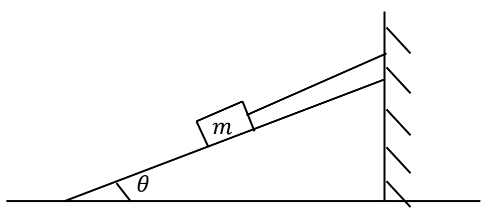
$$T\sin\theta + T\sin\theta - 539 = 0$$

$$2T\sin\theta = 539 = 0$$

$$T = \frac{539}{2 \times \sin 45}$$

$$T = 381N$$

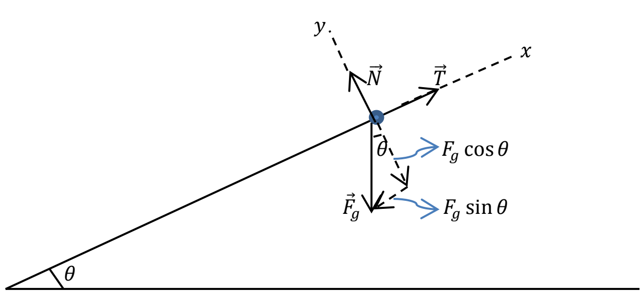
Example 17:



A cord holds stationary a block of mass $m = 15kg$ on a frictionless plane that is inclined at angle $\theta = 27^o$. (a) What is the magnitude of the tension $\vec{T}$ on the block from the cord and the normal force $\vec{N}$ on the block from the plane? (b)If the cord is cut, so that the body slides down the plane, calculate the acceleration of the body.

Solution:

(a) The free body diagram is sketched as



We note from Newton's 2nd law, $\vec{F}_{net} = m\vec{a}$

$$\vec{T} + \vec{N} + \vec{F}_g = m\vec{a} \quad \ldots \quad (1)$$

$$\vec{a} = 0 \quad (since\ the\ forces\ are\ in\ equilibrium)$$

$$\vec{T} + \vec{N} + \vec{F}_g = 0 \quad \ldots \quad (2)$$

$$solving\ in\ the\ cartesian\ coordinates, we\ have$$

x: $$T + 0 - F_g \sin\theta = 0$$

$$T = F_g \sin\theta = mg\sin\theta$$

$$T = 15 \times 9.8 \times \sin 27^o = 66.74N$$

y: $$0 + N - F_g \cos\theta = 0$$

$$N = F_g cos\theta = mg\cos\theta$$

$$N = 15 \times 9.8 \times \cos 27^o$$

$$N = 130.98N$$

$(b) Cutting\ the\ cord\ removes\ force\ \vec{T}\ from\ the\ block, so\ that\ the\ block\ slides\ along\ x\ axis$
$\therefore\ equation\ (2) becomes$

$$0 + 0 - F_g \sin\theta = ma$$

$$-mg\sin\theta = ma$$

$$a = -g\sin\theta$$

$$a = -9.8 \times \sin 27^o$$

$$a = -4.45\ m/s^2$$

5.0 KINETIC ENERGY, WORK, AND POWER

5.1 KINETIC ENERGY

The kinetic energy K is energy associated with the state of motion of a particle and is defined as:

$$K = \frac{1}{2}mv^2 \quad \ldots \quad 5.1$$

where m, is the mass of the particle and v is the speed of the particle which is well below the speed of light. Its S.I. unit is the joule (J).

Example 18:
A car of mass 1310kg being driven at a speed limit of 24.6 m/s has a kinetic energy of:

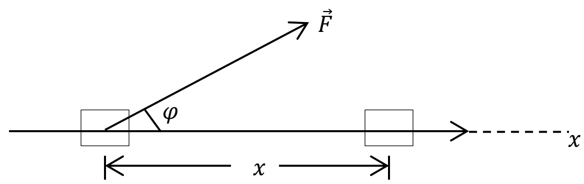
$$K_{car} = \frac{1}{2}mv^2 = \frac{1}{2}(1310)(24.6)^2 = 4.0 \times 10^5 J$$

5.2 WORK

Work W is the energy transferred to or from an object by means of a force acting on the object. Energy transferred to the object is positive work, and energy transferred from the object is negative work. It is a scalar quantity.

The work done by a force can be defined as the product of the magnitude of the displacement and the component of the force in the direction of the displacement. The unit of work in S.I. is the Joule (J).

Consider the figure below.



Work can be expressed mathematically as

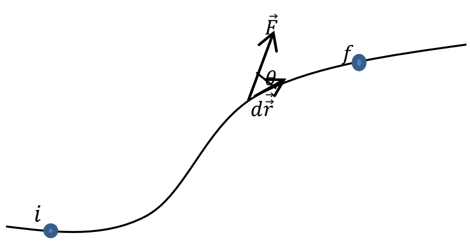
$$W = Fx\cos\varphi \quad \ldots \quad (5.2)$$

$$or$$

$$W = \vec{F}.\vec{x} \quad \ldots \quad (5.3)$$

Equation (5.3) is especially useful for calculating the work when $\vec{F}$ and $\vec{x}$ are given in unit vector notations.

Consider the figure below.



If a force displaces the particle through a distance dr, then the work done dW by the forced is

$$dW = \vec{F}.d\vec{r} \quad \ldots \quad (5.4)$$

The total work done by the force in moving the particle from initial point i to final point f will be

$$W = \int_i^f \vec{F}.d\vec{r} = \int_i^f F dr\cos\theta \quad \ldots \quad (5.5)$$

5.3 WORK-KINETIC ENERGY THEOREM

The work-kinetic energy theorem states that ***"when a body is acted upon by a force or resultant force, the work done by the force is equal to the change in kinetic energy of the body."***

Proof:

$$W = \int_i^f \vec{F}.d\vec{r} = \int_i^f Fdr\cos\theta$$

$$Force\ \vec{F}\ is\ in\ the\ direction\ of\ displacement\ d\vec{r}, \therefore\ \theta = 0$$

$$W = \int_i^f Fdr = \int_i^f madr$$

$$W = \int_i^f m\left(\frac{dv}{dt}\right)dr = \int_i^f m\left(\frac{dv}{dr}\frac{dr}{dt}\right)dr$$

$$W = \int_i^f m\frac{dv}{dr}vdr = \int_i^f mvdv = m\int_i^f vdv$$

$$W = m\left[\frac{v^2}{2}\right]_i^f = m\left[\frac{v_f^2}{2} - \frac{v_i^2}{2}\right]$$

$$W = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 \quad \ldots \quad (5.6)$$

$$W = K_f - K_i$$

$$W = \Delta K \quad \ldots \quad (5.7)$$

5.4 POWER

Power is the rate of doing work. If an amount of work ΔW is done in a small interval of time Δt, the power P is

$$P = \lim_{\Delta t \to 0}\frac{\Delta W}{\Delta t} = \frac{dW}{dt} \quad \ldots \quad (5.8)$$

Power is a scalar quantity and its unit is J/s which is also called the watt (W).

For a constant force

$$P = \frac{dW}{dt} = \vec{F}.\frac{d\vec{r}}{dt} = \vec{F}.\vec{v} \qquad (5.9)$$

$$since\ the\ instantaneous\ velocity\ \vec{v} = \frac{d\vec{r}}{dt}$$

If $\vec{F}$ acts at an angle θ to $\vec{v}$, then

$$P = \vec{F}.\vec{v} = Fv\cos\theta \quad \ldots \quad (5.10)$$

Example 19:
A body moves with velocity $\vec{v} = (5\hat{\imath} + 2\hat{\jmath} - 3\hat{k})m/s$ under the influence of a constant force $\vec{F} = (4\hat{\imath} + 3\hat{\jmath} - 2\hat{k})N$. Determine (a) the instantaneous power; (b) the angle between $\vec{F}$ and $\vec{v}$.

Solution:

(a) $\quad Power = \vec{F}.\vec{V}$

$$= (4\hat{\imath} + 3\hat{\jmath} - 2\hat{k}).(5\hat{\imath} + 2\hat{\jmath} - 3\hat{k}) = 20 + 6 + 6 = 32W$$

(b) $\quad P = FV\cos\theta$

$$\theta = \cos^{-1}\left[\frac{P}{Fv}\right]$$

$$F = |\vec{F}| = \sqrt{4^2 + 3^2 + (-2)^2} = \sqrt{29}$$

$$V = |\vec{v}| = \sqrt{5^2 + 2^2 + (-3)^2} = \sqrt{38}$$

$$\theta = \cos^{-1}\left[\frac{P}{Fv}\right] = \cos^{-1}\left[\frac{32}{\sqrt{29} \times \sqrt{38}}\right] = 15.43^o$$

Example 20:
A crate slides across an oily parking lot through a displacement $\vec{d} = (-3m)\hat{\imath}$ while a steady wind pushes against the crate with a force $\vec{F} = (2N)\hat{\imath} + (-6N)\hat{\jmath}$. (a) Calculate the work done by the wind, (b) if the crate has a kinetic energy of $10J$ at the beginning of the displacement $\vec{d}$, what is the kinetic energy at the end of $\vec{d}$?

Solution:

(a) $\quad W = \vec{F}.\vec{d} = (2\hat{\imath} - 6\hat{\jmath}).(-3\hat{\imath}) = (2\hat{\imath} - 6\hat{\jmath}).(-3\hat{\imath} + 0\hat{\jmath}) = -6J$

(b) $\quad W = \Delta K = K_f - k_i$

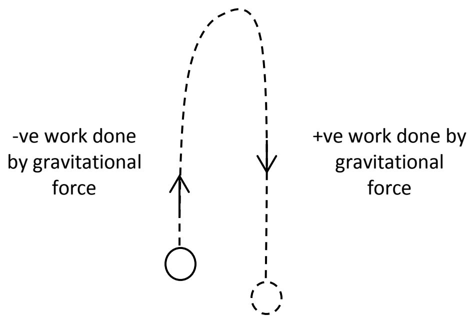
$$K_f = W + K_i = -6 + 10 = 4J$$

EXERCISE
A constant force $\vec{F} = (5\hat{\imath} + 3\hat{\jmath} - 2\hat{k})N$ moves a particle from position $\vec{r}_1 = (2\hat{\imath} - \hat{\jmath} + 4\hat{k})m$ to a position $\vec{r}_2 = (3\hat{\imath} + 5\hat{\jmath} + \hat{k})m$. Calculate the work done by the force. [$\boldsymbol{Ans = 29J}$]

6.0 POTENTIAL ENERGY

Potential energy U is the energy that can be associated with the configuration (or arrangement) of a system of objects that exerts forces on one another.

Consider a tennis ball thrown upward as shown in figure 6.1 below.



As the ball is thrown upward, the gravitational force does negative work on it by decreasing its kinetic energy. As the ball descends, the gravitational force does positive work on it by increasing its kinetic energy. Thus the change in potential energy is defined to equal the negative of work done. That is

$$\Delta U = -W \qquad \dots \qquad (6.1)$$

Recall that

$$W = \int_{x_i}^{x_f} F(x)dx$$

$$\therefore \ \Delta U = -\int_{x_i}^{x_f} F(x)dx \qquad \dots \qquad (6.2)$$

6.1 GRAVITATIONAL POTENTIAL ENERGY

Gravitational potential energy is the potential energy associated with a system consisting of Earth and a nearby particle.

$$\Delta U = -\int_{y_i}^{y_f} F(y)dy = -\int_{y_i}^{y_f} (-mg)dy = mg\int_{y_i}^{y_f} dy = mg[y]_{y_i}^{y_f} = mg(y_f - y_i)$$

$$\Delta U = mg\Delta y \qquad \dots \qquad (6.3)$$

Usually, reference point y_i is taken as zero.

$$\Delta U = mgy \qquad \dots \qquad (6.4)$$

Example 21:

A mass of $2kg$ hangs $5m$ above the ground. What is the gravitational potential energy of the mass-earth system at (i) the ground, (ii) a height of $3m$ above the ground, and (iii) a height of 6m.

Solution:

(i) $U = mg(y_f - y_i) = 2 \times 9.8 \times (5 - 0) = 98J$

(ii) $U = mg(y_f - y_i) = 2 \times 9.8 \times (5 - 3) = 39.2J$

(iii) $U = mg(y_f - y_i) = 2 \times 9.8 \times (5 - 6) = -19.6J$

6.2 ELASTIC POTENTIAL ENERGY

Elastic potential energy is the energy associated with the state of compression or extension of an elastic object. For a spring that exerts a force $f = -kx$ when its free end has displacement x, the elastic potential is

$$\Delta U = -\int_{x_i}^{x_f} F dx = -\int_{x_i}^{x_f} (-kx) dx = k\int_{x_i}^{x_f} x dx$$

$$= k\left[\frac{x^2}{2}\right]_{x_i}^{x_f} = \frac{1}{2}kx_f^2 - \frac{1}{2}kx_i^2 \qquad \ldots \qquad (6.5)$$

6.3 CONSERVATION OF MECHANICAL ENERGY

We recall from equation (5.7)

$$W = \Delta K \qquad \ldots \qquad (6.6)$$

Also, we recall from equation (6.1)

$$W = -\Delta U \qquad \ldots \qquad (6.7)$$

Combining equations (6.5) and (6.6)

$$\Delta K = -\Delta U$$

$$K_2 - K_1 = -(U_2 - U_1)$$

Rearranging

$$K_2 + U_2 = K_1 + U_1 \qquad \ldots \qquad (6.8)$$

In words,

$$\begin{pmatrix} sum\ of\ K.E\ and\ P.E\ for \\ any\ state\ of\ a\ system \end{pmatrix} = \begin{pmatrix} the\ sum\ of\ K.E\ and\ P.E\ for\ any \\ other\ state\ of\ the\ system \end{pmatrix}$$

Equation (6.8) is the conservation of mechanical energy equation.

The **principle of conservation of mechanical energy** states that: *In an isolated system, the kinetic energy and potential energy can change, but their sum, the mechanical energy E_{mec} of the system cannot change.* That is

$$\Delta E_{mec} = \Delta K + \Delta U = 0 \qquad \ldots \qquad (6.9)$$

7.0 **CENTER OF MASS**

The center of mass of a body or a system of bodies is the point that moves as though all of the masses were concentrated there and all external forces were applied there.

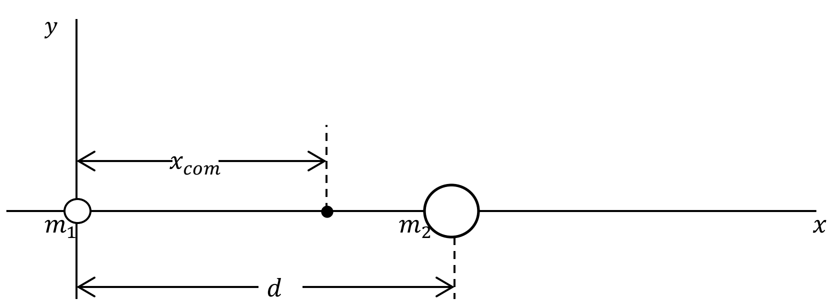


Figure 7.1

We define the position of the center of mass (com) of the two-particle separated by a distance d in figure 7.1 as:

$$x_{com} = \frac{m_2}{m_1 + m_2} d \quad \ldots \quad (7.1)$$

We arbitrarily chose the origin of the x axis to coincide with the particle of mass m_1.

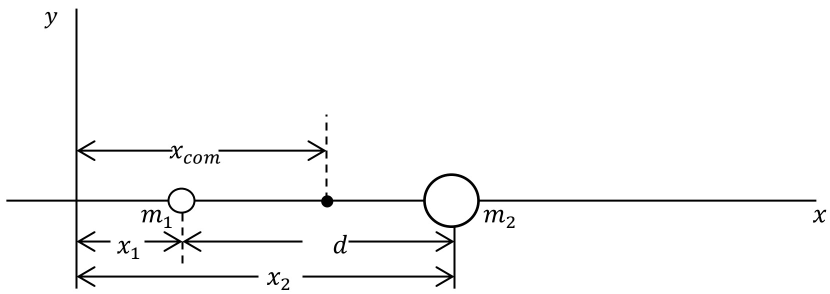


Figure 7.2

If the origin is located farther from the particles, the position of the center of mass is calculated as:

$$x_{com} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \quad \ldots \quad (7.2)$$

Generally, for n particles along x axis

$$x_{com} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \cdots + m_n x_n}{M} = \frac{1}{M} \sum_{i=1}^{n} m_i x_i \quad \ldots \quad (7.3)$$

where $M = m_1 + m_2 + m_3 + \cdots + m_n$.

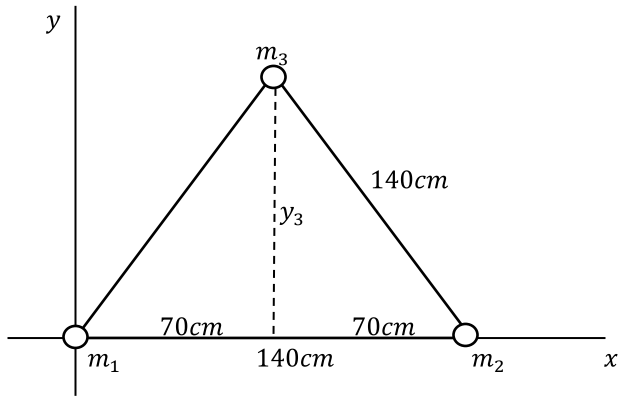
If the particles are distributed in three dimensions, the center of mass must be identified by three coordinates.

$$x_{com} = \frac{1}{M} \sum_{i=1}^{n} m_i x_i, \quad y_{com} = \frac{1}{M} \sum_{i=1}^{n} m_i y_i, \quad z_{com} = \frac{1}{M} \sum_{i=1}^{n} m_i z_i \quad \ldots \quad (7.4)$$

Example 22:

Three particles of mass $m_1 = 1.2kg, m_2 = 2.5kg$, and $m_3 = 3.4kg$ form an equilateral triangle of edge length $a = 140cm$. Where is the center of mass of this three-particle system?

Solution:



The distance y can be obtained using Pythagoras theorem

$$140^2 = 70^2 + y_3^2$$

$$y_3^2 = 140^2 - 70^2 = 19600 - 4900 = 14700$$

$$y_3 = \sqrt{14700} = 121cm$$

$$x_{com} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{M} = \frac{(1.2)(0) + (2.5)(140) + (3.4)(70)}{7.1} = 83cm$$

$$y_{com} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{M} = \frac{(1.2)(0) + (2.5)(0) + (3.4)(121)}{7.1} = 58cm$$

8.0 COLLISION

A Collision is an isolated event in which two or more bodies exert relatively strong forces on each other for a relatively short time.

8.1 ELASTIC AND INELASTIC COLLISION

A collision is said to be **elastic** if the total kinetic energy of the system of two colliding bodies is unchanged by the collision (i.e. conserved). Whereas, if the kinetic energy of the system is not conserved, such a collision is called an **inelastic collision**.

In every day collision such as collision of two cars, kinetic energy is not conserved, since some kinetic energy will be lost in form of heat and sound.

8.2 LINEAR MOMENTUM

In physics, momentum $\vec{p}$ is defined as the product of an object's mass and its velocity:

$$\vec{p} = m\vec{v} \qquad \ldots \qquad (8.1)$$

By Newton's second law of motion, momentum is related to force by

$$\vec{F} = \frac{d\vec{p}}{dt} \quad \ldots \quad (8.2)$$

8.3 LAW OF CONSERVATION OF LINEAR MOMENTUM

The law states that: ***In a closed, isolated system, the linear momentum of each colliding body may change but the total linear momentum $\vec{p}$ of the system cannot change, whether the collision is elastic or inelastic.***

$$\vec{P} = constant \quad or \quad \vec{P}_i = \vec{P}_f \quad \ldots \quad (8.3)$$

where i and f denote initial and final respectively.

8.4 ELASTIC COLLISIONS IN ONE DIMENSION

Consider the collision of two masses m_1 and m_2 along x axis.

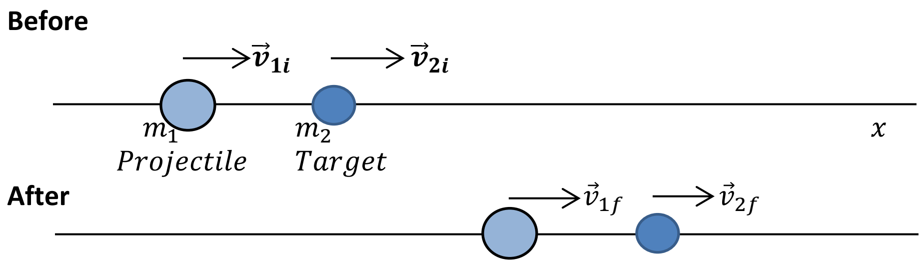


Figure 8.1

By conservation of linear momentum, we have

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \quad \ldots \quad (8.4)$$

Also, kinetic energy is conserved for elastic collision. So, we have

$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \quad \ldots \quad (8.5)$$

Equations 8.4 and 8.5 can be used simultaneously if there are two unknowns.

Example 23:

A truck of mass $5500kg$ moving with a velocity of $4.6m/s$ hits from behind a car of mass $1200kg$ on a straight line path with it. If the vehicles move with different velocities after collision, calculate the velocities of the truck and the car after collision if the car was originally stationary.

Solution:

$$m_T v_{Ti} + m_c v_{ci} = m_T v_{Tf} + m_c v_{cf} \quad \ldots \quad (i)$$

$$\frac{1}{2} m_T v_{Ti}^2 + \frac{1}{2} m_c v_{ci}^2 = \frac{1}{2} m_T v_{Tf}^2 + \frac{1}{2} m_c v_{cf}^2 \quad \ldots \quad (ii)$$

$$5500(4.6) + 1200(0) = 5500 v_{Tf} + 1200 v_{cf} \quad \ldots \quad (iii)$$

$$\frac{1}{2}(5500)(4.6)^2 + \frac{1}{2}(1200)(0)^2 = \frac{1}{2}(5500) v_{Tf}^2 + \frac{1}{2}(1200) v_{cf}^2 \quad \ldots \quad (iv)$$

Equations (iii) and (iv) reduces to

$$25300 = 5500v_{Tf} + 1200v_{cf} \quad \ldots \quad (v)$$

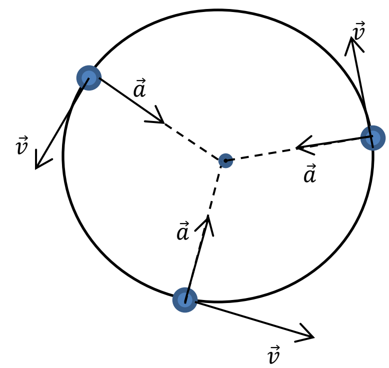
$$58190 = 2750v_{Tf}^2 + 600v_{cf}^2 \quad \ldots \quad (vi)$$

Solving equations (v) and (vi) simultaneously, we obtain

$$v_{cf} = 2.1m/s \quad and \quad v_{Tf} = 4.14m/s$$

9.0 UNIFORM CIRCULAR MOTION

A particle is said to be in uniform circular motion if it travels around a circle or a circular arc at a constant (uniform) speed.



$Figure$ 9.1

Figure 9.1 shows the relation between the velocity and acceleration vectors at different stages during uniform circular motion.

The following points must be noted:

(i) Although the speed is uniform, the velocity (a vector) changes in direction, therefore, a particle in uniform circular motion accelerates.

(ii) The velocity vector $\vec{v}$ is always directed tangent to the circle in the direction of the motion.

(iii) The acceleration is always directed radially inward, therefore the acceleration is called **centripetal acceleration**.

(iv) The magnitude of the centripetal acceleration is:

$$a = \frac{v^2}{r} \quad \ldots \quad (9.1)$$

where r is the radius of the circle and v is the speed of the particle.

(v) The particle travels the circumference of the circle in time

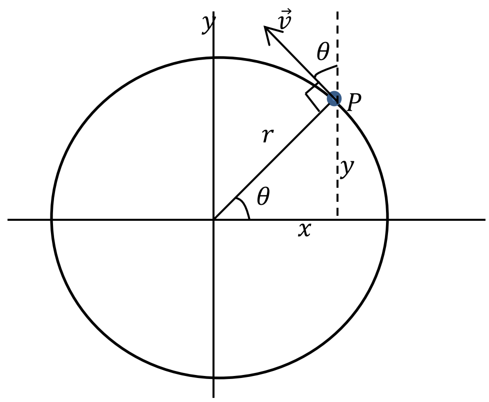
$$T = \frac{2\pi r}{v} \quad \ldots \quad (9.2)$$

where T is called the period of revolution.

EXERCISE

Prove that the centripetal acceleration is $a = \frac{v^2}{r}$.

Solution:



$$\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath}$$

$$\vec{v} = (-v\sin\theta)\hat{\imath} + (v\cos\theta)\hat{\jmath}$$

$$From\ diagram, \sin\theta = \frac{y}{r}, \quad \cos\theta = \frac{x}{r}$$

$$\vec{v} = \left(-v\frac{y}{r}\right)\hat{\imath} + \left(v\frac{x}{r}\right)\hat{\jmath}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{v}) = \left(-\frac{v}{r}\frac{dy}{dt}\right)\hat{\imath} + \left(\frac{v}{r}\frac{dx}{dt}\right)\hat{\jmath}$$

$$Note\ \frac{dy}{dt} = v_y = v\cos\theta \qquad and\ \frac{dx}{dt} = v_x = v\sin\theta$$

$$\vec{a} = \left(-\frac{v}{r}v\cos\theta\right)\hat{\imath} + \left(\frac{v}{r}v\sin\theta\right)\hat{\jmath}$$

$$a = |\vec{a}| = \sqrt{\left(-\frac{v}{r}v\cos\theta\right)^2 + \left(\frac{v}{r}v\sin\theta\right)^2}$$

$$= \sqrt{\left(\frac{v^2}{r}\right)^2(cos^2\theta) + \left(\frac{v^2}{r}\right)^2(sin^2\theta)}$$

$$= \sqrt{\left(\frac{v^2}{r}\right)^2(cos^2\theta + sin^2\theta)} = \sqrt{\left(\frac{v^2}{r}\right)^2} = \frac{v^2}{r}$$

$$\therefore a = \frac{v^2}{r}$$

10.0 **ROTATION**

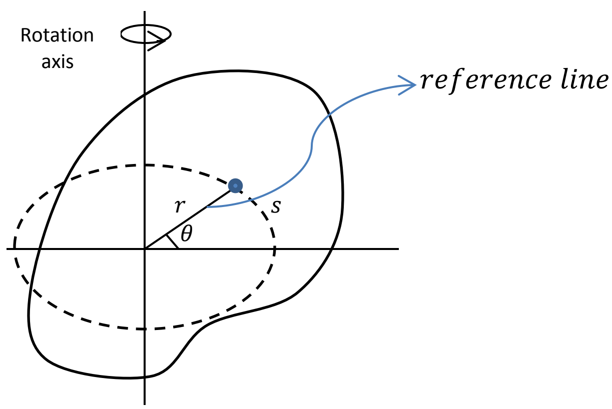
10.1 RIGID BODY

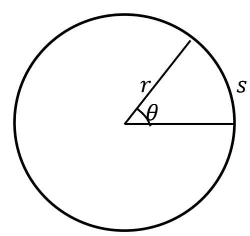
A **rigid body** is a body is a body that can rotate with all its part locked together and without a change in its shape.

10.2 ROTATION

Rotation occurs when every point of a rigid body moves in a circle whose center lies on the axis of rotation, and every point moves through the same angle during a particular time interval.

10.3 ANGULAR POSITION (θ)





Angular position θ is the angle of a reference line relative to a fixed direction, say x.

$$\theta = \frac{s}{r} \quad (rad) \qquad \ldots \qquad (10.1)$$

where s is the length of arc and r is the radius of the circle. θ is measured in radians (rad).

10.4 ANGULAR DISPLACEMENT

A body undergoes angular displacement if it changes angular position from θ_1 to θ_2. It is expressed as:

$$\Delta\theta = \theta_2 - \theta_1 \qquad \ldots \qquad (10.2)$$

10.5 ANGULAR VELOCITY AND SPEED

The **average angular speed** ω_{avg} is the ratio of the angular displacement $\Delta\theta$ to time Δt of a body undergoing rotation. It is expressed as:

$$\omega_{avg} = \frac{\Delta\theta}{\Delta t} \qquad \ldots \qquad 10.3$$

The **instantaneous angular velocity** $\vec{\boldsymbol{\omega}}$ of the body is:

$$\vec{\omega} = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \qquad \ldots \qquad (10.4)$$

The magnitude of the body's angular velocity is the angular speed ω.

10.6 ANGULAR ACCELERATION

The average angular acceleration is expressed as

$$\alpha_{avg} = \frac{\Delta\omega}{\Delta t} = \frac{\omega_2 - \omega_1}{t_2 - t_1} \quad \ldots \quad (10.5)$$

The instantaneous angular acceleration α of a body is expressed as:

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} \quad \ldots \quad (10.6)$$

10.7 RELATING THE ANGULAR AND LINEAR VARIABLES

Since for a rigid body, all particles make one revolution in the same amount of time, they all have the same angular speed ω. When a rigid body rotates, each particle in the body moves in its own circle about that axis.

Recall from equation 10.1.

$$The\ \boldsymbol{Position}\ S = \theta r \quad \ldots \quad (10.7)$$

$$\boldsymbol{Speed}\ v = \frac{dS}{dt} = \frac{d}{dt}(\theta r) = r\frac{d\theta}{dt} = r\omega$$

$$v = r\omega \quad \ldots \quad (10.8)$$

where v is the linear speed and ω is the angular speed.

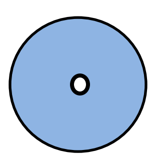
A rigid body in rotational motion will have two forms of acceleration.

(1) $\boldsymbol{Tangential\ acceleration}\ a_t = \frac{dv}{dt} = \frac{d}{dt}(\omega r) = r\frac{d\omega}{dt} = r\alpha \quad \ldots \quad (10.9)$

(2) $\boldsymbol{Radial\ acceleration}\ a_r = \frac{v^2}{r} = \frac{(r\omega)^2}{r} = r\omega^2 \quad \ldots \quad (10.10)$

10.8 KINETIC ENERGY OF ROTATION

We cannot apply the formula $k = \frac{1}{2}mv^2$ to rigid body such as compact disc because, the kinetic energy of its center of mass is zero ($m = 0\ at\ the\ center$).



Instead, we treat the compact disc as a collection of particles with different speeds.

$$k = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_3^2 + \cdots$$

$$k = \sum \frac{1}{2} m_i v_i^2$$
$$since\ v = \omega r$$
$$k = \sum \frac{1}{2} m_i (r_i \omega)^2$$
$$k = \frac{1}{2} \left(\sum m_i r_i^2 \right) \omega^2$$
The quantity in parenthesis is called the **rotational inertia** or **moment of inertia I** of the body with respect to the axis of rotation.

$$\therefore k = \frac{1}{2} I \omega^2 \quad \ldots \quad (10.11)$$

where

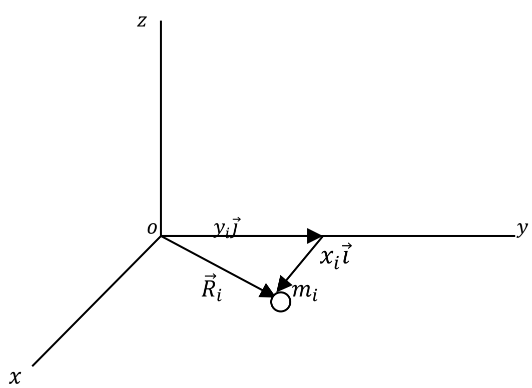
$$I = \sum m_i r_i^2 \quad \ldots \quad (10.12)$$

10.9 PERPENDICULAR AXES THEOREM

Statement of the theorem: The moments of inertia of any plane lamina about an axis normal to the plane of the lamina is equal to the sum of the moments of inertia about any two mutually perpendicular axis passing through the given axis and lying in the plane of the lamina. That is,

$$I_z = I_x + I_y \quad \ldots \quad (10.13)$$

Proof:



The position vector of particle with mass m_i in the xy plane is:

$$\vec{R}_i = x_i\vec{i} + y_i\vec{j}$$

So that, the moment of inertia of particle m_i is $m_i R_i^2$.

$\therefore$ *The total moment of inertia of all particles about z axis is*

$$I_z = \sum_{i=1}^{n} m_i R_i^2 = \sum_{i=1}^{n} m_i (x_i^2 + y_i^2)$$

$$= \sum_{i=1}^{n} m_i x_i^2 + \sum_{i=1}^{n} m_i\, y_i^2$$
$$I_z = I_x + I_y$$

11.0 GRAVITATION

Gravitation is the tendency of bodies to move toward each other. Isaac Newton proposed a force law that we call Newton's Law of gravitation.

11.1 NEWTON'S LAW OF GRAVITATION

The Law states that: ***Every particle attracts any other particle with a gravitational force whose magnitude is given by***

$$F = \frac{Gm_1 m_2}{r^2} \quad \begin{pmatrix} Newton's\ Law\ of \\ Gravitation \end{pmatrix} \quad \ldots \quad (11.1)$$

where m_1 and m_2 are the masses of the particle, r is the distance between them, and G is the gravitational constant with a value $G = 6.67 \times 10^{-11} N.m^2/kg^2$.

11.2 SUPERPOSITION PRINCIPLE

This is a principle that says ***the net force on a particle is the vectorial summation of forces of attraction between the particle and individual interacting particles***. Mathematically, the principle is expressed as

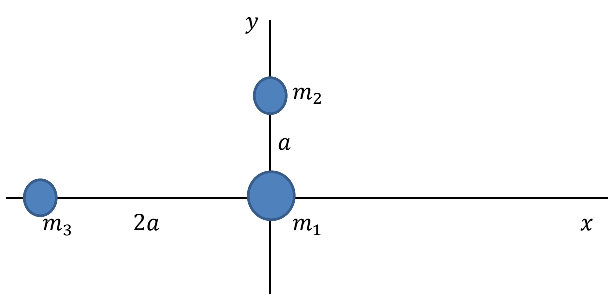
$$\vec{F}_{1,net} = \sum_{i=2}^{n} \vec{F}_{1i} = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14} + \cdots + \vec{F}_{1n} \quad \ldots \quad (11.2)$$

if there are n interacting particles. Here, $\vec{F}_{1,net}$ is the net force on particle 1.

For extended object, we write the principle as

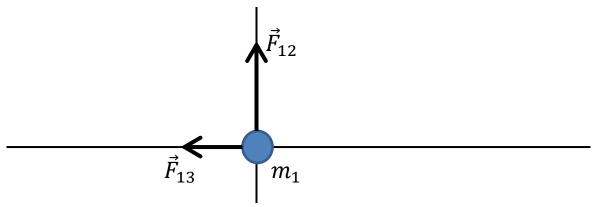
$$\vec{F}_1 = \int d\vec{F} \quad \ldots \quad (11.3)$$

Example 24:



The figure shows an arrangement of three particles. $m_1 = 6kg, m_2 = m_3 = 4kg$, and distance $a = 2cm$. What is the magnitude of the net gravitational force $\vec{F}_{1net}$ that acts on particle 1 due to other particles.

Solution:

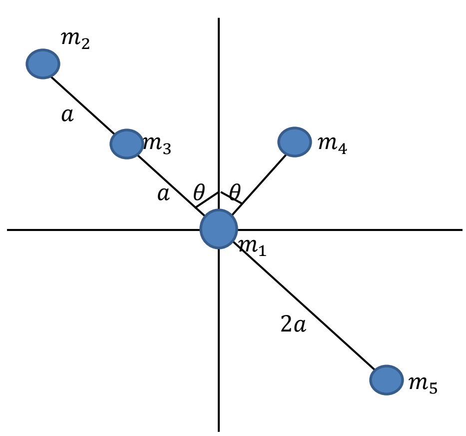


By superposition principle

$$\vec{F}_{1net} = \vec{F}_{12} + \vec{F}_{13}$$
$$\vec{F}_{12} = F_{12}\hat{\imath} + F_{12}\hat{\jmath} = F_{12}\hat{\imath}$$
$$\vec{F}_{13} = F_{13}\hat{\imath} + F_{13}\hat{\jmath} = -F_{13}\hat{\jmath}$$
$$F_{12} = \frac{Gm_1m_2}{a^2} = \frac{(6.67 \times 10^{11})(6)(4)}{(0.02)^2} = 4 \times 10^{-6}N$$
$$F_{13} = \frac{Gm_1m_2}{(2a)^2} = \frac{(6.67 \times 10^{11})(6)(4)}{(0.04)^2} = 1 \times 10^{-6}N$$
$$\vec{F}_{12} = F_{12}\hat{\imath} = 4 \times 10^{-6}N\hat{\imath}$$
$$\vec{F}_{13} = -F_{13}\hat{\jmath} = -1 \times 10^{-6}N\hat{\jmath}$$
$$\vec{F}_{1net} = \vec{F}_{12} + \vec{F}_{13} = 4 \times 10^{-6}N\hat{\imath} + (-1 \times 10^{-6}N\hat{\jmath})$$
$$F_{1net} = \sqrt{(F_{12})^2 + (-F_{13})^2}$$
$$F_{1net} = \sqrt{(4 \times 10^{-6})^2 + (-1 \times 10^{-6})^2}$$
$$F_{1net} = 4.1 \times 10^{-6}N$$

EXERCISE

(1)



Five particles are arranged as shown in the figure above. $m_1 = 8kg, m_2 = m_3 = m_4 = m_5 = 2kg, a = 3cm$, and $\theta = 30^o$. What is the gravitational force $\vec{F}_{1net}$ on particle 1 due to the other particles.

(2) What must be the separation between a $5.2kg$ particle and a $2.4kg$ particle for their gravitational attraction to have a magnitude of $2.3 \times 10^{-12} N$?

(3) In the figure below, two spheres of mass m and a third sphere of mass M form an equilateral triangle. The net gravitational force on that central sphere from the three other spheres is zero. (a) What is M in terms of m? (b) If we double the value of m_4, what is the magnitude of the net gravitational force on the central sphere?