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Introduction to Proof

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Contents

To the Instructor

These notes have been written for use in an introduction to proof class with a Calculus I prerequisite. The course is designated as writing intensive and satisfies one of the two writing requirements for the mathematics major. Most of the students that have taken this class at Berry College are mathematics majors and minors, but occasionally other science majors will take the course. At the beginning of the semester students often work in small groups one or two days per week until they begin to develop more familiarity with the techniques of proof. In these small groups, they sometimes work on constructing proofs that they might later present to the class. Other times, they work collaboratively on various kinds of worksheets that are designed to help them see the unity of mathematics and prepare them to transport the things they learn in the proof class to other courses. As they become more proficient at writing proofs, they spend less time working in groups and more time doing presentations at the board. By the end of the semester, each day of class is spent on student presentations.

In the notes that follow, there are **Question**s, **Exercise**s and **Problem**s that are meant for the students. The **Question**s are there to get the student started thinking about definitions or axioms before they start trying to prove anything. The solutions to the **Exercise**s are generally more routine and as such are intended to be used to stimulate class, or small group, discussion, without the student having to worry about constructing a formal proof. The **Problem**s are there for students to work on and create proofs. Most of these statements are true, though not all. The general instruction is meant to be "Prove, or disprove and salvage, if possible." That is, if the statement is true, provide a proof. If the statement is false, provide a counterexample and then find a related statement which is true and prove it.

To the Student

These notes are written to introduce you to the basic techniques of mathematical proof. There is no particular unifying content area, though you may later recognize some of the exercises as coming from number theory, abstract algebra, real analysis or topology. The primary goal of the course is to aid your development as a mathematically mature student by teaching you to write convincing proofs. In the process you will need to learn to use mathematical language correctly and precisely. The course is also designed to give you a glimpse of what it is that mathematicians do, and hopefully the way that mathematicians go about doing it. Many people think that mathematics is just about solving equations and fiddling about with numbers. However, equations are to mathematics as typing is to writing. Solving equations can provide a means to an end, but the act of solving an equation is not always the end in itself.

In your first calculus course, you learned about limits, continuity, differentiation and integration at an introductory level. Most likely the class was focused primarily on the development of calculus and how to apply calculus to solve problems. You may have seen some proofs, but they probably were not stressed too much to accommodate the non-mathematics majors in the audience. In more advanced mathematics courses however, the main thrust of the course is to understand the theoretical underpinnings of the topic at hand. (This is the stuff that makes many mathematicians really dig mathematics.) In classes like real analysis, abstract algebra and topology you will spend the majority of the semester learning about the concepts that give mathematics its robust structure. You will see proofs of theorems and prove some on your own. In this class you will learn what constitutes a rigorous mathematical proof and how to write one.

Throughout the pages that follow you will find various statements labeled as *Axiom*, *Definition*, *Exercise* and the occasional *Question*. Anything labeled as an Axiom is a concept we will accept as an *a priori* truth. That is, we will accept it tacitly without proof. But this is not to say that you should blindly accept it as gospel. Rather, you should give each axiom careful consideration and think about why it might be reasonable to accept it as true. You will likely discuss these axioms in class and hopefully see why we choose to include these assumptions in our system, but you will probably not debate their truth. They are part of the commonly accepted parlance of mathematics and we will use them accordingly.

With regard to the Definitions, we will define concepts rigorously whenever possible, but in mathematics not all concepts can be defined, owing to the limitations of language. Sometimes we will have to deal with undefinable terms, but in those cases we can ordinarily agree on what we mean without writing down a definition, because the truth of the axioms gives them meaning. When there is a definition of a concept you need to think about what it really means. To the person on the street, a definition is "a phrase that helps give an intuitive understanding of a word or idea." [17] However, to a mathematician definitions are the glue that binds a proof together. The real difference here is that when we use definitions as mathematicians, we must use them precisely and without any sort of *poetic license*. Thus, knowing and understanding the definitions are often the first steps in preparing to write convincing proofs. Keep in mind the words of the Dalai Lama:

> *Do not just pay attention to the words; Instead pay attention to meanings behind the words. But, do not just pay attention to meanings behind the words; Instead pay attention to your deep experience of those meanings.*
>
> – Tenzin Gyatso, The Fourteenth Dalai Lama

The Exercises are there for you to practice using the definitions and applying the axioms without having to write complete proofs. They are for practice, but you should give them the attention they deserve. Often they will provide insight into how an axiom might be used in a proof or how a definition can be applied to produce an integral piece of an argument. The Questions are much like the Exercises, but may require you to decide in advance whether you think the statements are true or not.

Besides the axioms, definitions and exercises you will see many statements labeled *Problem*. These are the essence of the course. These are the statements that you will be proving on your own and turning in for a grade. These are the nuggets of truth that you will reveal to yourself by creating their proofs.

There are several different methods of proof that we will examine. Some are direct and some are indirect. Some will count many things and some will attempt to show the existence of one thing. When all else fails, try to fall back on what you know. Please don't fall prey to the temptation to make up a proof method that seems to fit the situation. In the long run, this will only help you be confused about what you're trying to do. And since your main goal is to develop an understanding of how mathematics really works, you will only be hindering your progress. (Plus you'll likely hinder your grade in the process.) A word about proof style: your first goal should be to learn how to construct correct mathematical proofs. While the temptation will at first be to write short proofs with lots of numerical, graphical or intuitive arguments, you really should try to include a lot of detail and some discussion about the context of your reasoning. Don't worry too much about how good your proof sounds until you are certain that it really is a proof. Then, once you have begun to master the techniques of proof, you can think about how to make your proofs more efficient or more polished. With regard to pictures, a picture is **not a proof**. However, a good picture can be a valuable aid in helping you discover a proof and helping the reader understand your proof. So, if you think a picture would be helpful include one. If you're not sure whether a picture would be helpful, go ahead and include one anyway.

Here is an interesting, semi-cautionary note: there are some perfectly sensible mathematical statements that cannot be verified or proven false. In 1931 twenty-five year old German mathematician Kurt Gödel proved the following theorem that set the mathematical world on its ear:

Gödel's Incompleteness Theorem *Any sufficiently complex logical system will contain unverifiable and untestable statements.*

This doesn't say that proofs are hard to find, or that we are just not creative enough to find them, but rather that proofs are not there to be found. But there's no need to worry. Even though there are unprovable statements, there are plenty of statements that do have proofs. In this text we will concentrate on statements that have proofs with the goal of discovering some of the proofs. Note that there has been no mention of “the” proof, since there is not necessarily a single proof of a given statement. Hence, you need not find the “expected” proof on your own, but instead just an argument that really is a proof.[1]

Finally, throughout the material that follows, you are allowed to assume some common mathematical knowledge from your previous mathematics courses and anything that appears in the text before a problem you are working on, unless otherwise directed. This body of common information should be something that your class and instructor can agree on in advance. When using technical terms it is best to use the definitions given here. That is, even though you are allowed to assume certain ideas from other courses, use definitions as they are phrased here so that everyone is on the same page, and using the same mathematics. If you are seeing a definition of a familiar concept phrased a different way, this is a good time to begin seeing things from different perspectives and to practice being able to use different tools.

Notation

While mathematics is written in complete sentences, many times a mathematical symbol will take the place of a word or phrase. For example “$A = B$” is a mathematical sentence where A and B are nouns and “$=$” is the verb. Mathematicians use symbols to make their writing easier to understand and write. However, the use of symbols should never be a barrier to comprehension. Here is a short table of symbols that mathematicians often use in informal presentations, but not generally in writing. If your class includes a presentation component, this would be a good place to use these symbols, but you should be judicious about using them in the homework you hand in.

Symbol	Meaning	Symbol	Meaning
$\forall$	for all	$\exists$	there exists
$\Rightarrow$	implies	$\Leftarrow$	is implied by
iff	if and only if	$\Longleftrightarrow$	if and only if
$\ni$	such that	$\square$	end of a proof

[1]This liberates you from having to reproduce something that someone else already produced.

Chapter 1

Symbolic Logic – The thing that really separates us from the apes.

There was things which he stretched, but mainly he told the truth.
–Mark Twain, *Huckleberry Finn*

Somebody who thinks logically is a nice contrast to the real world.
–The Law of Thumb

How do humans communicate with each other? This question has many different possible answers. We are all certainly aware of the power of a facial expression or a hand gesture. However, each of these types of communication is often open to interpretation. One person may give the same hand gesture to two different people and those people may interpret it differently. This difference of interpretation may depend on many factors, too many in fact to go into here, but suffice it to say that mathematical proofs are not often constructed using facial expressions or hand gestures. Instead, we rely on written language. How then do we ensure that our language does not leave room for interpretation? This is sometimes a difficult chore, but in mathematical language we seldom run into this problem. The primary way we get around this is through the use of symbolic logic. According to Hale, "*Logic comprises the rules by which mathematicians operate, the "grammar" of the language.*" [9] That is, logic is the tool we will use to create proofs. A proof is built up of statements, each of which follows from the preceding statement or statements.

1.1 Statements

Definition 1.1. *A* **statement** *is a declarative sentence which is either true or false, but not both.*

Note. Here we mean that the truth value of the statement does not depend on the person who is reading it. That is, it is possible for everyone to agree on the same truth value without regard to personal opinion.

Example 1.2. The following are examples of statements:

1. The Canadian national anthem is called *O Canada*.

2. The moon is made of green cheese.
3. Today is Wednesday.
4. James K. Polk was the 11th President of the United States.

Example 1.3. The following are not examples of statements.

1. Are you from Montana?
2. He is six feet tall.
3. Chocolate is better than vanilla.
4. Go west young man.

Question 1.4. Why are the items in Example 1.3 not examples of statements?

Exercise 1.5. Determine which of the following are statements. For the ones that are not statements, explain why not. If so, determine the truth value of the statement.

1. Calvin Coolidge was the greatest American President.
2. The square root of a rational number is always a rational number.
3. Mixing yellow and red paint will give you orange paint.
4. Life is like a box of chocolates.
5. When will the Red Sox win the World Series?
6. This sentence is false.
7. A group of owls is called a parliament.
8. Every former President of the United States is buried in the United States.

1.2 Compound statements and logical connectives

Now that we have an idea of what a statement is, we need to see how to put them together to form more complex statements and proofs. Then we will be in the position to discuss the rules of logic as they apply to compound statements, which are statements formed by simpler statements using logical connectives or implications. This gives us the following axiom about the construction of compound statements.

Axiom 1.6. *If A and B are statements then so are:*

1. Not A.	$(\sim A)$	(negation)
2. A and B	$(A \wedge B)$	(conjunction)
3. A or B	$(A \vee B)$	(disjunction)
4. If A, then B.	$(A \Rightarrow B)$	(implication)

Given that we can combine statements to form new statements, we need to figure out how to determine the truth or falsity of a compound statement. In mathematics there is seldom much to argue about with regard to truth, but how do we figure it out? We'll start with the simplest way to form a new statement, the negation. As the name suggests, the negation of a statement has the opposite truth value. Here is the *truth table* for a negation.

A	$\sim A$
T	F
F	T

Since we're not focused on symbolic logic as a core of this course, we will restate this as an axiom.

Axiom 1.7. *If A is a statement with a given truth value, then $\sim A$ is a statement with the opposite truth value.*

Exercise 1.8. Write the negation of the following phrases.

1. Pi is a positive real number.
2. Georgia is the eleventh largest state.
3. Flatland State University has no major in paleontology.

It will be worth your while to start thinking about how to form the negation of a statement. In later sections we will see that the negation of a statement can sometimes be a helpful thing when trying to prove the statement. Once we have more machinery, we will be able to create more interesting examples of negation that may come in handy. Stay tuned.

Now we'll move on to the logical connectives **and** and **or**. We again use truth tables to define the truth of these compound statements from the truth of their constituent parts.

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

Notice the structure of the truth tables. The statement "A and B" is true only when both A and B are true. That is, an **and** statement is false if either part is false. Meanwhile, the statement "A or B" is true if either part of the statement is true. In spoken language, it is often the case that an **or** statement will only be true when one part or the other is true, but not both. For example in a restaurant there may be a menu option for `Soup or Salad`. This would imply that you could have one or the other, but probably not both without paying extra. This kind of connective is called an *exclusive or* and is commonly used in computer science. We will not use this connective in mathematical proofs.

For the sake of completeness and consistency, we include axioms that correspond to the truth tables for the logical connectives **and** and **or**, just as we included the truth table for negation.

Axiom 1.9. *Let A and B be statements. Then the statement $A \wedge B$ is true if and only if both A and B are true.*

Axiom 1.10. *Let A and B be statements. Then the statement $A \vee B$ is true unless both A and B are false.*

Exercise 1.11. Determine what information would be necessary to determine whether each of the following statements is true or false. Then, determine the truth values.

1. The capital of Spain is Barcelona and today is Wednesday.
2. The capital of Spain is Barcelona or today is Wednesday.
3. Yao Ming is short and Verne Troyer is tall.
4. The ratio of the circumference of a circle to its diameter is 3 or the capital of Indiana is Indianapolis.
5. Water boils at 100° Celsius and freezes at 0° Fahrenheit.
6. A boar is a kind of a pig and an uninteresting person.

Now we're ready to attempt to make our first proof. The following is a pair of statements that are known as DeMorgan's Laws. They display the relationship between negation and the logical connectives *and* and *or*. You may want to think of them as symbolic logic analogues of the distributive property that you are familiar with, though this metaphor is not quite precise. You will see other analogues of the distributive property very soon. In the exercises that follow, the instructions are to show that two statements are *logically equivalent*. By logically equivalent, we mean that two compound statements have the same truth values for each choice of truth value for the statements that make up the compound statement. In order to show this, you will need to construct a truth table. Once we have these new tools in place, we will be in a position to give the constituent statements a mathematical context and begin proving mathematical statements.

Problem 1.12 (DeMorgan's Laws). Let A and B be statements and establish the following:

1. $\sim (A \wedge B)$ is logically equivalent to $(\sim A) \vee (\sim B)$
2. $\sim (A \vee B)$ is logically equivalent to $(\sim A) \wedge (\sim B)$

Before we move on to other logical connectives we'll collect a few "algebraic" properties of the connectives we already have.

Problem 1.13 (The Commutative Property). Prove each of the following:

1. $A \wedge B$ is logically equivalent to $B \wedge A$
2. $A \vee B$ is logically equivalent to $B \vee A$

Problem 1.14 (The Associative Property). Prove each of the following:

1. $(A \wedge B) \wedge C$ is logically equivalent to $A \wedge (B \wedge C)$
2. $(A \vee B) \vee C$ is logically equivalent to $A \vee (B \vee C)$

Problem 1.15 (**The Distributive Property).** Prove each of the following:

1. $A \wedge (B \vee C)$ is logically equivalent to $(A \wedge B) \vee (A \wedge C)$
2. $A \vee (B \wedge C)$ is logically equivalent to $(A \vee B) \wedge (A \vee C)$

1.3 Implications

Now we'll examine compound statements of the form "if A, then B." In this statement the "if A" part is called the **antecedent** or **hypothesis** and A is called the **sufficient condition** for the implication. (Why?) The "then B" part is called the **consequent** or **conclusion** and B is called the **necessary condition** for the implication. (Why?) In mathematical proofs we will encounter this statement form often. The truth table for an implication is as follows:

A	B	$A \Rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

It seems to be counterintuitive that an implication could be true if the antecedent is false. Here's an example that may help you see why we might want to construct our truth table this way:

> Suppose that your uncle Ted tells you that if you get an A in this class, then he will pay for your next semester's tuition. If you get an A and he pays, then he told you the truth. On the other hand, if you get an A and he does not pay, then you can conclude that he lied to you. But if you do not get an A, then whether he pays or not, in neither case can you conclude that he lied to you. Thus, if we choose a truth table in which the antecedent of an implication being false goes with the statement form being true, then we are consistent with this idea.

As before, we now include an axiom that we can use when we are proving a statement in the form of an implication.

Axiom 1.16. *Let A and B be statements. Then the statement "If A, then B" is true unless A is true and B is false.*

Exercise 1.17. Determine the truth or falsity of the following statements.

1. If x is an integer, then $x^3 > 0$.
2. If y is a positive integer, then y can be written as a sum of powers of two.
3. If π is a rational number, then the area of a circle is $E = mc^2$.
4. If Homer Simpson is the President of the United States, then Marge Simpson is the Queen of England.
5. If Jupiter has more than three moons, then we live in the 20th century.

6. If the Earth is the center of the universe, then $3 = 5$.

As in non-mathematical English, there is often more than one way to say what you want to say. The following problem gives us an additional way to express the statement "*if A, then B*." We will see more about alternative ways to express a statement later. Stay tuned.

Problem 1.18. Prove that the following statements are equivalent:

1. If A, then B. $[A \Rightarrow B]$
2. Not A, or B. $[\sim A \vee B]$

Problem 1.19. Prove that the following statements are equivalent:

1. It is not the case that A implies B. $[\sim (A \Rightarrow B)]$
2. A and not B $[A \wedge \sim B]$

In the mathematical literature, and in logic texts, there are lots of other statements that carry the same meaning as the statement "*if A, then B*." These include:

A suffices for B	A only if B	B is necessary for A	B if A

You should think about why these statement forms are equivalent and try to become comfortable with using different, equivalent statement forms more or less interchangeably.

There are three natural statements that are closely associated with an implication. We can create these by reversing the direction of the implication and/or adding negations. The three statements are the **converse**, **inverse** and **contrapositive** of the implication and are given by the following.

Definition 1.20. *Suppose that $A \Rightarrow B$. The* **converse** *of $A \Rightarrow B$ is $B \Rightarrow A$. The* **inverse** *of $A \Rightarrow B$ is $(\sim A) \Rightarrow (\sim B)$. The* **contrapositive** *of $A \Rightarrow B$ is $(\sim B) \Rightarrow (\sim A)$.*

Example 1.21. Form the converse, inverse and contrapositive of the following statements:

1. If it rains, then the ground gets wet.
2. If a polygon has no diagonals, then it is a triangle.
3. If a function is differentiable, then it is continuous.
4. If fren is a glurb, then ramal will qapla.

Note: We will not often deal with the inverse of a statement. It is included here for the sake of completeness.

Given an implication, what do you think the relationship is among the truth tables for its converse, inverse and contrapositive? We can verify that an implication and its contrapositive are logically equivalent, while the inverse and converse may have different truth values than the original implication. The fact that a statement and its contrapositive are logically equivalent will often be useful to us when constructing proofs. More on this later.

Problem 1.22. Prove that an implication and its contrapositive are logically equivalent.

Problem 1.23. For a given implication, prove that its inverse and converse are logically equivalent.

Sometimes an implication and its converse are both true. In this case, we can make the statement "A if and only if B," or "A iff B," or $A \Longleftrightarrow B$. We call the two statements logically equivalent and this type of statement form is called a **biconditional equivalence**.

1.4 Quantifiers

It isn't very often that mathematicians make statements about isolated objects or cases. Often, if not usually, the statements concern every one of a certain kind of object or a statement asserts that it is possible to find an example of a certain thing. This gives rise to the need for **quantifiers**. There are two of these phrases: the universal quantifier **for all** ($\forall$), and the existential quantifier **there exists** ($\exists$). The use of the universal quantifier implies that a certain property is true for every object in a certain class and the existential quantifier states that an object possessing a certain property exists. Note that when we assert the existence of something in mathematics, we are not saying that only one of them exists. For example, if we say "there exists a flarn", we are making no claims about the uniqueness of the flarn, just that there is *at least one of them*. If we want to say something about uniqueness, we need to say it directly, such as "there exists a unique tweddle flarn."

Example 1.24. The following are examples of statements with quantifiers. Determine the truth value of each statement. Now try to form the negation of each statement.

1. All mammals have hair.
2. All rational numbers are natural numbers.
3. There exists a man from West Virginia.
4. There exists a real number whose square is not positive.

What happens when we negate a statement with a quantifier? That is, what is the effect of negation on quantifiers? If we negate a statement with a *for all* in it, then we're saying that it's not true that every object in a certain class possesses a certain property. In order for the negated statement to be true, we need only find one object for which the property is not true. On the other hand, when we negate a *there exists* statement, we're saying that no object possessing a certain property exists. If we denote the given property by $P(x)$, then the relationship between **for all** and **there exists** works as follows:

Example 1.25. The following pairs of statements are logically equivalent.

1. (a) "$P(x)$ is true for all x"
   (b) "There is no x for which $P(x)$ is not true."
2. (a) "There is some x for which $P(x)$ is true."
   (b) "It is not the case that $P(x)$ fails for all x."

Put another way, in a sort of extended logical system, a **for all** statement is like the conjunction of a very large number of simpler statements and a **there exists** statement is like the disjunction of a very large number of simpler statements.

Exercise 1.26. Rephrase the preceding comment in your own words.

Exercise 1.27. Write the negation of the following phrases.

1. All squares are rectangles.
2. Some squares are rectangles.
3. No squares are rectangles.

Chapter 2

Proof methods – Arguing isn't just for married people

> *"My dear Watson, try a little analysis yourself," said he with a touch of impatience. "You know my methods. Apply them, and it will be instructive to compare results."*
>
> –Arthur Conan Doyle, *The Sign of the Four*

In Section 1 our proofs were limited to a single type since the truth tables supplied all the meaning for the statements. This will not be the case once we give the statements some meaning in a mathematical context. That is, mathematics will give the statements meaning and we will use the tools we developed in the first chapter to give our new proofs an infrastructure. Most often we will attempt to phrase our mathematical statements in the form of an implication and then use our understanding of implications to prove these statements. The basic question then is: How do we go about applying the logical structure of an implication to prove a mathematical statement? If we can rephrase our statements in the form of an implication $[A \Rightarrow B]$, there are three basic methods we can appeal to in order to construct proofs. They are:

Direct proof: Assume A, follow logical deductions to conclude B.

Contrapositive: Assume $\sim B$ and follow logical deductions to conclude $\sim A$.

Contradiction: Assume A and $\sim B$, follow logical deductions to obtain a contradiction.

You will return to these methods over and over throughout the rest of your career in mathematics. They will be among the most often used tools in your toolbox. A word about proofs by contradiction. Since you are assuming two pieces of information, both A and $\sim B$, your proof will be weaker than if you chose another proof method. Also, since you can never be sure what the contradiction is that you will find, it is generally a better idea to choose a different proof method. Many students will use proof by contradiction as their default proof strategy, but this method really should be your last line of defense. A direct proof is a stronger argument and is often more constructive, so you should attempt to use those methods first. (Note that a proof by contrapositive is just a direct proof of the contrapositive of the statement.) Besides, knowing why something is true is a more powerful form of knowledge than just knowing that it isn't wrong.

We know that a proof should be written in a way that will convince someone of the truth or falsehood of a particular mathematical claim. Given that we believe that a statement is either true or false, how do we go about finding the argument that will convince our reader? Remember, it is not good form to purposefully try to conceal poor thinking. Here is a short list of hints that will help you find out how to begin forming your own proofs.

- **Know and understand the definitions** – As Carol Schumacher reminded us, definitions are at the heart of what mathematicians do. If you are unfamiliar with some technical term, look it up. There is no way that you can write a correct proof that you really understand, unless you know what everything means up front. There is **no substitute** for knowing the definitions.

- **Work out some examples** – Make sure what you're trying to prove has a chance to be true. While it isn't possible to prove many statements by providing examples (unless you're doing an existence proof) you should do enough examples to convince yourself that the statement you are working on might be true. Note that this is a good time to make sure that you are using the definitions correctly. Moreover, in the course of working out examples, you may stumble upon an idea that you can use in or for your proof. Inspiration sometimes comes from the strangest places.

- **Look for counterexamples** – If you have an idea that the statement you are trying to prove is not true, see if you can find a counterexample. Except in the case of existence proofs, you will generally be proving a statement with a universal quantifier. If you can find a case where the universal quantifier doesn't hold, then you have disproved the statement. While you can't prove something by finding a few examples, you can disprove a statement by finding a single counterexample.

- **Try to use standard proof methods** – Mimic what you know how to do. Start by trying to use a direct proof. If that doesn't work, try the contrapositive. Sometimes one statement may be easier to prove than the other because of the way a statement is phrased. Since any statement is logically equivalent to it's contrapositive, you may be able to rewrite the statement in a logically equivalent form that is more convenient to you to think about. When proving an "iff" statement, if you get stuck on one direction, then try the other one. This may sound too simple to have to write down, but sometimes the simple things are the easiest to forget.

- **Give your proof a skeleton** – Write down the statement, the word *Proof* and the assumptions. Then at the bottom of the page, write down the conclusion. Now see if you can write either a second line, based on the assumptions, or a statement from which the proof follows. Keep trying to work from both ends and you may just meet in the middle with a complete proof. This, perhaps, is not always the best strategy, but it can work.

- **Ockham's Razor** –All else being equal, the simplest solution is the best.[1]

[1] Don't worry about this one too much until you have developed some confidence in your proof writing. To begin with, just make sure that your argument really is a proof. After you have figured out how to write correct proofs, then you can start to think about polishing them.

In order to gain some practice using the methods discussed above, we will start with a few statements involving numerical properties that you are familiar with. For each of the Problems in this section, Prove, or Disprove and Salvage If Possible.

Definition 2.1. An integer n is **even** if $n = 2k$ for some integer k. An integer n is **odd** if $n = 2k+1$ for some integer k.

Problem 2.2. An integer is even if and only if its square is even.

Problem 2.3. If x and y are odd integers, then so is xy.

Problem 2.4. If n is an integer, then n^2+3n+2 is even.

Problem 2.5. If x, y, z are integers and $x+y$ is odd and $y+z$ is odd, then $x+z$ is odd.

Problem 2.6. If x and y are odd integers, then $x+y$ is even.

Problem 2.7. Suppose x and y are distinct integers. Then

$$(x+1)^2 = (y+1)^2$$

if and only if $x+y = -2$.

Problem 2.8. If x and y are distinct, positive, real numbers, then $\frac{x}{y} + \frac{y}{x} > 2$.

Chapter 3

Mathematical induction – The domino effect.

> *Music is the pleasure the human soul experiences from counting without being aware that it is counting.* –Gottfried Leibniz

In the previous section we described three methods of *deductive proofs*. Here we describe proof by induction. But first, what is the difference? According to the *American Heritage Dictionary of the English Language*:

- **deduction** is reasoning from the general to the specific, and
- **induction** is reasoning from the specific to the general.

Often in mathematics we wish to make generalizations about things we have observed. In fact, often we want to prove a general statement based on what we can prove about a particular incarnation of that statement. An example of this is when we wish to prove that a statement is true for an infinite number of cases. Since we can only prove any particular statement for a finite number of cases (because of the lack of time machines) we need some way to extend this idea. The following idea is a common tool used to accomplish this goal.

Axiom 3.1 (The Principle of Mathematical Induction). *Let S be a subset of the natural numbers $\mathbb{N}$. If:*

(i) $1 \in S$*, and* [*the base case*]

(ii) $k \in S$ *implies* $k+1 \in S$, [*the inductive step*]

then $S = \mathbb{N}$.

How can we use induction to prove a statement? First we need to be able to phrase our statement in terms of natural numbers. That is, we need to be able to identify some property that our statement is asserting about each of the elements of some set. We will call this property P and say that $P(k)$ is the corresponding statement about the natural number k. Then there are two things to do. First we need to prove the base case. That is, we need to prove that $P(1)$ is true.

> Note that when we write $P(k)$ we mean "the statement $P(k)$ is true," and not some polynomial P. For instance if we wish to prove that for every $n \in \mathbb{N}$, the sum $1+2+\cdots+n = \frac{n(n+1)}{2}$, then we can let $P(k)$ be the statement "the sum of the first k natural numbers is $\frac{k(k+1)}{2}$." However, we would not write
>
> $$P(k) = 1+2+\cdots+k = \frac{k(k+1)}{2}$$
>
> since $P(k)$ has a **truth value** (of either true or false) rather than a numerical value like $\frac{k(k+1)}{2}$.

Secondly, we would need to prove the implication in part (ii) of the axiom. That is, we prove the implication $P(k) \Rightarrow P(k+1)$, using an appropriate method. A good first step in a proof by induction is to identify what $P(k)$ represents. To give your proof some structure, and so that you don't fall into the trap of thinking of $P(k)$ as some function, it is a good idea to explicitly write down what P stands for.

Example 3.2. If n is a natural number, then the sum of the first n natural numbers is $\frac{n(n+1)}{2}$.

> Since we are making an assertion about an arbitrary natural number, we really mean that it is true for all natural numbers. Hence, this is a convenient place to use induction. (Of course, you should look at a few examples to convince yourself that this statement could be true before you start trying to prove it.)

$$\begin{aligned}
1+2+3+4+5+6 &= 21 = \frac{42}{2} = \frac{6\cdot 7}{2} \\
1+2+3+\cdots+21 &= 231 = \frac{462}{2} = \frac{21\cdot 22}{2} \\
1+2+3+\cdots+231 &= 26796 = \frac{53592}{2} = \frac{231\cdot 232}{2}
\end{aligned}$$

Proof. [By induction.] Let $P(n)$ be the statement "the sum of the first n natural numbers is $\frac{n(n+1)}{2}$." (Now we need to show that $P(1)$ is true.)[1] Since $1 = \frac{1(2)}{2}$ we can see that $P(1)$ is true.[2] (This takes care of the base case, now we need to prove the inductive step.)

Now assume that $P(k)$ is true for some natural number k. That is

$$1+2+\cdots+k = \frac{k(k+1)}{2}.$$

[1] The parenthetical remarks are not so much a part of the proof, but rather what you should be thinking as you are constructing a proof.

[2] Notice here that we haven't really added anything **to** anything else. We usually think of addition in terms of adding two things (like numbers) to each other to make another number. In this case we are "adding" only a single term, which makes the sum the same as the single number we have added. We often make conventions like this in mathematics, like defining 0! to be 1.

Then, to see that $P(k+1)$ is true, observe that:

$$\begin{aligned}
1+2+\cdots+k+(k+1) &= \frac{k(k+1)}{2}+k+1 && \text{(Why?)}\\
&= \frac{k^2+k}{2}+\frac{2k+2}{2} && \text{(Why?)}\\
&= \frac{k^2+3k+2}{2} && \text{(Why?)}\\
&= \frac{(k+1)(k+2)}{2} && \text{(Why?)}\\
&= \frac{(k+1)\big((k+1)+1\big)}{2}.
\end{aligned}$$

Thus $P(k+1)$ is true. Therefore, by the axiom of induction, the sum of the first n natural numbers is given by $\frac{n(n+1)}{2}$. □

Problem 3.3. If n is a natural number, then the sum of the first n even numbers is n^2+n.

Problem 3.4. If n is a natural number, then the sum of the first n odd numbers is n^2.

Problem 3.5. If n is a natural number, then the sum of the cubes of the first n natural numbers is equal to $\left[\frac{n(n+1)}{2}\right]^2$.

Problem 3.6. If n is a natural number, then $n! \geq 2^{n-1}$.

Problem 3.7. For all $n \in \mathbb{N}$, the expression n^2+n+41 is a prime number.

Problem 3.8. If a and r are real numbers with $r \neq 1$ and n is a natural number, then

$$a+ar+ar^2+\cdots+ar^n = \frac{a(r^{n+1}-1)}{r-1}.$$

Problem 3.9. If n is a natural number, then

$$2\cdot 6\cdot 10\cdot 14\cdots(4n-2) = \frac{(2n)!}{n!}.$$

Problem 3.10. If $x \in \mathbb{R}$ with $x > -1$ and n is a natural number, then

$$(1+x)^n \geq 1+nx.$$

Sometimes the base case does not correspond to the natural number 1. This may be the case whenever some statement is not true for some number of cases. The following three problems illustrate this.

Problem 3.11. If n is a natural number and $n \geq 4$, then $3^n > 2n^2+3n$. [Note that the inequality is false if $n < 4$.]

Problem 3.12. If n is a natural number and $n \geq 4$, then $n! > n^2$. [Note that the inequality is false if $n < 4$.]

There is another formulation of induction where we assume a bit more information at the beginning. This method is sometimes called *complete induction* or *strong induction.*

Axiom 3.13 (Strong Mathematical Induction). *Let $P(n)$ be a statement for each natural number n.*[3] *If:*

(i) $P(1)$*, and*

(ii) if m is a natural number, then $[P(j)$ for all $j \leq m] \Rightarrow P(m+1)$

then $P(n)$ is true for every $n \in \mathbb{N}$.

Note the difference between the original induction axiom and this one. In this case, we are not only assuming that $P(n)$ to show $P(n+1)$, but rather that $P(j)$ is true for all j from 1 to n.

Problem 3.14. Every natural number greater than 1 is either prime or the product of primes.

Problem 3.15. Let $a_1 = 1$, $a_2 = 3$, and for $n \geq 2$ let $a_n = a_{n-1} + a_{n-2}$. Show that $a_n < (7/4)^n$ for all natural numbers.

Problem 3.16. Let $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, and for each natural number greater than 3 define $a_n = a_{n-1} + a_{n-2} + a_{n-3}$. Prove that $a_n < 2^n$ for all $n \in \mathbb{N}$.

[3] Later, we will call this a function on the natural numbers, but since we haven't defined the word 'function' yet, we'll refer to $P(n)$ as a statement.

Chapter 4

Set theory – What do you call a box with an empty box in it?

A set is a many that allows itself to be thought of as a one. –Georg Cantor

Now that we have some machinery to begin doing proofs, we can start gathering some mathematical ideas. We'll start with the notion of a **set**. If you recall from the introduction we mentioned that we might have to deal with undefinable terms. The notion of a set affords us our first encounter with this issue. The problem lies in the structure of our language. Consider, for example the following attempt at a definition:

Attempt 4.1. *A* **set** *is a collection of objects that all share some common characteristic.*

This seems reasonable, but we hit a problem when we try to decide what the word *collection* means. We would need to appeal to a definition of the word collection which might include words like: bunch, group, conglomeration or cluster. In the end, because our language has only finitely many words, we would come back to the word *set* and our definition would, in effect, be circular. Additionally, there is the problem of what we mean by the word *object*, but you already get the picture. However, all is not lost. We can all essentially agree on what we mean by a *set* without having to worry too much about rigor. What we will do is use our intuition to understand some things and then use these new ideas to show that our intuition was right.

4.1 Notation and definitions

Mathematical convention is to denote sets by capital letters and the objects in the sets by lower case letters. We call the objects in the sets **elements** and write "$x \in S$" for the phrase "x is an element of S." If we consider an element y that is not in S, then we write $y \notin S$.

There are two standard ways to describe a set. The **roster method** is used to describe a set by simply listing the elements of the set, such as $C = \{\mathtt{red}, \mathtt{yellow}, \mathtt{blue}\}$. The **set builder method** describes a set by explaining the commonality that the elements possess. For example, $P = \{x \mid x \text{ is a primary color}\}$. Notice that the set C is the same as the set P, but we have described the set in two different ways.

One thing we will need to be careful about is where our sets live. Given a set, or collection of sets, we need to specify the context in which we're considering it, or them. That is, every set lives inside some **universal set**. For example, if we consider the set S to contain the last year of the twentieth century during which the Boston Red Sox won the world series, then $S = \{1918\}$ and our universal set U could be the years between 1901 and 2000. That is $U = \{1901, 1902, \ldots 1999, 2000\}$. Of course there may be many others.

Example 4.2. Here are some examples of sets. For each set, choose two larger sets which could be appropriate universal sets.

1. $A = \{\text{odd prime numbers less than } 10\}$ (Can you rewrite this set in a different way?)
2. $B = \{x \mid x^2 - 8x + 12 = 0\}$ (Can you rewrite this set in a different way?)
3. $C = \{\text{West Virginia}, \text{South Carolina}, \text{Ohio}, \text{Georgia}\}$
4. $D = \{\text{West Virginia}, \text{South Carolina}, \text{Ohio}, \text{Virginia}\}$

Just like numbers, sets are mathematical objects. Hence there are operations we can perform on sets to get new different sets. There are many operations we can perform on sets, but we will deal mainly with three. These are the most common and will suffice for most of our purposes. We'll begin with the definition of equality.

Definition 4.3. *If A and B are sets, then we say that A and B are* **equal***, and write $A = B$, precisely when A and B contain the same elements.*

Definition 4.4. *If A and B are sets, then we say that A is a* **subset** *of B, written $A \subseteq B$, if every element of A is an element of B.*

Definition 4.5. *We let $\varnothing$ denote the set that contains no elements and call $\varnothing$ the* **empty set***.*

Note: We can also write the empty set as $\{\}$, but not $\{\varnothing\}$. (Why?)

Definition 4.6. *If A and B are sets, then the* **union** *of A and B, written $A \cup B$, is the set of all elements that are in A or B.*

Definition 4.7. *If A and B are sets, then the* **intersection** *of A and B, written $A \cap B$, is the set of all elements that A and B have in common.*

Definition 4.8. *If A and B are sets and $A \cap B = \varnothing$, then A and B are said to be* **disjoint***.*

Exercise 4.9. For the sets in Example 4.2 form the following unions and intersections. Write the sets using set builder notation or the roster method.
1. $A \cup B$ 2. $A \cap B$ 3. $C \cup D$ 4. $C \cap D$ 5. $A \cup D$

Definition 4.10. *If A is a subset of a universal set U, then the* **complement** *of A, written A^c is the set of all elements of U that are not in A.*

Problem 4.11. If $A \subseteq B$, then $B^c \subseteq A^c$.

Definition 4.12. *If A and B are sets, then the* **set difference** *of A and B, or A minus B, written $A \setminus B$ is the set of all elements of A that are not in B.*

Problem 4.13. Let A and B be sets. Show that $A \setminus B = A \cap B^c$.

Example 4.14. Let $A = \{1,2,3,4,5,6,7,8,9,10\}$, $B = \{2,4,6,8,10\}$, $C = \{2,3,5,7\}$ and $D = \{3,5,7,9\}$. Define the following:
1. $A \setminus C$ 2. $B \cup C$ 3. $C \cap D$ 4. $B \cup D$

4.2 Venn diagrams

We can sometimes represent the operations on sets graphically. If we have two sets, then we can use the following **Venn diagram** to compute the set operations from the preceding section. Notice that the two circles break the universal set into four distinct regions. There are two regions that contain the elements in either A or B but not in both. (Regions 1 and 2.) The middle football shaped region (#3) contains the elements that are common to both A and B and the region outside (#4) contains the elements of the universal set that are in neither A nor B.

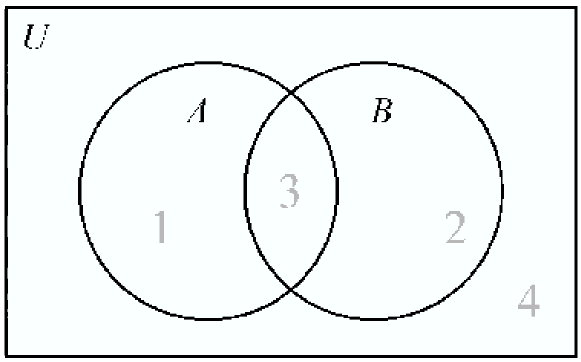

Figure 4.1. A Venn diagram for two sets A and B.

Exercise 4.15. Match the following sets with the regions of the Venn diagram above.
1. $A \cap B$ 2. $A \setminus B$ 3. $(A \cup B)^c$ 4. $B \setminus A$

We can extend this notion to Venn diagrams with three sets. This splits up the universal set unto eight distinct pieces.

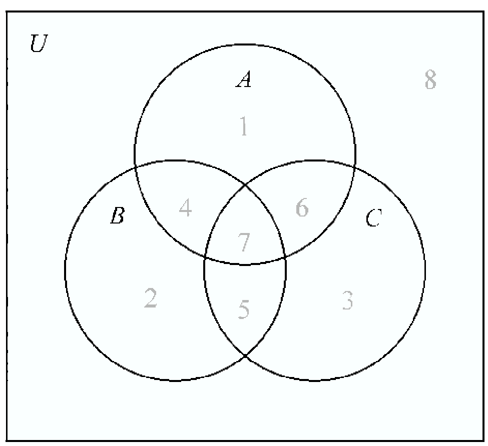

Figure 4.2. A Venn diagram for three sets A, B and C.

Exercise 4.16. Describe in words each of the eight regions in the three set Venn diagram.

Exercise 4.17. Use set notation to describe each of the regions in the three set Venn diagram.

Exercise 4.18. Draw a Venn diagram for each of the following sets.

1. $(A \cup B \cup C)^c$
2. $(A \cap B \cap C)^c$

Question 4.19. Can you find Venn diagrams for four or more sets?

4.3 Set operations

We can collect set operations that are analogous to the operations we had in Section 1 with regard to logical expressions. We can use Venn diagrams to get some intuition about proving statements about sets, though these pictures do not suffice to make a proof. The first four problems collect several facts about sets that should hopefully come as no surprise even though they might be counterintuitive at first glance.

Problem 4.20 (The Transitive Property). If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Problem 4.21. If A is a subset of the universal set U, then:

1. $A \cap U = A$
2. $A \cup \varnothing = A$

Problem 4.22. If A is a subset of the universal set U, then:

1. $A \subseteq A$
2. $\varnothing \subseteq A$

Problem 4.23. If A is a subset of the universal set U, then:

1. $A \cup A^c = U$
2. $A \cap A^c = \varnothing$

The following axiom will provide us a convenient technique to use when attempting to prove that two, possibly different looking, sets are the same set. It will come up often during this section.

Axiom 4.24. *Let A and B each be a set. Then $A = B$ iff $A \subseteq B$ and $B \subseteq A$.*

The theorems that follow are set theory analogues to statements you know about logical connectives. That they would hold for sets as well should be straightforward to see and their proofs are similar to those in the logic section.

For the following four theorems, suppose A, B and C are subsets of a universal set U.

Problem 4.25 (The Associative Property). Prove the following statements:

1. $A \cup (B \cup C) = (A \cup B) \cup C)$
2. $A \cap (B \cap C) = (A \cap B) \cap C)$

Problem 4.26 (The Commutative Property). Prove the following statements:

1. $A \cup B = B \cup A$
2. $A \cap B = B \cap A$

Problem 4.27 (The Distributive Property I). Prove the following statements:

1. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
2. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Problem 4.28 (The Distributive Property II). Prove the following statements:

1. $(A \cap B)^c = A^c \cup B^c$
2. $(A \cup B)^c = A^c \cap B^c$

In the following four problems, A, B and C are subsets of a universal set U.

Problem 4.29. $A = (A \setminus B) \cup (A \cap B)$

Problem 4.30. $A \setminus (A \setminus B) = A \cap B$

Problem 4.31. $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$

Problem 4.32. $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$

4.4 Set products and power sets

We can use existing sets to create new sets in other ways besides combining them in the context of an existing universal set like we did in the last section. Recall the Cartesian plane you learned about in high school algebra. How did you think about this? Probably as just the set of ordered pairs of real numbers. We can extend this idea to other pairs of sets. We'll start with the following definition:

Definition 4.33. If X and Y are sets, then the **set product** or **cross product** of X and Y, written $X \times Y$, is defined as the set of all ordered pairs of the form (x,y) such that $x \in X$ and $y \in Y$. That is

$$X \times Y = \{(x,y) \mid x \in X, y \in Y\}.$$

If $X = Y$, then we can write $X \times X$ as X^2.

Note that this definition fits our intuitive understanding of an ordered pair, and this will suffice for our purposes. However, when the full rigor of set theory is brought to bear, we define the ordered pair (x,y) by the set $\Big\{\{x\},\{x,y\}\Big\}$.

Exercise 4.34. Explain why the definition of ordered pair (x,y) from above captures the idea of an ordered pair.

Example 4.35. Let $S = \{a,b,c\}$ and $T = \{1,2\}$. Then the set $S \times T$ is given by:

$$S \times T = \{(a,1),(a,2),(b,1),(b,2),(c,1),(c,2)\}.$$

Exercise 4.36. For the sets $A = \{\square, \triangle, \bigcirc\}$ and $B = \{\bullet, \bar{\bullet}, \underline{\bullet}\}$ form the products $A \times A$, $A \times B$, $B \times A$ and $B \times B$. Is $A \times B = B \times A$? Why?

We will see this idea again when we start talking about functions and relations. It is the foundation of how these concepts work and is the basis for many other structures in algebra, analysis, topology and geometry.

Problem 4.37. Let A, B, C and D be sets. If $A \subseteq B$ and $C \subseteq D$, then $(A \times C) \subseteq (B \times D)$.

Problem 4.38. Let A, B and C be sets. Then $(A \cup B) \times C = (A \times B) \cup (B \times C)$.

Problem 4.39. Let A, B and C be sets. Then $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$.

Problem 4.40. Let A, B, C and D be sets. Then $(A \setminus B) \times (C \setminus D) = (A \times C) \setminus [(A \times D) \cup (B \times C)]$.

Exercise 4.41. Given three sets A, B and C, how do you think the set $A \times B \times C$ should be defined? How would this set compare to the set $A \times (B \times C)$? $(A \times B) \times C$?

Exercise 4.42. Let $A = \{a, b, c\}$, $B = \{1, 2\}$ and $C = \{\text{red}, \text{blue}, \text{green}\}$. How many elements should there be in the set $A \times B \times C$? Form the set $A \times B \times C$.

Exercise 4.43. Give a rigorous definition of an ordered triple (x, y, z).

Exercise 4.44. Suppose A, B and C are sets. Does $(A \times B) \times C = A \times (B \times C)$? Explain.

Now we move to the idea of a power set. This is another method of creating new sets from old. We'll start with a definition.

Definition 4.45. If P is a set, then the **power set** of S, written $\mathscr{P}(S)$, is the set of all subsets of S. That is $\mathscr{P}(S) = \{B \mid B \subseteq S\}$.

Example 4.46. Let $S = \{a, b, c\}$. Then

$$\mathscr{P}(S) = \Big\{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\Big\}.$$

Exercise 4.47. If S is a set with n elements, then $\mathscr{P}(S)$ has 2^n elements.

Problem 4.48. Prove that $S = T$ if and only if $\mathscr{P}(S) = \mathscr{P}(T)$.

In the following three problems, let A and B be sets. Prove, or Disprove and Salvage if possible:

Problem 4.49. $\mathscr{P}(A \cap B) = \mathscr{P}(A) \cap \mathscr{P}(B)$

Problem 4.50. $\mathscr{P}(A \cup B) = \mathscr{P}(A) \cup \mathscr{P}(B)$

Problem 4.51. $\mathscr{P}(A \setminus B) = \mathscr{P}(A) \setminus \mathscr{P}(B)$.

Exercise 4.52. In your own words, explain why the operation of creating a set product corresponds to multiplication of natural numbers and the operation of finding a power set corresponds to exponentiation of natural numbers.

4.5 Index sets and set operations

Often it is necessary to consider a large number of sets at once. In this case, when we may have more sets than letters, it is useful to have another way to describe the sets in question. We'll start with some notation and then some examples.

Definition 4.53. Let T be a set. If $\mathscr{T} = \{T_\alpha \mid \alpha \in A\}$ is a set of subsets of T, then we call $\mathscr{T}$ a **family** of sets. The set A is an **index set** and each of the elements of A is an **index**.

Example 4.54. Let $A_1 = \{a,b,c,d\}$ and $A_2 = \{d,e,f\}$. Then we can make the following observations:

1. The family $\mathscr{A}$ consists of two sets: $\mathscr{A} = \{A_1, A_2\}$.
2. Since there are two sets, we need two indices in our index set. That is $I = \{1,2\}$ will work for an index set.
3. If $i \in I$, then $i = 1$ or $i = 2$.
4. We can say that each set is a member of the family, but not a subset of the family. That is $A_i \in \mathscr{A}$ for each i, but $A_i \nsubseteq \mathscr{A}$ for either i.

Exercise 4.55. For the sets in Example 4.54 find $\bigcup\limits_{i\in I} A_i$ and $\bigcap\limits_{i\in I} A_i$.

Example 4.56. Consider the sets $[0,1]$, $[0,\frac{1}{2}]$, $[0,\frac{1}{3}]$,.... We can "list" these sets by means of an index set. Notice that each set is an interval that starts at zero and goes up to the reciprocal of a natural number. Thus we can index these sets by naming them as follows

$$I_1 = [0,1], \quad I_2 = \left[0,\frac{1}{2}\right], \quad I_3 = \left[0,\frac{1}{3}\right] \dots$$

where the index set is the natural numbers. In general we can say that $I_n = [0,\frac{1}{n}]$.

Example 4.57. Consider the set of circles centered at the origin each of whose radius is less than or equal to 1. We can define each set as follows: pick a radius, r, and the circle is

$$\left\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = r^2\right\}.$$

The interval $(0,1]$ can thus serve as an index set. Then we can name each set as follows

$$C_r = \left\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = r^2\right\}.$$

Note that in each of the previous two examples the index sets are infinite sets. This need not always be the case, but it is common.

Exercise 4.58. For the sets in Example 4.57 determine $\bigcup\limits_{r\in(0,1]} C_r$ and $\bigcap\limits_{r\in(0,1]} C_r$.

Exercise 4.59. Define a family of sets $\mathscr{F}$, which can be indexed by the natural numbers, so that:

$$\bigcup_{n\in\mathbb{N}} F_n = \mathbb{R} \qquad \text{and} \qquad \bigcap_{n\in\mathbb{N}} F_n = \varnothing.$$

Chapter 5

Functions and relations – The ins and outs of who knows who and favoritism

> *"The foundation for calculus, indeed for all of modern mathematics, is the language of functions and sets. As preparation for the long road of calculus, the student must have internalized these ideas so that he/she can use them comfortably and naturally. The student must be able to readily apprehend statements (both oral and written) made in this basic mathematical language and, more importantly, the student must also be able to formulate his/her own statements."*
>
> –Brian Blank & Steven Krantz, *Calculus*

You are likely familiar with the concept of a function. It is what drives the ideas in calculus and the algebra you have studied since high school. In this section we will make precise what you have a good understanding of already. We will begin by defining a relation and then using this idea to formulate the notion of a function. What you intuitively understand the word relation to mean is exactly what it means mathematically.

5.1 Relations and equivalence relations

Example 5.1. Consider the people in a class on the first day of the semester. Each student in the class may or may not know the first name of another student in the class. Mathematically we can describe this social phenomenon using a relation. Specifically, if we want to describe the fact that Cedric knows Alfred's first name, then we can use an ordered pair. We can write the ordered pair (Cedric, Alfred) to indicate that Cedric knows Alfred's first name. (*Why do we require the order?*) Hence, the relation **knows the first name of** is the set of ordered pairs (x,y) where person x knows person y's first name.

Definition 5.2. Let S and T be sets. A **relation** $\mathscr{R}$ on S and T is a subset of $S \times T$. That is $\mathscr{R} \subset S \times T$. If $\mathscr{R}$ is a relation, then we can write $(s,t) \in \mathscr{R}$ or $s\mathscr{R}t$ to indicate that (s,t) is an element of the relation. If (s,t) is an element of a relation, then we say that "s is related to t." Sometimes we will use mathematical shorthand and write $s \sim t$.

Exercise 5.3. Create a relation $\mathscr{K}$ that describes the situation in Example 5.1. Use a hypothetical class of at least six people where no one knows everyone's first name on the first day of class.

Exercise 5.4. Define a relation $\mathscr{W}$ on $\mathbb{N}\times\mathbb{N}$ by $(m,n)\in\mathscr{R}$ if $2m<n<3m+6$. Which of the following ordered pairs are in $\mathscr{W}$?
a) $(2,6)$ b) $(5,21)$ c) $(7,31)$ d) $(3,5)$ e) $(4,16)$
Explain your answers.

Definition 5.5. Suppose a relation $\mathscr{R}$ is defined on S and T. Then the **domain** of $\mathscr{R}$ is the set of all $s\in S$ such that $(s,t)\in\mathscr{R}$ for some $t\in T$. The set T is called the **codomain** of the relation and the **range** of $\mathscr{R}$ is the set of all $t\in T$ such that $(s,t)\in\mathscr{R}$ for some $s\in S$.

Exercise 5.6. Let $S=\{1,2,\dots 12\}$ and $T=\{1,2,\dots,30\}$. Define a relation $\mathscr{D}$ on S and T by $(s,t)\in\mathscr{D}$ if $s=3t-1$. Enumerate $\mathscr{D}$ and list the elements of the range of $\mathscr{D}$.

N.B. If $S=T$ then we can say that $\mathscr{R}$ is a relation defined on S without any ambiguity.

Exercise 5.7. Define a relation $\mathscr{F}$ on $\mathbb{Z}$ by the condition $(m,n)\in\mathscr{F}$ if $m-n$ is divisible by five. Show the following:

1. If $\ell\in\mathbb{Z}$, then $(\ell,\ell)\in\mathscr{F}$.
2. If $(m,n)\in\mathscr{F}$, then $(n,m)\in\mathscr{F}$.
3. If $(\ell,m)\in\mathscr{F}$ and $(m,n)\in\mathscr{F}$, then $(\ell,n)\in\mathscr{F}$.

Definition 5.8. Let $\sim$ be a relation on a set X. Then

1. $\sim$ is said to be **reflexive** if for all $x\in X$ we have $(x,x)\in\sim$. That is $x\sim x$.
2. $\sim$ is said to be **symmmetric** if for all $x,y\in X$ if $x\sim y$, then $y\sim x$.
3. $\sim$ is said to be **transitive** if for all $x,y,z\in X$, if $x\sim y$ and $y\sim z$, then $x\sim z$.

Problem 5.9. Let $X=\{1,2,3,4,5\}$.

1. Define a relation $\mathscr{R}$ on X that is reflexive, but not symmetric or transitive.
2. Define a relation $\mathscr{S}$ on X that is symmetric, but not reflexive or transitive.
3. Define a relation $\mathscr{T}$ on X that is transitive, but not reflexive or symmetric.

Problem 5.10. Let $Y=\{a,b,c,d,e\}$. Define a relation $\mathscr{A}$ on Y that is reflexive and symmetric, but not transitive. If this is not possible, explain why not.

Problem 5.11. Let $Y=\{a,b,c,d,e\}$. Define a relation $\mathscr{B}$ on Y that is reflexive and transitive, but not symmetric. If this is not possible, explain why not.

Problem 5.12. Let $Y=\{a,b,c,d,e\}$. Define a relation $\mathscr{C}$ on Y that is transitive and symmetric, but not reflexive. If this is not possible, explain why not.

Problem 5.13. Let $Z=\{\bigcirc,\square,\triangle,\bigstar,\diamondsuit\}$. Define a relation $\mathscr{E}$ on Z, with more than five elements, that is reflexive, symmetric and transitive.

Definition 5.14. A relation $\mathscr{P}$ defined on a set A is called an **equivalence relation** if $\mathscr{P}$ is reflexive, symmetric and transitive. The usual notation for an equivalence relation is $\sim$.

Problem 5.15. Define a relation $\mathscr{P}$ on $\mathbb{Z}$ by $(a,b) \in \mathscr{P}$ if a and b are both even or both odd. Prove that $\mathscr{P}$ is an equivalence relation.

Definition 5.16. If $\sim$ is an equivalence relation on a set S and $x \in S$, then the **equivalence class** of x, written E_x or $[x]$ is given by the set $\{y \in S \mid x \sim y\}$.

Exercise 5.17. What are the equivalence classes in Exercise 5.7? Problem 5.15?

Problem 5.18. Define a relation $\sim$ on $\mathbb{R}$ by, $x \sim y$ if $x - y$ is rational. Prove that $\sim$ is an equivalence relation.

Problem 5.19. Suppose $\sim$ is an equivalence relation on a set S. If $a \sim b$ for some $a, b \in S$, then $E_a = E_b$.

Problem 5.20. Suppose $\mathscr{R}$ is an equivalence relation on a set A. Prove that if s and t are elements of A, then either $E_s \cap E_t = \varnothing$ or $E_s = E_t$.

The punchline of Theorem 5.20 is that the equivalence classes of an equivalence relation *partition* the set A into pairwise disjoint subsets.

Problem 5.21. Consider the relation $\sim$ on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ defined by $(a,b) \sim (c,d)$ if $a \cdot d = b \cdot c$. Prove that $\sim$ is an equivalence relation.

Question 5.22. Can you describe the equivalence classes in Problem 5.21 in terms of familiar mathematical objects?

Problem 5.23. Let $S = \mathbb{R} \times \mathbb{R}$ and define a relation $\sim$ on S by $(a,b) \sim (c,d)$ if $d - b = 4(c - a)$. Show that $\sim$ is an equivalence relation.

Question 5.24. Can you describe the equivalence classes in Problem 5.23 in terms of familiar mathematical objects?

5.2 Order relations

Definition 5.25. Let $\sim$ be a relation on a set X. Then $\sim$ is said to be **antisymmetric** if for all $x, y \in X$, if $x \sim y$ and $y \sim x$, then $x = y$.

Definition 5.26. A relation $\leq$ on a set A is called a **partial ordering** if $\leq$ is reflexive, antisymmetric and transitive. A set A together with a partial ordering is called a **partially ordered set** or **poset**.

Notice that the symbol we have used for a partial ordering $(\leq)$ is a symbol that you are probably familiar with. This is because the usual notion of "*less than or equal to*" represents a partial ordering. Similarly, the idea of "*greater than or equal to*" also represents a partial ordering.

Exercise 5.27. Show that the usual meaning of $\leq$ is a partial ordering.

Problem 5.28. Let B be a set. Show that the relation $\subseteq$ on $\mathscr{P}(B)$ is a partial ordering.

Problem 5.29. Suppose that A and B are sets with partial orderings $<_A$ and $<_B$ respectively. Define a relation $<$ on $A \times B$ by defining $(a_1, b_1) < (a_2, b_2)$ if $a_1 <_A a_2$ and $b_1 <_B b_2$. Show that $<$ is a partial ordering on $A \times B$.

5.3 Functions

Now that we have a firm grasp on the idea of a relation and the structure of set products, we can move on to the idea of a function. You have likely spent most of your time in mathematics courses studying something about functions and you probably already have a working idea of what a function is and how it works.

Exercise 5.30. Consider the relation $\mathscr{R}$ on the set $A = \{1,2,3,4\}$ given by

$$\mathscr{R} = \{(1,2),(2,3),(3,4),(4,1),(3,3)\}.$$

Sketch these points on a graph. Do you think that $\mathscr{R}$ is a function? Why or why not?

Even though you could probably pick a function out of a lineup, what does the word really mean with respect to mathematics? Here's a definition of function paraphrased from a precalculus text that shall remain nameless:

> *A* **function** *f from a set A to a set B is a rule that assigns to each element x in A exactly one element y in B.*

What's wrong with this definition? Recall our discussion of how to define a set. We encounter the same difficulty here. What is this mysterious "rule" and how does it do the assigning? For that matter, what does it even mean to assign one element to another? It would probably be easy to get into a deep philosophical discussion about the nature of truth, but then we would be far off the track. However, all is not lost. From the preceding definition we can ferret out the idea that we somehow want to deal with pairs of things, often numbers. This easily leads us to the notion of a relation, since a relation is a set of ordered pairs. In fact, we will use the ideas we learned about relations to make the notion of a function precise.

Definition 5.31. Suppose f is a relation on X and Y. Then we say that f is a **function** if

1. for each $x \in X$, there exists a $y \in Y$ such that $(x,y) \in f$, and (who knows who)
2. if $(x,y) \in f$ and $(x,z) \in f$, then $y = z$. (favoritism)

If f is a function from X to Y, then we write $f : X \to Y$, and may say that "f maps X into Y". If $(x,y) \in f$, then we will often write the familiar $y = f(x)$.

How might we paraphrase Definition 5.31? The first part says that each element of X must be paired with some element of Y (who knows who) and the second part says that each element of X can be paired with only one element of Y (favoritism).

Definition 5.32. Suppose $f : X \to Y$ is a function. Then

1. The set X is called the **domain** of f.
2. The set Y is called the **codomain** of f.
3. The **range** of f is the set $\{y \in Y \mid y = f(x) \text{ for some } x \in X\}$.
4. If $A \subseteq X$, then the **image** of A is the set $\{y \in Y \mid y = f(x) \text{ for some } x \in A\}$, denoted by $f[A]$.

5. If $B \subseteq Y$, then the **preimage** if B is the set $\{x \in X \mid f(x) \in B\}$, denoted by $f^{-1}[B]$.

Note that the range of a function is a subset of the codomain, but not necessarily the same set as the codomain. If the range equals the codomain then the function is called "onto." We will come back to this shortly. Also, note the square brackets in the notation for the image of a set. This is an indication that the action of the function produces a subset of the codomain rather than just an element of the codomain.

Exercise 5.33. Let $A = \{a,b,c,d,e\}$ and $B = \{a,e,i,o,u\}$ and suppose f is a relation on $A \times B$ given by $f = \{(a,i),(b,i),(c,a),(d,i),(e,e)\}$. Does the relation f define a function from A to B? What is the range of f? Does the range equal the codomain?

Exercise 5.34. Let $A = \{\heartsuit, \clubsuit, \spadesuit, \diamondsuit\}$, $B = \{\bigcirc, \odot, \oplus, \otimes\}$ and $C = \{\triangle, \nabla, \triangleleft, \triangleright\}$. Which of the following define functions? If not, why not? For each relation that is a function, what is the domain? The range?

1. $\{(\heartsuit,\nabla),(\spadesuit,\nabla),(\diamondsuit,\nabla),(\clubsuit,\nabla)\}$
2. $\{(\nabla,\bigcirc),(\triangle,\oplus),(\triangleleft,\otimes),(\triangleright,\odot)\}$
3. $\{(\bigcirc,\odot),(\odot,\oplus),(\oplus,\otimes),(\otimes,\bigcirc)\}$
4. $\{(\heartsuit,\bigcirc),(\diamondsuit,\nabla),(\clubsuit,\oplus),(\spadesuit,\triangleleft)\}$
5. $\{(\triangle,\spadesuit),(\triangleleft,\heartsuit).(\triangleright,\clubsuit)\}$

Problem 5.35. Suppose $f : X \to Y$ and let $A \subseteq X$ and $B \subseteq X$. Prove:

1. If $A \subseteq B$, then $f[A] \subseteq f[B]$.
2. $f[A \cup B] = f[A] \cup f[B]$

Problem 5.36. Suppose $f : X \to Y$ and let $A \subseteq X$ and $B \subseteq X$. Prove:

1. $f[A \cap B] \subseteq f[A] \cap f[B]$. Give an example where equality fails.
2. $f[A \setminus B] \supseteq f[A] \setminus f[B]$. Give an example where equality fails.

Problem 5.37. Suppose $f : X \to Y$ and let $C \subseteq Y$ and $D \subseteq Y$. Prove:

1. If $C \subseteq D$, then $f^{-1}[C] \subseteq f^{-1}[D]$.
2. $f^{-1}[C \cup D] = f^{-1}[C] \cup f^{-1}[D]$

Problem 5.38. Suppose $f : X \to Y$ and let $C \subseteq Y$ and $D \subseteq Y$. Prove:

1. $f^{-1}[C \cap D] = f^{-1}[C] \cap f^{-1}[D]$
2. $f^{-1}[C \setminus D] = f^{-1}[C] \setminus f^{-1}[D]$

Functions are often used to model some sort of behavior, either of a physical system or to predict the behavior of a social system. In mathematics there are other uses of functions, namely we often use functions to compare mathematical structures. In order to make these comparisons more meaningful, we need two more definitions.

Definition 5.39. Suppose $f : X \to Y$ is a function. Then

1. f is **onto** (or **surjective**) if for every $y \in Y$, there exists an $x \in X$ such that $y = f(x)$.
2. f is **one-to-one**, or 1-1 (or **injective**) if $f(a) = f(b)$ implies $a = b$.

A function f that is both one-to-one and onto is called a **one-to-one correspondence** or a **bijection** or is said to be **bijective**.

With these two notions of kinds of functions we can determine things like whether two sets have the same size or whether two different looking mathematical objects have the same inherent structure.

Example 5.40. Suppose $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = 3x + 5$. Is f one-to-one? Onto?

Solution. *Pick $y \in \mathbb{R}$ and consider $x = \frac{y-5}{3}$. Then*

$$f(x) = f\left(\frac{y-5}{3}\right) = 3\left(\frac{y-5}{3}\right) + 5 = y - 5 + 5 = y.$$

So f is onto. Moreover, if $f(a) = f(b)$ then we see $3a + 5 = 3b + 5$. That is $3a = 3b$ or $a = b$, and so f is one-to-one.

Problem 5.41. Give an example of each of the following and prove the properties you claim.

1. A function $f : \mathbb{R} \to \mathbb{R}$ that is one-to-one but not onto.
2. A function $f : \mathbb{R} \to \mathbb{R}$ that is onto but not one-to-one.

Problem 5.42. Give an example of each of the following and prove the properties you claim.

1. A function $f : \mathbb{R} \to \mathbb{R}$ that is neither one-to-one nor onto.
2. A function $f : \mathbb{R} \to \mathbb{R}$ that is both one-to-one and onto and not of the form $f(x) = ax + b$.

Exercise 5.43. How do the graphs of your functions in Problems 5.41 and 5.42 help you understand the concepts of one-to-one and onto?

Problem 5.44. Let $A = \{a, b, c, d, e\}$ and $B = \{w, x, y, z\}$. Is there a one-to-one function from A into B? Explain.

Problem 5.45. Let $C = \{k, \ell, m, n\}$ and $D = \{p, q, r, s, t\}$. Is there an onto function from C into D? Explain.

Exercise 5.46. What, in general, can you say about the relative sizes of the domain and codomain of an onto function? A one-to-one function?

Problem 5.47. Suppose $f : X \to Y$ and let $A \subseteq X$ and $B \subseteq Y$. Then

1. If f is one-to-one, then $f^{-1}[f[A]] = A$.
2. If f is onto, then $f[f^{-1}[B]] = B$.

NB: Now that you have become familiar with the idea of a function we will write a function as a formula, (like $f(x) = 3x+4$), given that we know the domain and range of the function, whenever possible. Keep in mind, however, that it is not always possible, nor even desirable, to write a function in this form.

Just like we could combine sets to form sets, we can combine functions to create new functions. There are many different ways to combine functions, some familiar and some unfamiliar. We will start with four that are likely familiar and then look at the rigorous definition of an old friend from calculus. We begin with a definition.

Definition 5.48. Let $f,g : X \to Y$ be functions and assume that Y is a set in which the following operations make sense. Then the following are also functions:

1. $f+g$ defined by $(f+g)(x) = f(x)+g(x)$ for all $x \in X$
2. $f-g$ defined by $(f-g)(x) = f(x)-g(x)$ for all $x \in X$
3. $f \cdot g$ defined by $(f \cdot g)(x) = f(x) \cdot g(x)$ for all $x \in X$
4. $\dfrac{f}{g}$ defined by $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$ for all $x \in X$ such that $\frac{f(x)}{g(x)} \in Y$.

Notice that each of parts (a) through (d) of the preceding definition defines a (possibly different) function whose domain is a subset of X.

Question 5.49. Why is the phrase "possibly different" in the preceding definition? Can it ever be the case that any of the functions $(f+g)$, $(f-g)$, $f(\cdot g)$ or $\left(\frac{f}{g}\right)$ are the same as either f or g?

Question 5.50. What are the possibilities for the codomains of the functions in Definition 5.48? Can any of them be anything other than Y?

Example 5.51. Suppose f and g are functions with the same domain and codomain. What is $f+g$?

Solution. *Since f and g are functions, each can be represented by a set of ordered pairs. Then we can write $f+g$ as*

$$f+g = \{(x, y+y') \mid (x,y) \in f \text{ and } (x,y') \in g\}.$$

Exercise 5.52. Express each of the other three new functions in Definition 5.48 as a set of ordered pairs.

Exercise 5.53. Let $f,g : \mathbb{R} \to \mathbb{R}$ be functions defined by $f(x) = x^3+x^2$ and $g(x) = 4x-9$. Find $f+g$, $f \cdot g$ and $\frac{f}{g}$. Do all of these functions have the same domain?

In practice the operations you just looked at are the most convenient ones to deal with. They behave largely like numbers do, and you have used these operations many times in the past. The next operation, that of composition, is one you have also seen before, but it is often the most difficult one for students to get the hang of.

Definition 5.54. Suppose $f : X \to Y$ and $g : Y \to Z$ are functions. Then we can define the composite function $g \circ f : X \to Z$ by $(g \circ f)(x) = g(f(x))$ for all $x \in X$.

Note that since $f[X] \subseteq Y$, we know that each output of f is a valid input for g.

Exercise 5.55. Suppose $f : X \to Y$ and $g : Y \to Z$ are functions. Express the composite function $g \circ f$ as a set of ordered pairs.

Exercise 5.56. Suppose $f : S \to T$ and $g : X \to Y$ are functions. What condition would we need in order to form the composite function $f \circ g : X \to T$?

Exercise 5.57. Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions defined by $f(x) = x^3 + x^2$ and $g(x) = 4x - 9$. Find $f \circ g$ and $f \circ g$. What can you conclude about the commutativity of the operation $\circ$?

Problem 5.58. Suppose $f : X \to Y$ and $g : Y \to Z$ are functions.

1. If $g \circ f$ is one-to-one, prove that f must be one-to-one.
2. Find an example where $g \circ f$ is one-to-one, but g is not one-to-one.

Problem 5.59. Suppose $f : X \to Y$ and $g : Y \to Z$ are functions.

1. If $g \circ f$ is onto, prove that g must be onto.
2. Find an example where $g \circ f$ is onto, but f is not onto.

Problem 5.60. Suppose $f : X \to Y$ and $g : Y \to Z$ are functions. If f and g are one-to-one, then so is $g \circ f$.

Problem 5.61. Suppose $f : X \to Y$ and $g : Y \to Z$ are functions. If f and g are onto, then so is $g \circ f$.

Sometimes if the domain and codomain of one function are the codomain and domain of another function, the two functions can undo the action of each other. To make this precise we have the following definition:

Definition 5.62. Suppose $f : X \to Y$ and $g : Y \to X$ are functions such that $(g \circ f)(x) = x$ for all $x \in X$ and $(f \circ g)(y) = y$ for all $y \in Y$. Then we say that g is the **inverse** of f and write $g = f^{-1}$.

Problem 5.63. Consider sets A, B and C given by $A = \{1,2,3,4,5\}$, $B = \{a,b,c,d,e\}$ and $C = \{\clubsuit, \heartsuit, \diamondsuit, \spadesuit, \bigstar, ✠\}$. Further define $f : A \to B$ and $g : B \to C$ by

$$\begin{aligned} f &= \{(1,b),(2,d),(3,a),(4,c),(5,c)\}, \text{and} \\ g &= \{(a,\bigstar),(b,✠),(c,\clubsuit),(d,\diamondsuit),(e,\heartsuit)\}. \end{aligned}$$

1. Find $g \circ f$.
2. Find g^{-1}.
3. Determine whether f is invertible. Explain why or why not.

Exercise 5.64. Suppose $f : A \to B$ is a function whose inverse exists. Express f^{-1} as a set of ordered pairs given that $f = \{(a, f(a)) \mid a \in A\}$.

Note that even though we use the same notation for the inverse image of a set and the inverse of a function, they are not the same thing. The inverse of a function is another function and the inverse image of a set is another set. Even though we can think of a function as a set, there is a significant difference. The inverse image of a set is a set that always exists – even if it is the empty set – but the inverse of a function doesn't always exist. In fact, the inverse of a function will only exist if the function is both one-to-one and onto.

Exercise 5.65. In light of the preceding discussion, describe the difference between $f^{-1}(y)$ and $f^{-1}[\{y\}]$.

Problem 5.66. Suppose $f : A \to B$ is a bijection. Prove that:

1. $(f^{-1} \circ f)(a) = a$ for all $a \in A$
2. $(f \circ f^{-1})(b) = b$ for all $b \in B$

Appendix A

Mathematical Writing [by Neal Carothers]

Mathematics is suffering from a bad public image. This is actually a recent phenomenon: Fifty years ago mathematics had no public image at all, let alone a poor one. But a lot has changed in those fifty years. Calculus, for example, was at one time taught only to highly specialized scientists and engineers—now it's taught in most high schools. Mathematics effects a larger portion of our society than ever before and yet, somehow, elicits more disdain than excitement. Our society is becoming more mathematically literate, but evidently no more sympathetic to mathematics. Why?

Critics point to a growing inability (or unwillingness) of mathematics teachers to communicate. Those most able to express their interest in mathematics are apparently failing to do so. And if one generation of teachers does a poor job in communicating mathematical ideas, the next generation of teachers suffers. In other words, love and enthusiasm for mathematics is contracted from our teachers, much like a virus. Only those infected can pass it on.

At the heart of society's misconceptions about mathematics is the failure to recognize mathematics as a *human* endeavor. Human beings study mathematics because it pleases them to do so. Not because it builds better mouse traps. We study mathematics for the same reasons that we study art, or music, or literature. That mathematics is frequently useful to engineers and businessmen is typically of more interest to engineers and businessmen than mathematicians. In much the same way that art, music, and literature are useful to an advertiser producing television commercials, mathematics is useful to engineers and businessmen; but its utility is peripheral to the real object of study. The unenlightened often complain about this point of view.

Our challenge as mathematicians is to communicate the elegance and beauty of mathematics to the unenlightened without relying on its utility as a crutch. Your goal as a student of mathematics is to learn its language and its culture well enough to meet this challenge. The alternative is an unfulfilled life in which your work is misunderstood and unappreciated. One of the goals of this course is to help you avoid this bleak prospect by sharing a few ideas about how mathematics is written and how it is spoken.

The following suggestions may prove useful in improving your communication skills.

1. Mathematics is written in *complete sentences.* Any mathematically literate reader with the ability to translate the symbols should be able to understand each statement. A student taking this course in, say, Montana should be able to read and understand your writing. You are *not* writing simply for my benefit! It might help if you imagined that you were writing for the benefit of some mythical person (who may not have access to a particular reference or textbook).
2. It's polite, both to me and to our mythical friend, to include the *statement* of the problem (or the theorem you're about to prove) along with your solution (or proof). This not only makes the solution self-contained, and so easier to read, it also acts as a reminder of just what it is that you need to do.
3. Proofread, edit, rewrite, proofread, edit, rewrite, ... Try reading your solution aloud. Does it make sense? Is it clumsy or confusing? If not, then delete a few offensive lines, or add a few extra lines of clarification. A proof is judged first and foremost by its clarity. Elegance and simplicity are icing on the cake; they can only be introduced after a proof is "fully baked."
4. Be direct. Although proofs by contradiction are often short and "slick", you should avoid them, when possible, in favor of direct, more easily understood proofs. When contradiction seems the only logical course of action, then say so: A proof by contradiction should begin by announcing itself to the reader.
5. Write. You will notice that most "professional" proofs are absolutely thick with prose. Very few intelligible proofs are written using only mathematical symbols.[1]

[1] For more on this see Appendix B *Comments on Style.*

Appendix B

Comments on Style [by James R. Munkres]

The problems are to be written out carefully and correctly, in good mathematical style. This means:[1]

1. Write in complete sentences.

2. Punctuate! (Correctly, if possible.[2])

3. Avoid such abbreviations as $\exists$, $\forall$, $\wedge$, $\vee$, s.t., $\ni$, w.r.t, and similar vulgarisms.[3] All are acceptable in informal mathematical conversations, or in a research paper in Logic. In mathematics research journal or texts they are not allowed by editors. [There are a few horrendous exceptions. Here is an example, quoted from a textbook on topology:

   > "Let $f : [0,\Omega) \to [0,\Omega)$ be s.t. $f(\alpha) < \alpha$ for all $\alpha \geq$ some α_0. Then $\exists \beta_0 \forall \beta \exists \alpha \geq \beta : f(\alpha) \leq \beta_0$."

   Most mathematicians find this sentence unreadable "as is"; mentally they translate it into the English language. It is an example of bad mathematical style.]

4. About the symbols $\Rightarrow$ (implies) and $\Leftarrow$ (is implied by), there is some disagreement among mathematics editors as to their acceptability. They are coming into more wide-spread use, in any case. In this course, they will be acceptable.[4]

5. Try to steer a middle course between too much detail and not enough. Give reasons for your answers sufficient to convince the reader that your argument is correct and that you understand why it is correct. But don't bore the reader (and get writer's

[1] The footnotes are mine, not Munkres'.

[2] Mathematicians don't like to be picky about this, we're not the grammar police. However, the quality of your writing does reflect on you and as such you should try to adhere to the established rules you learned in your writing classes.

[3] These abbreviations will be acceptable if you are doing presentations at the board. However, in your written solutions you should generally avoid them.

[4] Moreover, if you are doing an *if and only if* proof, it is a good idea to use these markers to indicate to the reader which direction of the proof you are about to do.

cramp[5]) by checking each tiny detail labor-ously in writing. At one extreme of style (bad) are those texts written so concisely that the reader much fill in most of the details himself. At the other extreme (also bad) are the problem set solutions written by your most conscientious fellow student, from which it is almost impossible to extract the basic idea, because of the wealth of detail included! Try to hit somewhere in the middle.

6. Incidentally, an illegible proof is incorrect by definition! [6]

7. A common error is to write in what I call "stream of consciousness" style, a la William Faulkner. When you finish a thought, stop, put down a period, and take a good breath before you begin the next sentence (with a capital letter, please).[7]

[5]or carpal tunnel syndrome

[6]You can eliminate the possibility of this problem if you prepare your proofs in LaTeX.

[7]It may be a good idea to really embrace the spirit of this if you are doing presentations at the board. As you are writing down your proof, you should explain it line by line as you are transcribing it on the board. If you write it all down in advance and then explain it, your classmates may not be able to keep up with what you are doing if they are trying to write down what you did while they listen to your explanation. Of course, you should abide by the rules set down by your teacher.

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