سوال ۱

$P(X) = \frac{e^{-\alpha}\alpha^x}{x!} \xrightarrow{\alpha=\lambda*T} \frac{e^{-\lambda*T}(\lambda*T)^x}{x!}$

$T = t_a + t_b, t_a \to Gamma(\beta,\theta) \to P(X,T) = \frac{e^{-\lambda*T}(\lambda*T)^x}{x!}$

$P(X, t_a + t_b) = P_X(X, t_a) * P_T(t_a) + P_X(X, T - t_a) * P_T(T - t_a)$

$P_T(t_a) = \frac{\beta*\theta*(\beta*\theta*t_a)^{\beta-1}e^{-\beta\theta t_a}}{\Gamma(\beta)}$

$P(X, t_a{+}t_b) = \frac{e^{-\lambda_1*t_a}(\lambda_1*t_a)^x}{x!} \times \frac{\beta*\theta*(\beta*\theta*t_a)^{\beta-1}e^{-\beta\theta t_a}}{\Gamma(\beta)} + \frac{e^{-\lambda_2*(T-t_a)}(\lambda_2*(T-t_a))^x}{x!} \times \frac{\beta*\theta*(\beta*\theta*(T-t_a))^{\beta-1}e^{-\beta\theta(T-t_a)}}{\Gamma(\beta)}$

$\xrightarrow[\theta=5,\Gamma(\beta)=2]{\beta=2,\lambda_1=3,\lambda_2=2} P(X, t_a) = \frac{(50t_a e^{-10t_a})*e^{-3*t_a}*(3*t_a)^x + (50(10-t_a)e^{-10(10-t_a)})*e^{-2*(10-t_a)}*(2*(10-t_a))^x}{x!}$

$\xrightarrow{\text{marginalize } t_a} P(X) = \int_0^{10}(\frac{(50t_a e^{-10t_a})*e^{-3*t_a}*(3*t_a)^x + (50(10-t_a)e^{-10(10-t_a)})*e^{-2*(10-t_a)}*(2*(10-t_a))^x}{x!})dt_a$

Aproximate answer = 0.0005

سوال ۲

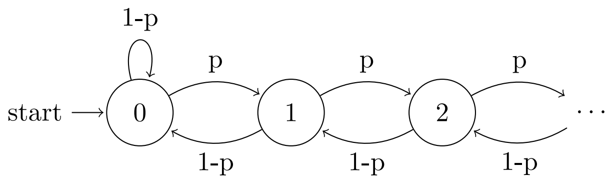
We split our probability into two parts, we consider that the device will get broken in the first 5 hours and then remain good for the rest of the 45 hours.
$P(X < 50) = P(0 \leq X \leq 5|extensive\_time) + P(5 \leq X \leq 50|normal\_time)$

$\lambda_{extensive} = 1/50, \lambda_{normal} = 1/500 \rightarrow F_{extensive} = 1 - e^{-5*1/50}, F_{normal} = 1 - e^{-45*1/500}$

$F_{noraml} + F_{extensive} = F_{total} = 0.0952 + 0.0865 = 0.1817 \rightarrow$ The machine will get broken in 18.17% in the first 50 hours of its work

سوال ۳

۱.



۲.

Lets write our probabilities according to π_0

$\pi_0 = (1-p)\pi_0 + (1-p)\pi_1 \rightarrow \pi_1 \Rightarrow \pi_1 = \frac{p}{1-p} * \pi_0$

$\pi_1 = (p)\pi_0 + (1-p)\pi_2 \rightarrow (1-p)\pi_2 \Rightarrow \pi_2 = \frac{p}{1-p} * \pi_0 - p\pi_0 = \frac{p^2}{(1-p)^2} * \pi_0$

$\pi_2 = (p)\pi_1 + (1-p)\pi_3 \rightarrow (1-p)\pi_3 \Rightarrow \pi_3 = (\frac{p}{1-p})^2 * \pi_0 - p\pi_1 = \frac{p^3}{(1-p)^3} * \pi_0$

$\vdots$

$\pi_{i-1} = (p)\pi_{i-2} + (1-p)\pi_i \rightarrow (1-p)\pi_i \Rightarrow \pi_i = (\frac{p}{1-p})^{i-1} * \pi_0 - p\pi_{i-1} = \frac{p^i}{(1-p)^i} * \pi_0$

$\pi_0 + \pi_1 + \pi_2 + \cdots + \pi_i + \cdots + \pi_n = 1 \rightarrow (1 + \frac{p}{1-p} + (\frac{p}{1-p})^2 + \cdots + (\frac{p}{1-p})^n)\pi_0 = 1$

$\frac{p}{1-p} = a \rightarrow a^0 + a^1 + \cdots + a^n = \frac{a^n - 1}{a-1} \xrightarrow{n \rightarrow \text{inf}} = \frac{1}{1-a} \rightarrow \frac{1}{1-a}\pi_0 = 1 \rightarrow \pi_0 = 1 - a = 1 - \frac{p}{1-p}$

$\pi_i = (\frac{p}{1-p})^i * (1 - \frac{p}{1-p})$

۳.

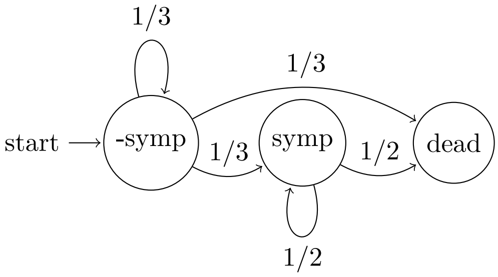
We know that : $0 \leq \pi_i \leq 1$

$\pi_i = (\frac{p}{1-p})^i * (1 - \frac{p}{1-p}) \xrightarrow{0 \leq i} 0 \leq 1 - \frac{p}{1-p} \leq 1$

We know that with probability 1, we go to state 0, because its the starting state :

$0 < 1 - \frac{p}{1-p} \rightarrow 2p < 1 \rightarrow p < \frac{1}{2}$

سوال ۴



$$\begin{array}{c|ccc} & -symp & symp & dead \\ \hline -symp & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ symp & 0 & \frac{1}{2} & \frac{1}{2} \\ dead & 0 & 0 & 1 \end{array}$$

$\pi_{-symp} = 1/3 * \pi_{-symp}$

$\pi_{symp} = 1/3 * \pi_{-symp} + 1/2 * \pi_{symp}$

$\pi_{dead} = 1/3 * \pi_{-symp} + 1/2 * \pi_{symp} + \pi_{dead}$

$\pi_{dead} + \pi symp + \pi_{-symp} = 1 \rightarrow \pi_{dead} = 1, \pi_{symp} = \pi_{-symp} = 0$

As we have seen, in infinity, the patient would likely(prob = 1) be dead.

$E(-symp) = 1 + 1/3 * E(-symp) + 1/3 * E(symp) + 1/3 * E(dead)$

$E(symp) = 1 + 1/2 * E(symp) + 1/2 * E(dead)$

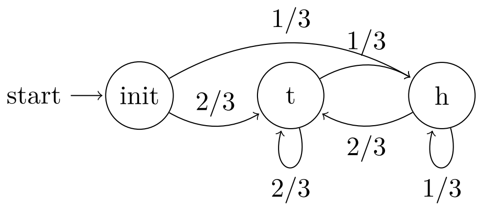
$E(dead) = 0$

Now we solve the equations to find our expected values.
$E(symp) = 1 + 1/2 * E(symp) \rightarrow E(symp) = 2$

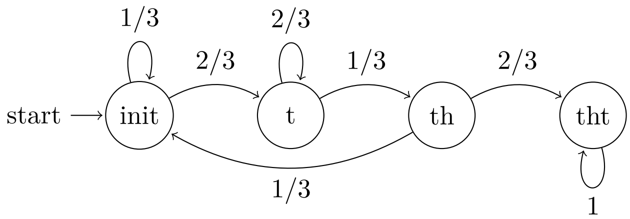
$E(-symp) = 1 + 1/3 * E(-symp) + 1/3 * 2 \rightarrow E(symp) = 5/2$

This means if we start with no syptom and each transition between states takes 1 day, the patient will likely die in 5/2 days.

سوال ۵



Now we change this change this chain to fullfil the question.



$E(init) = 1 + 1/3 * E(init) + 2/3 * E(t)$

$E(t) = 1 + 2/3 * E(t) + 1/3 * E(th)$

$E(th) = 1 + 2/3 * E(tht) + 1/3 * E(init)$

$E(tht) = E(init) - 33/4, E(th) = E(init) - 9/2, E(t) = E(init) - 3/2$

We know that if we get into E(tht), there will remain no step to take $\rightarrow E(tht) = 0$

$E(init) = 33/4, E(th) = 15/4, E(t) = 27/4$

So if we are in init, the expected steps to get to tht would be 33/4 !!!