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We should see the strategies and their outcomes.
If probability $Player_1 = P, Player_2 = Q, Player_3 = R \rightarrow$
$E(U_1(T,(q_1,q_2),(r_1,r_2))) = 3*q_1*r_1 + 1*q_2*r_1 + 0*q_1*r_2 - 4*q_2*r_2$
$E(U_1(B,(q_1,q_2),(r_1,r_2))) = -4*q_1*r_1 + 0*q_2*r_1 + 1*q_1*r_2 + 2*q_2*r_2$
$E(U_2((p_1,p_2),L,(r_1,r_2))) = 3*p_1*r_1 + 1*p_2*r_1 + 0*p_1*r_2 - 4*p_2*r_2$
$E(U_2((p_1,p_2),R,(r_1,r_2))) = -4*p_1*r_1 + 0*p_2*r_1 + 1*p_1*r_2 + 2*p_2*r_2$
$E(U_3((p_1,p_2),(q_1,q_2),X)) = -2*p_1*q_1 + 2*p_1*q_2 + 2*p_2*q_1 + 0*p_2*q_2$
$E(U_3((p_1,p_2),(q_1,q_2),Y)) = 0*p_1*q_1 + 2*p_1*q_2 + 2*p_2*q_1 - 2*p_2*q_2$

Now we know that :

$$E(U_1(T,...,...)) = E(U_1(B,...,...))$$
$$E(U_2(...,L,...)) = E(U_2(...,R,...))$$
$$E(U_3(...,...,X)) = E(U_3(...,...,Y))$$

$3*q_1*r_1 + q_2*r_1 - 4*q_2*r_2 = -4*q_1*r_1 + q_1*r_2 + 2*q_2*r_2 \Rightarrow 7q_1r_1 + q_2r_1 - q_1r_2 - 6q_2r_2 = 0$
$3*p_1*r_1 + 1*p_2*r_1 - 4*p_2*r_2 = -4*p_1*r_1 + 1*p_1*r_2 + 2*p_2*r_2 \Rightarrow 7p_1r_1 + p_2r_1 - p_1r_2 - 6p_2r_2 = 0$
$-2*p_1*q_1 + 2*p_1*q_2 + 2*p_2*q_1 = 2*p_1*q_2 + 2*p_2*q_1 - 2*p_2*q_2 \Rightarrow 2p_2q_2 - 2p_1q_1 = 0$
$p_1 + p_2 = 1, q_1 + q_2 = 1, r_1 + r_2 = 1$

Now we have 6 equations that we need to solve :
We use determinant to solve the equations

$$\vdots$$

$$p_1 = p_2 = \frac{1}{2}, q_1 = q_2 = \frac{1}{2}, r_1 = \frac{7}{15}, r_2 = \frac{8}{15}$$

$$MixedStrategy = ((\frac{T}{2}, \frac{B}{2}), (\frac{L}{2}, \frac{R}{2}), (\frac{7X}{15}, \frac{8Y}{15}))$$

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First we see that for player 2, A is a strictly dominated strategy so we remove it.

$$\begin{array}{ccc} & & q & 1-q \\ & & B & C \\ p_1 & X & -1,0 & 4,0 \\ p_2 & Y & 2,2 & 3,1 \\ 1-p_1-p_2 & Z & 2,4 & 3,3 \end{array}$$

Now we classify it.
$Q = (q, 1-q), P = (p_1, p_2, 1-p_1-p_2)$
$E(U_1(X,Q)) = -1 * q + 4 * (1-q) = 4 - 5q$
$E(U_1(Y,Q)) = 2 * q + 3 * (1-q) = 3 - q$
$E(U_1(Z,Q)) = 2 * q + 3 * (1-q) = 3 - q$
Now if we solve the equations above we would see that :
$\begin{cases} q > \frac{1}{4} & Y, Z \rightarrow \text{X is dominated} \\ q \leq \frac{1}{4} & X \rightarrow \text{Y,Z are dominated} \end{cases}$
So now we would put the question in two ways :
Case 1 : q > 1/4

$$\begin{array}{cccc} & & q & 1-q \\ & & B & C \\ p_2 & Y & 2,2 & 3,1 \\ 1-p_2 & Z & 2,4 & 3,3 \end{array} \Longrightarrow \text{C is dominated by B so q = 1} \Longrightarrow \begin{array}{ccc} & & 1 \\ & & B \\ p_2 & Y & 2,2 \\ 1-p_2 & Z & 2,4 \end{array} \Rightarrow Mixed = ((p_2Y, (1-p2)Z), (B))$$

Case 2 : q ≤ 1/4

$$\begin{array}{cccc} & & q & 1-q \\ & & B & C \\ p_1 & X & -1,0 & 4,0 \end{array} \Longrightarrow Mixed = ((X), (qB, (1-q)C)), q \leq \frac{1}{4}$$

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\begin{array}{ccccccc}
 & 1 & 2 & 3 & \dots & n & \\
1 & \omega_{11}-W_1 & \omega_{12} & \dots & & \omega_{1n} & p_1 \\
2 & \omega_{21} & \omega_{22}-W_2 & \dots & & \vdots & p_2 \\
3 & \omega_{31} & \omega_{32} & \dots & & \vdots & \vdots \\
\vdots & & & & & & \\
n & \omega_{n1} & \omega_{n2} & \dots & \dots & \omega_{nn}-W_n & p_n \\
Min & \omega_{11}-W_1 & & \dots & \dots & q_n &
\end{array}
$$

We know player 1, plays each node with probability p_i and player 2 plays with probability p_j so the expected payoff of player 1 is $\omega_{ij} - W_i \times p_j$.
now we need to find that for each i and j what is the expected payoff :

$\sum_i W_i * \sum_j [p_j * (\omega_{ij} - W_i * p_j)]$ thus the expression is written as $\sum_i W_i * \sum_j p_j * \omega_{ij} - W_i^2 * \sum_j (p_j^2) \xrightarrow{p_j \leq 1 \to p_j^2 \leq 1} \leq \sum_i W_i * \sum_j p_j * \omega_{ij} - W_i^2$

Now we could simplify the expression and see that :
$\sum_i W_i * \sum_j p_j * \omega_{ij} - W_i^2 = \sum i \sum j (p_i * p_j * \omega_i j)$
so we would conclude that the expected payoff of player 1 is the expected value of edges and the game is 0 value.

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The game is 0 value so there exists an equilibrim that both players payoffs are 0. Now like the first part, think as if $probability_{p1} = X, probability_{p2} = Y$
$E(P1 = x^T Ay) \xrightarrow{min_x max_y x^T Ay = 0} max_y = 0 \Rightarrow A^T x \leq 0, 0 \leq x, 0 \leq y, \sum_i x_i = 1, \sum_j y_j = 1$
Now the above problem is simply an LP with constraints and because that this game is a zero-sum game, the optimal values are the same as first problem.
Because we found an optimal point, there exists a solution in which $A^T x = 0 \Rightarrow x^T A = 0$

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We first need to draw the game matrix, as we know this game is a zero sum game and it is also a symmetric matrix, so our guess would be that the probabilities should be the same.
Now we prove our guess :

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		1	2	3	...	k	
	1	$-1,1$	$0,0$	...			m_1
Majid	2	$0,0$	$-1,1$	...		$\vdots$	m_2
	3	$0,0$	$0,0$	...		$\vdots$	$\vdots$
	$\vdots$						
	k	$0,0$	$0,0$	...	...	$-1,1$	m_k
		h_1	h_2	...	...	h_k	

Now we write the equations :
$E(U_1(1,H)) = -1 * h_1 + 0 * h_2 + ... + 0 * h_k = -h_1$
$E(U_1(2,H)) = 0 * h_1 - 1 * h_2 + ... + 0 * h_k = -h_2$
$\vdots$
$E(U_1(k,H)) = 0 * h_1 + 0 * h_2 + ... - 1 * h_k = -h_k$
$-h_1 = -h_2 = -h_3 = ... = -h_k \xrightarrow{h_1+h_2+...+h_k=1} h_1 = h_2 = ... = h_k = \frac{1}{k}$

$E(U_2(M,1)) = 1 * m_1 + 0 * m_2 + ... + 0 * m_k = m_1$
$E(U_2(M,2)) = 0 * m_1 + 1 * m_2 + ... + 0 * m_k = m_2$
$\vdots$
$E(U_2(M,m)) = 0 * m_1 + 0 * m_2 + ... + 1 * m_k = m_k$
$m_1 = m_2 = m_3 = ... = m_k \xrightarrow{m_1+m_2+...+m_k=1} m_1 = m_2 = ... = m_k = \frac{1}{k}$

If any of the probabilities are less than $\frac{1}{k}$ then it would be strictly dominated by mixed strategy of other strategies and we know that no strategy in this game is dominated likewise if any of probabilities are more than $\frac{1}{k}$ then they would dominate the other strategies and we know there is no dominating strategy in this game because of the nature of the game.
So to conclude :

$$MixedStrategy = (\tfrac{1}{k} \times (1,2,3,...,k), \tfrac{1}{k} \times (1,2,3,...,k))$$

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\begin{array}{ccccccccc}
 & & \text{P2} & & & & & & \\
 & & 1 & 2 & 3 & \dots & n & & \\
 & 1 & 0,0 & 1,2 & \dots & & 1,n & p_1 & \\
\text{P1} & 2 & 2,1 & 0,0 & \dots & & \vdots & p_2 & \\
 & 3 & 3,1 & 3,2 & \dots & & \vdots & \vdots & \\
 & \vdots & & & & & & & \\
 & n & n,1 & n,2 & \dots & \dots & 0,0 & p_n & \\
 & & q_1 & q_2 & \dots & \dots & q_n & &
\end{array}
\xrightarrow{n=5}
\begin{array}{ccccccc}
 & 1 & 2 & 3 & 4 & 5 & \\
1 & 0,0 & 1,2 & 1,3 & 1,4 & 1,5 & p_1 \\
2 & 2,1 & 0,0 & 2,3 & 2,4 & 2,5 & p_2 \\
3 & 3,1 & 3,2 & 0,0 & 3,4 & 3,5 & p_3 \\
4 & 4,1 & 4,2 & 4,3 & 0,0 & 4,5 & p_4 \\
5 & 5,1 & 5,2 & 5,3 & 5,4 & 0,0 & p_5 \\
 & q_1 & q_2 & q_3 & q_4 & q_5 &
\end{array}
$$

We get a little help from part 2 of this question, as we know and prove in part 2, for number in range 1..C, the formula is : $C = \lfloor n - \frac{\sqrt{8n+1}-1}{2} \rfloor \xrightarrow{n=5} C = 2$
We know that strateges 1 and 2 are strictly dominated by mixed strategies of (3,4,5) so :

$$
\begin{array}{ccccc}
 & 3 & 4 & 5 & \\
3 & 0,0 & 3,4 & 3,5 & p_3 \\
4 & 4,3 & 0,0 & 4,5 & p_4 \\
5 & 5,3 & 5,4 & 0,0 & p_5 \\
 & q_3 & q_4 & q_5 &
\end{array}
$$

Now we right the equations and see if any of the strategies are strictly dominated by
$E(U_1(3,Q)) = 0 * q_3 + 3 * q_4 + 3 * q_5 = 3 * (q_4 + q_5)$
$E(U_1(4,Q)) = 4 * q_3 + 0 * q_4 + 4 * q_5 = 4 * (q_3 + q_5)$
$E(U_1(5,Q)) = 5 * q_3 + 5 * q_4 + 0 * q_5 = 5 * (q_3 + q_4)$
$\xrightarrow[E(U_1(3,Q))=E(U_1(4,Q))=E(U_1(5,Q))]{q_3+q_4+q_5=1} q_3 = \frac{7}{47}, q_4 = \frac{17}{47}, q_5 = \frac{23}{47}$
We know that the game is symmetric so $q_3 = p_3, q_4 = p_4, q_5 = p_5 \Rightarrow$

$$MixedStrategy = ((\tfrac{7}{47}, \tfrac{17}{47}, \tfrac{23}{47}) \times (3,4,5)), (\tfrac{7}{47}, \tfrac{17}{47}, \tfrac{23}{47}) \times (3,4,5)))$$

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For this part we assume the matrix below,

$$
\begin{array}{cccccccc}
 & & & \text{P2} & & & & \\
 & & 1 & 2 & 3 & ... & n & \\
 & 1 & 0,0 & 1,2 & ... & & 1,n & p_1 \\
\text{P1} & 2 & 2,1 & 0,0 & ... & & \vdots & p_2 \\
 & 3 & 3,1 & 3,2 & ... & & \vdots & \vdots \\
 & \vdots & & & & & & \\
 & n & n,1 & n,2 & ... & ... & 0,0 & p_n \\
 & & q_1 & q_2 & ... & ... & q_n &
\end{array}
$$

Now lets see, think as if there exists a probability, p_i and $1 \leq i \leq C$ where $p_i > 0$:

$$
\begin{array}{ccccccccc}
 & & & \text{P2} & & & & & \\
 & & 1 & 2 & ... & i & ... & n & \\
 & 1 & 0,0 & 1,2 & ... & 1,i & ... & 1,n & p_1 \\
\text{P1} & 2 & 2,1 & 0,0 & ... & & \vdots & & p_2 \\
 & 3 & 3,1 & 3,2 & ... & & \vdots & & \vdots \\
 & \vdots & & & & & & & \\
 & i & i,1 & i,2 & ... & 0,0 & ... & i,n & p_i \\
 & \vdots & & & & & & & \\
 & n & n,1 & n,2 & ... & ... & 0,0 & & p_n \\
 & & q_1 & q_2 & ... & ... & q_n & &
\end{array}
$$

For p_i to be more than 0 :
$\forall j \in [C+1, n], j \neq i : E_{\text{Mixed}}(U_1(j, ...)) \leq E_{\text{Pure}}(U_1(i, ...))$
$(I) \forall j \in [C+1, n] : (C+1) * p_{(\text{C+1})} + ... + n * p_n - j * p_j \leq i$

(II)We also know that the game is symmetric, thus :
$E(U_2(C+1, P)) = (C+1) * (p_{(\text{C+2})} + p_{(\text{C+3})} + ... + p_n)$
$E(U_2(C+2, P)) = (C+2) * (p_{(\text{C+1})} + p_{(\text{C+3})} + ... + p_n)$
$E(U_2(C+3, P)) = (C+3) * (p_{(\text{C+1})} + p_{(\text{C+2})} + ... + p_n)$
$\vdots$
$E(U_2(n, P)) = (n) * (p_{(\text{C+1})} + p_{(\text{C+2})} + ... + p_{(\text{n-1})})$

$E(U_2(C+1, P)) = E(U_2(C+2, P)) = E(U_2(C+3, P)) = ... = E(U_2(n, P))$
$p_{(\text{C+1})} + p_{(\text{C+2})} + ... + p_n = 1$

By solving the equation above, we would see that for all k, k' in range [C+1,n], if $k' < k \Rightarrow$

$p_{k'} < p_k$.
Now j could at most be equal to n, so according to $(I) \rightarrow p_{C+1} * (C+1+C+2+...+n-1) \leq (C+1)*p_{C+1}+...+(n-1)*p_{n-1} \rightarrow p_{C+1}*(\frac{(n-1-C-1+1)(n-1+C+1)}{2}) \leq i \leq C = \lfloor n - \frac{\sqrt{8n+1}-1}{2} \rfloor$

Now by simplifing the expression we could see that the above equation is wrong and thus we have a contradiction.
So with proof by contradiction we have proven that there does not exist any p_i which $1 \leq i \leq C$ therfore $p_i = 0$ and we could strongly conclude that numbers [1,C] will not be played by player 1.
The game is symmetric so the same goes for player 2 and thus we know that no player would play numbers [1,C].