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$$\begin{array}{ccc} & x_2 & y_2 \\ x_1 & ③,⑦ & 6,6 \\ y_1 & 2,2 & ⑦,③ \end{array}$$

(x1,x2) and (y1,y2) are the nash equilibriums.

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First we see that for player 2, A is a strictly dominated strategy so we remove it.

$$\begin{array}{cccc} & & q & 1-q \\ & & B & C \\ p_1 & X & -1,0 & 4,0 \\ p_2 & Y & 2,2 & 3,1 \\ 1-p_1-p_2 & Z & 2,4 & 3,3 \end{array}$$

Now we classify it.
$Q=(q,1-q), P=(p_1,p_2,1-p_1-p_2)$
$E(U_1(X,Q)) = -1*q+4*(1-q) = 4-5q$
$E(U_1(Y,Q)) = 2*q+3*(1-q) = 3-q$
$E(U_1(Z,Q)) = 2*q+3*(1-q) = 3-q$
Now if we solve the equations above we would see that :
$\begin{cases} q > \frac{1}{4} & Y,Z \rightarrow \text{X is dominated} \\ q \leq \frac{1}{4} & X \rightarrow \text{Y,Z are dominated} \end{cases}$
So now we would put the question in two ways :
Case 1 : q > 1/4

$$\begin{array}{cccc} & & q & 1-q \\ & & B & C \\ p_2 & Y & 2,2 & 3,1 \\ 1-p_2 & Z & 2,4 & 3,3 \end{array} \Longrightarrow \text{C is dominated by B so q = 1} \Longrightarrow \begin{array}{ccc} & & 1 \\ & & B \\ p_2 & Y & 2,2 \\ 1-p_2 & Z & 2,4 \end{array} \Rightarrow Mixed = ((p_2Y,(1{-}p2)Z),(B))$$

Case 2 : q ≤ 1/4

$$\begin{array}{cccc} & & q & 1-q \\ & & B & C \\ p_1 & X & -1,0 & 4,0 \end{array} \Longrightarrow Mixed = ((X),(qB,(1-q)C)), q \leq \frac{1}{4}$$

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We could see that for player 2, if a2 >= 1, then the payoff would be less or equal to a2 = 0, so a2 >= 1 strategies are weakly dominated by a2 = 0 strategy, hence we should cut them off.
Now that we know that player 2 strategies are limited to only $0 \leq a_2 \leq 1$, we should find the payoffs :
If player 1 plays x_1 , the best possible move for player 2 would be $-\frac{a_1}{2} + \frac{1}{2}$ and if player 2 plays a_2, the best possible move for player 1 would be $\frac{a_2}{2}$.
If player 2 starts off with 1, player 1 would choose $\frac{1}{2}$ and if player 1 chooses $\frac{1}{2}$, player 2 would choose $\frac{1}{4}$ and so on ...
$a_2 = 1 \Rightarrow a_1 = \frac{1}{2} \Rightarrow a_2 = \frac{1}{4} \Rightarrow a_1 = \frac{1}{8} \Rightarrow ...$
If you get the limit of this equation you will see that $x_1 \to 0.2, x_2 \to 0.4$

سوال ۳

We first need to draw the game matrix, as we know this game is a zero sum game and it is also a symmetric matrix, so our guess would be that the probabilities should be the same.
Now we prove our guess :

$$
\begin{array}{ccccccccc}
 & & & & \text{Hamid} & & & & \\
 & & & 1 & 2 & 3 & \ldots & k & \\
 & & 1 & -1,1 & 0,0 & \ldots & & & m_1 \\
\text{Majid} & & 2 & 0,0 & -1,1 & \ldots & & \vdots & m_2 \\
 & & 3 & 0,0 & 0,0 & \ldots & & \vdots & \vdots \\
 & & \vdots & & & & & & \\
 & & k & 0,0 & 0,0 & \ldots & \ldots & -1,1 & m_k \\
 & & & h_1 & h_2 & \ldots & \ldots & h_k &
\end{array}
$$

Now we write the equations :
$E(U_1(1, H)) = -1 * h_1 + 0 * h_2 + ... + 0 * h_k = -h_1$
$E(U_1(2, H)) = 0 * h_1 - 1 * h_2 + ... + 0 * h_k = -h_2$
$\vdots$
$E(U_1(k, H)) = 0 * h_1 + 0 * h_2 + ... - 1 * h_k = -h_k$
$-h_1 = -h_2 = -h_3 = ... = -h_k \xrightarrow{h_1+h_2+...+h_k=1} h_1 = h_2 = ... = h_k = \frac{1}{k}$

$E(U_2(M, 1)) = 1 * m_1 + 0 * m_2 + ... + 0 * m_k = m_1$
$E(U_2(M, 2)) = 0 * m_1 + 1 * m_2 + ... + 0 * m_k = m_2$
$\vdots$
$E(U_2(M, m)) = 0 * m_1 + 0 * m_2 + ... + 1 * m_k = m_k$
$m_1 = m_2 = m_3 = ... = m_k \xrightarrow{m_1+m_2+...+m_k=1} m_1 = m_2 = ... = m_k = \frac{1}{k}$

If any of the probabilities are less than $\frac{1}{k}$ then it would be strictly dominated by mixed strategy of other strategies and we know that no strategy in this game is dominated likewise if any of probabilities are more than $\frac{1}{k}$ then they would dominate the other strategies and we know there is no dominating strategy in this game because of the nature of the game.
So to conclude :

$$MixedStrategy = (\frac{1}{k} \times (1, 2, 3, ..., k), \frac{1}{k} \times (1, 2, 3, ..., k))$$

سوال ۴

If we choose a profile in which a = (e,e,e,...), then we have nash equillibrium.
The proof is that $u_i = e$ and by mathamatical proof we show that for each player, utility of derivation of this profile is less that the original.
We assume that there exists one player j that chooses $e^{'}$:

$$\begin{cases} e^{'} < e & \rightarrow min_j e_j = e^{'} \rightarrow u_j = 2e^{'} - e^{'} < e \otimes \\ e^{'} > e & \rightarrow min_j e_j = e \rightarrow u_j = 2e - e^{'} < 2e - e \otimes \end{cases}$$

So we see that no other player would want to change the strategy, so a = (e,e,e,...) is a nash equillibrium.
Now we need to show that no other nash equillibrium points exist in this game.
We assume that there is a equillibrium $a = (e, e, e^{'}, e, ...)$ which at least one player has chosen a differenct strategy $e^{'}$ and the profile is also a nash equillibrium.

$$\begin{cases} e^{'} < e & \rightarrow min_j e_j = e^{'} \rightarrow u_j = 2e^{'} - e^{'} < e \\ e^{'} > e & \rightarrow min_j e_j = e \rightarrow u_j = 2e - e^{'} < 2e - e \end{cases} \Rightarrow \text{player with strategy } e^{'} \text{ will choose strategy e}$$

to get better utility hence the original point $a = (e, e, e^{'}, e, ...)$ was not a nash equillibrium.

To describe a game we would need : Players, Action Set, Utilities
To simlify the question, we name Hamid : player 1 and Masoud : player 2
Players : $N = \{1, 2\}$
Action Set : $a_1 = (ACEG, ADEG, ADFG)$, $a_2 = (BCEH, BDFH, BDEH)$
Utilities : $u_1(a_1, a_2)$ = cost for player 1 , $u_2(a_1, a_2)$ = cost for player 2

Now we draw the utility matrix :

	$BCEH$	$BDFH$	$BDEH$
$ACEG$	$-13, -10$	$-11, -5$	$-11, -7$
$ADEG$	$-8, -8$	$-8, -5$	$-10, -9$
$ADFG$	$-5, -8$	$-6, -6$	$-5, -7$

And now to remove the dominated strategies :

	$\cancel{BCEH}$	$BDFH$	$\cancel{BDEH}$
$\cancel{ACEG}$	$-13, -10$	$-11, -5$	$-11, -7$
$\cancel{ADEG}$	$-8, -8$	$-8, -5$	$-10, -9$
$ADFG$	$-5, -8$	$-6, -6$	$-5, -7$

	$BDFG$
$ADFG$	$-6, -6$

So the nash equilibrium would be : (ADFG,BDFG) ✓

We want the nash equilibrium to happen in (ACEG,BCEH) so it should not be dominated.
-10 should become more than -5 and -13 should become more than -5.
cost of multiple passes from CE < 2
cost of single pass from CE $<$ cost of multiple passes from CE < 2
In this case if our costs are integers and we dont have negative costs, then the only feasable solution would be CE = (0,1).
Now we draw our utility matrix once again :

	$BCEH$	$BDFH$	$BDEH$
$ACEG$	$-7, -4$	$-6, -5$	$-6, -7$
$ADEG$	$-8, -3$	$-8, -5$	$-10, -9$
$ADFG$	$-5, -3$	$-6, -6$	$-5, -7$

As we could see, ACEG is weakly dominated by ADFG so point (ACEG,BCEH) could not be a nash equilibrium.