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We should see the strategies and their outcomes.
If probability $Player_1 = P, Player_2 = Q, Player_3 = R \rightarrow$
$E(U_1(T,(q_1,q_2),(r_1,r_2))) = 3*q_1*r_1 + 1*q_2*r_1 + 0*q_1*r_2 - 4*q_2*r_2$
$E(U_1(B,(q_1,q_2),(r_1,r_2))) = -4*q_1*r_1 + 0*q_2*r_1 + 1*q_1*r_2 + 2*q_2*r_2$
$E(U_2((p_1,p_2),L,(r_1,r_2))) = 3*p_1*r_1 + 1*p_2*r_1 + 0*p_1*r_2 - 4*p_2*r_2$
$E(U_2((p_1,p_2),R,(r_1,r_2))) = -4*p_1*r_1 + 0*p_2*r_1 + 1*p_1*r_2 + 2*p_2*r_2$
$E(U_3((p_1,p_2),(q_1,q_2),X)) = -2*p_1*q_1 + 2*p_1*q_2 + 2*p_2*q_1 + 0*p_2*q_2$
$E(U_3((p_1,p_2),(q_1,q_2),Y)) = 0*p_1*q_1 + 2*p_1*q_2 + 2*p_2*q_1 - 2*p_2*q_2$

Now we know that :

$$E(U_1(T,...,...)) = E(U_1(B,...,...))$$
$$E(U_2(...,L,...)) = E(U_2(...,R,...))$$
$$E(U_3(...,...,X)) = E(U_3(...,...,Y))$$

$3*q_1*r_1 + q_2*r_1 - 4*q_2*r_2 = -4*q_1*r_1 + q_1*r_2 + 2*q_2*r_2 \Rightarrow 7q_1r_1 + q_2r_1 - q_1r_2 - 6q_2r_2 = 0$
$3*p_1*r_1 + 1*p_2*r_1 - 4*p_2*r_2 = -4*p_1*r_1 + 1*p_1*r_2 + 2*p_2*r_2 \Rightarrow 7p_1r_1 + p_2r_1 - p_1r_2 - 6p_2r_2 = 0$
$-2*p_1*q_1 + 2*p_1*q_2 + 2*p_2*q_1 = 2*p_1*q_2 + 2*p_2*q_1 - 2*p_2*q_2 \Rightarrow 2p_2q_2 - 2p_1q_1 = 0$
$p_1 + p_2 = 1, q_1 + q_2 = 1, r_1 + r_2 = 1$

Now we have 6 equations that we need to solve :
We use determinant to solve the equations

$$\vdots$$

$$p_1 = p_2 = \frac{1}{2}, q_1 = q_2 = \frac{1}{2}, r_1 = \frac{7}{15}, r_2 = \frac{8}{15}$$

$$MixedStrategy = ((\frac{T}{2}, \frac{B}{2}), (\frac{L}{2}, \frac{R}{2}), (\frac{7X}{15}, \frac{8Y}{15}))$$

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First we see that for player 2, A is a strictly dominated strategy so we remove it.

$$\begin{array}{cccc} & & q & 1-q \\ & & B & C \\ p_1 & X & -1,0 & 4,0 \\ p_2 & Y & 2,2 & 3,1 \\ 1-p_1-p_2 & Z & 2,4 & 3,3 \end{array}$$

Now we classify it.
$Q = (q, 1-q), P = (p_1, p_2, 1-p_1-p_2)$
$E(U_1(X, Q)) = -1 * q + 4 * (1-q) = 4 - 5q$
$E(U_1(Y, Q)) = 2 * q + 3 * (1-q) = 3 - q$
$E(U_1(Z, Q)) = 2 * q + 3 * (1-q) = 3 - q$
Now if we solve the equations above we would see that :
$\begin{cases} q > \frac{1}{4} & Y, Z \rightarrow \text{X is dominated} \\ q \leq \frac{1}{4} & X \rightarrow \text{Y,Z are dominated} \end{cases}$
So now we would put the question in two ways :
Case 1 : q > 1/4

$$\begin{array}{cccc} & & q & 1-q \\ & & B & C \\ p_2 & Y & 2,2 & 3,1 \\ 1-p_2 & Z & 2,4 & 3,3 \end{array} \Longrightarrow \text{C is dominated by B so q = 1} \Longrightarrow \begin{array}{ccc} & & 1 \\ & & B \\ p_2 & Y & 2,2 \\ 1-p_2 & Z & 2,4 \end{array} \Rightarrow Mixed = ((p_2 Y, (1-p2)Z), (B))$$

Case 2 : q ≤ 1/4

$$\begin{array}{cccc} & & q & 1-q \\ & & B & C \\ p_1 & X & -1,0 & 4,0 \end{array} \Longrightarrow Mixed = ((X), (qB, (1-q)C)), q \leq \frac{1}{4}$$

سوال ۲

We could see that for player 2, if a2 >= 1, then the payoff would be less or equal to a2 = 0, so a2 >= 1 strategies are weakly dominated by a2 = 0 strategy, hence we should cut them off.
Now that we know that player 2 strategies are limited to only $0 \leq a_2 \leq 1$, we should find the payoffs :
If player 1 plays x_1 , the best possible move for player 2 would be $-\frac{a_1}{2} + \frac{1}{2}$ and if player 2 plays a_2, the best possible move for player 1 would be $\frac{a_2}{2}$.
If player 2 starts off with 1, player 1 would choose $\frac{1}{2}$ and if player 1 chooses $\frac{1}{2}$, player 2 would choose $\frac{1}{4}$ and so on ...
$a_2 = 1 \Rightarrow a_1 = \frac{1}{2} \Rightarrow a_2 = \frac{1}{4} \Rightarrow a_1 = \frac{1}{8} \Rightarrow ...$
If you get the limit of this equation you will see that $x_1 \to 0.2, x_2 \to 0.4$

سوال ۳

We first need to draw the game matrix, as we know this game is a zero sum game and it is also a symmetric matrix, so our guess would be that the probabilities should be the same.
Now we prove our guess :

$$
\begin{array}{ccccccccc}
 & & \multicolumn{6}{c}{\text{Hamid}} & \\
 & & 1 & 2 & 3 & ... & k & & \\
 & 1 & -1,1 & 0,0 & ... & & & m_1 \\
\text{Majid} & 2 & 0,0 & -1,1 & ... & & \vdots & m_2 \\
 & 3 & 0,0 & 0,0 & ... & & \vdots & \vdots \\
 & \vdots & & & & & & \\
 & k & 0,0 & 0,0 & ... & ... & -1,1 & m_k \\
 & & h_1 & h_2 & ... & ... & h_k &
\end{array}
$$

Now we write the equations :

$E(U_1(1,H)) = -1 * h_1 + 0 * h_2 + ... + 0 * h_k = -h_1$

$E(U_1(2,H)) = 0 * h_1 - 1 * h_2 + ... + 0 * h_k = -h_2$

$\vdots$

$E(U_1(k,H)) = 0 * h_1 + 0 * h_2 + ... - 1 * h_k = -h_k$

$-h_1 = -h_2 = -h_3 = ... = -h_k \xrightarrow{h_1+h_2+...+h_k=1} h_1 = h_2 = ... = h_k = \frac{1}{k}$

$E(U_2(M,1)) = 1 * m_1 + 0 * m_2 + ... + 0 * m_k = m_1$

$E(U_2(M,2)) = 0 * m_1 + 1 * m_2 + ... + 0 * m_k = m_2$

$\vdots$

$E(U_2(M,m)) = 0 * m_1 + 0 * m_2 + ... + 1 * m_k = m_k$

$m_1 = m_2 = m_3 = ... = m_k \xrightarrow{m_1+m_2+...+m_k=1} m_1 = m_2 = ... = m_k = \frac{1}{k}$

If any of the probabilities are less than $\frac{1}{k}$ then it would be strictly dominated by mixed strategy of other strategies and we know that no strategy in this game is dominated likewise if any of probabilities are more than $\frac{1}{k}$ then they would dominate the other strategies and we know there is no dominating strategy in this game because of the nature of the game.
So to conclude :

$$MixedStrategy = (\frac{1}{k} \times (1, 2, 3, ..., k), \frac{1}{k} \times (1, 2, 3, ..., k))$$

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\begin{array}{c}
\text{P2} \\
\text{P1}\;
\begin{array}{c|ccccc|c}
 & 1 & 2 & 3 & \dots & n & \\
\hline
1 & 0,0 & 1,2 & \dots & & 1,n & p_1 \\
2 & 2,1 & 0,0 & \dots & & \vdots & p_2 \\
3 & 3,1 & 3,2 & \dots & & \vdots & \vdots \\
\vdots & & & & & & \\
n & n,1 & n,2 & \dots & \dots & 0,0 & p_n \\
\hline
 & q_1 & q_2 & \dots & \dots & q_n &
\end{array}
\end{array}
\xrightarrow{n=5}
\begin{array}{c|ccccc|c}
 & 1 & 2 & 3 & 4 & 5 & \\
\hline
1 & 0,0 & 1,2 & 1,3 & 1,4 & 1,5 & p_1 \\
2 & 2,1 & 0,0 & 2,3 & 2,4 & 2,5 & p_2 \\
3 & 3,1 & 3,2 & 0,0 & 3,4 & 3,5 & p_3 \\
4 & 4,1 & 4,2 & 4,3 & 0,0 & 4,5 & p_4 \\
5 & 5,1 & 5,2 & 5,3 & 5,4 & 0,0 & p_5 \\
\hline
 & q_1 & q_2 & q_3 & q_4 & q_5 &
\end{array}
$$

We get a little help from part 2 of this question, as we know and prove in part 2, for number in range 1..C, the formula is : $C = \lfloor n - \frac{\sqrt{8n+1}-1}{2} \rfloor \xrightarrow{n=5} C = 2$
We know that strateges 1 and 2 are strictly dominated by mixed strategies of (3,4,5) so :

$$
\begin{array}{c|ccc|c}
 & 3 & 4 & 5 & \\
\hline
3 & 0,0 & 3,4 & 3,5 & p_3 \\
4 & 4,3 & 0,0 & 4,5 & p_4 \\
5 & 5,3 & 5,4 & 0,0 & p_5 \\
\hline
 & q_3 & q_4 & q_5 &
\end{array}
$$

Now we right the equations and see if any of the strategies are strictly dominated by
$E(U_1(3,Q)) = 0 * q_3 + 3 * q_4 + 3 * q_5 = 3 * (q_4 + q_5)$
$E(U_1(4,Q)) = 4 * q_3 + 0 * q_4 + 4 * q_5 = 4 * (q_3 + q_5)$
$E(U_1(5,Q)) = 5 * q_3 + 5 * q_4 + 0 * q_5 = 5 * (q_3 + q_4)$
$\xrightarrow[E(U_1(3,Q))=E(U_1(4,Q))=E(U_1(5,Q))]{q_3+q_4+q_5=1} q_3 = \frac{7}{47}, q_4 = \frac{17}{47}, q_5 = \frac{23}{47}$
We know that the game is symmetric so $q_3 = p_3, q_4 = p_4, q_5 = p_5 \Rightarrow$

$$MixedStrategy = ((\tfrac{7}{47}, \tfrac{17}{47}, \tfrac{23}{47}) \times (3,4,5)), (\tfrac{7}{47}, \tfrac{17}{47}, \tfrac{23}{47}) \times (3,4,5)))$$

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