Geometric Origin of Scaling

in Large Traffic Networks

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Seminar zum FOKUS-Forschungsmodul

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World Airport Network of 4069 airports connected over 25.453 links
O'Danleyman *et al.* (2011)

- Understanding large traffic networks is key in order to predict human mobility patterns, epidemic spreading and global trade
- A lot of real networks are scale-free networks, i.e. they possess a power-law degree distributions $p(k) \sim k^{-\gamma}$

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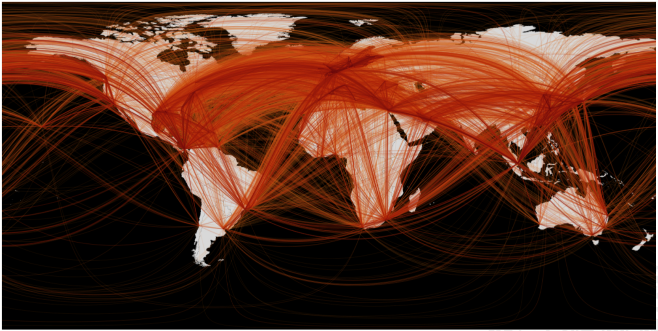
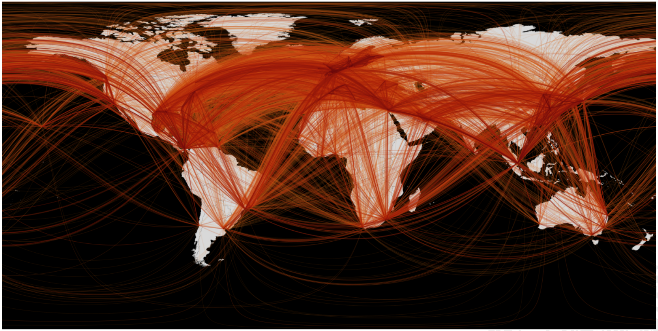


Global Cargo-Ship Network of 951 ports along 25.819 routes

O'Danleyman *et al.* (2011)

- Preferential attachment à la Barabási and Albert (2009) can lead to scaling in traffic networks
- Simple, robust model can show the geometric origin of relations between scaling exponents

Motivation



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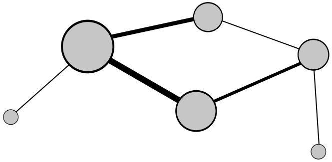
Outline

Introduction

Variables of interest for undirected graphs



adjacency matrix A_{ij}

degree of node $k_i = \sum_{j \neq i} A_{ij}$

weight of link w_{ij}

strength of node $s_i = \sum_{j \neq i} w_{ij}$

distance strength of node $s_i^d = \sum_{j \in n(i)} d_{ij}$

$n(i) =$ set of ports connected to i

Observed scaling

Study	β	β^d	α
World Airport Network (WAN) Barrat *et al.* (2004)	1.5 ± 0.1		0.5 ± 0.1
North American Airport Network Barrat *et al.* (2005)	1.7	1.4	
Airport Network of India Bagler (2008)	1.43 ± 0.06		
Passenger Airport Network of China Zhang *et al.* (2010)	1.58		
Cargo Airport Network of China Zhang *et al.* (2010)	2.2		
Network of Global Cargo Ship Movements (NGCSM) Kaluza *et al.* (2010)	1.46 ± 0.1		
WAN and NGCSM O'Danleyman *et al.* (2011)	1.33		
World Wide Maritime Transportation Network Hu and Zhu (2009)	1.3		

Scaling exp.

$$s_i \sim k_i^{\beta}$$

$$s_i^d \sim k_i^{\beta_d}$$

$$w_{ij} \sim (k_i k_j)^{\alpha}$$

Model assumptions

1 Each port has a hidden variable called fitness x

2 Probability of travel between two ports is $p(i \to j) = x_i x_j$

3 Link between ports will only exist if earnings cover costs

1 Each port has a hidden variable called fitness x

- x models socio-economic factors like population, number of companies, etc.
- $x \in [0, \infty)$ is a random number drawn from an arbitrary probability distribution $\rho(x)$

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- For undirected graphs this translates to weights $w_{ij} = x_i x_j$. w_{ij} should be interpreted as average fluxes between ports.
- Fluctuations around this rule can be implemented

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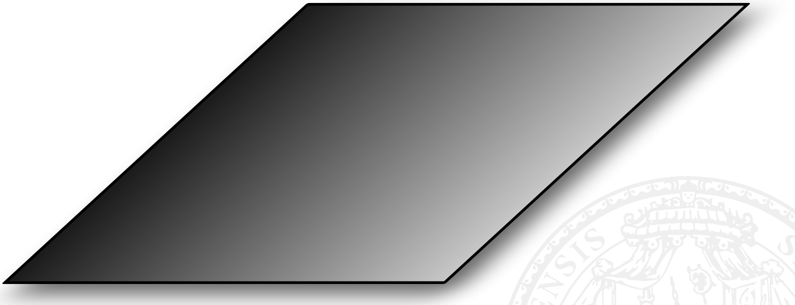
Model assumptions

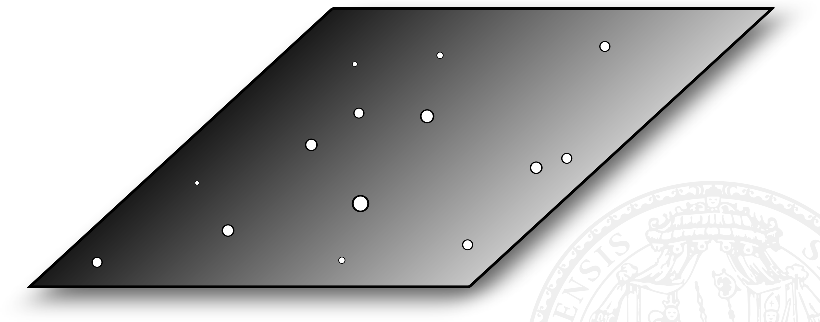
1. Each port has a hidden variable called fitness x
2. Probability of travel between two ports is $p(i \to j) = x_i x_j$
3. Link between ports will only exist if earnings cover costs
   - Earnings from volume of traffic and costs of travel are modelled by linear functions $f(x_i x_j)$ and $c(r_{ij})$
   - The linear relation between costs and travel distance is empirically verified for planes by W. M. Swan, N. Adler (2006)
   - The adjacency matrix A_{ij} becomes $\Theta\left(f(x_i x_j) - c(r_{ij})\right)$

Infinite 2D plane

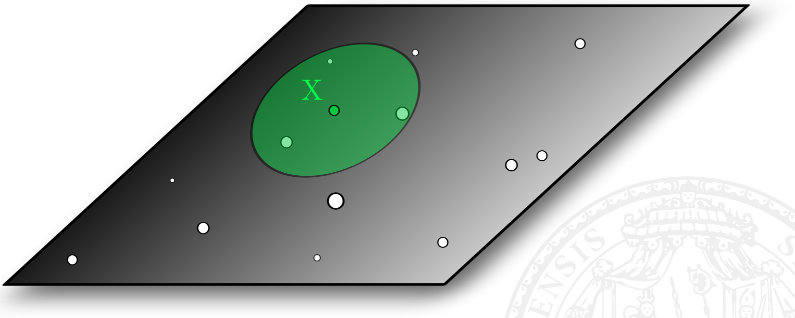
Infinite 2D plane



Imagine an infinite plane



Spread randomly ports as spatial Poisson process and assign random fitnesses



Calculate the ensemble average by summing up all connected ports for each subsystem

$$\begin{aligned}
\langle k(\mathrm{X})\rangle &= \sigma \int \mathrm{d}A \, \left\langle \Theta\left(f(x\mathrm{X}) - c(r)\right)\right\rangle_\rho \\
&= \sigma \int_0^\infty \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta \, r \int_0^\infty \mathrm{d}x \, \rho(x)\, \Theta\left(f(x\mathrm{X}) - c(r)\right) \\
&= 2\pi \int_0^\infty \mathrm{d}x \, \rho(x) \int_0^{c^{-1}(f(x\mathrm{X}))} \mathrm{d}r \, r \\
&= \pi\sigma \int_0^\infty \mathrm{d}x \, \rho(x) \left[c^{-1}\left(f(x\mathrm{X})\right)\right]^2
\end{aligned}$$

And for $f(x) = c(x)$

$$\langle k(\mathrm{X})\rangle = \pi\sigma\mathrm{X}^2 \underbrace{\int_0^\infty \mathrm{d}x \, \rho(x) x^2}_{=M_2(\rho)}$$

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Expected strength and distance strength of a node

$s_i = \sum_{j \neq i} w_{ij} \qquad s_i^d = \sum_{j \in n(i)} d_{ij}$

$$\langle s(\mathrm{X}) \rangle = \sigma \int \mathrm{d}A \; \langle x\mathrm{X}\,\Theta\left(x\mathrm{X} - r\right)\rangle_\rho$$
$$= 2\pi\sigma\mathrm{X} \int \mathrm{d}x \; \rho(x) x \int_0^{x\mathrm{X}} \mathrm{d}r \; r$$
$$= \pi\sigma\mathrm{X}^3 \underbrace{\int \mathrm{d}x \; \rho(x) x^3}_{M_3(\rho)}$$
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Proportionality

$$\langle k \rangle (\mathrm{X}) \sim \mathrm{X}^2 \qquad \Rightarrow \langle s \rangle \sim \langle k \rangle^{3/2}$$
$$\langle s \rangle (\mathrm{X}) \sim \mathrm{X}^3 \qquad \left\langle s^d \right\rangle \sim \langle k \rangle^{3/2}$$
$$\left\langle s^d \right\rangle (\mathrm{X}) \sim \mathrm{X}^3 \qquad \langle w_{ij} \rangle \sim \left(\langle k_i \rangle \langle k_j \rangle \right)^{1/2}$$

So far, we have no limitations on ρ. But for applications, it is constrained by some observed $p_0(k)$. To conserve the probability, we have to fulfill

$$\rho(x)\mathrm{d}x = p(k)\mathrm{d}k$$
$$\Rightarrow \rho(x) = \underbrace{\frac{\mathrm{d}k}{\mathrm{d}x}}_{\sim x} p\Big(\underbrace{k(x)}_{\sim x^2}\Big)$$

For power law distributions $p_0(k) \sim k^{-\xi}$

$$\rho(x) \sim x^{1-2\xi}$$

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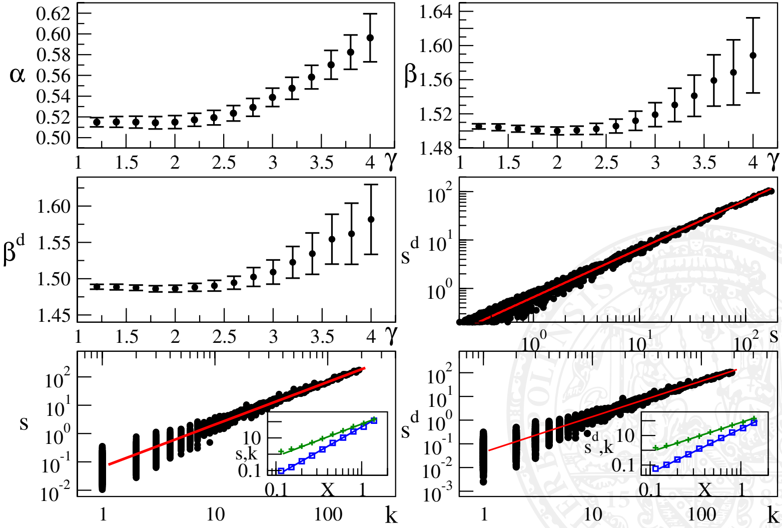
Simulations and real networks

$s_i \sim k_i^{\beta}$ $\qquad s_i^d \sim k_i^{\beta_d}$ $\qquad w_{ij} \sim (k_i k_j)^{\alpha}$ $\qquad \rho \sim x^{-\gamma}$



Simulations on finite 2D plane ($\gamma = 2.6$) M. Popović *et al.* (2012)

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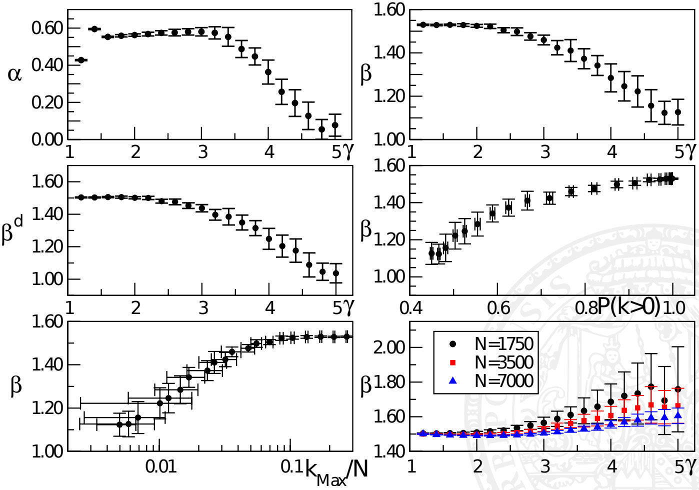

Simulations with real locations of 18745 American airports and on a sphere with $\sigma = N/4\pi$

M. Popović *et al.* (2012)

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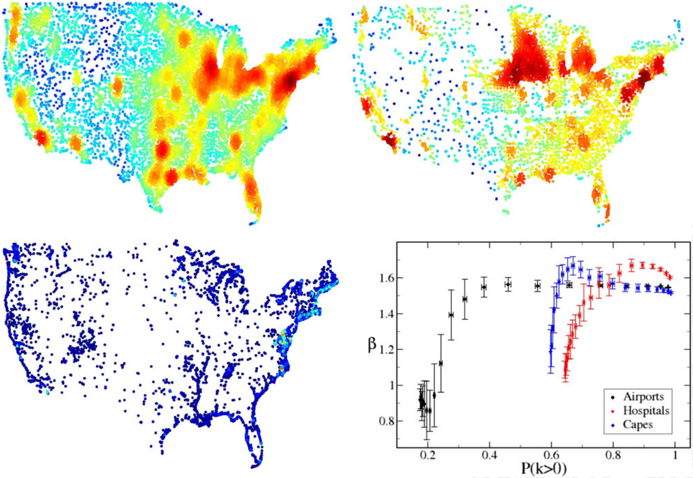

Locations of 18745 airports, 13385 hospitals and 11938 capes.

$x_{\text{min}} = 0.01$, $x_{\text{max}} = 0.65$, $N_{\text{runs}} = 20$ M. Popović *et al.* (2012)

Definition by Geographic Names Information System (GNIS)

Cape

Projection of land extending into a body of water (lea, neck, peninsula, point)

Obviously, capes form large traffic networks.

Networks on a sphere

How about the *real* geometry on which ports lie?

Basically, the integrals are the same

$$\langle k(\mathrm{X})\rangle = \sigma R^2 \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \sin\theta \mathrm{d}\theta \int_{R\theta/\mathrm{X}}^{\infty} \rho(x)\mathrm{d}x \ ,$$

$$\langle s(\mathrm{X})\rangle = \sigma R^2 \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \sin\theta \mathrm{d}\theta \int_{R\theta/\mathrm{X}}^{\infty} \mathrm{X}x\rho(x)\mathrm{d}x \ ,$$

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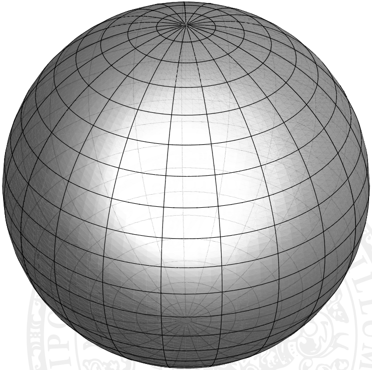
where we have set the origin of the coordinate system to a pole.



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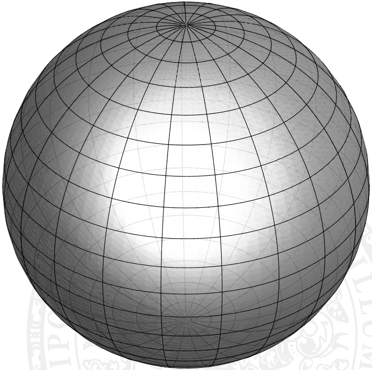

Calculating the expected degree

$$\langle k(\mathrm{X})\rangle = \sigma R^2 \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \sin\theta \mathrm{d}\theta \int_{R\theta/\mathrm{X}}^{\infty} \rho(x)\mathrm{d}x$$

By performing the angular integrations, we obtain

$$\langle k(\mathrm{X})\rangle = 2\pi\sigma R^2 \left[\int_0^{\pi} \mathrm{d}\theta\ \sin\theta \int_0^{\infty} \mathrm{d}x\ \rho(x)\Theta\left(x\mathrm{X} - \theta R\right)\right] \qquad \Leftrightarrow \theta \stackrel{!}{<} x\mathrm{X}/R$$

$$= 2\pi\sigma R^2 \left[\int_0^{\pi} \mathrm{d}\theta\ \sin\theta \left(\underbrace{\int_0^{\pi R/\mathrm{X}} \mathrm{d}x}_{x\mathrm{X}/R<\pi} + \underbrace{\int_{\pi R/\mathrm{X}}^{\infty} \mathrm{d}x}_{x\mathrm{X}/R>\pi}\right) \rho(x)\Theta\left(x\mathrm{X} - \theta R\right)\right]$$

$$= 2\pi\sigma R^2 \left[\int_0^{\pi R/\mathrm{X}} \mathrm{d}x\ \rho(x)\left(1 - \cos\frac{x\mathrm{X}}{R}\right) + \int_{\pi R/\mathrm{X}}^{\infty} \mathrm{d}x\ \rho(x)\cdot 2\right]$$

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Second integral may be neglected for distributions which fall off fast enough.

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By performing the angular integrations, we obtain

$$\begin{aligned}
\langle k(\mathrm{X})\rangle &= 2\pi\sigma R^2 \left[\int_0^{\pi} \mathrm{d}\theta\ \sin\theta \int_0^{\infty} \mathrm{d}x\ \rho(x)\Theta\left(x\mathrm{X} - \theta R\right)\right] \qquad \Leftrightarrow \theta \overset{!}{<} x\mathrm{X}/R \\
&= 2\pi\sigma R^2 \left[\int_0^{\pi} \mathrm{d}\theta\ \sin\theta \Bigg(\underbrace{\int_0^{\pi R/\mathrm{X}} \mathrm{d}x}_{x\mathrm{X}/R<\pi} + \underbrace{\int_{\pi R/\mathrm{X}}^{\infty} \mathrm{d}x}_{x\mathrm{X}/R>\pi}\Bigg)\rho(x)\Theta\left(x\mathrm{X} - \theta R\right)\right] \\
&= 2\pi\sigma R^2 \left[\int_0^{\pi R/\mathrm{X}} \mathrm{d}x\ \rho(x)\left(1 - \cos\frac{x\mathrm{X}}{R}\right) + \int_{\pi R/\mathrm{X}}^{\infty} \mathrm{d}x\ \rho(x)\cdot 2\right] \\
&= 2\pi\sigma R^2 \left[\int_0^{\infty} \mathrm{d}x\ \rho(x)\left(1 - \cos\frac{x\mathrm{X}}{R}\right) + \int_{\pi R/\mathrm{X}}^{\infty} \mathrm{d}x\ \rho(x)\left(1 + \cos\frac{x\mathrm{X}}{R}\right)\right].
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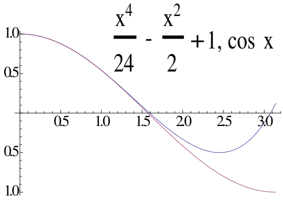

If $\rho(x)\to 0$ for $\frac{x\mathrm{X}}{R} > \pi$, we may expand

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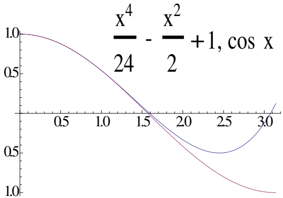

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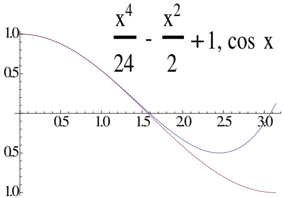

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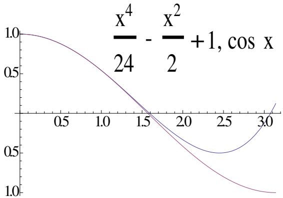

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one obtains

$$\frac{\mathrm{d}\,\langle s\rangle}{\mathrm{d}\,\langle k\rangle} = \frac{5\mathrm{X}M_3\left(\frac{3}{5} - \frac{\mathrm{X}^2 M_5}{12R^2 M_3}\right)}{2M_2\left(1 - \frac{\mathrm{X}^2 M_4}{6M_2R^2}\right)},$$

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$$\approx \frac{3}{2}\frac{M_3(\rho)}{M_2(\rho)}\mathrm{X} - \frac{M_4(\rho)}{4R^2 M_2(\rho)}\left[\frac{5}{6}\frac{M_5(\rho)}{M_4(\rho)} - \frac{M_3(\rho)}{M_2(\rho)}\right]\mathrm{X}^3$$

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This can be integrated with the condition that $\langle s \rangle = 0$ when $\langle k \rangle = 0$. By expressing X through $\langle k \rangle$, the exponents are calculated

$$\langle s \rangle = \frac{M_3(\rho)}{\sqrt{\pi\sigma} M_2^{3/2}(\rho)} \langle k \rangle^{3/2} + \frac{M_4(\rho)}{24 R^2 M_2^{5/2}(\rho)(\pi\sigma)^{3/2}} \left[3\frac{M_3(\rho)}{M_2(\rho)} - 2\frac{M_5(\rho)}{M_4(\rho)} \right] \langle k \rangle^{5/2}$$

And similarly $\left\langle s^d \right\rangle \sim \langle k \rangle^{3/2} + \alpha \langle k \rangle^{5/2}$ in leading order.

Wether corrections to the infinite exponents are negligible or not depends on the system.

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Further discussion

- Realistically, we can expect that some links are stronger or weaker than $x_i x_j$
- Politcal reasons might cause lower traffic than expected from fitness
- Model this by new random variable η from $p(\eta)$
- $w_{ij} = x_i \eta_{ij} x_j$ and $\langle \eta \rangle = 1$ leads to

$$\langle k(\mathrm{X}) \rangle = \pi \sigma M_2(\rho) M_2(p) \mathrm{X}^2$$

  and analogously for $\langle s \rangle$, $\langle s^d \rangle$
- Global scaling exponents are not seeing these kinds of fluctuations

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$$k_{\mathrm{nn}}(k) = \frac{1}{N_k} \sum_{i|k_i=k} \frac{1}{k} \sum_j A_{ij} k_j = \left\langle \frac{1}{k} \sum_j A_{ij} k_j \right\rangle_{\{i\}|k_i=k}$$

where N_k is number of ports with degree k

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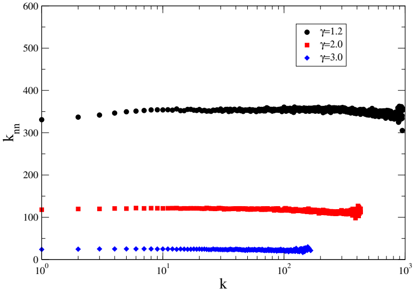
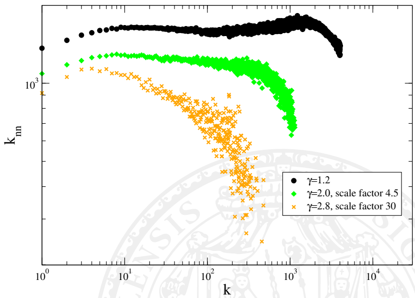


Flat finite plane



Locations of 18745 airports

M. Popović *et al.* (2012)

With the differential element $dV_D = r^{D-1} C_D dr$ follows

$$\langle k(\mathrm{X})\rangle = \sigma C_D \int_0^\infty r^{D-1} dr \int_0^\infty \Theta\left(x\mathrm{X} - r\right)\rho(x)dx, \qquad \to \langle k\rangle \sim \mathrm{X}^D$$

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and hence

$$\beta = \beta^d = \frac{D+1}{D} \quad \text{and} \quad \alpha = \frac{1}{D},$$

which might be of interest for the casual space traveller.

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$$\langle s^d(\mathrm{X})\rangle = \sigma C_D \int_0^\infty r^{D-1} dr \int_0^\infty r\Theta\left(x\mathrm{X} - r\right)\rho(x)dx \qquad \to \left\langle s^d\right\rangle \sim \mathrm{X}^{D+1}$$

and hence

$$\beta = \beta^d = \frac{D+1}{D} \quad \text{and} \quad \alpha = \frac{1}{D},$$

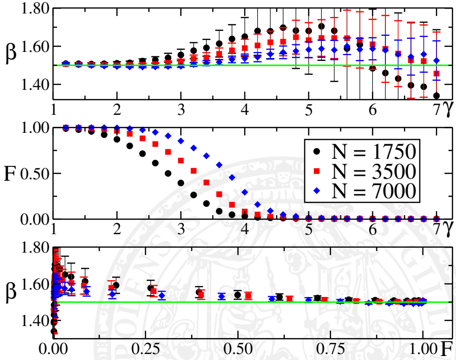
which might be of interest for the casual space traveller.

We have seen that infinite exponents are only well reproduced for $\gamma < 3$.

If the giant component breaks down, fluctuations of subgraphs dominate the behaviour of β.

The 'finite size' effect probably results from not keeping $\langle k \rangle$ fixed.

F is size of giant component.



M. Popović *et al.* (2012)

Simulations on a finite plane. Same picture for the sphere.

- Degree distributions of real networks can be modelled by introducing a fitness
- The adjacency matrix can be represented by a Heaviside function which compares fitnesses (earnings) to distances (costs)
- Relationships between scaling exponents set constraints on the distributions
- But they are quite robust with respect to the heterogeneity of the spatial distribution or the curvature of the plane

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