Linear Algebra

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1) Matrix

② Types of Matrix

(i) Square Matrix

(ii) Principle Sub-Matrix

(iii) Diagonal Matrix → Scalar Matrix
→ Unit Matrix or Identity Matrix

(iv) NULL Matrix (Need not be square) {O}

(v) Upper triangular Matrix {include diagonal}

(vi) Lower triangular Matrix { " " }

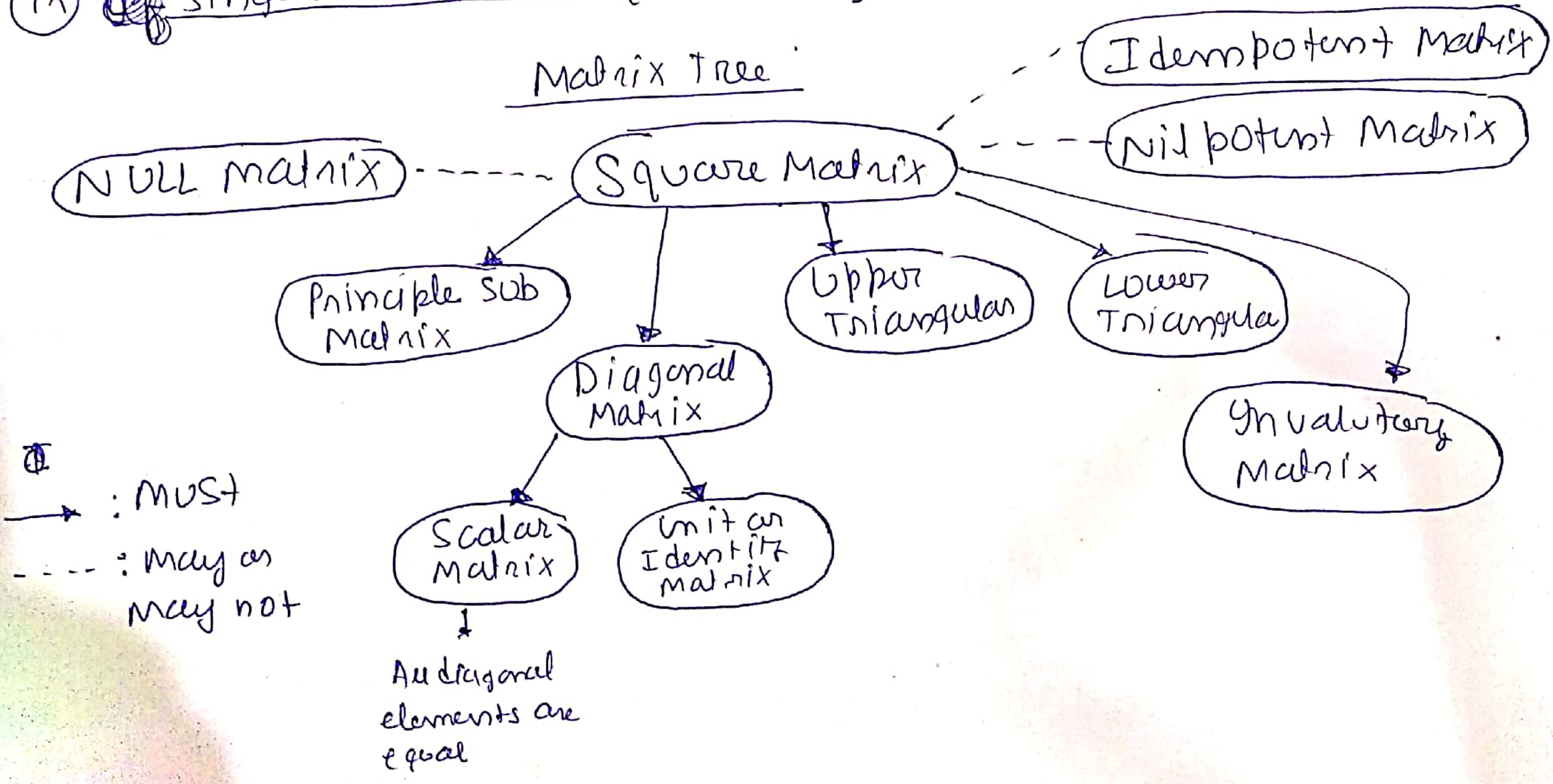
(vii) Idempotent Matrix ($A^2 = A$)

(viii) Involuntary Matrix ($A^2 = I$)

(viii) Nilpotent Matrix ($A^x = O$) {Smallest index x is always x which make A^x, NULL Matrix}

(ix) Singular Matrix ($|A| = 0$)

Matrix Tree



③ Special ~~[illegible]~~ Properties

m : rows
n : columns

Rectangular Matrix

- Square Matrix $\boxed{m = n}$ — Possible operation
  - Inverse (A^{-1})
  - Adjoint adj A
  - Cofactor cof A
  - Determinant |A|
  - Eigen value & Eigen vector
  - Rank ρ
- Any other Rectangular Matrix but not square $\boxed{m \neq n}$ — possible operation
  - Rank ρ

④ Equality of two matrix { Size of A = Size of B ; $a_{ij} = b_{ij}$ }

⑤ Addition of two matrix { size of A = size of B }

ⓐ ~~[illegible]~~ Associative ⓑ Commutative

ⓒ Identity element: 0 ⓓ Inverse: (−A)

⑥ ~~[illegible]~~ Subtraction of two matrix { size of A = size of B }

$\boxed{A - B \Rightarrow A + (-B)}$

ⓐ Neither commutative nor associative

⑦ Multiplication of matrix by scalar { $A \cdot k = k \cdot A = [k \cdot A]_{m \times n}$ }

⑧ Multiplication of two matrices { no. of column A = no. of row of B }

$A = [a_{ij}]_{m \times n}$ $\quad B = [b_{jk}]_{n \times p}$

* Not commutative

$C = [C_{ik}]_{m \times p}$ $\quad C_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$

* Associative

⑨ Trace of a matrix { Σ diag (A) } { Sum of principle diagonal }

Transpose of a Matrix {A' or A^T}

* $(AB)^T = B^T \cdot A^T$
* $(ABC)^T = C^T \cdot B^T \cdot A^T$

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(11) Conjugate of a Matrix {$\bar{A}$}

Matrix obtain by replacing conjugate of complex number only.

ex: $A = \begin{bmatrix} 2+3i & 4-7i & 8 \\ -i & 6 & 9+i \end{bmatrix}$ ⇒ $\bar{A} = \begin{bmatrix} 2-3i & 4+7i & 8 \\ +i & 6 & 9-i \end{bmatrix}$

* $\bar{A} = A$ {if A is real matrix}

  $\bar{A} = -A$ {if A is purely imaginary}

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(12) Transpose of Conjugate Matrix (A^θ) {$(\bar{A})^T$}

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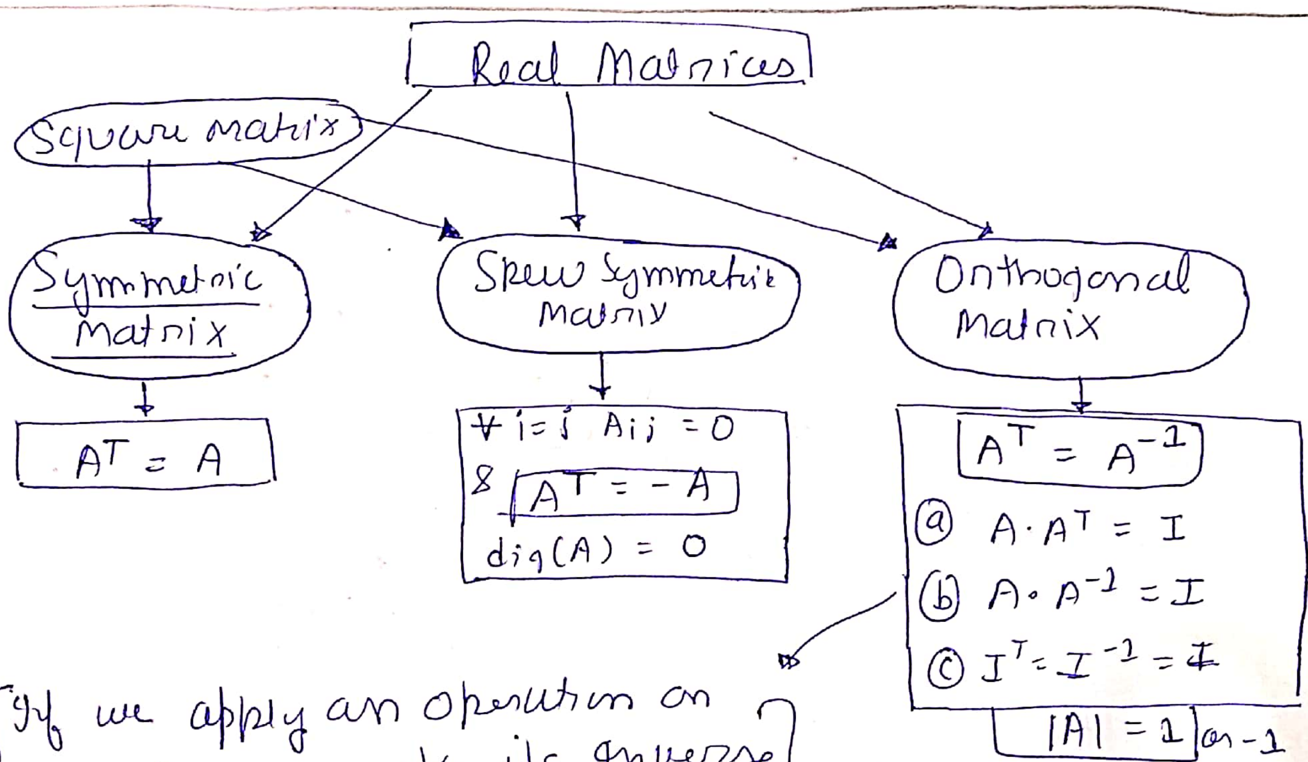
(13)

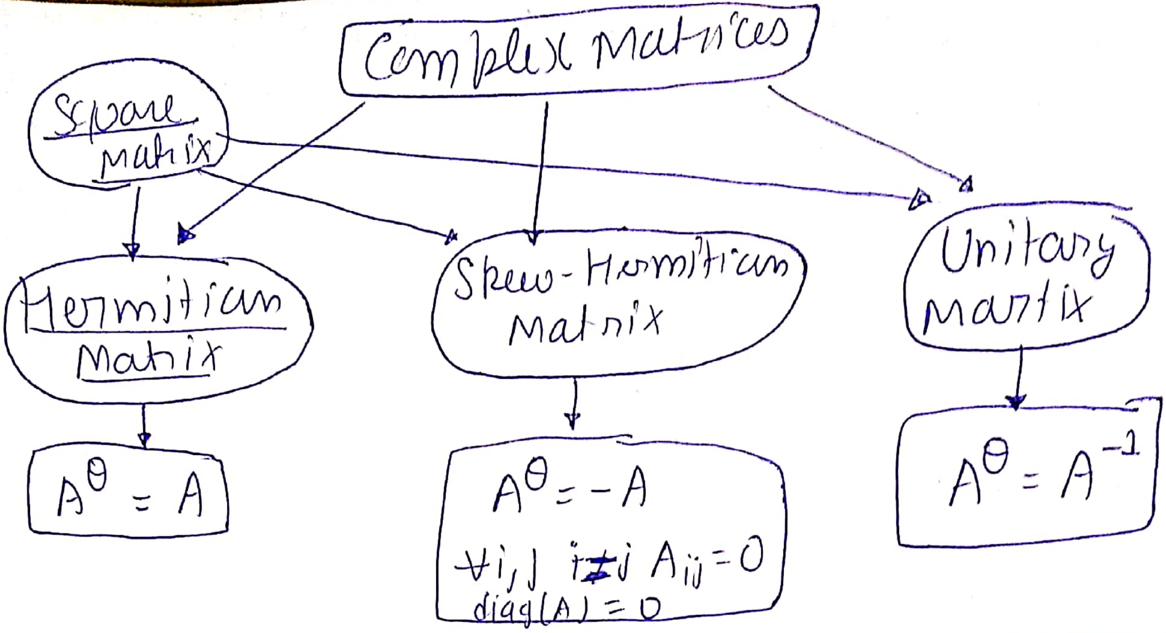


{ If we apply an operation on an element with its inverse we get Identity element of it.

If we apply operation on of any element with its identity we are going to get same element again }

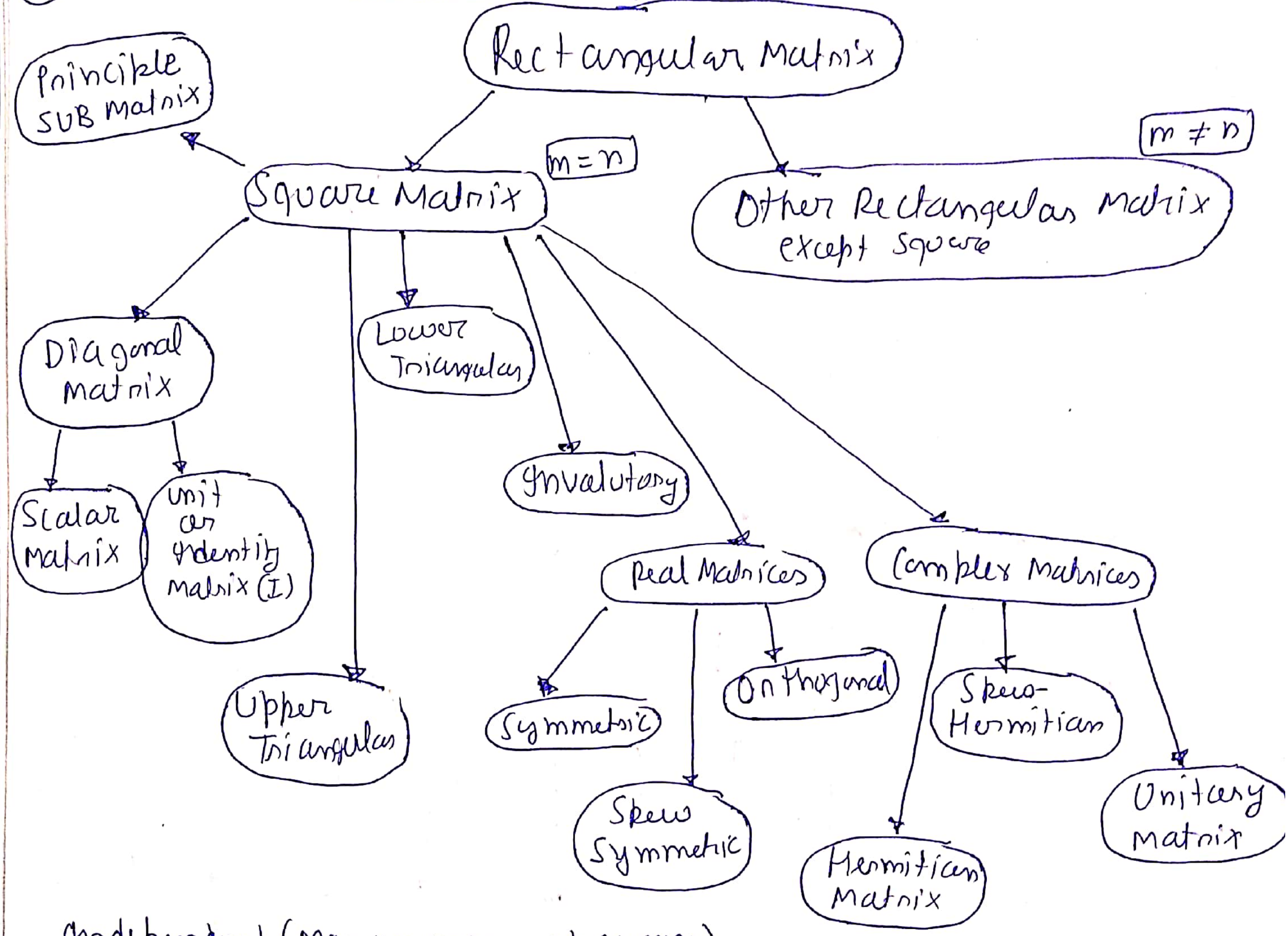
{ Group Theory }

(14)



(15)

Papa Tree of Matrices



Independent (May or may not square)

* NULL Matrix
* Idempotent Matrix
* Nilpotent Matrix

Special Products

Operation	Square	Non-square
(i) Determinant	✓	×
(ii) Inverse	✓	×
(iii) Adjoin	✓	×
(iv) Cofactor	✓	×
(v) Eigen Val & Vec.	✓	×
(vi) Rank	✓	✓

Determinants

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \longrightarrow \begin{array}{c}\text{Sign matrix} \\ \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}\end{array}$$

Normal method

Δ or $|A|$ (Determinant) (Expand by 1st Row) $= a_{11}(a_{22} \cdot a_{33} - a_{23} \cdot a_{32}) - a_{12}(a_{21} \cdot a_{33} - a_{23} \cdot a_{31}) + a_{13}(a_{21} \cdot a_{32} - a_{22} \cdot a_{31})$

Other method {only for 3x3}

$$D = \begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{33} \\ - & - & -\,+ & + & + \end{array}$$

$= (a_{11} a_{22} a_{33}) + (a_{12} \cdot a_{23} \cdot a_{31}) + (a_{13} \cdot a_{21} \cdot a_{33}) - (a_{13} \cdot a_{22} \cdot a_{31}) - (a_{11} \cdot a_{23} \cdot a_{32}) - (a_{12} \cdot a_{21} \cdot a_{33})$

(17) Minors (M)

ex: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & \cancel{a_{22}} & \cancel{a_{23}} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

minor of $a_{21} = \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix} = M_{21}$

(18) Cofactors ($A_{ij} = (-1)^{i+j} \cdot M_{ij}$) {Row$(a_{11}\ a_{12}\ a_{13})$ × Cof$(a_{11}\ a_{12}\ a_{13})$ = $|A|$}

ex: $A_{12} = (-1)^{1+2} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$

Important

* The sum of product of any row or column with its corresponding cofactor gives $\boxed{\text{Determinant of Matrix}}$

Ans: $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 6 & 1 \\ 2 & 0 & 2 \end{bmatrix}$, $Cof(A) = \begin{bmatrix} 12 & 4 & 12 \\ -4 & 2 & 4 \\ 2 & -1 & 8 \end{bmatrix}$

$|A| = 1(1 \times 12) + (2 \times 4) + (0 \times -12) = 20$ (Δ of A)

(19) Adjoin of Matrix { $adj A = \left(Cof(A)\right)^T$ }

when all element of A is replace by its cofactor, then its Transpose gives its Adjoin.

Properties:

(a) $A \times adj(A) = |A| \times I$

Multiply by A^{-1} & divide by $|A|$

$A^{-1} \times A \times \dfrac{adj A}{|A|} = \dfrac{|A|}{|A|} \times I \times A^{-1}$

(b) $A^{-1} = \dfrac{adj(A)}{|A|}$

ex:

$A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 6 & 1 \\ 2 & 0 & 2 \end{bmatrix}$, $Cof(A) = \begin{bmatrix} 12 & 4 & -12 \\ -4 & 2 & 4 \\ 2 & -1 & 8 \end{bmatrix}$, $\left(Cof(A)\right)^T = (adj A) = \begin{bmatrix} 12 & -4 & 2 \\ 4 & 2 & -1 \\ -12 & 4 & 8 \end{bmatrix}$

$A \times adj A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 6 & 1 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 12 & -4 & 2 \\ 4 & 2 & -1 \\ -12 & 4 & 8 \end{bmatrix} = \begin{bmatrix} (12+8+0) & (-4+4+0) & (2-2+0) \\ (-12+24-12) & (4+12+4) & (-2-6+8) \\ (24+0-24) & (-8+0+8) & (4+0+16) \end{bmatrix}$

$= \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix} = 20 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A| \cdot I$

Trick to find Adjoin

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Step 1:

a_{11} a_{12} a_{13} a_{11} a_{12}

a_{21} a_{22} a_{23} a_{21} a_{22}

a_{31} a_{32} a_{33} a_{31} a_{33}

Step 2: & Step 3:

a_{11}	a_{12}	a_{13}	a_{11}	a_{12}
a_{21}	a_{22}	a_{23}	a_{21}	a_{22}
a_{31}	a_{32}	a_{33}	a_{31}	a_{32}
a_{11}	a_{12}	a_{13}	a_{11}	a_{12}
a_{21}	a_{22}	a_{23}	a_{21}	a_{22}

Expand horizontally & write values vertically

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Properties of Determinant

(i) Values of Determinant not change when rows and columns are interchange. $|A^T| = |A|$

(ii) If any two rows or columns are interchange then determinant is multiply by -1.

(iii) If A be n-row square matrix, & K be any scalar then

$$\boxed{|K \cdot A| = K^n |A|}$$

(iv) ex:
$$\begin{vmatrix} k \cdot a_{11} & k \cdot a_{12} \\ k \cdot a_{21} & k \cdot a_{22} \end{vmatrix}_{2\times 2} = (k^2 \cdot a_{11} \cdot a_{22} - k^2 \cdot a_{12} \cdot a_{21})$$

$$= k^2 \cdot (a_{11} \cdot a_{22} - a_{12} \cdot a_{21})$$

(iv) (a) $|AB| = |A| \cdot |B|$

(b) $|A^n| = (|A|)^n$

ex: $|A^n| = |A * A * A * \cdots n| = |A| \cdot |A| \cdot |A| \cdots n$

$= (|A|)^n$

(c) $\boxed{|A^{-1}| = \frac{1}{|A|}}$

ex: $|A \cdot A^{-1}| = |I| = 1$

$|A| \cdot |A^{-1}| = 1$

$$\boxed{|A^{-1}| = \frac{1}{|A|}}$$

(v) $A \cdot adj(A) = |A| \cdot I$ so, $A_{n\times n}$

$|adj(A)| = |A|^{n-1}$

$|Adj(Adj A)| = |A|^{(n-1)^2}$

(21) Properties of Inverse

(i) $(AB)^{-1} = B^{-1} \cdot A^{-1}$

(ii) $(ABC)^{-1} = C^{-1} \cdot B^{-1} \cdot A^{-1}$

(iii) $A_{n\times n}$ is nonsingular then $(A^T)^{-1} = (A^{-1})^T$

$(A^\theta)^{-1} = (A^\theta)^{-1}$

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(22) Rank of Matrix

* It is used to find maximum no. of linearly independent row or column in matrix.

  Linearly Independent means we can not able to write any row or column with help of combination of other row and column.

* A square $(A_{n\times n})$ matrix is non singular if its rank is equal to n.
* If matrix is non singular then its ~~[illegible]~~ inverse exsists.

Properties :

(a) If A is NULL Matrix, then $\rho(A) = 0$

(b) If A is non-zero matrix (atleast one element must not zero) then $\rho(A) \geq 1$

(c) If A is identity matrix, then $\rho(A) = n$ (order of Identity Mat)

(d) If A is matrix of order $m \times n$ then

$$\rho(A) = \begin{cases} < (\min(m,n)) & , \text{if } \Delta = 0 \\ = \min(m,n) & , \text{if } \Delta \neq 0 \end{cases}$$

(e) $\rho(AB) \leq \min(\rho(A), \rho(B))$

(23) Nullity of Matrix : No. of n dimension column vector which give result as 0, after product with matrix

$$\boxed{\text{Nullity} = \cancel{\rho(\text{Rank of matrix})}\ \text{No. of column} - \rho(\text{Rank of Matrix})}$$

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Homogeneous & Non Hogeneous Linear equation (9)

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0\\
a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n &= 0\\
\vdots\quad \vdots \quad \vdots \quad \vdots\\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0
\end{aligned}$$

Homogeneous

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{bmatrix}_{m\times n} \qquad B = \begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix}_{m\times 1}$$

m-equations & n unknown.

$$X = \begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}_{n\times 1}$$

Coefficient Matrix

$[x_1=0, x_2=0, \ldots x_n=0]^t$ is always solution for it. (trivial solution)

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2\\
a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n &= b_3\\
\vdots\\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

Non Hogeneous

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{bmatrix}_{m\times n}$$

Coefficient matrix

$$B = \begin{bmatrix}b_1\\b_2\\\vdots\\b_m\end{bmatrix}_{m\times 1} \qquad X = \begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}_{n\times 1}$$

: Homogeneous only follow this path because of one of the trivial solution 0, system never in inconsistent state.

A system of linear equation with n variable $AX = B$

agumented matrix ↑

↓ Find Rank of A & $(A|B)$

- $\rho(A) = \rho(A|B)$ → System Consistent
  - $\rho(A) = \rho(A|B) = n$ → unique solution
  - $\rho(A) = \rho(A|B) < n$ → Infinite no. of Solution
- $\rho(A) \neq \rho(A|B)$ → System inconsistent → no Solution

Rules:

(a) If $(A|B)$ is rectangular matrix then dont include last column (To represent both rank)

(b) If $(A|B)$ is square matrix then include both and then find rank of $(A|B)$ & A.

25) Gaussian Jordan Elimination Method to Solve Linear equations

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

(i) Make agumented Matrix $= [AB] = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$

(ii) Always keep a_{11} as 1.
If not try to make it 1.

(iii) (a) Take a_{11} and make all its below element i.e $a_{21}, a_{31} \ldots$ as 0.

(b) Take a_{22} and make all its below element i.e $a_{32}, a_{42}, \ldots$ as 0.

(c) Same step as above for all diagonal element.

(d) Swap rows to keep it in order if require.

(iv) After step (iii) we get Upper triangular Matrix no start subtituting variable from bottom.

ex: $\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1' \\ 0 & a_{22} & a_{23} & b_2' \\ 0 & 0 & a_{33} & b_3' \end{array}\right] \Rightarrow$

$$a_{33} \cdot z = b_3'$$
$$a_{22}y + a_{32}z = b_2'$$
$$a_{11}x + a_{12}y + a_{13}z = b_1'$$

Eigen value and Eigen Vector

$$\boxed{A \cdot X = \lambda \cdot X}$$

here, λ is eigen value & X is eigen vector.

(i) Characteristic equation

(a) $|A - \lambda I| = 0$

Roots of it is called as eigen value, latent root or characteristic root or proper value.

(b) Value which satisfy:

$$|A - \lambda I| \cdot X = 0$$

is known as eigen vector of matrix.

(ii) Properties

(a) Sum of eigen value is equal to Trace of matrix.

$\lambda_1 + \lambda_2 + \cdots = a_{11} + a_{22} + a_{33} \cdots$

(b) Product of eigen value is equal to determinent of matrix: $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdots = |A|$

(c) Eigen value of Symmetric Matrix is purely real.

(d) Eigen value of Skew-Symmetric Matrix is purely Imaginary.

(e) Eigen value of Hermitian matrix is purely real.

(f) Eigen value of Skew-Hermitian matrix is purely imaginary.

(g) If matrix is either lower, upper or diagonal then its principle diagonal elements are eigen values.

(h) $\boxed{\lambda^n = A^n}$, $\boxed{\frac{1}{\lambda} = A^{-1}}$

(i) ADA^{-1}, $A = \begin{bmatrix} \updownarrow & \updownarrow & \\ V_1 & V_2 & \cdots \\ \updownarrow & \updownarrow & \end{bmatrix}$ } eigen vector $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ } eigen value.

(iii) Find eigen vector

Step 1: Find eigen value using characteristic eqn

i.e $|A-\lambda I| = 0$

Step 2: use each eigen value, to find its respective eigen vector.

$$\begin{bmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

(b) Find it if options are given:

* Try to find using properties
* If eigen vector is given then use option:

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\text{Given}} = \underbrace{\lambda}_{\text{eigen value}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\text{eigen vector}}$$

we are going to get same eigen vector again.

(iv) Scientific ~~Equation~~ Root method

eqn: $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

then

Step 1: Find 1st root using hit and trial method

So we get $\boxed{\lambda = 1}$

Step 2:

multiply root & value and write it down below second value

(1×-6)

1	1	-6	11	-6	coefficient of eqn
		+1	-5	6	
	1	-5	6	0	

1st root

Copy First value as it is

→ we must get 0 here otherwise our considered root is wrong

$$\Rightarrow \boxed{(\lambda-1)(\lambda^2-5\lambda+6)}$$

Cailey Hamilton theoreme

Any square matrix satisfy its own characteristic.

$$|A - \lambda I| = 0$$

$$\lambda^2 + x \cdot \lambda + c = 0$$

$\boxed{\lambda = A}$

Characteristic eqn : $\boxed{A^2 + x \cdot A + C \cdot I = 0}$

After putting A, LHS = RHS

(ii) Application of Cailey Hamilton Thm

(a) Finding inverse

ex: $A^2 - 3A - 10I = 0$

$$I = \frac{1}{10}[A^2 - 3A]$$

$$A^{-1} \cdot I = \frac{1}{10}[A \cdot A \cdot A^{-1} - 3 \cdot A \cdot A^{-1}]$$

$$\boxed{A^{-1} = \frac{1}{10}[A - 3I]}$$

(b) Finding High Power of Matrix

If A is nxn Matrix, any power of A can be written as polynomial of maximum degree (n-1). $A_{2\times 2}$ (n=2)

ex: $A^2 - 3A - 10I = 0$ --- (i)

$A^2 = 3A + 10I$ --- (ii)

$A^3 = 3A^2 + 10A$

$\boxed{A^3 = 3(3A + 10I) + 10A = 19A + 30I}$

$\boxed{A^4 = 19A^2 + 30A = 19(3A + 10I) + 30A}$

(c) Express any matrix polynomial in A of $n \times n$ as Polynomial of degree $(n-1)$ in A by using Cayley Hamilton Theorem.

ex: $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ Express $2A^5 - 3A^4 + A^2 - 4I$ as linear polynomial of A.

Char eqn $|A - \lambda I|$ & putting $\lambda = A$ $= A^2 - 5A + 7I$

$A^2 = 5A - 7I$

$A^3 = 5A^2 - 7A$

$A^4 = 5A^3 - 7A^2$

$A^5 = 5A^4 - 7A^3$

Now,

$2A^5 - 3A^4 + A^2 - 4I = 2(5A^4 - 7A^3) \cancel{+} 3A^4 + A^2 - 4I$

$= \boxed{138A - 403I}$ {Linear Form}

Characteristic eqn

2x2 : $\lambda^2 - B_1\lambda + \det(A) = 0$ B_1 : Trace of A

3x3 : $\lambda^3 - B_1\lambda^2 + B_2\lambda - \det(A) = 0$ B_2 : Ninja rule i.e $|A_{11} + A_{22} + A_{33}|$ (cofactor sum of these element)

Trivial or Non trivial

$Ax = B$, For this case $B = 0$

$Ax = 0$

(i) Trivial : $|A| \neq 0$, $\rho(A) =$ No. of variable

(ii) NonTrivial : $|A| = 0$, $\rho(A) <$ No. of variable.