Literature Review

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2D Discrete Fourier Transform

- Fourier transform of a 2D signal defined over a discrete finite 2D grid of size MxN/RxC.

Or equivalently

- Fourier transform of a 2D set of samples forming a bidimensional sequence.
- As in the 1D case, 2D-DFT, though a self-consistent transform, can be considered as a mean of calculating the transform of a 2D sampled signal defined over a discrete grid.

2D Discrete Fourier Transform

- m and n are discrete spatial variables. (units:pixels)
- k and l are integers representing frequency.

2D Discrete Fourier Transform (DFT)

$$F[k,l] = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m,n] e^{-j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

where $k = 0,1,\ldots,M-1$ and $l = 0,1,\ldots,N-1$

Inverse DFT

$$f[m,n] = \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} F[k,l] e^{j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

Periodicity

$$f[m,n] = \sum_{k=0}^{M-1}\sum_{l=0}^{N-1} F[k,l]e^{j2\pi\left(\frac{k}{M}m+\frac{l}{N}n\right)}$$

$$f[m+M,n+N] = \sum_{k=0}^{M-1}\sum_{l=0}^{N-1} F[k,l]e^{j2\pi\left(\frac{k}{M}(m+M)+\frac{l}{N}(n+N)\right)}$$

$$= \sum_{k=0}^{M-1}\sum_{l=0}^{N-1} F[k,l]e^{j2\pi\left(\frac{k}{M}m+\frac{l}{N}n\right)}\cancelto{1}{e^{j2\pi\left(\frac{k}{M}M+\frac{l}{N}N\right)}}$$

$$= f[m,n]$$

$$\sum_{r=0}^{R-1}\sum_{c=0}^{C-1}\phi^*_{k_r,k_c}[r,c]\phi_{l_r,l_c}[r,c] = \sum_{r=0}^{R-1}\sum_{c=0}^{C-1}(e^{j\frac{2\pi k_r}{R}r}\cdot e^{j\frac{2\pi k_c}{C}c})\cdot(e^{-j\frac{2\pi l_r}{R}r}\cdot e^{-j\frac{2\pi l_c}{C}c})$$

$$= \sum_{r=0}^{R-1} e^{j\frac{2\pi(k_r-l_r)}{R}r}\sum_{c=0}^{C-1} e^{j\frac{2\pi(k_c-l_c)}{C}c} = \begin{cases} RC & if k_r = l_r \ \ and\ k_c = l_c \\ 0 & otherwise \end{cases} \qquad (0 \le k_r, l_r < R, 0 \le k_c, l_c < C)$$

Orthogonality

THE INNER PRODUCT OF THE 2D DFT BASIS FUNCTIONS IS GIVEN HERE.

Properties

Property	Expression(s)
Fourier transform	$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$
Inverse Fourier transform	$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}$
Polar representation	$F(u,v) = \lvert F(u,v)\rvert e^{-j\phi(u,v)}$
Spectrum	$\lvert F(u,v)\rvert = \left[R^2(u,v) + I^2(u,v)\right]^{1/2}$, $R = \text{Real}(F)$ and $I = \text{Imag}(F)$
Phase angle	$\phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$
Power spectrum	$P(u,v) = \lvert F(u,v)\rvert^2$
Average value	$\bar{f}(x,y) = F(0,0) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)$
Translation	$f(x,y)e^{j2\pi(u_0x/M+v_0y/N)} \Leftrightarrow F(u-u_0, v-v_0)$ $f(x-x_0, y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi(ux_0/M+vy_0/N)}$ When $x_0 = u_0 = M/2$ and $y_0 = v_0 = N/2$, then $f(x,y)(-1)^{x+y} \Leftrightarrow F(u-M/2, v-N/2)$ $f(x-M/2, y-N/2) \Leftrightarrow F(u,v)(-1)^{u+v}$

Properties

Conjugate symmetry	$F(u,v) = F^*(-u,-v)$ $\lvert F(u,v)\rvert = \lvert F(-u,-v)\rvert$
Differentiation	$\dfrac{\partial^n f(x,y)}{\partial x^n} \Leftrightarrow (ju)^n F(u,v)$ $(-jx)^n f(x,y) \Leftrightarrow \dfrac{\partial^n F(u,v)}{\partial u^n}$
Laplacian	$\nabla^2 f(x,y) \Leftrightarrow -(u^2+v^2)F(u,v)$
Distributivity	$\Im[f_1(x,y)+f_2(x,y)] = \Im[f_1(x,y)] + \Im[f_2(x,y)]$ $\Im[f_1(x,y)\cdot f_2(x,y)] \neq \Im[f_1(x,y)]\cdot\Im[f_2(x,y)]$
Scaling	$af(x,y) \Leftrightarrow aF(u,v),\ f(ax,by) \Leftrightarrow \dfrac{1}{\lvert ab\rvert}F(u/a, v/b)$
Rotation	$x = r\cos\theta \quad y = r\sin\theta \quad u = \omega\cos\varphi \quad v = \omega\sin\varphi$ $f(r,\theta+\theta_0) \Leftrightarrow F(\omega,\varphi+\theta_0)$
Periodicity	$F(u,v) = F(u+M,v) = F(u,v+N) = F(u+M,v+N)$ $f(x,y) = f(x+M,y) = f(x,y+N) = f(x+M,y+N)$
Separability	See Eqs. (4.6-14) and (4.6-15). Separability implies that we can compute the 2-D transform of an image by first computing 1-D transforms along each row of the image, and then computing a 1-D transform along each column of this intermediate result. The reverse, columns and then rows, yields the same result.

Properties

Property	Expression(s)
Computation of the inverse Fourier transform using a forward transform algorithm	$\frac{1}{MN} f^*(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F^*(u, v) e^{-j2\pi(ux/M + vy/N)}$ This equation indicates that inputting the function $F^*(u, v)$ into an algorithm designed to compute the forward transform (right side of the preceding equation) yields $f^*(x, y)/MN$. Taking the complex conjugate and multiplying this result by MN gives the desired inverse.
Convolution†	$f(x, y) * h(x, y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n) h(x - m, y - n)$
Correlation†	$f(x, y) \circ h(x, y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f^*(m, n) h(x + m, y + n)$
Convolution theorem†	$f(x, y) * h(x, y) \Leftrightarrow F(u, v) H(u, v)$; $f(x, y) h(x, y) \Leftrightarrow F(u, v) * H(u, v)$
Correlation theorem†	$f(x, y) \circ h(x, y) \Leftrightarrow F^*(u, v) H(u, v)$; $f^*(x, y) h(x, y) \Leftrightarrow F(u, v) \circ H(u, v)$

Impulse	$\delta(x, y) \Leftrightarrow 1$
Gaussian	$A\sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2(x^2+y^2)} \Leftrightarrow Ae^{-(u^2+v^2)/2\sigma^2}$
Rectangle	$\mathrm{rect}[a, b] \Leftrightarrow ab\dfrac{\sin(\pi ua)}{(\pi ua)}\dfrac{\sin(\pi vb)}{(\pi vb)}e^{-j\pi(ua+vb)}$
Cosine	$\cos(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow \frac{1}{2}[\delta(u + u_0, v + v_0) + \delta(u - u_0, v - v_0)]$
Sine	$\sin(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow j\frac{1}{2}[\delta(u + u_0, v + v_0) - \delta(u - u_0, v - v_0)]$

Common DFT Pairs

2D DFT using 1D DFT

Alternatively we can start with columns and then do rows as well

$$F[k_r, k_c] = \frac{1}{RC}\sum_{r=0}^{R-1}\sum_{c=0}^{C-1} f[r,c]\cdot e^{-j\left(\frac{2\pi k_r}{R}r+\frac{2\pi k_c}{C}c\right)}$$

$$= \underbrace{\frac{1}{R}\sum_{r=0}^{R-1}\underbrace{\left(\frac{1}{C}\sum_{c=0}^{C-1} f[r,c]\cdot e^{-j\frac{2\pi k_c}{C}c}\right)}_{\text{first, obtain the DFT for each row}}\cdot e^{-j\frac{2\pi k_r}{R}r}}_{\text{then, take the DFT of each resulting columns}}$$

2D DFT using 1D DFT

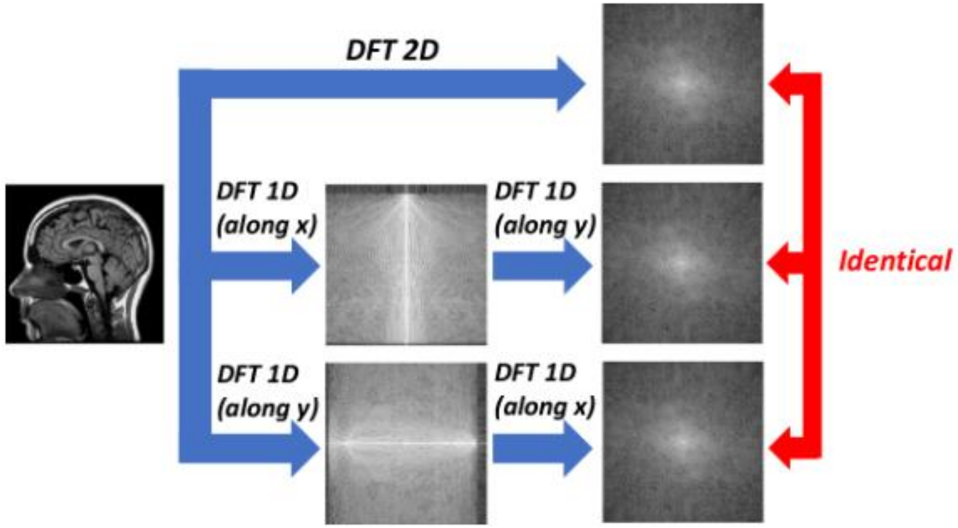
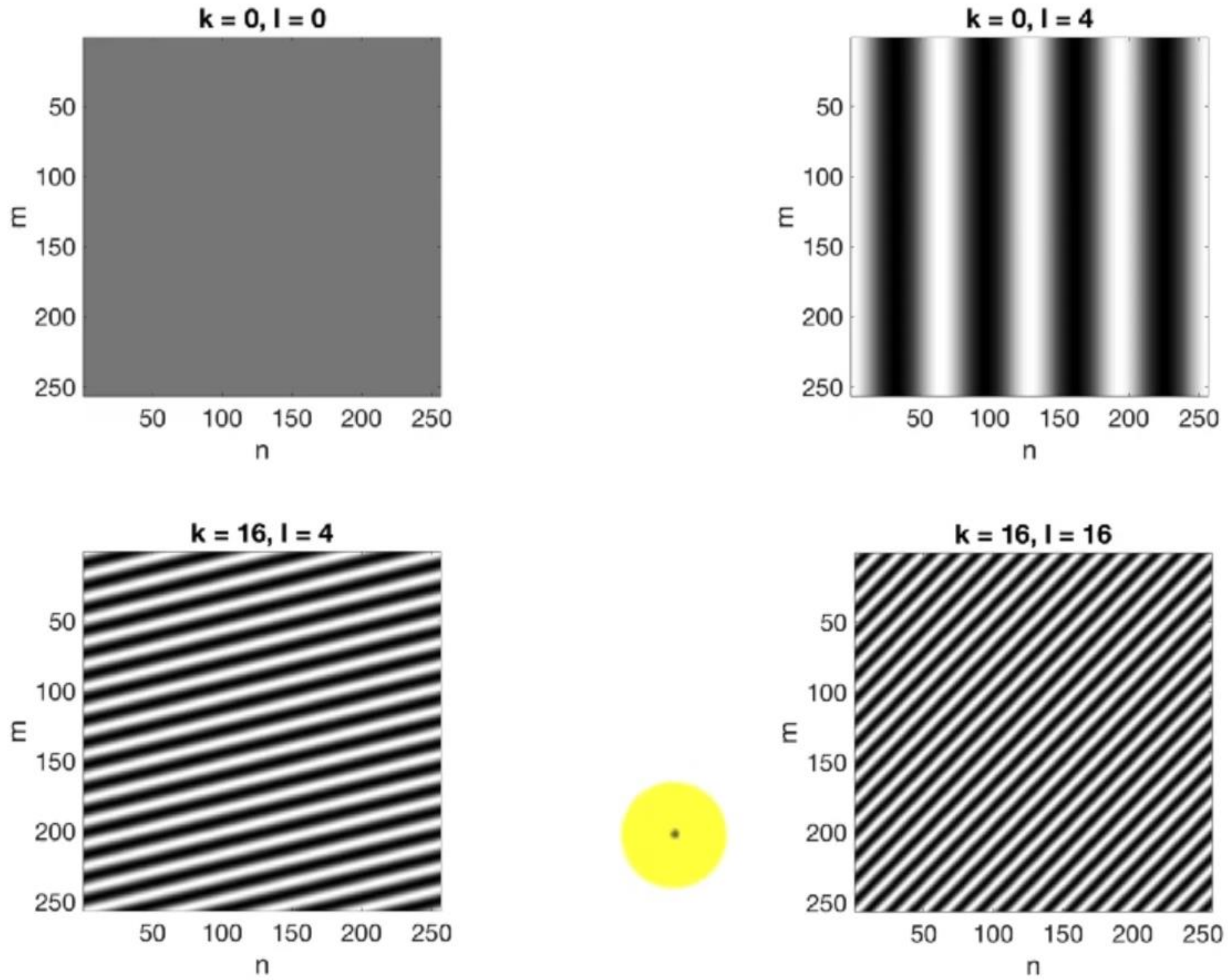
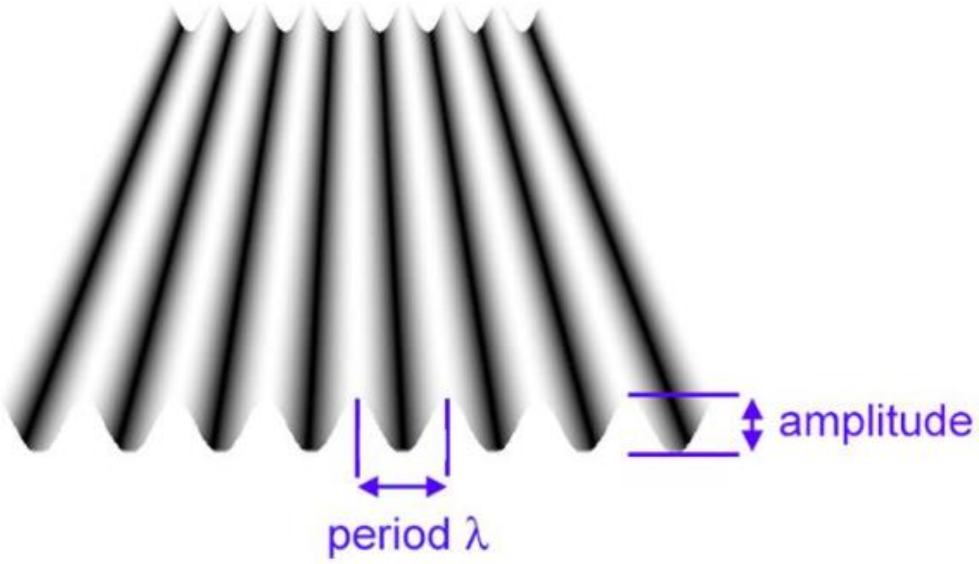


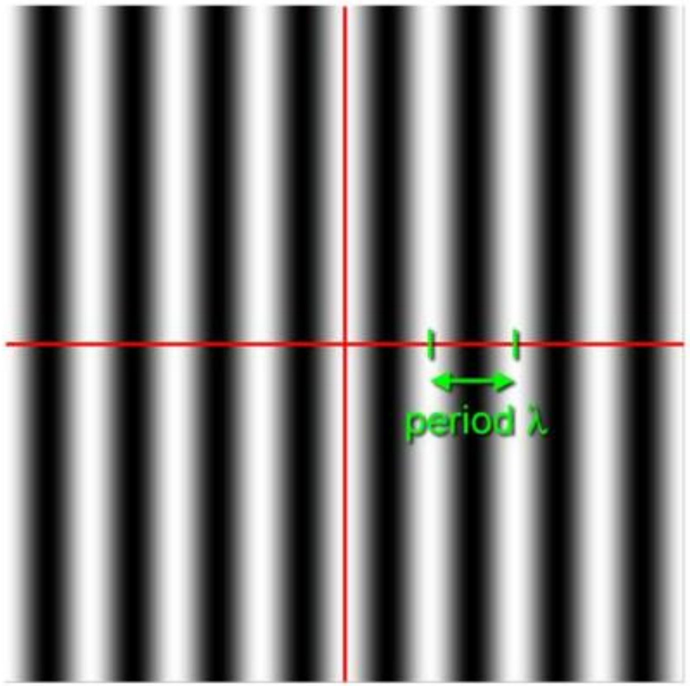
Figure 12.1: The 2D DFT can be equivalently decomposed as a 1D DFT along one dimension, followed by a 1D DFT along the other dimension. This equivalence extends to N-D DFTs, which can be decomposed into a concatenation of 1D DFTs along each of the N dimensions.

Complex Sinusoids

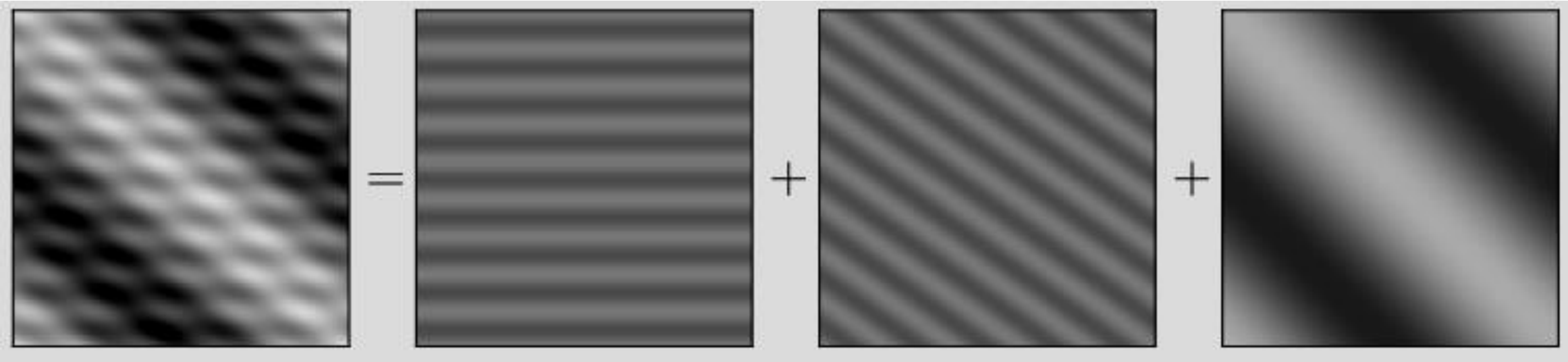
- Black stripes represent troughs and white stripes represent peaks.
- Slope=l/k.







Complex Sinusoid



Complex Sinusoids

In this example, the image is simple enough to be decomposed by using only three oscillations.

Example

$$F[k_r, k_c] = \frac{1}{RC}\sum_{r=0}^{R-1}\sum_{c=0}^{C-1}\delta[r]\delta[c]\cdot e^{-j\left(\frac{2\pi k_r}{R}r+\frac{2\pi k_c}{C}c\right)}$$

$$= \frac{1}{RC}e^{-j\left(\frac{2\pi k_r}{R}0+\frac{2\pi k_c}{C}0\right)}$$

$$= \frac{1}{RC}$$

$$\boxed{\delta[r]\delta[c] \overset{DFT}{\Longrightarrow} \frac{1}{RC}}$$

References

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- https://homepages.inf.ed.ac.uk/rbf/HIPR2/fourier.htm
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