` `*Economics of Production Function with Two Input* **and** *One Output*· (*Quadratic* **and** *Cobb Douglas Production Function*)

 # *Author: Bijesh Mishra, PhD Student of Natural Resource Ecology and Management, Forest Resource Economics, Oklahoma State Univeristy, Stillwater, OK.*

Input Side Economics: of Production Function with Two Input and One Output (Quadratic and Cobb Douglas Production Function)

 # Profit Maximization, Cost Minimization Subejct to Output Constraints and Output Maximization Subject to Budget Constraints, Duality and Comprative Statics

 # Some terminologies to understand:
 # Orinary Input Demand Functions:
 # Constrained Input Demand Functions:
 # Expenditure Demand Functions:
 # Supply Function:
 # Profit Function:
 # Cost Function:
 # Conditional Production Function:

restart;
#Use this line to enter the value of variables as given in the exam.
$r1 := w; r2 := r; p := p; b := 0; Co := Co; yo := yo;$
 # Input 1 Cost, Input 2 Cost, Output Price, Fixed Cost, Constrained Investment or Budget, Constrained Output.

$x1 := h; x2 := k; b1 := 80; b2 := 75; c1 := -4; c2 := -3; d1 := -4; a := 0;$
#Change x1 and x2 and Coefficients x1, x2, $x1^2$, $x2^2$, $x1 \cdot x2$, *Constant* **for** *quadratic function.*
$m := m; n := n; Z := Z;$ *#Change m, n and Z to change Cobb Douglas production function.*

 # Cobb Douglas Production Function:

$Z := Z; m := m; n := n;\ cobb := Z \cdot x1^m x2^n;$

 # Quadratic Production function (Two Input One Output):

$b1 := b1; b2 := b2; c1 := c1; c2 := c2; d1 := d1; a := a; x1 := x1; x2 := x2; quad := b1 \cdot x1 + b2 \cdot x2 + c1 \cdot x1^2 + c2 \cdot x2^2 + d1 \cdot x1 \cdot x2 + a;$
 # x1= Input 1, x2 = Input 2. b1, b2, c1, c2, d1, are coefficients of inputs and their interactions, a is a constant.
$y := quad;$ ***#Change production function (quad or cobb).***
$VC := r1 \cdot x1 + r2 \cdot x2; FC := b; TotalCost := VC + FC;$
 # Cost Functions: Variable Cost (VC), Fixed Cost (FC), Total Cost (TC)

$TVP := p \cdot y; Profit\_Function := TVP - TotalCost;$ *# TVP = Total Value Product, Profit_fun = Profit.*

$TPP := y;\ APP1 := simplify\left(\frac{y}{x1}\right);\ APP2 := simplify\left(\frac{y}{x2}\right);\ AVP1 := simplify(APP1 \cdot p);\ AVP2 := simplify(APP2 \cdot p);$

Total Physical Product, Average Physical Productivity of Factor x1, Average Physical Productivity of Factor x2, Average Value productivity of Factor x1, Average Value Productivity of Factor x2

$f1 := \frac{\partial}{\partial x1}(y);\quad f2 := \frac{\partial}{\partial x2}(y);\ MVP1 := p \cdot f1;\ MVP2 := p \cdot f2;\ MFC1 := MVP1;\ MFC2 := MVP2;$

Marginal Physical Productivity (MPP) of Factor x1**,** Marginal Physical Productivity of Factor x2, Marginal Value Productivity $\cdot$ (MVP) of Factor x1, Marginal Value Productivity of Factor x2, Marginal Factor Cost $\cdot$ (MFC) of Factor x1, Marginal Factor Cost of Factor x2.

$f11 := \frac{\partial}{\partial x1}(f1);\ f22 := \frac{\partial}{\partial x2}(f2);\ f12 := \frac{\partial^2}{\partial x1\, \partial x2}(y);\ f21 := \frac{\partial^2}{\partial x2\, \partial x1}(y);$

Second Order Conditions of f1, f2.
#f11 & f22 < 0 ⇒Diminishing Marginal Physical Returns.
#f12 & f21 are Factor Inerdependence ($f12$ **or** $f21 < 0 \Rightarrow$ competitive; $f12$ **or** $f21 = 0 \Rightarrow$ independent; $f12$ **or** $f21 > 0 \Rightarrow$ complementary).

$MRTS := simplify\left(\left(\frac{f1}{f2}\right)\right);$

Marginal Rate of Technical Substitution $(MRTS_{21})$ If MRTS is simplified **by** equating with constant, k, that gives isolcline at MRTS = k.

$MarginalRevenue := \frac{f1}{f2};$ # $\left(\frac{r1}{r2} = \frac{f1}{f2}\right)$ Marginal Revenue (MR) = Marginal Rate of Technical Substitution $(MRTS_{21})$

$\left(\left(\frac{r1}{f1} = \frac{r2}{f2}\right)\right)$

$MarginalRevenue\_PerfectCompetition := p;$ #Only in perfectly competitive market.

$Slope\_Of\_Isoquants\_at\_a\_Point := eval(MarginalRevenue, [x1 = x1, x2 = x2]);$

Slope of Isoquants at a Point. Replace x1 **and** x2 **by** given values of x1, **and** x2.

$SOC := simplify(f2 \cdot f2 \cdot f11 - 2 \cdot f1 \cdot f2 \cdot f12 + f1 \cdot f1 \cdot f22);$

Second Order Conditions: Border Hessian Condition.

$Curvature := simplify\left(\left(\frac{1}{f2^3}\right) \cdot SOC\right);$ # Curvature. This curvature is derived **from** production function. Gives change **in** slope of the isoquant **and** used **to** determine convexity of the isoqunt.

Also, if you know the equation of isoquant, then curvature is the second derivative of the equation for isoquant.

$Curvature\_at\_a\_Point := eval(Curvature, [x1 = x1, x2 = x2]);$ # ***Curvature at a Point. Replace*** x1 **and** x2 **by** given values of x1, **and** x2.

$Elasticity\_of\_Factor\_Subsitution := simplify\left(\frac{f1 \cdot f2 \cdot (f2 \cdot x2 + f1 \cdot x1)}{(x1 \cdot x2) \cdot (2 \cdot f1 \cdot f2 \cdot f12 - f1 \cdot f1 \cdot f22 - f2 \cdot f2 \cdot f11)}\right);$

Elasticity of Factor Subsitution (σ):

$ElasticityOfFS := simplify(eval(Elasticity\_of\_Factor\_Subsitution, [x1 = x1, x2 = x2]));$

Enter value of x1 and x2 as desired to calculate elasticity of factor subsitution.

$Elasticity\_x1 := \frac{f1}{APP1};$ # Elasticity of Production for Factor x1. MPP1 divided by APP1.

$Elasticity\_x2 := \frac{f2}{APP2}$; # Elasticity of Production for Factor x2. (MPP2 divided by APP2).

$Functional\_Coefficient := simplify(Elasticity\_x1 + Elasticity\_x2)$; #Elasticity of Production or Function Coefficient of Production Function.

Elasticity of Production or Functional Coefficient (ε): Also gives Return to Scale (= 1 $\Rightarrow$ Constant); (> 1 $\Rightarrow$ Increasing); OR· (< 1 $\Rightarrow$ Decreasing).

$Functional\_Coefficient\_Alternative := simplify\left(\frac{(f1 \cdot x1 + f2 \cdot x2)}{y}\right)$;

#Functional_Coefficient_Alternative and Functional_Coefficient are same, calculated differently

 # Isoquants:

$Two\_Isoquants\_x1 := solve(y = v, x1)$; #v is constant.

$Two\_Isoquants\_x2 := solve(y = v, x2)$; #v is constant.

 # Stages of Productions for Two Inputs:

$Stage\_I\_II\_Boundary\_Input\_x1 := solve(Elasticity\_x1 = 1, x1)$; #use positive value

$Stage\_II\_III\_Boundary\_Input\_x1 := solve(Elasticity\_x1 = 0, x1)$; #use positive value

$Stage\_I\_II\_Boundary\_Input\_x2 := solve(Elasticity\_x2 = 1, x2)$; #use positive value.

$Stage\_II\_III\_Boundary\_Input\_x2 := solve(Elasticity\_x2 = 0, x2)$; #use positive value.

$Inflection\_Point\_Input\_x1 := solve(f1 = APP1, x1)$; #Inflection Point for input X1

$Inflection\_Point\_Input\_x2 := solve(f2 = APP2, x2)$; #Inflection Point for input X2

 # Homogenous function:

if one common value in term of t^r is obtained for entire equation, the funciton is homogenous of degree "r" and has constant proportion of return to scale of r. The r value is also equal to functional ceofficient (**ε**). The degree of homogenity of MPPs **and** APPs of homogenous production functions are "r-1".

$Homogen := simplify(eval(y, [x1 = t \cdot x1, x2 = t \cdot x2]))$;

 # Ridgelines x1 and x2:

$Ridgeline\_x1 := solve(f1 = 0, x1)$; **# MPP1 = 0 and Stage III begins for input x2.**

$Ridgeline\_x2 := solve(f2 = 0, x2)$; **# MPP2 = 0 and stage III begins for input x1.**

 # Profit Maximization (Without Any Constraints) and Duality with Hottlings Lemma:

profit := *Profit_Function*;

pf1 := *diff* (*profit*, *x1*); *pf11* := *diff* (*pf1*, *x1*); # *pf11* < 0 *for profit max.*

pf2 := *diff* (*profit*, *x2*); *pf22* := *diff* (*pf2*, *x2*); # *pf22* < 0 *for profit max.*

pf12 := *diff* (*pf1*, *x2*); *pf21* := *diff* (*pf2*, *x1*);

EP_p_x1 := *solve*(*pf1* = 0, *x1*); *# Expansion Path x1 Or Pseudo scale line x1.*

EP_p_x2 := *solve*(*pf2* = 0, *x2*); *# Expansion Path x2 Or Pseudo scale line x2.*

x2s_profit := *simplify*(*solve*((*eval*(*pf2*, *x1* = *EP_p_x1*)) = 0, *x2*));

 # *X2Star:* *Profit Maximizing Level of Input x2* Or *Ordinary Input Demand Function x2.*

x1s_profit := *simplify*(*eval*(*EP_p_x1*, *x2* = *x2s_profit*));

 # *X1Star:* *Profit Maximizing Level of Input x1* Or *Ordinary Input Demand Function x1.*

MaxProfOut := *simplify*(*eval*(*y*, [*x1* = *x1s_profit*, *x2* = *x2s_profit*]));

 # Profit Maximizing Level of Output.

ProfitStar := *simplify*(*eval*(*profit*, [*x1* = *x1s_profit*, *x2* = *x2s_profit*]));

 # *ProfitStar:* *Maximum Profit Function, Indirect Profit Function.*

CostStar_MaxProfit := *simplify*(*eval*(*TotalCost*, [*x1* = *x1s_profit*, *x2* = *x2s_profit*]));

 # *CostStar:* *Cost at Maximum profit.*

$\frac{r1}{pf1} = \frac{r2}{pf2};\ \frac{r1}{f1} = \frac{r2}{f2};\ \frac{r1}{r2} = \frac{f1}{f2};\ \frac{r1}{r2} = \frac{pf1}{pf2};$

BHessian_Profit_Max := (*pf11*·*pf22* − *pf12*·*pf21*);

#SOC Condition: *Determinant of Boarder Hessian Matrix > 0 for maximization.*

$Curvature\_ProfitMax := -\left(\frac{\mathrm{abs}(pf11 \cdot pf22 - pf12 \cdot pf21)}{pf2^3}\right);$

#Strictly quasi concave in perfect competition.

Curvature_ProfitMax_Value := *simplify*(*eval*(*Curvature_ProfitMax*, [*x1* = *x1s_profit*, *x2* = *x1s_profit*]));

***Duality: Production Function* to *Indirect Conditional Profit Function* in *Price Space* and *Back* to *Production Function*.**

 # Duality related to Hotllings Lemma follows same procedure of profit maximization without any constraints until we derive Indirect Profit Function (ProfitStar).

 # So, this section is a contuination from profit maximization for duality. This section derive original production function from where we started using duality approach.

 # To go back to production function we need to use ***Hottlings Lemma*** *i.e. Partial derivative of indirect profit function with respect to p gives supply function.*

```
Hottlings_Lemma_SupplyFunction := simplify(diff(ProfitStar, p));
    #Hottlings Lemma gives supply function.
Ordinary_Input_Demand_Function_x1 := simplify(-diff(ProfitStar, r1));
    #This is same as X1Star from profit maximization.
Ordinary_Input_Demand_Function_x2 := simplify(-diff(ProfitStar, r2));
    #This is same as X2Star from profit maximization.
Duality_Hottling_r1 := solve(Ordinary_Input_Demand_Function_x2 = x2, r1);
Duality_Hottling_r2 := solve(Ordinary_Input_Demand_Function_x1 = x1, r2);
Duality_Hottling_r2Star := simplify(solve(eval(Ordinary_Input_Demand_Function_x1, [r1
    = Duality_Hottling_r1]) = x1, r2));
Duality_Hottling_r1Star := simplify(eval(Duality_Hottling_r1, [r2 = Duality_Hottling_r2Star]));
Original_Production_Function_Hottlings := simplify(eval(Hottlings_Lemma_SupplyFunction, [r1
    = Duality_Hottling_r1Star, r2 = Duality_Hottling_r2Star]));
```

 # Comparative Static of Profit Maximization Without Any Constraints:

$$Own\_Price\_Effect\_X1Star\_Profit := \left(\frac{f22}{p\cdot(f11\cdot f22 - f12\cdot f12)}\right);$$

$Own\_Price\_Effect\_X1S\_Profit\_Alternative := diff(x1s\_profit, r1);$ # Own Price Effect in x1.

$diff(x1s\_profit, r1)$ when $dr2 = dp = 0$; For maximum profit, Own Price Effect < 0.

$$Output\_Price\_Effect\_X1Star\_Profit := simplify\left(\frac{-f1\cdot f22 + f2\cdot f12}{p\cdot(f11\cdot f22 - f12\cdot f12)}\right);$$

$Output\_Price\_Effect\_X1S\_Profit\_Alternative := simplify(diff(x1s\_profit, p));$

Output Price Effect in x1

diff(x1s_profit, p) when dr1 = dr2 = 0. #I have to specify which x1, I want to use.

$$Own\_Price\_Effect\_X2S\_profit := \left(\frac{f11}{p\cdot(f11\cdot f22 - f12\cdot f12)}\right);$$

$Own\_Price\_Effect\_X2S\_Profit\_Alternative := diff(x2s\_profit, r2);$ #` `Own Price Effect **in** x2.

$diff(x2s\_profit, r2)$ when $dr1 = dp = 0$.

$$Output\_Price\_Effect\_X2Star\_Profit := simplify\left(\frac{-f2\cdot f11 + f1\cdot f12}{p\cdot(f11\cdot f22 - f12\cdot f12)}\right);$$

$Output\_Price\_Effect\_X2S\_Profit\_Alternative := simplify(diff(x2s\_profit, p));$

Output Price Effect in x2 # diff(X2S_profit, p) when dr1 =dr2 = 0.

#I have to specify which x2, I want to use.

$$Cross\_Price\_Effect\_X1S\_Profit := \left(-\frac{f12}{p\cdot(f11\cdot f22 - f12\cdot f12)}\right); Cross\_Price\_Effect\_X2S\_Profit := \left(-\frac{f21}{p\cdot(f11\cdot f22 - f12^2)}\right);$$

$Cross\_Price\_Effect\_X1S\_Profit\_alternative := diff(x2s\_profit, r1);$

$Cross\_Price\_Effect\_X2S\_Profit\_Alternative := diff(x1s\_profit, r2)$ #Cross Price Effect.

Cross Price Effect implies $diff(x1s\_profit, r2)$ when $dr1 = dp = 0$ **for** X1S_Profit & $diff(x2s\_profit, r1)$ when $dr2 = dp = 0$ **for** X2S_Profit. Both are same.

 # Economic Interdependence of Factors:

If cross effects i.e. diff(x1s_profit, r2) and diff(x1s_profit, r2) are

< 0 means two factors are complementary.

= 0 means two factors are independent.

> 0 means two factors are competitive.

 # Cost Minimization subject to Output Constraints (Min C st (yo - y)): Least Cost Combination of Two Factors of Production.

#***Conditional Factor Demands*** *is defined as relationship between quantity of factor used and factor price holding output constant. It reflects cost minimizing movements along with an isoquant as factor price changes.*

$Cost := TotalCost;$

$LC := Cost + \lambda \cdot (yo - y);$ *# Lagrangean function. λ is lagrangean multiplier.*

$LCf1 := diff(LC, x1); LCf11 := diff(LCf1, x1);$ *#FOC and SOC of lagrangean function wrt x1*

$LCf2 := diff(LC, x2); LCf22 := diff(LCf2, x2);$ *#FOC and SOC of lagraangean function wrt x2*

$LCf12 := diff(LCf1, x2); LCf21 := diff(LCf2, x1);$

 # Cross differentiation of LCf1 and LCf2 wrt x2 and x1 respectively. Gives interdependence of factors.

$LCF\lambda := diff(LC, \lambda);$ *#FOC of lagraangean function wrt λ.*

$LCf1\lambda := solve(LCf1, \lambda);$ ***# λ from LCf1.***

$LCf2\lambda := solve(LCf2, \lambda);$ ***# λ from LCf2.***

$\frac{r1}{LCf1} = \frac{r2}{LCf2}; \frac{r1}{f1} = \frac{r2}{f2}; \frac{r1}{r2} = \frac{f1}{f2}; \frac{r1}{r2} = \frac{LCf1}{LCf2};$

$EP\_C\_x1 := solve(LCf1\lambda = LCf2\lambda, x1);$ ***# Expansion path X1***

$EP\_C\_x2 := solve(LCf1\lambda = LCf2\lambda, x2);$ ***# Expansion Path X2***

$x2s\_cost := (solve((eval(LCF\lambda, x1 = EP\_C\_x1)), x2));$

#X2Star: *Cost Minimizing Input x2 Demand Function, [Verify : Constrained Input Deman Function x2.] #Use small positive value*

$x1s\_cost := (eval(EP\_C\_x1, x2 = x2s\_cost));$ ***# X1Star:*** *Cost Minimizing Input x1 Demand Function . [Verify : Constrained Input Deman Function x1.] #use small positive value.*

$CostStar := (r1 \cdot x1s\_cost + r2 \cdot x2s\_cost + b);$ ***#CostStar:*** *Minimum Cost* **for** *the production of given level of output. Indirect Conditional Cost Function.*

$ystar\_cost := (eval(y, [x1 = x1s\_cost, x2 = x2s\_cost]));$

#Ystar: *Output level produced. This should be equal to given level of output.*

$LCf1\lambda Star := (eval(LCf1\lambda, [x1 = x1s\_cost, x2 = x2s\_cost]));$ ***#λ1Star*** *Lagrangean Multiplier #Use positive value #Marginal Cost*

$LCf2\lambda Star := (eval(LCf2\lambda, [x1 = x1s\_cost, x2 = x2s\_cost]));$ ***# λ2Star*** *Lagrangean multiplier #Use positive value. Marginal Cost*

$BHessian\_Cost\_Min := simplify(\lambda \cdot (LCf1 \cdot LCf1 \cdot LCf22 - 2 \cdot LCf1 \cdot LCf2 \cdot LCf12 + LCf2 \cdot LCf2 \cdot LCf11));$ ***#SOC Condition***

#SOC Condition: Minimum Cost = Determinant of Boarder Hessian Matrix < 0. If > 0, then cost is Maximum.

$\# BHessian\_Cost\_Min\_Value := eval(BHessian\_Cost\_Min, [x1 = x1s\_cost, x2 = x2s\_cost, \lambda = LCf1\lambda Star]);$

$Curvature\_CostMin := simplify\left(-\left(\frac{\mathrm{abs}(BHessian\_Cost\_Min)}{\lambda \cdot LCf2^3}\right)\right);$

#Curvature` *`Strictly quasi-concave production function or Convex Isoquant; SOC always holds*

$\# Corvature\_costMin\_Value := eval(Curvature\_CostMin, [x1 = x1s\_cost, x2 = x2s\_cost, \lambda = LCf1\lambda Star]);$

***Duality: Production Function* to *Indirect Conditional Cost Function* in *Price Space* and *back* to *Production Function*.**

 # Duality related to Shephard Lemma follows same procedure of cost minimization until we derive Indirect Cost Function (CostStar).

 # So, this section is a contuination from cost minimization for duality. This section derive original production function from where we started using duality approach.

 # To go back to production function we need to use ***Shepard*** *Lemma.*

*# Duality related to Shephard lemma follows same procedure of cost minimization until we derive indirect cost function (**CostStar**).*

 # So, this section is a continuation from cost−minimization for duality starting from original production function to indirect conditional cost function and back to original production function.

```
ConstantOutput_Input_Demand_x1 := (diff(CostStar, r1));
    # This should be equal to x1s_cost. Constant-Output input Demand Function x1.
ConstantOutput_Input_Demand_x2 := diff(CostStar, r2);
    # This should be equal to x1s_cost.  Constant-Output input Demand Function x2.
```

 # Note: We have to use concept that price ratio is equal to some constant, m,. i.e. m as ratio of constant output input demands r1 and r1. Then we solve for r1, subsitute resulting r1 in both constant output input demand functions r1 and r2 and solve for m from r1 and r2, then set them m equal to find original production function. (y0 is the y function).

```
sr1 := m·r2;
m1 := solve(eval(ConstantOutput_Input_Demand_x1, [r1 = sr1]) = x1, m); #m1 := eval(m1Init, n = 1);
m2 := solve(eval(ConstantOutput_Input_Demand_x2, [r1 = sr1]) = x2, m); #m2 := eval(m2Init, n = 1);

Original_Production_Function_ShephardLemma := simplify(solve(m1[1] = m2[1], yo));
```

 # Comparative Statics of Cost Minimization Model Or Conditional Factor Demands: *What is the relationship between quantities of factors used* **and** *factors prices when output is constant?*

$$Own\_Price\_Effect\_X1S\_Cost := \left(\frac{-f2 \cdot f2}{LCf1\lambda Star \cdot (2 \cdot f1 \cdot f2 \cdot f12 - f1 \cdot f1 \cdot f22 - f2 \cdot f2 \cdot f11)} \right);$$

$$Own\_Price\_Effect\_X1S\_Cost\_Alternative = diff(x1s\_cost, r1);$$

 # This is equivalent to $diff(x1s\_cost, r1)$ *when* $dr2 = dy = 0$ *This value is negative • after*

$$Output\_Effect\_X1S\_Cost := \frac{f2 \cdot f12 - f1 \cdot f22}{LCf1\lambda Star \cdot (2 \cdot f1 \cdot f2 \cdot f12 - f1 \cdot f1 \cdot f22 - f2 \cdot f2 \cdot f11)};$$

 # This is equivalent to $diff(x1s\_cost, y)$ *when* $dr1 = dr2 = 0$.

 # If this value > 0, then we have normal factor. And, if this value < 0, we have inferior factor. Verify Formula

$$Output\_Effect\_X1S\_Cost\_Alternative := (diff(x1s\_cost, yo));$$

$$Own\_Price\_Effect\_X2S\_Cost := \left(\frac{-f1 \cdot f1}{LCf2\lambda Star \cdot (2 \cdot f1 \cdot f2 \cdot f12 - f1 \cdot f1 \cdot f22 - f2 \cdot f2 \cdot f11)} \right);$$

$$Own\_Price\_Effect\_X2S\_Cost\_Alternative := diff(x2s\_cost, r2);$$

 # This is equivalent to $diff(x2s\_cost, r2)$ *when* $dr1 = dy = 0$.

$$Output\_Effect\_X2S\_Cost := \left(\frac{f1 \cdot f21 - f2 \cdot f11}{LCf2\lambda Star \cdot (2 \cdot f1 \cdot f2 \cdot f12 - f1 \cdot f1 \cdot f22 - f2 \cdot f2 \cdot f11)} \right);$$

 # This is equivalent to $diff(x2s\_cost, y)$ *when* $dr1 = dr2 = 0$.

 # If this value > 0, then we have normal factor. And, if this value < 0, we have inferior factor.Verify Formula.

$$Output\_Effect\_X2S\_Cost\_Alternative := diff(x2s\_cost, y);$$

$$Cross\_Price\_Effect\_X1S\_Cost := \left(\frac{f1 \cdot f2}{LCf1\lambda Star \cdot (2 \cdot f1 \cdot f2 \cdot f12 - f1 \cdot f1 \cdot f22 - f2 \cdot f2 \cdot f11)} \right);$$

$$Cross\_Price\_Effect\_X1S\_Cost\_Alternative := diff(x1s\_cost, r2);$$

 # This is equivalent to $diff(x1s\_cost, r2)$ *when* $dr1 = dy = 0$.

$$Cross\_Price\_Effect\_X2S\_Cost := \left(\frac{f2 \cdot f1}{LCf2\lambda Star \cdot (2 \cdot f1 \cdot f2 \cdot f12 - f1 \cdot f1 \cdot f22 - f2 \cdot f2 \cdot f11)} \right);$$

$$Cross\_Price\_Effect\_X2S\_Cost\_Alternative := diff(x2s\_cost, r1);$$

 # This is equivalent to $diff(x2s\_cost, r1)$ *when* $dr2 = dy = 0$.

 # Maximize Output (Utility) subect to Budget Constraints (Expenditure) (Max y st (Co - C)):

$Ly := y + \mu\cdot(Co - Cost)$; *# Lagrangean function.*
$Lyf1 := diff(Ly, x1)$; $Lyf11 := diff(Lyf1, x1)$; *#FOC and SOC of Lagrangean function wrt x1.*
$Lyf2 := diff(Ly, x2)$; $Lyf22 := diff(Lyf2, x2)$; *#FOC and SOC of Lagrangean function wrt x2.*
$Lyf12 := diff(Lyf1, x2)$; $Lyf21 := diff(Lyf2, x1)$;
#Cross SOCs of Lyf1 and Lyf2 wrt x2 and x2 respectively.
$Ly\mu := diff(Ly, \mu)$; *# FOC of Lagrangean function wrt μ.*
$Lyf1\mu := solve(Lyf1 = 0, \mu)$; *#Solve for μ using Lyf1μ.*
$Lyf2\mu := solve(Lyf2 = 0, \mu)$; *#Solve for μ using Lyf1μ.*
$EP\_Ly\_x1 := solve(Lyf1\mu = Lyf2\mu, x1)$;
 # Equate μ from above two equations and solve for x1. Expansion path x1.
$EP\_Ly\_x2 := solve(Lyf1\mu = Lyf2\mu, x2)$;
 # Equate μ from above two equations and solve for x2. Expansion path x2.
$x2s\_expd := solve((eval(Ly\mu, x1 = EP\_Ly\_x1)), x2)$;
#X2Star: *Demand function of input x2 to maximize the output subject to budget constraints. Ordinary Input Demand Function x2.*
$x1s\_expd := simplify(eval(EP\_Ly\_x1, x2 = x2s\_expd))$;
#X1Star: *Demand function of input x1 to maximize the output subject to budget constraints. Ordinary Input Demand Function x1.*
$ystar\_expd := simplify(eval(y, [x1 = x1s\_expd, x2 = x2s\_expd]))$; ***#YStar:*** *Supply function . Indirect Production Function, Total output produced under constrained budget* **or** *income.*
$CostStar\_expd := simplify(eval(Cost, [x1 = x1s\_expd, x2 = x2s\_expd]))$;
#CostStar: *Total cost to produce given amount of output.*
$Ly\mu Star\_Lyf1\mu := simplify(eval(Lyf1\mu, [x1 = x1s\_expd, x2 = x2s\_expd]))$;
#Lagrangean Multiplier Star (LyμStar_Lyf1μ = LyμStar_Lyf2μ)
#Lagrangean Multiplier Star is Marginal product (increase in output per unit cost) under constrained budget condition. LyμStar_Lyf1μ = LyμStar_Lyf2μ.
$Ly\mu Star\_Lyf1\mu := simplify(eval(Lyf2\mu, [x1 = x1s\_expd, x2 = x2s\_expd]))$;
#Lagrangean Multiplier Star (LyμStar_Lyf1μ = LyμStar_Lyf2μ)`.`
$BHessian\_Output\_Max := simplify(\mu\cdot(Lyf1\cdot Lyf1\cdot Lyf22 - 2\cdot Lyf1\cdot Lyf2\cdot Lyf12 + Lyf2\cdot Lyf2\cdot Lyf11))$;
 # SOC Conditions
#SOC Condition: Determinant of Boarder Hessian Matrix > 0 for Output Maximization.
$BHessian\_Output\_Max\_Value := simplify(eval(BHessian\_Output\_Max, [x1 = x1s\_expd, x2 = x2s\_expd, \lambda = Lyf1\mu Star]))$;
$Curvature\_Output\_Max := simplify\left(-\left(\frac{\mathrm{abs}(BHessian\_Output\_Max)}{\mu\cdot Lyf2^3}\right)\right)$; ***#Curvature***
$Corvature\_Output\_Max\_Value := simplify(eval(Curvature\_Output\_Max, [x1 = x1s\_expd, x2 = x2s\_expd, \lambda = Lyf1\mu Star]))$;
 # Homogenity of Demand Functions: Factor demand functions are homogenous of degree zero.
 # Comparative Statisc of Output Maximization: # Not Demonstrated in Note and not in book as well.

 # *Duality Roy's Identity: Production Function* to *Indirect Conditional Porduction Function* in *Price Space* and *Expenditure Space* and *back* to *Production Function*.

 # Roy's Identity follows same steps until maximize output (Utility) st Budget constraints (income) until we derive indirect production function or supply function (ystar_expd).

 # So, this section contimues after the derivation of ystar_expd above. The Roy's Identity is the ration of diff(ystar_expd, r1) to diff(ystar_expd, Co) gives -x1star (notice negative sign), OR diff(ystar_expd, r2) to diff(ystar_expd, dCo) gives -x2star (notice negative sign).

 # Here production function represents Utility function and Co represents income constraints.

$$RoysIdentity\_x1star := simplify\left(-\frac{diff(ystar\_expd, r1)}{diff(ystar\_expd, Co)} \right);$$

 # Expenditure Input Demand Function. This is same as x1s_expd. ***# h***

$$RoysIdentity\_x2star := simplify\left(-\frac{diff(ystar\_expd, r2)}{diff(ystar\_expd, Co)} \right);$$

 # Expenditure Input Demand Function. This is same as x2s_expd. ***# k***

$E\_from\_x1 := solve(RoysIdentity\_x1star = x1, Co);$

#Expenditure (Co) from Roys Identity x1Star # E from h

$E\_from\_x2 := solve(RoysIdentity\_x2star = x2, Co);$

#Expenditure (Co) from Roys Identity x2Star # E from k

$Roys\_Identity\_r1star := solve(E\_from\_x1 = E\_from\_x2, r1);$

 # This is price of Input x1. This is equivalent to r1. # w

$Roys\_Identity\_r2star := solve(E\_from\_x1 = E\_from\_x2, r2);$

 # This is price of Input x1. This is equivalent to r2. # r

$Original\_Production\_Function\_RoysId := (eval(ystar\_expd, [Co = E\_from\_x2, r1 = Roys\_Identity\_r1star[3]]));$

$reduce(Original\_Production\_Function\_RoysId);$

$Original\_Production\_Function\_RoysId2 := simplify(eval(ystar\_expd, [Co = E\_from\_x1, r2 = Roys\_Identity\_r2star[3]]));$

 # Output Side Economics: of Production Function with Two Input and One Output (Quadratic and Cobb Douglas Production Function)

<u># Cost Minimization subject to Output Constraints (Min C st (yo - y)): Least Cost Combination of Two Factors of Production. Necessary for output side economics.</u>