**#` `Economics of Production Function with Two Input and One Output·** $($**Quadratic and Cobb Douglas Production Function**$)$

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**# Input Side Economics: of Production Function with Two Input and One Output (Quadratic and Cobb Douglas Production Function)**
*# Profit Maximization, Cost Minimization Subejct to Output Constraints and Output Maximization Subject to Budget Constraints, Duality and Comprative Statics*

*# Some terminologies to understand:*
*# Orinary Input Demand Functions:*
*# Constrained Input Demand Functions:*
*# Expenditure Demand Functions:*
*# Supply Function:*
*# Profit Function:*
*# Cost Function:*
*# Conditional Production Function:*

*restart*;
*#Use this line to enter the value of variables as given in the exam.*
$r1 := w$; $r2 := r$; $p := p$; $b := 0$; $Co := Co$; $yo := yo$;
     *# Input 1 Cost, Input 2 Cost, Output Price, Fixed Cost, Constrained Investment or Budget, Constrained Output.*

$x1 := h$; $x2 := k$; $b1 := 80$; $b2 := 75$; $c1 := -4$; $c2 := -3$; $d1 := -4$; $a := 0$;
     *#Change x1 and x2 and Coefficients x1, x2, $x1^2$, $x2^2$, x1·x2, Constant* **for** *quadratic function.*
$m := m$; $n := n$; $Z := Z$; *#Change m, n and Z to change Cobb Douglas production function.*
**# Cobb Douglas Production Function:**

$Z := Z$; $m := m$; $n := n$; $cobb := Z \cdot x1^m x2^n$;
**# Quadratic Production function (Two Input One Output):**
$b1 := b1$; $b2 := b2$; $c1 := c1$; $c2 := c2$; $d1 := d1$; $a := a$; $x1 := x1$; $x2 := x2$; $quad := b1 \cdot x1 + b2 \cdot x2 + c1 \cdot x1^2 + c2 \cdot x2^2 + d1 \cdot x1 \cdot x2 + a$ ;
*# x1= Input 1, x2 = Input 2. b1, b2, c1, c2, d1, are coefficients of inputs and their interactions, a is a constant.*
$y := quad$; **#Change production function (quad or cobb).**
$VC := r1 \cdot x1 + r2 \cdot x2$; $FC := b$; $TotalCost := VC + FC$;
     *# Cost Functions: Variable Cost (VC), Fixed Cost (FC), Total Cost (TC)*

$TVP := p \cdot y$; $Profit\_Function := TVP - TotalCost$;  *# TVP = Total Value Product, Profit_fun = Profit.*

$$TPP := y;\ APP1 := simplify\left(\frac{y}{x1}\right);\ APP2 := simplify\left(\frac{y}{x2}\right);\ AVP1 := simplify(APP1 \cdot p)\ ;\ AVP2 :=$$

$$simplify(APP2 \cdot p);$$

<span style="color:red"># *Total Physical Product, Average Physical Productivity of Factor x1, Average Physical Productivity of Factor x2, Average Value productivity of Factor x1, Average Value Productivity of Factor x2*</span>

$$f1 := \frac{\partial}{\partial x1}(y);\quad f2 := \frac{\partial}{\partial x2}(y);\ MVP1 := p \cdot f1; MVP2 := p \cdot f2;\ MFC1 := MVP1; MFC2 := MVP2;$$

<span style="color:red"># *Marginal Physical Productivity (MPP) of Factor x1, Marginal Physical Productivity of Factor x2, Marginal Value Productivity · (MVP) of Factor x1, Marginal Value Productivity of Factor x2, Marginal Factor Cost · (MFC) of Factor x1, Marginal Factor Cost of Factor x2.*</span>

$$f11 := \frac{\partial}{\partial x1}(f1);\ f22 := \frac{\partial}{\partial x2}(f2);\ f12 := \frac{\partial^2}{\partial x1\ \partial x2}(y);\ f21 := \frac{\partial^2}{\partial x2\ \partial x1}(y);$$

<span style="color:red"># *Second Order Conditions of f1, f2.*
*#f11 & f22 < 0 ⇒ Diminishing Marginal Physical Returns.*
*#f12 & f21 are Factor Inerdependence (f12* **or** *f21 < 0 ⇒ competitive; f12* **or** *f21 = 0 ⇒ independent; f12* **or** *f21 > 0 ⇒ complementary).*</span>

$$MRTS := simplify\left(\left(\frac{f1}{f2}\right)\right);$$

<span style="color:red"># *Marginal Rate of Technical Subsitution $\left(MRTS_{21}\right)$ If MRTS is simplified* **by** *equating with constant, k, that gives isolcline at MRTS = k.*</span>

$$MarginalRevenue := \frac{f1}{f2};\ \#\left(\frac{r1}{r2} = \frac{f1}{f2}\right) Marginal\ Revenue\ (MR)$$

<span style="color:red">= *Marginal Rate of Technical Subsitution* $\left(MRTS_{21}\right)$</span>

$$\left(\left(\frac{r1}{f1} = \frac{r2}{f2}\right)\right)$$

$MarginalRevenue\_PerfectCompetition := p;$ <span style="color:red">#*Only in perfectly competitive market.*</span>

$Slope\_Of\_Isoquants\_at\_a\_Point := eval(MarginalRevenue, [x1 = x1, x2 = x2]);$

<span style="color:red"># *Slope of Isoquants at a Point. Replace x1* **and** *x2* **by** *given values of x1,* **and** *x2.*</span>

$SOC := simplify(\ f2 \cdot f2 \cdot f11 - 2 \cdot f1 \cdot f2 \cdot f12 + f1 \cdot f1 \cdot f22);$

<span style="color:red"># *Second Order Conditions: Border Hessian Condition.*</span>

$$Curvature := simplify\left(\left(\frac{1}{f2^3}\right) \cdot SOC\right);$$ <span style="color:red"># *Curvature. This curvature is derived* **from** *production function. Gives change* **in** *slope of the isoquant* **and** *used* **to** *determine convexity of the isoqunt.*

# *Also, if you know the equation of isoquant, then curvature is the second derivative of the equation for isoquant.*</span>

$Curvature\_at\_a\_Point := eval(Curvature, [x1 = x1, x2 = x2]);$ <span style="color:red">**# *Curvature at a Point. Replace* x1 *and* x2 **by** *given values of x1,* **and** *x2.*</span>

$$Elasticity\_of\_Factor\_Subsitution := \frac{f1 \cdot f2 \cdot (f2 \cdot x2 + f1 \cdot x1)}{(x1 \cdot x2) \cdot (2\ \cdot f1 \cdot f2 \cdot f12 - f1 \cdot f1 \cdot f22 - f2 \cdot f2 \cdot f11)};$$

<span style="color:red"># *Elasticity of Factor Subsitution (σ):*</span>

$ElasticityOfFS := eval(Elasticity\_of\_Factor\_Subsitution, [x1 = x1, x2 = x2]);$

<span style="color:red"># *Enter value of x1 and x2 as desired to calculate elasticity of factor subsitution.*</span>

$$Elasticity\_x1 := \frac{f1}{APP1};$$ <span style="color:red"># *Elasticity of Production for Factor x1. MPP1 divided by APP1.*</span>

$$Elasticity\_x2 := \frac{f2}{APP2}; \text{ # } \textit{Elasticity of Production for Factor x2. (MPP2 divided by APP2).}$$

$Functional\_Coefficient := simplify(Elasticity\_x1 + Elasticity\_x2); \text{ #}\textit{Elasticity of Production}$
   or *Function Coefficient of Production Function.*
   # *Elasticity of Production or Functional Coefficient (ε): Also gives Return to Scale ( = 1*
   $\Rightarrow$ *Constant*); $( > 1 \Rightarrow$ *Increasing*); $OR \cdot ( < 1 \Rightarrow$ *Decreasing*).

$$Functional\_Coefficient\_Alternative := simplify\left( \frac{(f1 \cdot x1 + f2 \cdot x2)}{y} \right);$$

   #*Functional_Coefficient_Alternative and Functional_Coefficient are same, calculated differently*

**# Isoquants:**
$Two\_Isoquants\_x1 := solve(y = v, x1); \text{ #}\textit{v is constant.}$
$Two\_Isoquants\_x2 := solve(y = v, x2); \text{ #}\textit{v is constant.}$
**# Stages of Productions for Two Inputs:**
$Stage\_I\_II\_Boundary\_Input\_x1 := solve(Elasticity\_x1 = 1, x1); \text{ #}\textit{use positive value}$
$Stage\_II\_III\_Boundary\_Input\_x1 := solve(Elasticity\_x1 = 0, x1); \text{ #}\textit{use positive value}$
$Stage\_I\_II\_Boundary\_Input\_x2 := solve(Elasticity\_x2 = 1, x2); \text{ #}\textit{use positive value.}$
$Stage\_II\_III\_Boundary\_Input\_x2 := solve(Elasticity\_x2 = 0, x2); \text{ #}\textit{use positive value.}$
$Inflection\_Point\_Input\_x1 := solve(f1 = APP1, x1); \text{ #}\textit{Inflection Point for input X1}$
$Inflection\_Point\_Input\_x2 := solve(f2 = APP2, x2); \text{ #}\textit{Inflection Point for input X2}$
**# Homogenous function:**

# *if one common value in term of* ${}^t r^t$ *is obtained* for *entire equation,*
   *the funciton is homogenous of degree* "r" and *has constant proportion of* return to *scale of r*
   *. The r value is also equal* to *functional ceofficient* $(\varepsilon)$. *The degree of homogenity of MPPs*
   **and** *APPs of homogenous production functions are* "r-1".

$Homogen := simplify(eval(y, [x1 = t \cdot x1, x2 = t \cdot x2]));$
**# Ridgelines x1 and x2:**
$Ridgeline\_x1 := solve(f1 = 0, x1); \text{ }$ **# MPP1 = 0 and Stage III begins for input x2.**
$Ridgeline\_x2 := solve(f2 = 0, x2); \text{ }$ **# MPP2 = 0 and stage III begins for input x1.**

$profit := Profit\_Function;$

$pf1 := diff(profit, x1); pf11 := diff(pf1, x1); \# pf11 < 0\ for\ profit\ max.$

$pf2 := diff(profit, x2); pf22 := diff(pf2, x2); \# pf22 < 0\ for\ profit\ max.$

$pf12 := diff(pf1, x2); pf21 := diff(pf2, x1);$

$EP\_p\_x1 := solve(pf1 = 0, x1); \#$ *Expansion Path x1 Or Pseudo scale line x1.*

$EP\_p\_x2 := solve(pf2 = 0, x2); \#$ *Expansion Path x2 Or Pseudo scale line x2.*

$x2s\_profit := simplify(solve((eval(pf2, x1 = EP\_p\_x1)) = 0, x2));$

   *# X2Star: Profit Maximizing Level of Input x2 Or Ordinary Input Demand Function x2.*

$x1s\_profit := simplify(eval(EP\_p\_x1, x2 = x2s\_profit));$

   *# X1Star: Profit Maximizing Level of Input x1 Or Ordinary Input Demand Function x1.*

$MaxProfOut := simplify(eval(y, [x1 = x1s\_profit, x2 = x2s\_profit]));$

   *# Profit Maximizing Level of Output.*

$ProfitStar := simplify(eval(profit, [x1 = x1s\_profit, x2 = x2s\_profit]));$

   *# ProfitStar: Maximum Profit Function, Indirect Profit Function.*

$CostStar\_MaxProfit := simplify(eval(TotalCost, [x1 = x1s\_profit, x2 = x2s\_profit]));$

   *# CostStar: Cost at Maximum profit.*

$\dfrac{r1}{pf1} = \dfrac{r2}{pf2}; \dfrac{r1}{f1} = \dfrac{r2}{f2}; \dfrac{r1}{r2} = \dfrac{f1}{f2}; \dfrac{r1}{r2} = \dfrac{pf1}{pf2};$

$BHessian\_Profit\_Max := (pf11 \cdot pf22 - pf12 \cdot pf21);$

   *#SOC Condition: Determinant of Boarder Hessian Matrix > 0 for maximization.*

$Curvature\_ProfitMax := -\left( \dfrac{abs(pf11 \cdot pf22 - pf12 \cdot pf21)}{pf2^3} \right);$

   *#Strictly quasi concave in perfect competition.*

$Curvature\_ProfitMax\_Value := eval(Curvature\_ProfitMax, [x1 = x1s\_profit, x2 = x1s\_profit]);$

# **Duality: Production Function to Indirect Conditional Profit Function in Price Space and Back to Production Function.**

\# *Duality related to Hotllings Lemma follows same procedure of profit maximization without any constraints until we derive Indirect Profit Function* ($ProfitStar$).

\# *So, this section is a contuination from profit maximization for duality. This section derive original production function from where we started using duality approach.*

\# *To go back to production function we need to use* **Hottlings Lemma** *i.e. Partial derivative of indirect profit function with respect to p gives supply function.*

$Hottlings\_Lemma\_SupplyFunction := simplify(diff(ProfitStar, p));$
> *#Hottlings Lemma gives supply function.*

$Ordinary\_Input\_Demand\_Function\_x1 := simplify(-diff(ProfitStar, r1));$
> *#This is same as X1Star from profit maximization.*

$Ordinary\_Input\_Demand\_Function\_x2 := simplify(-diff(ProfitStar, r2));$
> *#This is same as X2Star from profit maximization.*

$Duality\_Hottling\_r1 := solve(Ordinary\_Input\_Demand\_Function\_x2 = x2, r1);$

$Duality\_Hottling\_r2 := solve(Ordinary\_Input\_Demand\_Function\_x1 = x1, r2);$

$Duality\_Hottling\_r2Star := simplify(solve(eval(Ordinary\_Input\_Demand\_Function\_x1, [r1 = Duality\_Hottling\_r1]) = x1, r2));$

$Duality\_Hottling\_r1Star := simplify(eval(Duality\_Hottling\_r1, [r2 = Duality\_Hottling\_r2Star]));$

$Original\_Production\_Function := simplify(eval(Hottlings\_Lemma\_SupplyFunction, [r1 = Duality\_Hottling\_r1Star, r2 = Duality\_Hottling\_r2Star]));$

# Comparative Static of Profit Maximization Without Any Constraints:

$$Own\_Price\_Effect\_X1Star\_Profit := \left( \frac{f22}{p \cdot (f11 \cdot f22 - f12 \cdot f12)} \right);$$

$Own\_Price\_Effect\_X1S\_Profit\_Alternative := diff(x1s\_profit, r1);$ # Own Price Effect in x1.

# $diff(x1s\_profit, r1)$ when $dr2 = dp = 0$; For maximum profit, Own Price Effect < 0.

$$Output\_Price\_Effect\_X1Star\_Profit := simplify\left( \frac{-f1 \cdot f22 + f2 \cdot f12}{p \cdot (f11 \cdot f22 - f12 \cdot f12)} \right);$$

$Output\_Price\_Effect\_X1S\_Profit\_Alternative := simplify(diff(x1s\_profit, p));$

# Output Price Effect in x1

# diff(x1s_profit, p) when dr1 = dr2 = 0. #I have to specify which x1, I want to use.

$$Own\_Price\_Effect\_X2S\_profit := \left( \frac{f11}{p \cdot (f11 \cdot f22 - f12 \cdot f12)} \right);$$

$Own\_Price\_Effect\_X2S\_Profit\_Alternative := diff(x2s\_profit, r2);$ #``Own Price Effect in x2.

# $diff(x2s\_profit, r2)$ when $dr1 = dp = 0$.

$$Output\_Price\_Effect\_X2Star\_Profit := simplify\left( \frac{-f2 \cdot f11 + f1 \cdot f12}{p \cdot (f11 \cdot f22 - f12 \cdot f12)} \right);$$

$Output\_Price\_Effect\_X2S\_Profit\_Alternative := simplify(diff(x2s\_profit, p));$

# Output Price Effect in x2 # diff(X2S_profit, p) when dr1 =dr2 = 0.

# #I have to specify which x2, I want to use.

$$Cross\_Price\_Effect\_X1S\_Profit := \left( -\frac{f12}{p \cdot (f11 \cdot f22 - f12 \cdot f12)} \right); Cross\_Price\_Effect\_X2S\_Profit := \Big($$

$$-\frac{f21}{p \cdot (f11 \cdot f22 - f12^2)} \Big);$$

$Cross\_Price\_Effect\_X1S\_Profit\_alternative := diff(x2s\_profit, r1);$

$Cross\_Price\_Effect\_X2S\_Profit\_Alternative := diff(x1s\_profit, r2)$ #Cross Price Effect.

# Cross Price Effect implies $diff(x1s\_profit, r2)$ when $dr1 = dp = 0$ for X1S_Profit & $diff(x2s\_profit,$
$r1)$ when $dr2 = dp = 0$ for X2S_Profit. Both are same.

# Economic Interdependence of Factors:
# If cross effects i.e. diff(x1s_profit, r2) and diff(x1s_profit, r2) are
# < 0 means two factors are complementary.
# = 0 means two factors are independent.
# > 0 means two factors are competitive.

#*Conditional Factor Demands* is defined as relationship between quantity of factor used and factor price holding output constant. It reflects cost minimizing movements along with an isoquant as factor price changes.

$Cost := TotalCost;$

$LC := Cost + \lambda \cdot (yo - y);$  # *Lagrangean function.* $\lambda$ *is lagrangean multiplier.*

$LCf1 := diff(LC, x1); LCf11 := diff(LCf1, x1);$ #*FOC and SOC of lagrangean function wrt x1*

$LCf2 := diff(LC, x2); LCf22 := diff(LCf2, x2);$ #*FOC and SOC of lagraangean function wrt x2*

$LCf12 := diff(LCf1, x2); LCf21 := diff(LCf2, x1);$
   # *Cross differentiation of LCf1 and LCf2 wrt x2 and x1 respectively.* Gives interdependence of factors.

$LCF\lambda := diff(LC, \lambda);$ #*FOC of lagraangean function wrt* $\lambda$.

$LCf1\lambda := solve(LCf1, \lambda);$ # $\lambda$ *from LCf1.*

$LCf2\lambda := solve(LCf2, \lambda);$ # $\lambda$ *from LCf2.*

$$\frac{r1}{LCf1} = \frac{r2}{LCf2}; \frac{r1}{f1} = \frac{r2}{f2}; \frac{r1}{r2} = \frac{f1}{f2}; \frac{r1}{r2} = \frac{LCf1}{LCf2};$$

$EP\_C\_x1 := solve(LCf1\lambda = LCf2\lambda, x1);$ # *Expansion path X1*

$EP\_C\_x2 := solve(LCf1\lambda = LCf2\lambda, x2);$ # *Expansion Path X2*

$x2s\_cost := solve((eval(LCF\lambda, x1 = EP\_C\_x1)), x2);$
   #*X2Star:* Cost Minimizing Input x2 Demand Function, $[Verify :$
   Constrained Input Deman Function x2.$]$ #Use small positive value

$x1s\_cost := eval(EP\_C\_x1, x2 = x2s\_cost);$ # *X1Star:* Cost Minimizing Input x1 Demand Function. $[Verify :$ Constrained Input Deman Function x1.$]$ #use small positive value.

$CostStar := r1 \cdot x1s\_cost + r2 \cdot x2s\_cost + b;$ #*CostStar:* Minimum Cost **for** the production of given level of output. Indirect Conditional Cost Function.

$ystar\_cost := eval(y, [x1 = x1s\_cost, x2 = x2s\_cost]);$
   #*Ystar:* Output level produced. This should be equal to given level of output.

$LCf1\lambda Star := eval(LCf1\lambda, [x1 = x1s\_cost, x2 = x2s\_cost]);$ #$\lambda 1Star$ Lagrangean Multiplier
   #Use positive value #Marginal Cost

$LCf2\lambda Star := eval(LCf2\lambda, [x1 = x1s\_cost, x2 = x2s\_cost]);$ # $\lambda 2Star$ Lagrangean multiplier
   #Use positive value. Marginal Cost

$BHessian\_Cost\_Min := simplify(\lambda \cdot (LCf1 \cdot LCf1 \cdot LCf22 - 2 \cdot LCf1 \cdot LCf2 \cdot LCf12 + LCf2 \cdot LCf2 \cdot LCf11));$ #*SOC Condition*
   #SOC Condition: Minimum Cost = Determinant of Boarder Hessian Matrix < 0. If > 0, then cost is Maximum.

# $BHessian\_Cost\_Min\_Value := eval(BHessian\_Cost\_Min, [x1 = x1s\_cost, x2 = x2s\_cost, \lambda = LCf1\lambda Star]);$

$$Curvature\_CostMin := simplify\left(-\left(\frac{abs(BHessian\_Cost\_Min)}{\lambda \cdot LCf2^3}\right)\right);$$

   #*Curvature*` `Strictly quasi-concave production function or Convex Isoquant; SOC always holds

# $Corvature\_costMin\_Value := eval(Curvature\_CostMin, [x1 = x1s\_cost, x2 = x2s\_cost, \lambda = LCf1\lambda Star]);$

**# *Duality: Production Function* to *Indirect Conditional Cost Function* in *Price Space* and *back to Production Function.***

# *Duality related to Shephard Lemma follows same procedure of cost minimization until we derive Indirect Cost Function* (*CostStar*).

# *So, this section is a contuination from cost minimization for duality. This section derive original production function from where we started using duality approach.*

# *To go back to production function we need to use* **Shepard** *Lemma.*

# *Duality related to Shephard lemma follows same procedure of cost minimization until we derive indirect cost function* (***CostStar***).

# *So, this section is a continuation from cost−minimization for duality starting from original production function to indirect conditional cost function and back to original production function.*

$ConstantOutput\_Input\_Demand\_x1 := diff(CostStar, r1);$
    *# This should be equal to x1s_cost. Constant-Output input Demand Function x1.*
$ConstantOutput\_Input\_Demand\_x2 := diff(CostStar, r2);$
    *# This should be equal to x1s_cost.  Constant-Output input Demand Function x2.*

# *Note: We have to use concept that price ratio is equal to some constant, m,. i.e. m as ratio of constant output input demands r1 and r1. Then we solve for r1, subsitute resulting r1 in both constant output input demand functions r1 and r2 and solve for m from r1 and r2, then set them m equal to find original production function. (y0 is the y function).*

$sr1 := m \cdot r2;$
$m1 := solve(eval(ConstantOutput\_Input\_Demand\_x1, [r1 = sr1]) = x1, m); \#m1 := eval(m1Init, n = 1);$
$m2 := solve(eval(ConstantOutput\_Input\_Demand\_x2, [r1 = sr1]) = x2, m); \#m2 := eval(m2Init, n = 1);$

$Original\_Production\_Function\_ShephardLemma := simplify(solve(m1[1] = m2[1], yo));$

$Own\_Price\_Effect\_X1S\_Cost := \dfrac{-f2 \cdot f2}{LCf1\lambda Star \cdot (2 \cdot f1 \cdot f2 \cdot f12 - f1 \cdot f1 \cdot f22 - f2 \cdot f2 \cdot f11)}$ ;

$Own\_Price\_Effect\_X1S\_Cost\_Alternative = diff(x1s\_cost, r1);$

*# This is equivalent to  diff(x1s\_cost, r1) when dr2 = dy = 0 This value is negative • after*

$Output\_Effect\_X1S\_Cost := \dfrac{f2 \cdot f12 - f1 \cdot f22}{LCf1\lambda Star \cdot (2 \cdot f1 \cdot f2 \cdot f12 - f1 \cdot f1 \cdot f22 - f2 \cdot f2 \cdot f11)}$ ;

*# This is equivalent to diff(x1s\_cost, y) when dr1 = dr2 = 0.*

*# If this value > 0, then we have normal factor. And, if this value < 0, we have inferior factor. Verify Formula*

$Output\_Effect\_X1S\_Cost\_Alternative := diff(x1s\_cost, y);$

$Own\_Price\_Effect\_X2S\_Cost := \dfrac{-f1 \cdot f1}{LCf2\lambda Star \cdot (2 \cdot f1 \cdot f2 \cdot f12 - f1 \cdot f1 \cdot f22 - f2 \cdot f2 \cdot f11)}$ ;

$Own\_Price\_Effect\_X2S\_Cost\_Alternative := diff(x2s\_cost, r2);$

*# This is equivalent to diff(x2s\_cost, r2) when dr1 = dy = 0.*

$Output\_Effect\_X2S\_Cost := \dfrac{f1 \cdot f21 - f2 \cdot f11}{LCf2\lambda Star \cdot (2 \cdot f1 \cdot f2 \cdot f12 - f1 \cdot f1 \cdot f22 - f2 \cdot f2 \cdot f11)}$ ;

*# This is equivalent to diff(x2s\_cost, y) when dr1 = dr2 = 0.*

*# If this value > 0, then we have normal factor. And, if this value < 0, we have inferior factor. Verify Formula.*

$Output\_Effect\_X2S\_Cost\_Alternative := diff(x2s\_cost, y);$

$Cross\_Price\_Effect\_X1S\_Cost := \dfrac{f1 \cdot f2}{LCf1\lambda Star \cdot (2 \cdot f1 \cdot f2 \cdot f12 - f1 \cdot f1 \cdot f22 - f2 \cdot f2 \cdot f11)}$ ;

$Cross\_Price\_Effect\_X1S\_Cost\_Alternative := diff(x1s\_cost, r2);$

*# This is equivalent to diff(x1s\_cost, r2) when dr1 = dy = 0.*

$Cross\_Price\_Effect\_X2S\_Cost := \dfrac{f2 \cdot f1}{LCf2\lambda Star \cdot (2 \cdot f1 \cdot f2 \cdot f12 - f1 \cdot f1 \cdot f22 - f2 \cdot f2 \cdot f11)}$ ;

$Cross\_Price\_Effect\_X2S\_Cost\_Alternative := diff(x2s\_cost, r1);$

*# This is equivalent to  diff(x2s\_cost, r1) when dr2 = dy = 0.*

$Ly := y + \mu \cdot (Co - Cost);$ # *Lagrangean function.*
$Lyf1 := diff(Ly, x1); Lyf11 := diff(Lyf1, x1);$ #*FOC and SOC of Lagrangean function wrt x1.*
$Lyf2 := diff(Ly, x2); Lyf22 := diff(Lyf2, x2);$ #*FOC and SOC of Lagrangean function wrt x2.*
$Lyf12 := diff(Lyf1, x2); Lyf21 := diff(Lyf2, x1);$
   #*Cross SOCs of Lyf1 and Lyf2 wrt x2 and x2 respectively.*
$Ly\mu := diff(Ly, \mu);$ # *FOC of Lagrangean function wrt $\mu$.*
$Lyf1\mu := solve(Lyf1 = 0, \mu);$ #*Solve for $\mu$ using Lyf1$\mu$.*
$Lyf2\mu := solve(Lyf2 = 0, \mu);$ #*Solve for $\mu$ using Lyf1$\mu$.*
$EP\_Ly\_x1 := solve(Lyf1\mu = Lyf2\mu, x1);$
   # *Equate $\mu$ from above two equations and solve for x1. Expansion path x1.*
$EP\_Ly\_x2 := solve(Lyf1\mu = Lyf2\mu, x2);$
   # *Equate $\mu$ from above two equations and solve for x2. Expansion path x2.*
$x2s\_expd := solve((eval(Ly\mu, x1 = EP\_Ly\_x1)), x2);$
   #***X2Star:*** *Demand function of input x2 to maximize the output subject to budget constraints. Ordinary Input Demand Function x2.*
$x1s\_expd := eval(EP\_Ly\_x1, x2 = x2s\_expd);$
   #***X1Star:*** *Demand function of input x1 to maximize the output subject to budget constraints. Ordinary Input Demand Function x1.*
$ystar\_expd := eval(y, [x1 = x1s\_expd, x2 = x2s\_expd]);$
   #***YStar:*** *Supply function. Total output produced under constrained budget.*
$CostStar\_expd := eval(Cost, [x1 = x1s\_expd, x2 = x2s\_expd]);$
   #***CostStar:*** *Total cost to produce given amount of output.*
$Ly\mu Star\_Lyf1\mu := eval(Lyf1\mu, [x1 = x1s\_expd, x2 = x2s\_expd]);$
   #***Lagrangean Multiplier Star (Ly$\mu$Star_Lyf1$\mu$ = Ly$\mu$Star_Lyf2$\mu$)***
   *#Lagrangean Multiplier Star is Marginal product (increase in output per unit cost) under constrained budget condition. Ly$\mu$Star_Lyf1$\mu$ = Ly$\mu$Star_Lyf2$\mu$.*
$Ly\mu Star\_Lyf1\mu := eval(Lyf2\mu, [x1 = x1s\_expd, x2 = x2s\_expd]);$
   #***Lagrangean Multiplier Star (Ly$\mu$Star_Lyf1$\mu$ = Ly$\mu$Star_Lyf2$\mu$)***`.`
$BHessian\_Output\_Max := simplify(\mu \cdot (Lyf1 \cdot Lyf1 \cdot Lyf22 - 2 \cdot Lyf1 \cdot Lyf2 \cdot Lyf12 + Lyf2 \cdot Lyf2 \cdot Lyf11));$
   # ***SOC Conditions***
 *#SOC Condition: Determinant of Boarder Hessian Matrix > 0 for Output Maximization.*
$BHessian\_Output\_Max\_Value := simplify(eval(BHessian\_Output\_Max, [x1 = x1s\_expd, x2 = x2s\_expd, \lambda = Lyf1\mu Star]));$

$Curvature\_Output\_Max := simplify\left(-\left(\dfrac{\text{abs}(BHessian\_Output\_Max)}{\mu \cdot Lyf2^3}\right)\right);$ #***Curvature***

$Corvature\_Output\_Max\_Value := eval(Curvature\_Output\_Max, [x1 = x1s\_expd, x2 = x2s\_expd, \lambda = Lyf1\mu Star]);$
*# Homogenity of Demand Functions: Factor demand functions are homogenous of degree zero.*
*# Comparative Statisc of Output Maximization: # Not Demonstrated in Note and not in book as well.*

# *Output Side Economics: of Production Function with Two Input and One Output (Quadratic and Cobb Douglas Production Function)*

# *Cost Minimization subject to Output Constraints (Min C st (yo - y)): Least Cost Combination of Two Factors of Production. Necessary for output side economics.*