# Abstract

This paper presents an improved context-adaptive anisotropic diffusion model for image denoising. It is based on a new diffusion coefficient and alternate direction explicit fractional order differential operator. Differential of fractional order is applied to weaken the staircasing effect and preserves fine characteristic whereas the new diffusion coefficient can protect edges and fine characteristics from getting over-smoothed. Advantage of ADE is that it is more stable than classical euler explicit scheme [1]. Comparative experimental results validate that the proposed model successfully denoises the image and preserves fine characteristics of image.

# Introduction

The images of the real world objects mostly contains noise during the time when the images are captured or transmitted. The noise can be more intensive during the acquisition phase if the appliances used are having the low resolution sensors. The main objective of image denoising is to remove the noise while retaining the sharp features of the image such as edges, corners etc. Till now, no method has been developed which preserves the features of the image absolutely. The traditional image denoising methods are the iterating techniques. The noise is reduced up to a certain extent using these methods, however, the edges gets dislocated. In order to tackle this problem, Witkin and Koenderick introduced the scale space representation [2], based on which Perona and Malik introduced a new definition of space-scale through Anisotropic Diffusion [3]. It is non-linear partial differential equation based diffusion process. In this model, isotropic diffusion equation expressed with linear heat equation is replaced with an anisotropic diffusion. Anisotropic diffusion process can efficiently smoothen noise while preserving the boundaries and texture information within the image only if its crucial parameters are estimated and applied correctly. The set of parameter consists of conductance function, gradient threshold parameter and stopping parameter. This set defines the behavior and extent of diffusion process [4]. Few years back, fractional calculus became important to foundational research and engineering application. Fractional calculus suggests the methods to integrate and differentiate function to fractional or non-integral orders. There exists three popular definitions for fractional calculus, given by Grunwald-Letnikov(G-L), Caputo and Riemann-Liouville[5]. Two of these, R-L and G-L are most frequently used definitions in digital image processing. Pu et al.[6] proved that methods based on fractional differential preserves low-frequency contour features in smooth regions. They also proved that fractional differentiation retains high-frequency marginal features in regions that particularly have large gray-level variabilities, and can also enhance the texture details in those regions which do not have significant gray-level variabilities.

Mostly all the researches in anisotropic diffusion are oriented towards understanding the diffusivity function and its characteristic, then improving the overall performance of the anisotropic diffusion process for image denoising. Catte et al.[7] proposed a regularised version of PM diffusion process to address the ill-posed issue. He suggested that the image gradient I can be replaced with a smoothed gradient (G\*I ) for computing the values of diffusivity function. Considering the viewpoint of robust statistics, Black et al.[8] suggested another diffusivity function using Tukeys Biweight concept. This diffusion process could preserve sharp boundaries and better continuity of edges. However there are many limitations to the above mentioned models. First the staircasing effects still exists. Secondly the texture information is not recovered and in many existing methods gradient threshold parameter is pre-set manually.

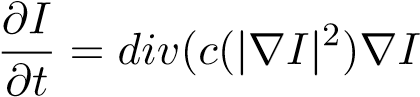
The remaining paper is organised as follows, In Sec 3,4 analysis of improved Perona-Malik model and ADE scheme is presented. In Sec 5 fractional Perona-Malik model using ADE is presented in detail and its numerical implementation are described. Experimental results are shown in Sec 6 and finally conclusion is drawn in Sec 6.

# Related theories

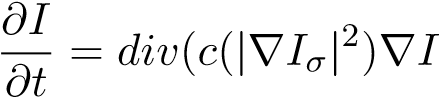
3.1. **Improved Perona Malik model.** Let *I*0 be the image with noise which needs to be restored. Mathematically *I*0 : Ω → R represent a noisy image and its obtained by following process

*I*0 = *I* + *η* (1)

Here the noise *η* is additive white gaussian noise with known mean and standard deviation. The image domain Ω → R2 is a bounded domain, a rectangle usually. The original image is recovered by solving the following anisotropic diffusion equation :

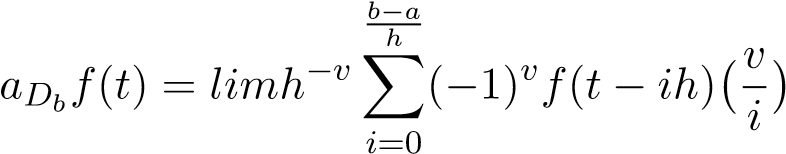
) (2)

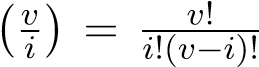
with initial condition *I*(*x,y,*0) = *I*0(*x,y*) where ∇ gradient operator, *div* - divergence, |*.*| Euclidean norm. *c*(|∇*I*|) denotes diffusion coefficient. It is positive and non-increasing over |∇*I*|. Speed of diffusion is controlled by diffusion coefficient. In ideal situation diffusion should be fast in smooth region and it should be very slow or ideally stop in the edge region. Therefore, *c*(|∇*I*|) satisfies the following criteria’s: *lim*∇*I*→0*c*(|∇*I*|) = 1 and *lim*∇*I*→∞*c*(|∇*I*|) = 0 Although the Perona-Malik model performs well, It is mathematically ill-posed. This is also called PeronaMalik paradox. It had few theoretical and practical issues. For these issues many people have been trying to study the Perona-Malik equation. In 1992, Catte et al. .[7] proposed a PDE, which is known as the Regularized PM model. This leads to the following model

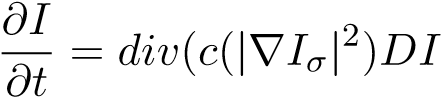
) (3)

where *Iσ* = *Gσ* ∗ *I* is the smoothed version of image. *I* convolved by a Gaussian smoothing kernel *Gσ*. Catte et al. had proved the existence, uniqueness and regularity of a solution. And this regularization belongs to spatial regularization.

3.2. **Fractional Order Anisotropic Diffusion.** As of now there do not exists any unified formula to define fractional calculus. Many mathematicians have analysed the problem at hand from different point of view and have obtained different definitions of fractional calculus. Of all the definitions, three definitions, namely Grünwald–Letnikov (G-L) [9], Riemann-Liouville(R-L) and Caputo[10] are very popular. Since G-L uses only one coefficient and is less complex, it made its way easily in image processing.

 (4)

where the integral part of  and  is binomial coefficient. Due to the limitations of integral order differential, fractional order differential is introduced in improved perona malik model. The improved perona malik model is now re-written as:

) (5)

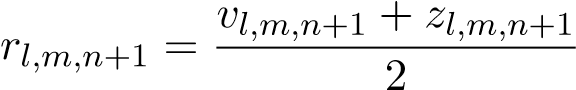
where *DI* = (*DxvI,DyvI*); *DxvI* and *DyvI* denotes fractional differential of order *v* about the space variable *x* and *y* respectively.

3.3. **Alternating direction explicit method.** Alternating direction explicit method (ADE) is a finite difference method which was introduced by Saulev in 1964 in order to solve the nonlinear and time dependent partial differential equations (PDEs). ADE method has applications in numerous fields such as wave propagation, hydrology and heat transfer etc. The work of Saulev was extended by the authors Larkin in 1964, Bakarat and Clark in 1966 and Leung and Osher in 2005. In recent past, many authors have applied the ADE method in various applications. The 2-D scheme for the ADE method is given by

(1 + *w* + *c*2)*vl,m,n*+1 = *vl*−1*,m,n*+1 + *vl*+1*,m,n* − (1 − *w* + *c*2)*vl,m,n* + *c*2[*vl,m*−1*,n*+1 + *vl,m*+1*,n*] (6)

(1 + *w* + *c*2)*zl,m,n*+1 = *zl*−1*,m,n* + *zl*+1*,m,n*+1 − (1 − *w* + *c*2)*zl,m,n* + *c*2[*zl,m*+1*,n*+1 + *zl,m*−1*,n*] (7)

Using equations (6) and (7) at each time level and averaging the results, we get

 (8)

The variables on the right side of equation (6) and (7) at time level *n* are replaced by *r*0*s*. where *vl,m,n* ≈ *u*(*l*∆*x,m*∆*y,n*∆*t*) and *zl,m,n* ≈ *u*(*l*∆*x,m*∆*y,n*∆*t*)

# Proposed Fractional Order ADE Anisotropic Diffusion

The fractional order Regularised P-M model is rewritten as :-

∂*I* /∂t = div(C(|∇*I*σ|2) Dv*I*) (9)

where Dv*I* = (Dvx *I*, Dvy *I*); Dvx *I* and Dvy *I* denotes fractional differential of order v about the space variable x and y respectively. To solve equation (9) we have discretised using a scheme similar to as suggested by Perona and Malik and modified it using the proposed ADE scheme Which is described below.

Numerical scheme suggested by Perona and Malik model[3]:-

I(ij)**t+1**= I**t** + lambda\*(cN \* ∇N + cS \* ∇S + cW \* ∇W+ cE \* ∇E) (10)

Which is modified by introducing the fractional order differential :-

I(ij)**t+1**= I**t** + lambda\*(cN \* ∇**v** N + cS \* ∇**v** S + cW \* ∇**v** W+ cE \* ∇**v**E) (11)

C(x) = 1 / (1 + (∇Iσ/K) **2** ) (12)

Where v denotes the order of fractional order. K can be calculated by the formula described below :-

K = Ko + K1 \* exp( -abs (∇x – med(∇x)) / max(∇x) - min(∇x)) (13)

Where Ko and K1 decides the speed of convergence and can be set manually. It has been observed experimentally that varying K0 in the range [K0 , K0 + K1] helps to achieve better convergence speed. This gradient threshold also makes this model semi-adaptive.

Meanwhile if when we use equation(12) for calculating values of diffusion coeffecients of four direction, when gradient is convoluted with Gaussian kernel, the values of coeffecients are less affected by noise and so values calculated are more correct and diffusion affects the edges less and so more information is preserved. This equation (12) can be further broken down into two parts, In the first step diffusion should only happen in forward direction, which we call forward sweep whereas the in second step diffusion should only occur in backward direction which we call backward sweep. Meanwhile in first step cN, cE remains same as calculated earlier, but when we calculate cW, cS in the next step they are improved values as they are being calculated from lesser noisy image obtained after applying forward sweep in the first step. This is modified ADE scheme.

Step 1:- I(ij)**t+1/2**= I**t** + lambda\*(cN \* ∇**v** N + cE \* ∇**v** E) **FORWARD DIFFERENCE**

Step 2:- Now, take gradient of image obtained from step 1, call it ∇’**v**

I(ij)**t+1**= I**t+1/2** + lambda\*(cW \* ∇’**v** W + cS \* ∇’**v** S) **BACKWARD DIFFERENCE**

Where I(ij)**t+1** is the final image after One iteration

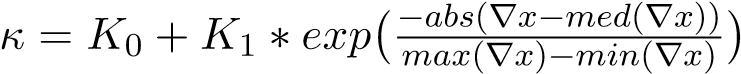
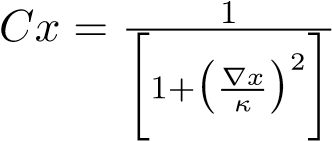
**Algorithm 1** Computational Algorithm for image denoising

**Step 1:** Input *I*0 as the initial image or noisy image, *λ*, *K*0, *K*1, *niter*.

**Step 2: Calculate** ∇*vN* and ∇*vE*

**Step 3:** Apply gaussian filter on *I*0.

**Step 4: Calculate** ∇*N* and ∇*E*. Using them calculate *cN*, *cE* using formula:

where  

**Step 5:** Update *I*, denoised image as same as size of *I*0 , by given formula: *It/*2 = *It/*2 +*λ*(*cN* ∗ ∇*vN* + *cE* ∗ ∇*vE*)

**Step 6: Calculate** ∇*vS* and ∇*vW*

**Step 7:** Apply gaussian filter on *It/*2

**Step 8: Calculate** ∇*S*, ∇*W* for *It/*2, using that calculate *cS*, *cW* using formula of *Cx* as mentioned in **Step 4**.

**Step 9: Calculate** *It* by the given formula: *It/*2 = *It/*2 + *λ*(*cS* ∗ ∇*vS* + *cW* ∗ ∇*vW*)

# Simulation and Results

Table 1. Comparison of the results obtained with algorithm P-M and the **proposed algorithm** with noise variance 0.01

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Image |  | Noisy | **P-M** | **FrADE** |
| Lena | PSNR | 20.2425 | 26.0871 | 26.6080 |
|  | SSIM | 0.4293 | 0.7189 | 0.7588 |
| TissueFibres | PSNR | 20.3593 | 30.9028 | 31.4278 |
|  | SSIM | 0.4079 | 0.8914 | 0.8925 |
| Body | PSNR | 21.1277 | 29.2295 | 30.5050 |
|  | SSIM | 0.2686 | 0.6621 | 0.7667 |
| CTScan | PSNR | 21.5546 | 29.8334 | 33.4245 |
|  | SSIM | 0.2512 | 0.7244 | 0.8262 |

Table 2. Comparison of the results obtained with algorithm P-M and the **proposed algorithm** with noise variance 0.02

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Image |  | Noisy | **P-M** | **FrADE** |
| Lena | PSNR | 17.3948 | 23.7220 | 24.6391 |
|  | SSIM | 0.3229 | 0.6074 | 0.6618 |
| TissueFibres | PSNR | 17.5766 | 28.7278 | 29.5922 |
|  | SSIM | 0.2876 | 0.8476 | 0.8509 |
| Body | PSNR | 18.2152 | 26.5696 | 27.2473 |
|  | SSIM | 0.2025 | 0.6208 | 0.7161 |
| CTScan | PSNR | 18.6684 | 26.9371 | 30.0087 |
|  | SSIM | 0.1671 | 0.6336 | 0.7161 |

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