Games

Multi-agent interaction

Up to this point:

This week:

Multi-agent interaction

Up to this point:

- Stochastic Uncertainty

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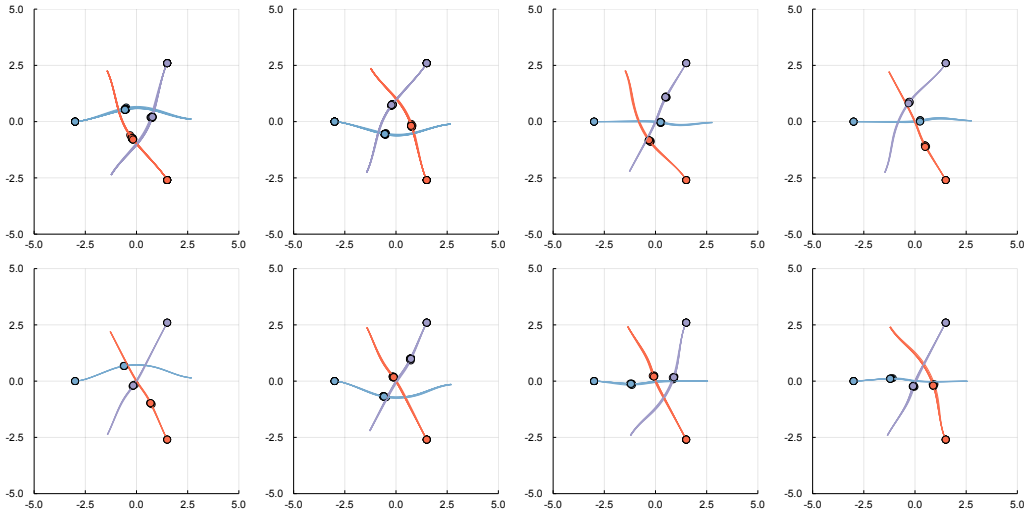
This week:

- Interaction Uncertainty
- Game Solution Concepts



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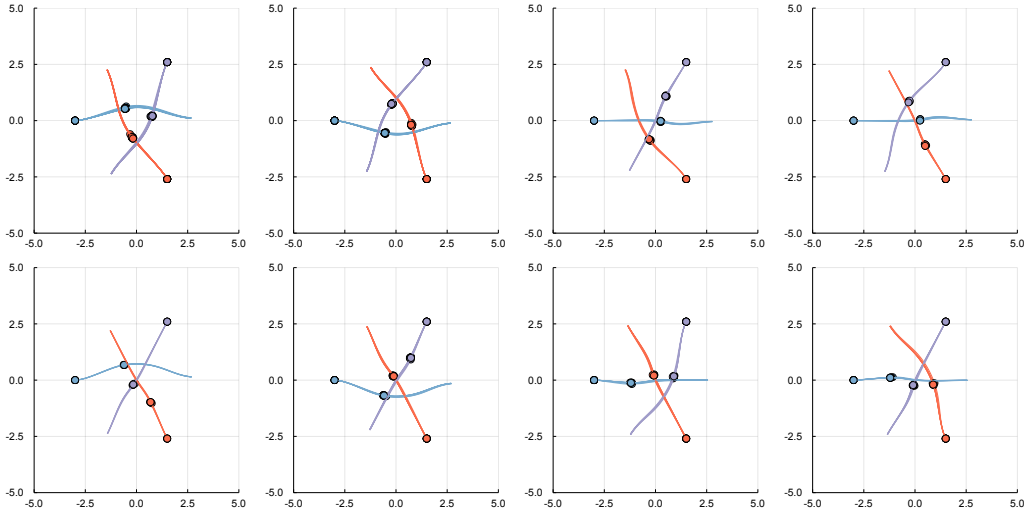

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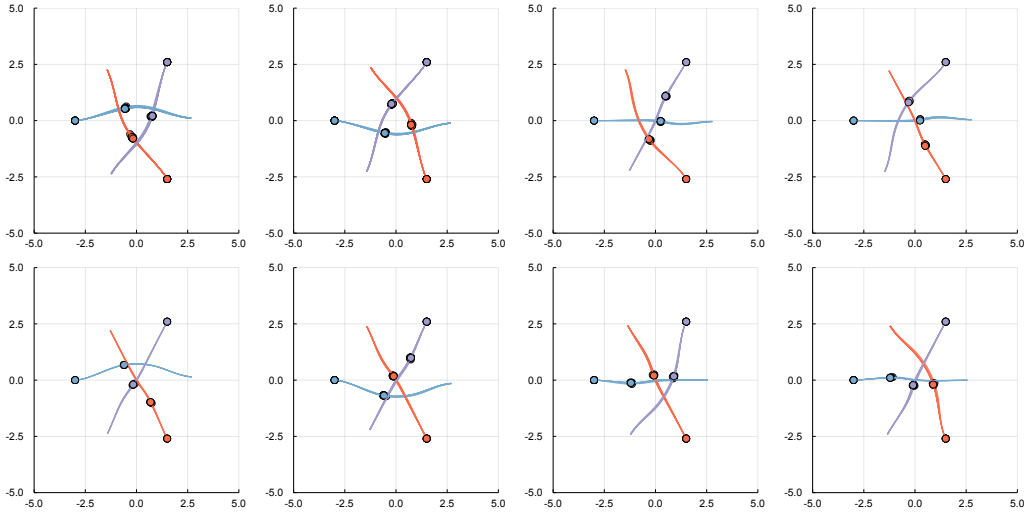

Kriegspiel

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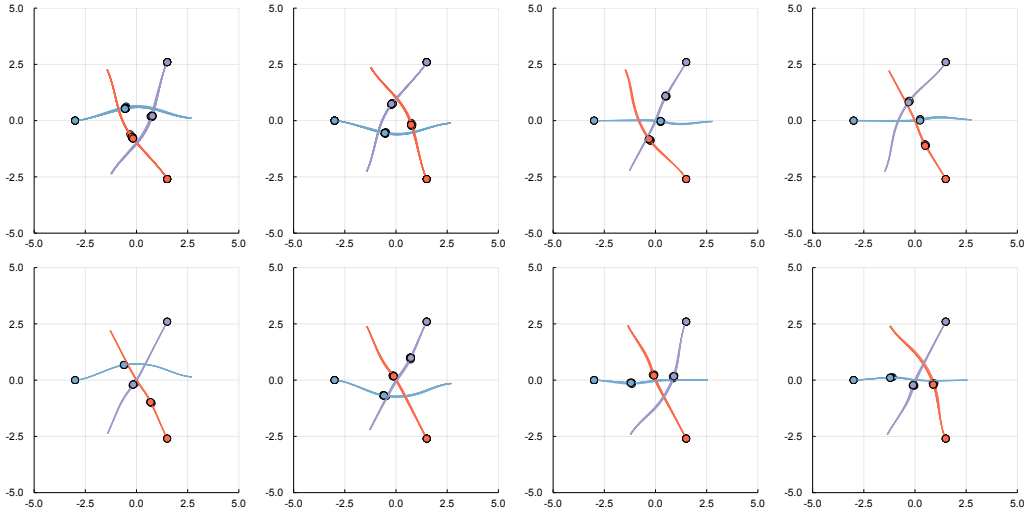

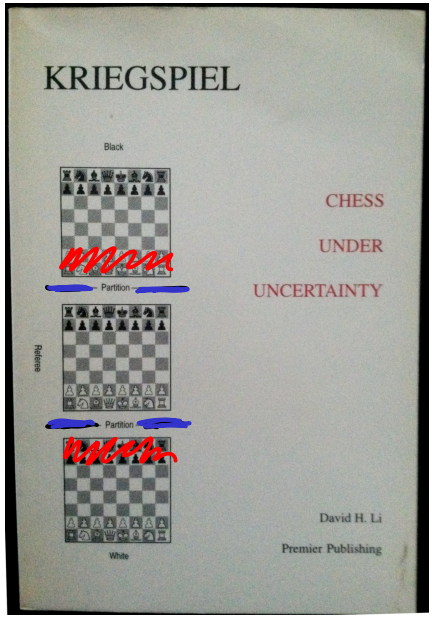

Kriegspiel

Outline

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1. Matrix Games
    1. Linear Programming
    2. Regret Matching

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    2. α-β Pruning
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Sources:

- *AI: A Modern Approach* by Russel and Norvig
- "An Introduction to Counterfactual Regret Minimization in Games" by Neller and Lanctot

Game Solution Concepts: Nash and Correlated Equilibria

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Rock Paper Scissors

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Rock Paper Scissors

	R	P	S
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Battle of the Sexes

		Gary	
		M	G
Monica	M	$2, 1$	$0, 0$
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- Correlated Equilibrium: Strategy profile with a shared random source, where no player has an incentive to deviate

1.5

Calculating Mixed-Strategy Equilibria

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$$\sigma_{\text{Gary}}(M) = x$$

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$$U_{\text{Monica}}(\sigma_{\text{Gary}}, M) = 2x,$$

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By hand

$$\sigma_{\text{Gary}}(M) = x \quad \frac{1}{3}$$

$$U_{\text{Monica}}(\sigma_{\text{Gary}}, M) = 2x, \quad \frac{2}{3}$$

$$U_{\text{Monica}}(\sigma_{\text{Gary}}, G) = 1 - x \quad \frac{2}{3}$$

If $x = 1/3$, these are equal, so Monica is indifferent. By symmetry, $\sigma = \{(\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3}\}$ is a Nash Equilibrium.

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Monica \ Gary	M	G
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$$\begin{aligned}
\underset{\pi, U}{\text{minimize}} \quad & \sum_i \left(U^i - U^i(\boldsymbol{\pi}) \right) \\
\text{subject to} \quad & U^i \geq U^i(a^i, \boldsymbol{\pi}^{-i}) \text{ for all } i, a^i \\
& \sum_{a^i} \pi^i(a^i) = 1 \text{ for all } i \\
& \pi^i(a^i) \geq 0 \text{ for all } i, a^i
\end{aligned}$$

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Regret Matching

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For action profile $\mathbf{a}$, player i's *regret* for not playing action a' is:

$$R(a_i') = U(a_i', a_{-i}) - U(\mathbf{a})$$

that is, the utility missed out on by not playing a'.

Positive regret means that action would have been better.

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Turn-taking Games

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Terminology

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Turn-taking Games

Terminology

- Max and Min players
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- zero-sum
- perfect information

Minimax Trees

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MDP Expectimax Tree

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$$V(s) = \max_{a \in \mathcal{A}} \left(R(s, a) + \mathrm{E}[V(s')] \right)$$

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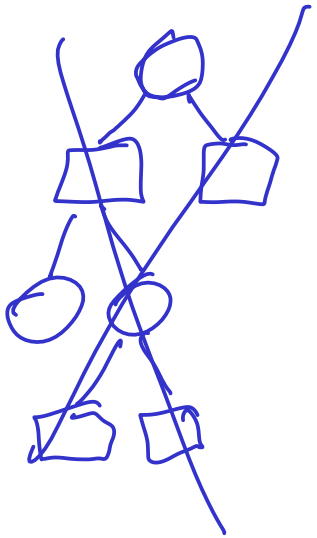
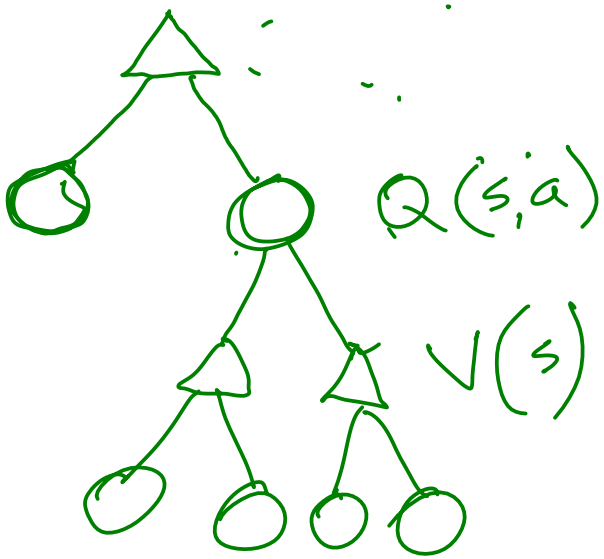
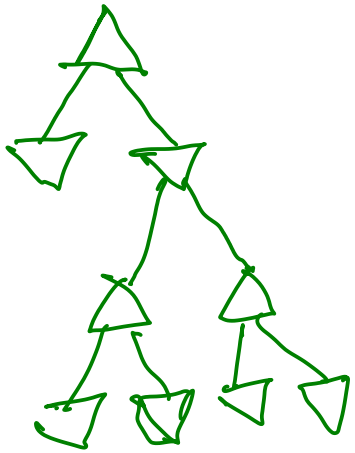




Minimax Tree

$$V_1(s) = \max_{a \in \mathcal{A}_1} \left(R(s,a) + \min_{a' \in A_2} (R(s',a') + V(s'')) \right)$$



Tree Backup Example

Tree Backup Example

```
function MINIMAX-DECISION(state) returns an action
  return arg max_{a ∈ ACTIONS(s)} MIN-VALUE(RESULT(state, a))
```

```
function MAX-VALUE(state) returns a utility value
  if TERMINAL-TEST(state) then return UTILITY(state)
  v ← −∞
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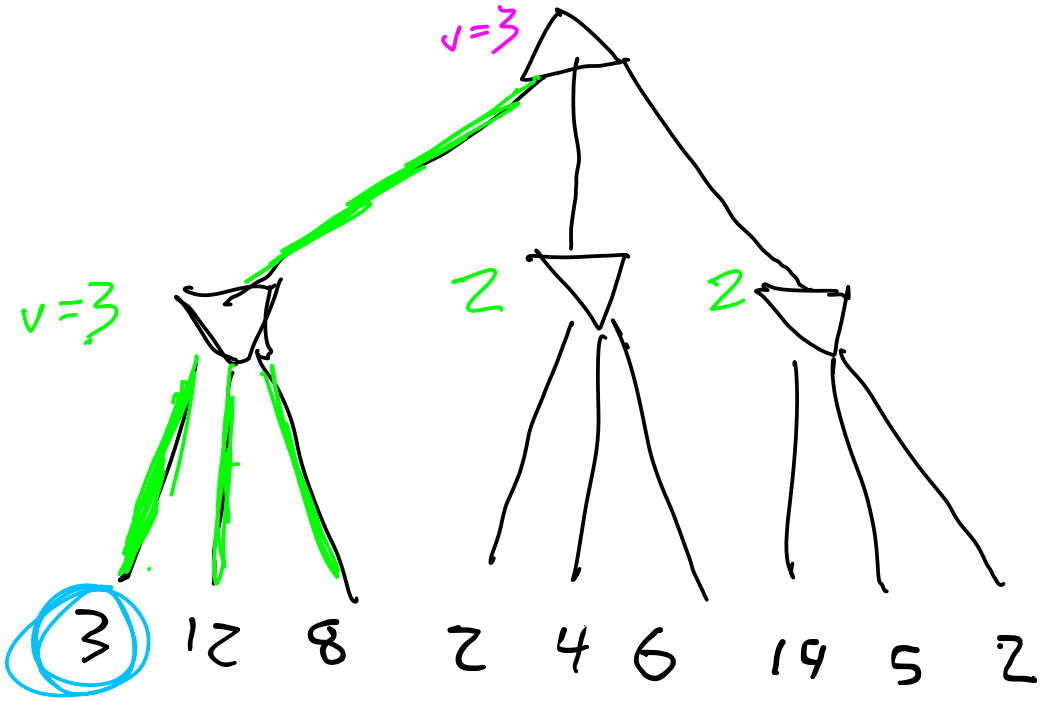

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```

3 12 8 2 4 6 14 5 2

Why is this harder than an MDP? (think back to sparse sampling)

Alpha-Beta Pruning

```
function ALPHA-BETA-SEARCH(state) returns an action
  v ← MAX-VALUE(state, −∞, +∞)
  return the action in ACTIONS(state) with value v
```

```
function MAX-VALUE(state, α, β) returns a utility value
  if TERMINAL-TEST(state) then return UTILITY(state)
  v ← −∞
  for each a in ACTIONS(state) do
    v ← MAX(v, MIN-VALUE(RESULT(s,a), α, β))
    if v ≥ β then return v
    α ← MAX(α, v)
  return v
```

```
function MIN-VALUE(state, α, β) returns a utility value
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  v ← +∞
  for each a in ACTIONS(state) do
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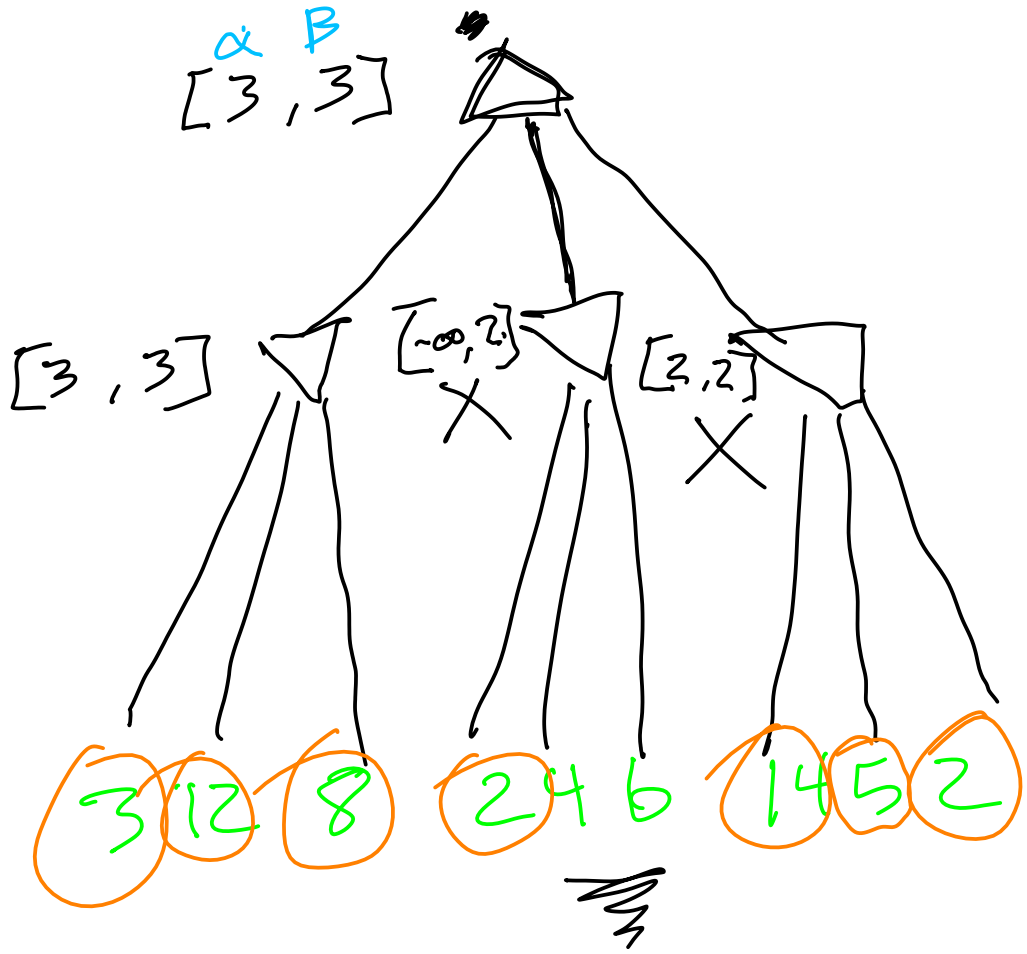
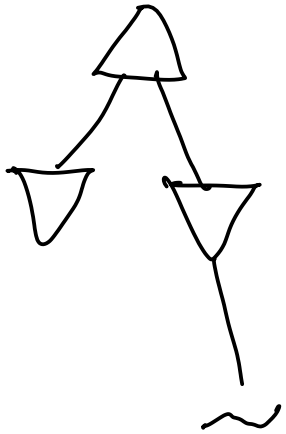
    if v ≤ α then return v
    β ← MIN(β, v)
  return v
```



Evaluation Functions



Evaluation Functions

Examples of evaluation functions:

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- Rollout using default policy

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- Rollout using default policy
- Neural Network
- Weighted combination of features (e.g. number of Pawns, Knights, etc.)

Deep Blue

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- Routinely reached depth 14, sometimes up to 40
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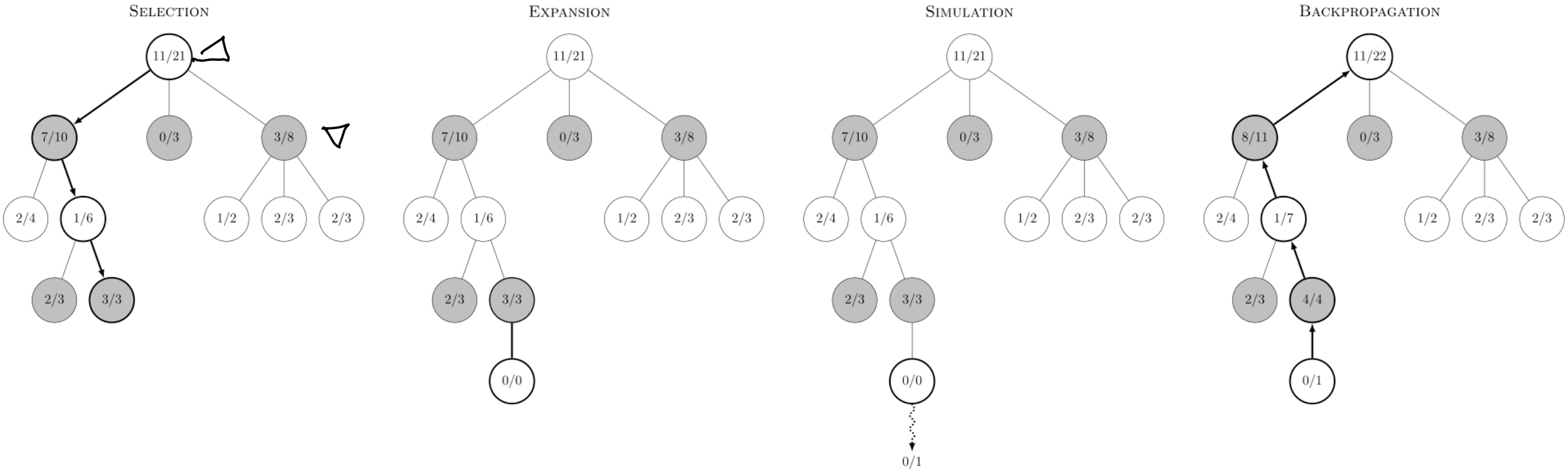
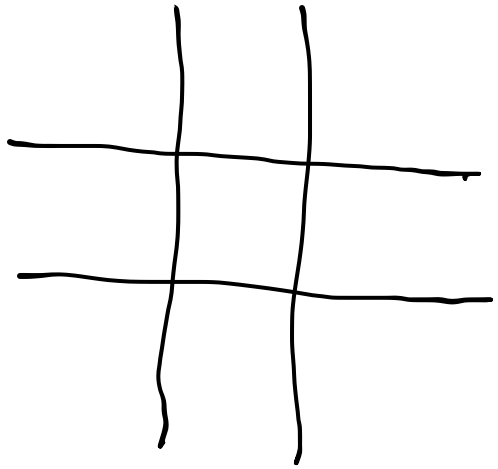
- Opening book of 4000 positions
- Database of 700,000 grandmaster games

MCTS for Games



Break Exercise



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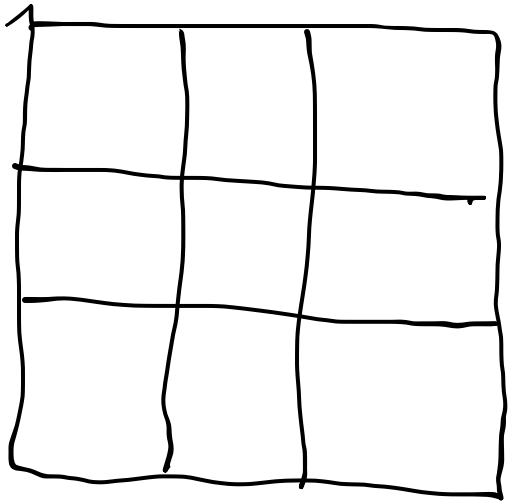

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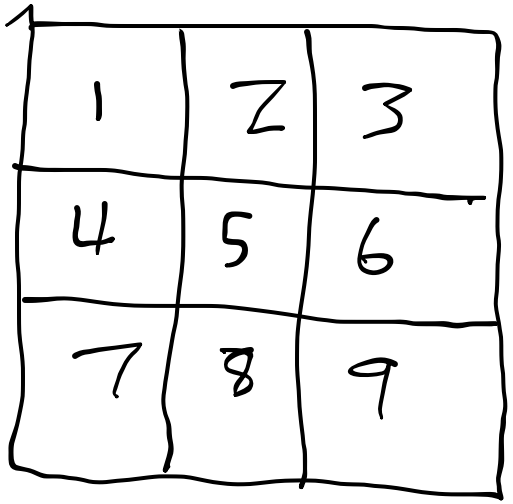

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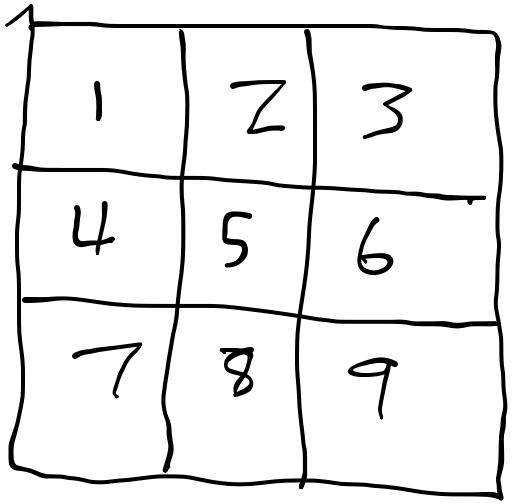
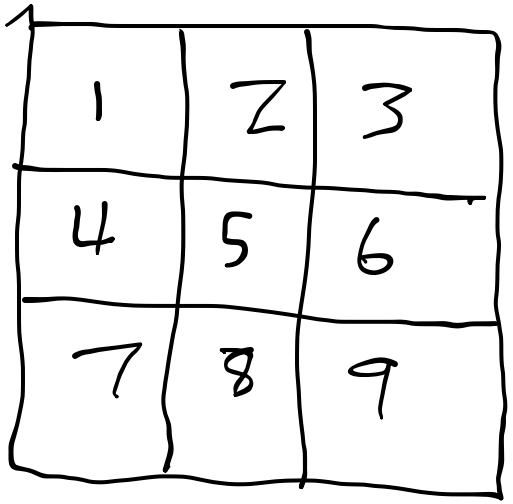


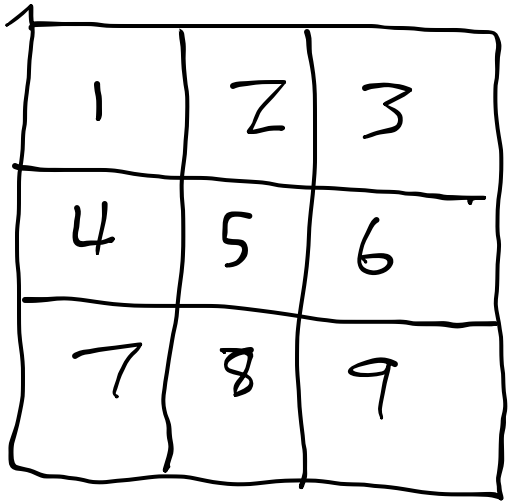
- Start at 5

Break Exercise



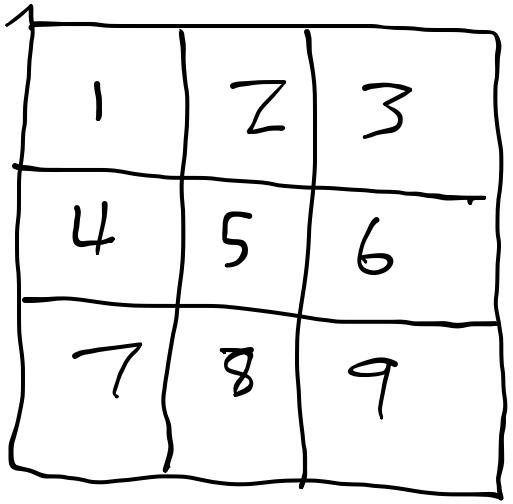
- Start at 5
- Max can move up/down

Break Exercise



- Start at 5
- Max can move up/down
- Min moves left/right

Break Exercise



- Start at 5
- Max can move up/down
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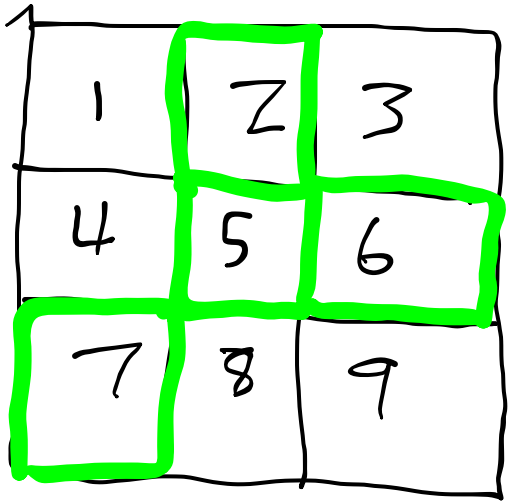
- Each player gets 2 turns

Break Exercise



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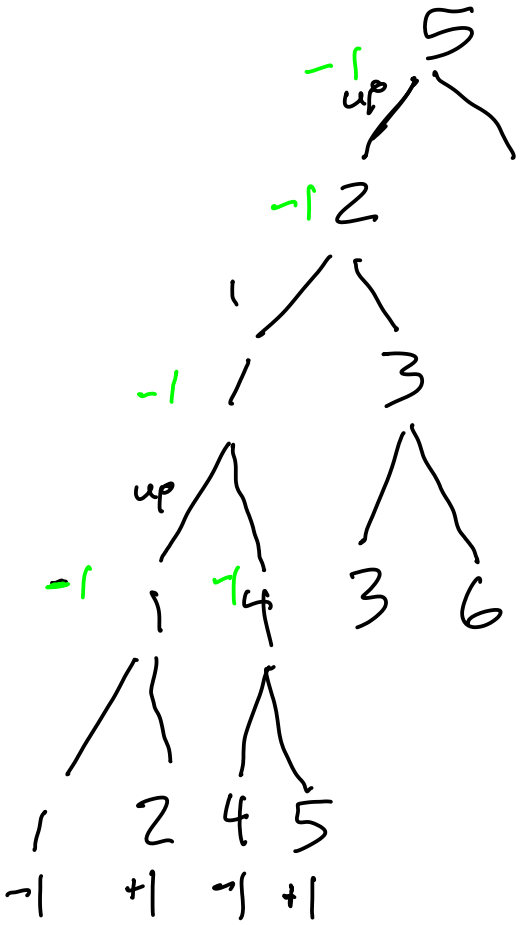
- On green squares, Max wins



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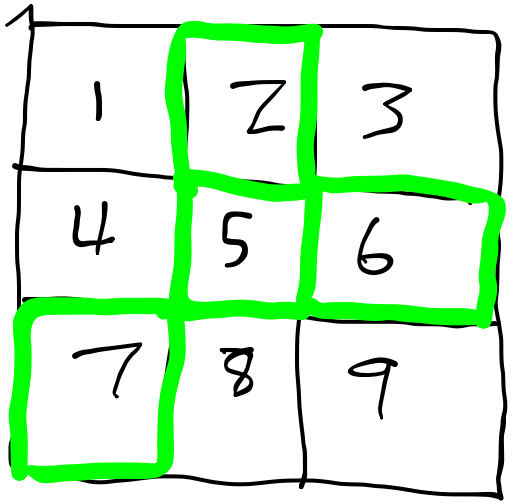

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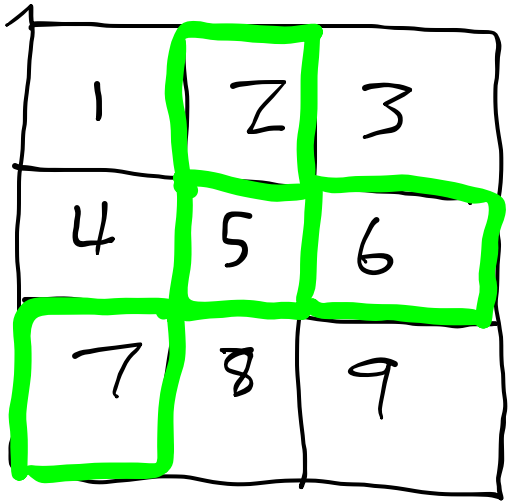

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Bonus: what if 5 was not green?

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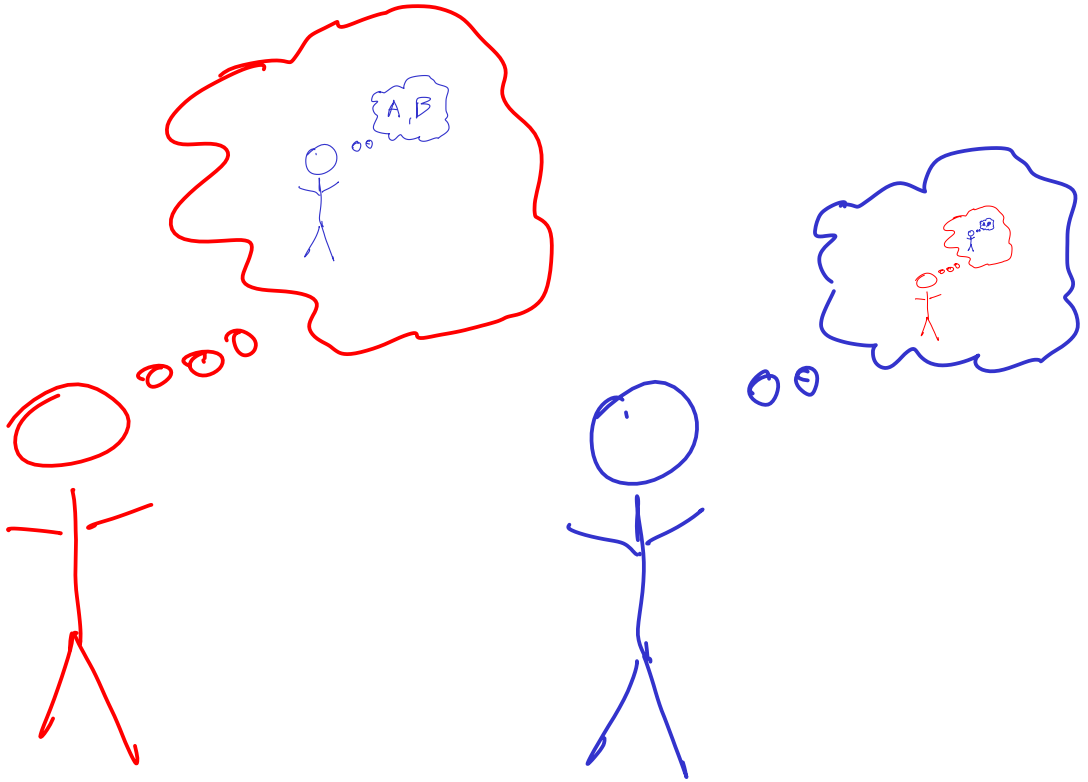

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- $U(h)$ (payoff for each leaf node in the game tree)

King-Ace Poker Example

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- 4 Cards: 2 Aces, 2 Kings

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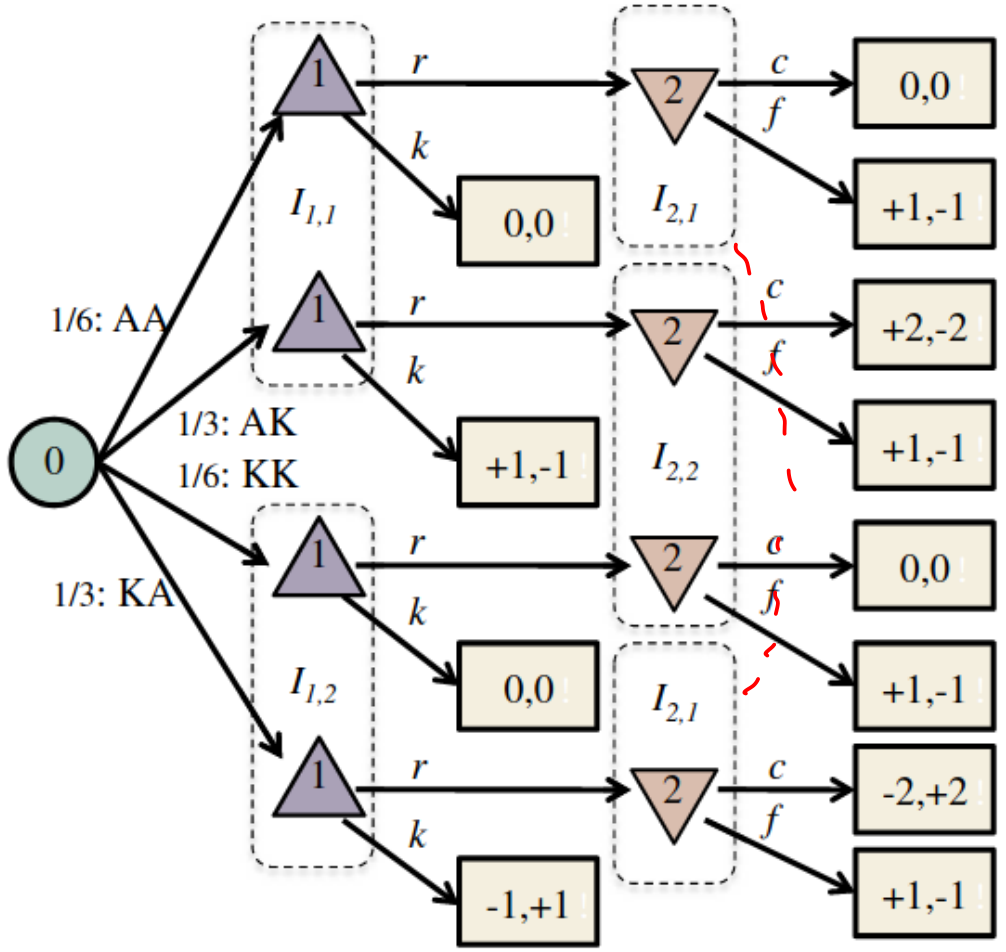
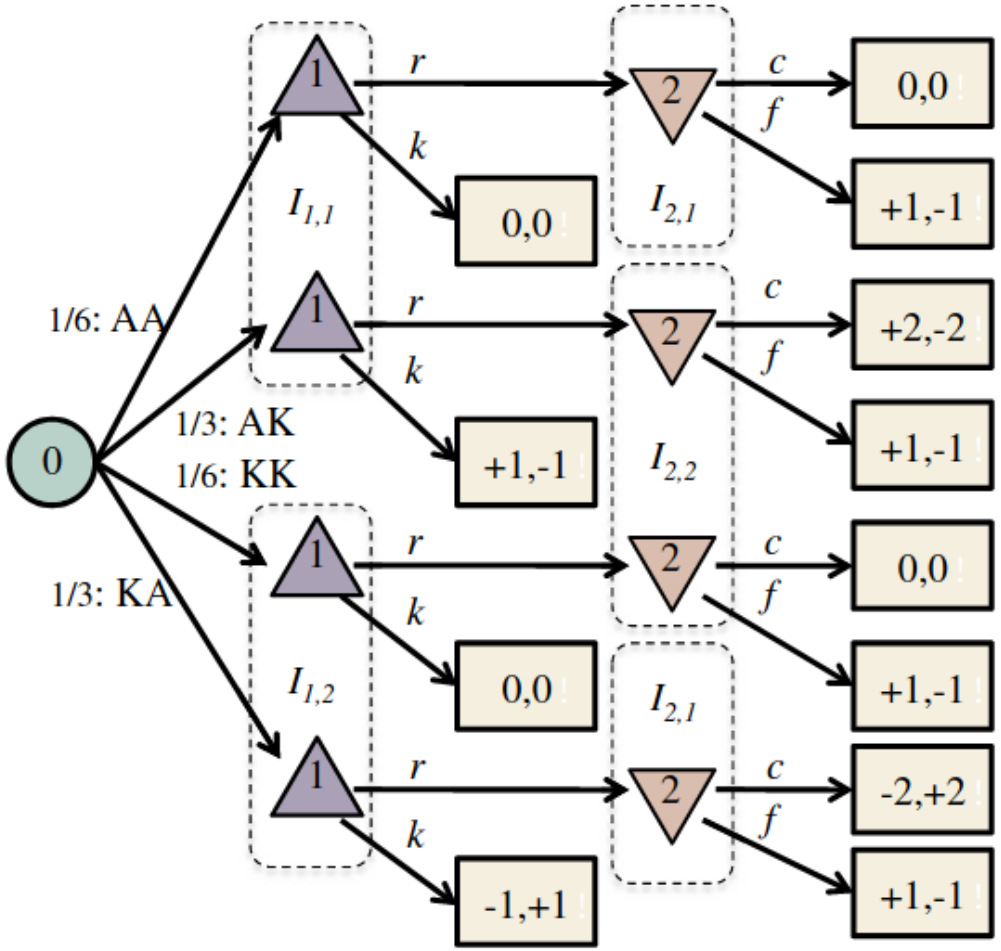


Extensive to Matrix Form



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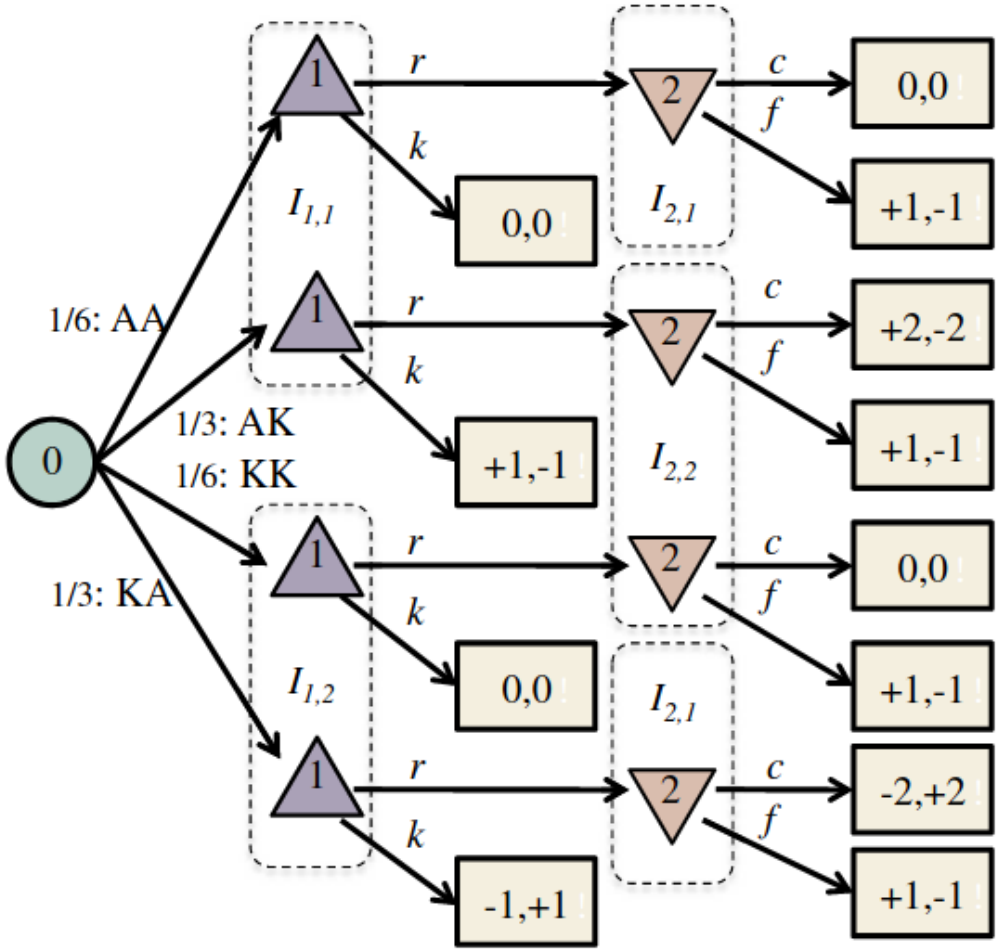

	2:*cc*	2:*cf*	2:*ff*	2:*fc*
1:*rr*	0	-1/6	1	7/6
1:*kr*	-1/3	-1/6	5/6	2/3
1:*rk*	1/3	**0**	1/6	1/2
1:*kk*	0	**0**	0	0

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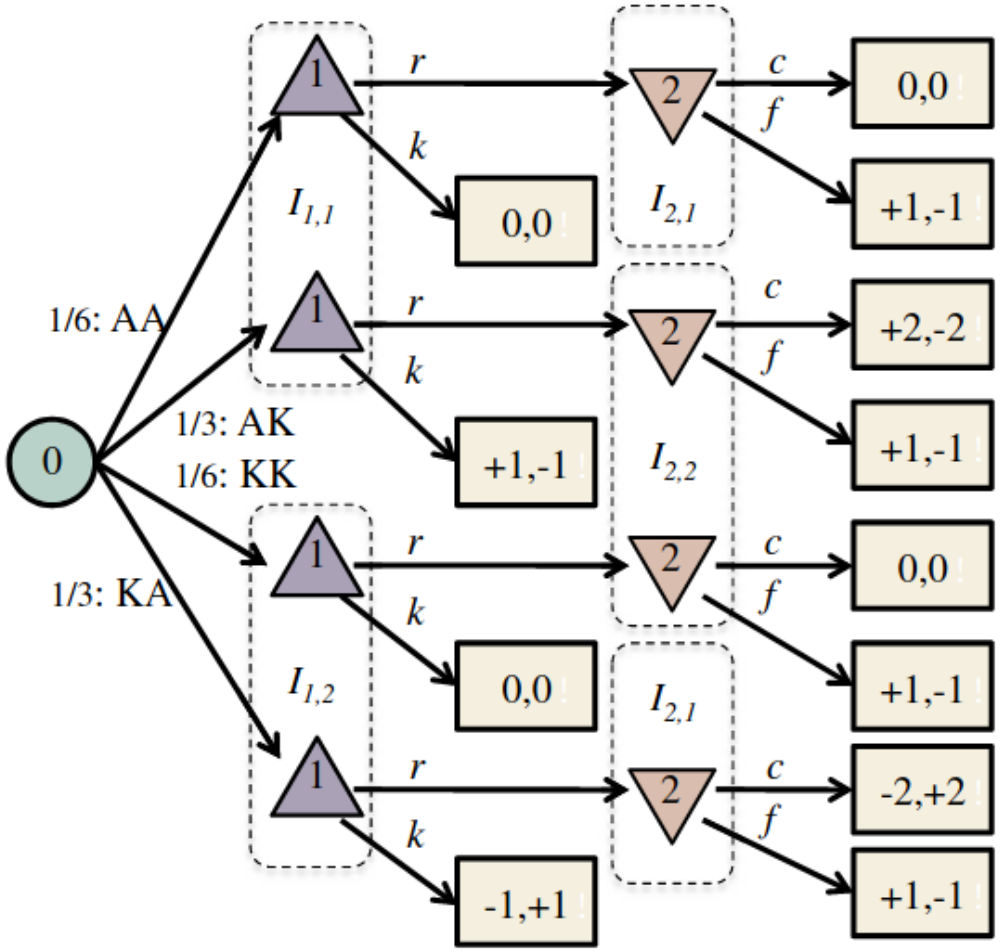

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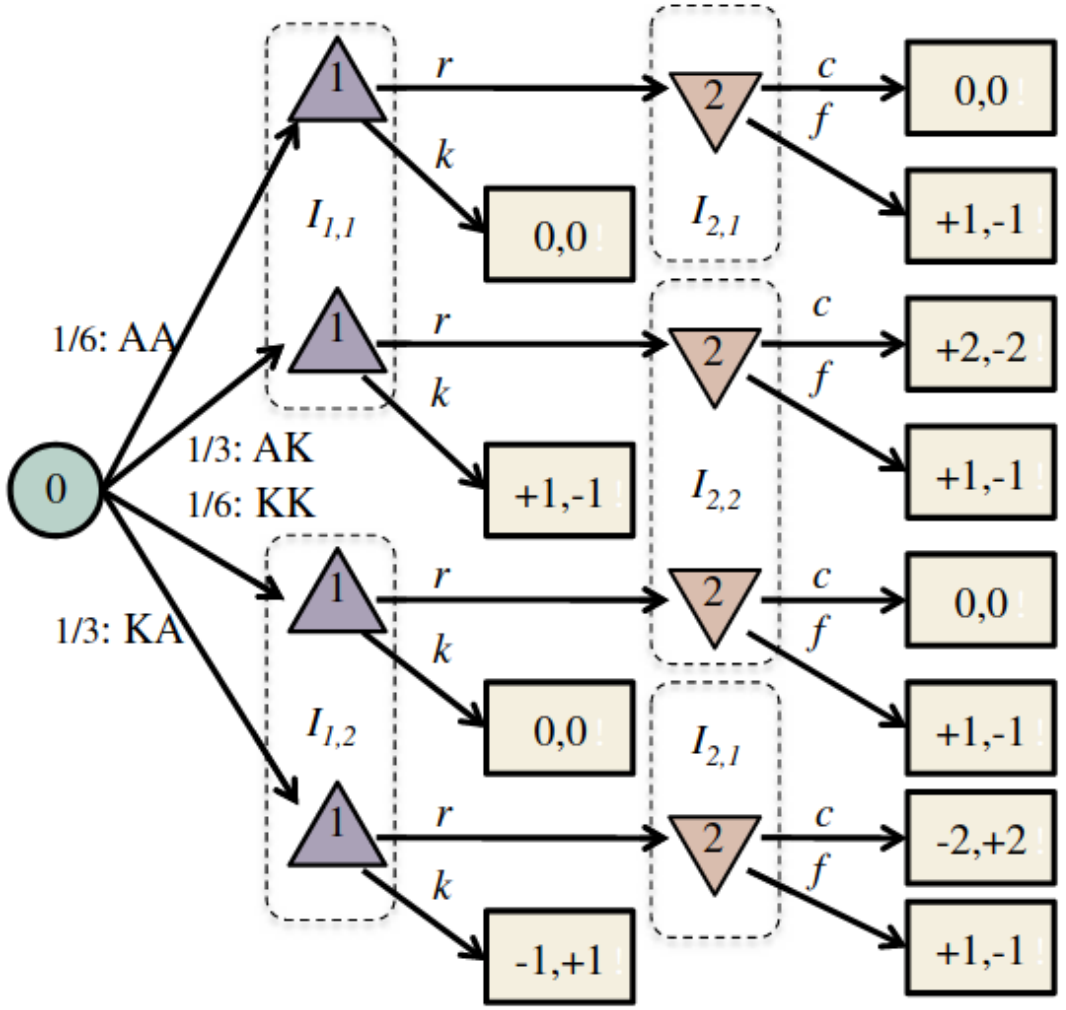
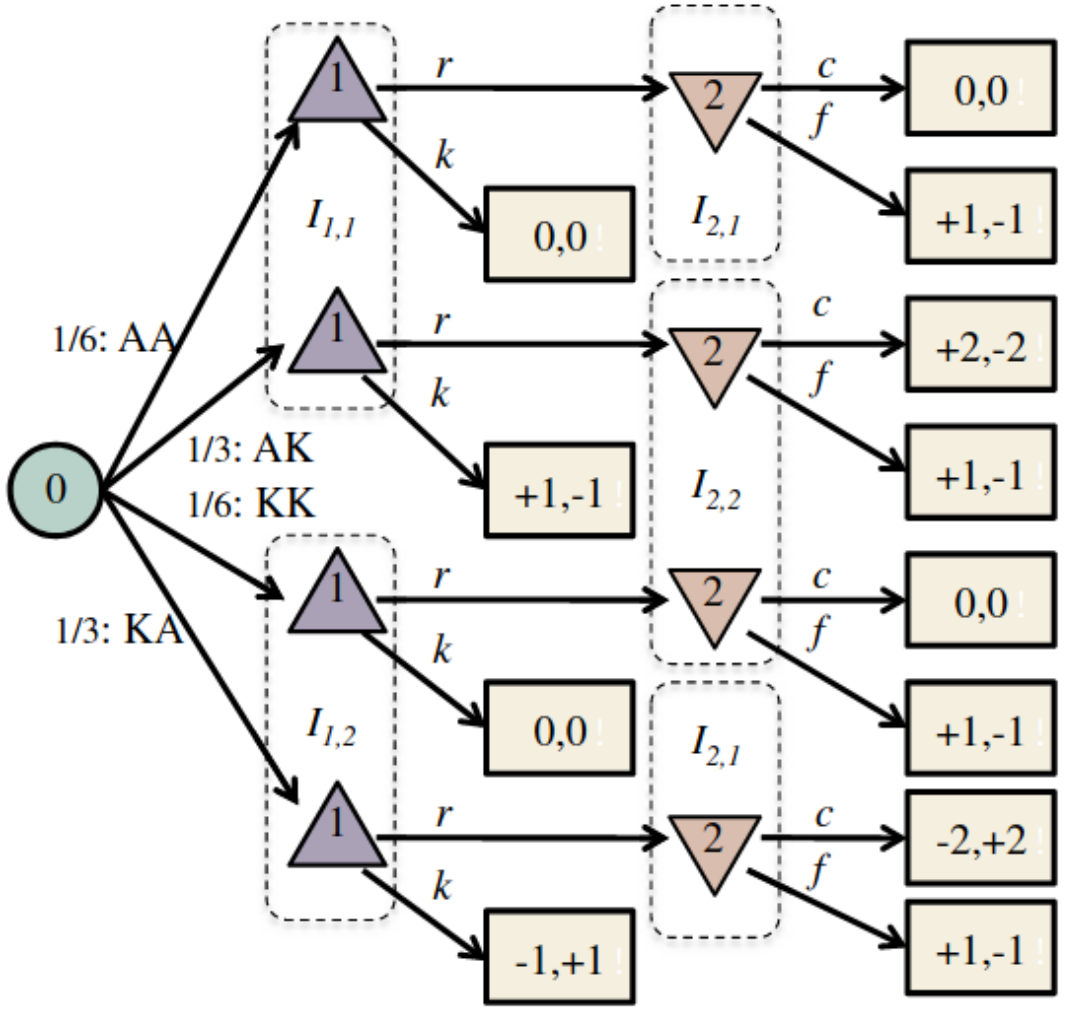
Exponential in number of info states!

Counterfactual Regret Minimization



Counterfactual Regret Minimization

1: Initialize cumulative regret tables: $\forall I, r_I[a] \leftarrow 0$.
2: Initialize cumulative strategy tables: $\forall I, s_I[a] \leftarrow 0$.
3: Initialize initial profile: $\sigma^1(I, a) \leftarrow 1/|A(I)|$



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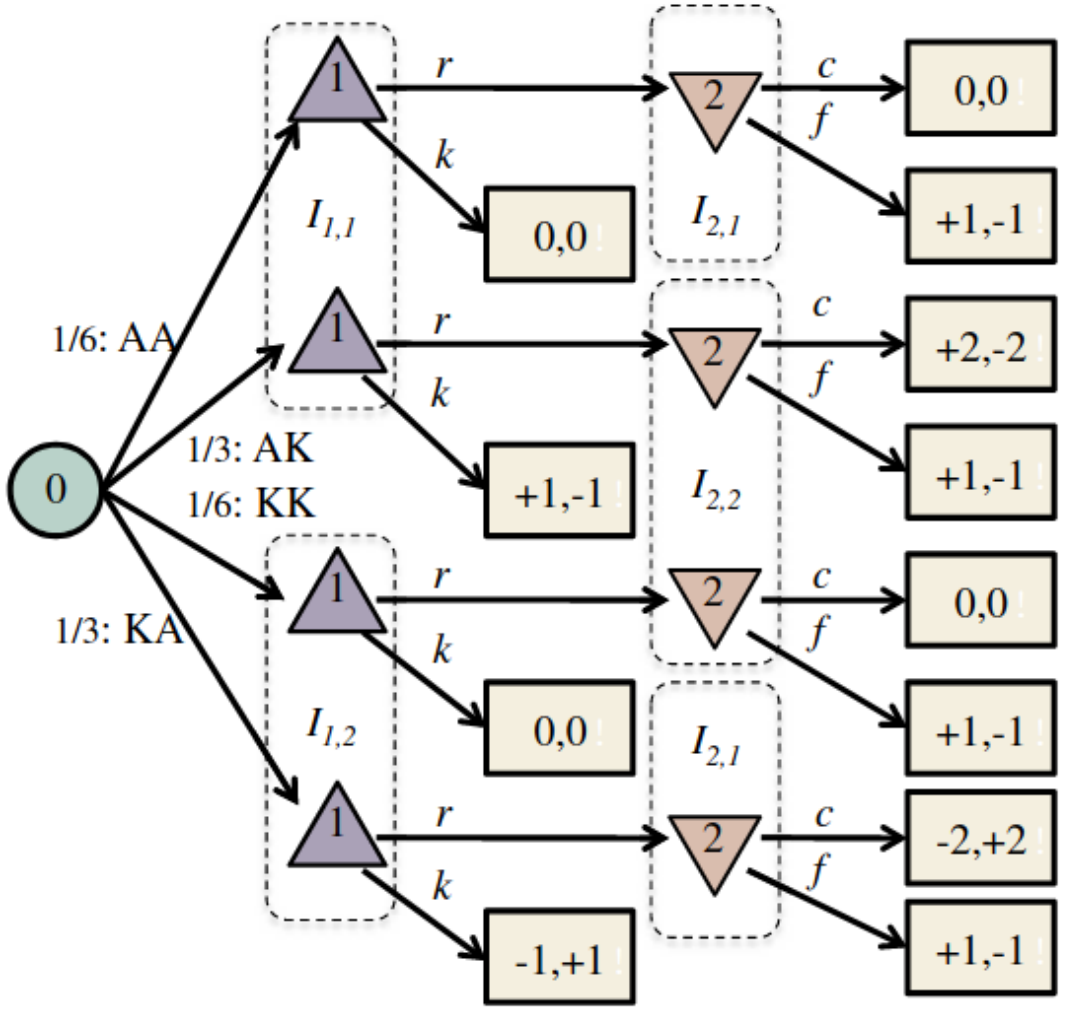
32: **function** Solve():
33: **for** $t = \{1, 2, 3, \ldots, T\}$ **do**
34: **for** $i \in \{1, 2\}$ **do**
35: CFR($\emptyset$, i, t, 1, 1)
36: **end for**
37: **end for**



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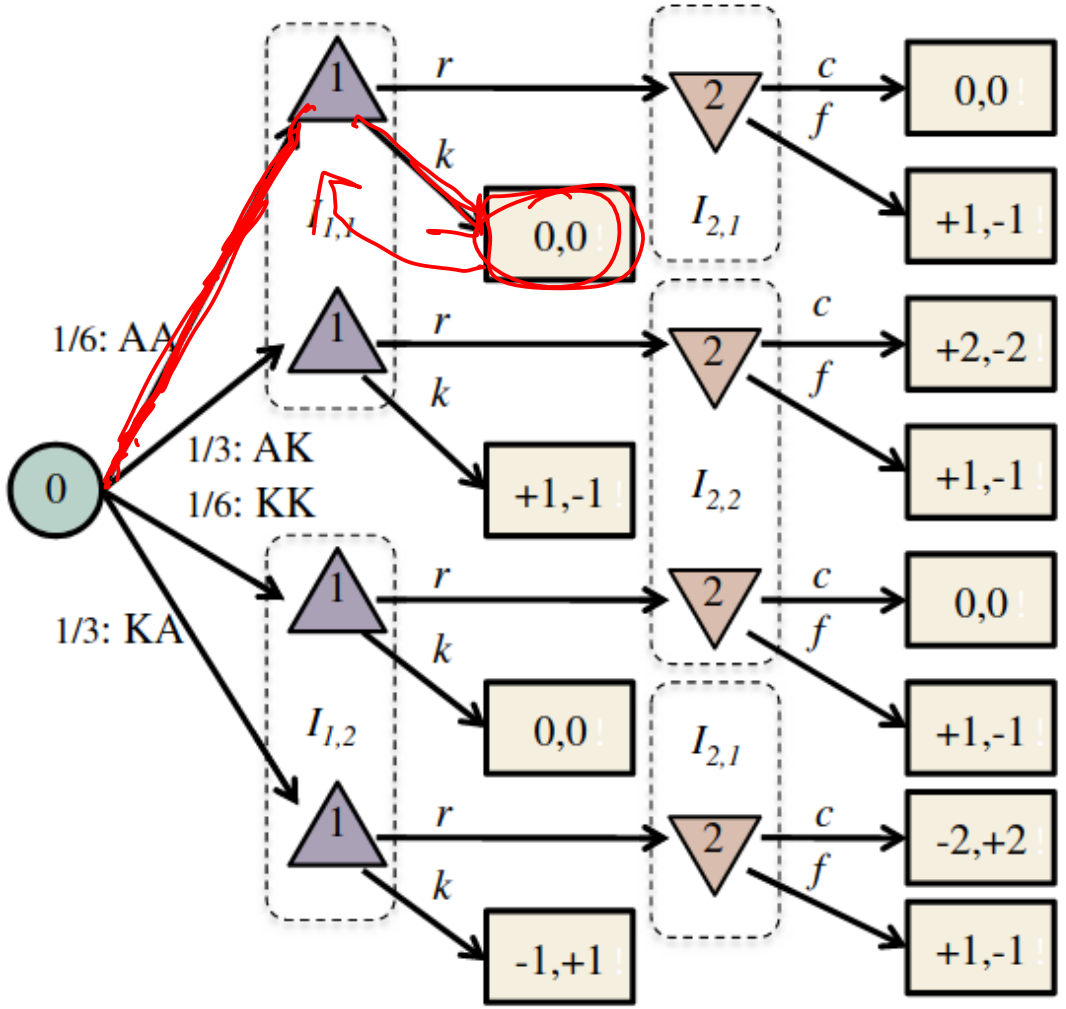
function CFR(h, i, t, π_1, π_2):
if h is terminal then
    return u_i(h)
else if h is a chance node then
    Sample a single outcome a ~ σ_c(h, a)
    return CFR(ha, i, t, π_1, π_2)
end if
Let I be the information set containing h.
v_σ ← 0
v_{σ_{I→a}}[a] ← 0 for all a ∈ A(I)
for a ∈ A(I) do
    if P(h) = 1 then
        v_{σ_{I→a}}[a] ← CFR(ha, i, t, σ^t(I, a) · π_1, π_2)
    else if P(h) = 2 then
        v_{σ_{I→a}}[a] ← CFR(ha, i, t, π_1, σ^t(I, a) · π_2)
    end if
    v_σ ← v_σ + σ^t(I, a) · v_{σ_{I→a}}[a]
end for
if P(h) = i then
    for a ∈ A(I) do
        r_I[a] ← r_I[a] + π_{-i} · (v_{σ_{I→a}}[a] − v_σ)
        s_I[a] ← s_I[a] + π_i · σ^t(I, a)
    end for
    σ^{t+1}(I) ← regret-matching values
end if
return v_σ

function Solve():
for t = {1, 2, 3, ..., T} do
    for i ∈ {1, 2} do
        CFR(∅, i, t, 1, 1)
    end for
end for
```



Kuhn Poker Example

3 Cards: 1, 2, and 3

Sequential Actions			Payoff
Player 1	Player 2	Player 1	
pass	pass		+1 to player with higher card
pass	bet	pass	+1 to player 2
pass	bet	bet	+2 to player with higher card
bet	pass		+1 to player 1
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Solution (found by hand in 1950s):

- Player 1 may pass on a 3 with arbitrary probability γ, but then they must bet on a 1 w.p. $\gamma/3$ and bet on a 2 in the second round w.p. $\gamma/3 + 1/3$
- Player 2 must bet 1/3 of the time when holding a 1 after a pass and bet 1/3 of the time when holding a 2 with a bet

Matrix to Extensive Form

Matrix to Extensive Form

	R	P	S
R	$0, 0$	$-1, 1$	$1, -1$
P	$1, -1$	$0, 0$	$-1, 1$
S	$-1, 1$	$1, -1$	$0, 0$

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- Strategies for imperfect information games often involve stochastic "bluffing"
- Counterfactual regret minimization can be used to solve extensive-form games