Introduction to Cosmology

HW6

to be handed by Friday, Dec. 4, 2015

Problem 1: Scalar Field in Cosmology

In cosmology, scalar fields are often used to drive the inflation of the universe. And it can also be used to mimic the dark energy (quintessence model) or dark matter (axion dark matter), hence be an alternative to the cosmological constant or cold dark matter. The action for a real scalar field can be written as

$$S_\varphi = \int \mathcal{L}_\varphi \sqrt{-g}\, d^4 x = \int \left[-\frac{1}{2} g^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi - V(\varphi) \right] \sqrt{-g}\, d^4 x, \tag{1}$$

where $\mathcal{L}_\varphi$ is the Lagrangian for the scalar field. Vary the action S_φ with respect to the metric $g^{\mu\nu}$, we can obtain the energy-momentum tensor for the scalar field:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_\varphi}{\delta g^{\mu\nu}} = \nabla_\mu \varphi \nabla_\nu \varphi + g_{\mu\nu} \mathcal{L}_\varphi. \tag{2}$$

(a) Treat the scalar field as an effective perfect fluid and compare Eq. (2) with the energy-momentum tensor for perfect fluid:

$$T_{\mu\nu} = U_\mu U_\nu \rho + g_{\mu\nu} p. \tag{3}$$

Derive the expression of the effective energy density ρ_φ and the effective pressure p_φ for the scalar field. What if we consider the case of a homogeneous and isotropic universe? (**Hint**: the four velocity can be taken as $U_\mu = -\frac{\nabla_\mu \varphi}{\sqrt{-g^{\alpha\beta} \nabla_\alpha \varphi \nabla_\beta \varphi}}$.)

(b) From the result of Problem (a), derive the equation of state $\omega = p/\rho$ for scalar field in the case of a homogeneous and isotropic universe. If we want to use the scalar field to mimic the dark energy, what conditions must be satisfied?

Problem 2: Horizon Problem

The temperature of cosmic microwave background radiation (CMBR) observed today is highly isotropic, which suggests that those regions of the universe must be causally connected (or in equilibrium) at earlier time. If we assume that there is no inflation at early stage, how big is the causally connected patch of the CMB? Show that a causally connected patch is only several square degrees in the fully sky, which is inconsistent with the isotropy of CMBR. That problem is called the "horizon problem". What is the main idea of inflation to solve this problem?

Problem 3: Flatness Problem (taken from "An Introduction to the Science of Cosmology", Derek Raine, E. G. Thomas, p177)

If Ω is of order unity today, show that, at the Planck time $t_p \sim 10^{-43}$s

$$|1 - \Omega| \lesssim 10^{-60}, \tag{4}$$

and that the radius of curvature, $R = c\, H^{-1} |\Omega - 1|^{-1/2}$ exceeded the Hubble radius by 30 orders of magnitude.

Problem 4: Problems in "Physical Foundations of Cosmology" by Mukhanov

In chapter 5, do problems 5.2 and 5.4.