Introduction to Cosmology

HW2

to be handed in until Friday noon, Nov. 6, 2014

Problem 1: Friedmann Equations

To know the expansion history of the universe, we need to solve the Friedmann equations:

$$\begin{aligned} H^2 &= \frac{8\pi G}{3}\rho - \frac{k}{a^2}, && (1) \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3p). && (2) \end{aligned}$$

Here $H \equiv \frac{\dot{a}}{a}$. Depending on the contents of the universe (dust, radiation, vacuum energy,...), the expansion of the universe is very different. Thus in order to solve the Friedmann equations, we also need the information about the matter contents, i.e. the equation of state (EoS): $p = \omega\rho$. Then we can solve these three equations to get the time dependence of the scale factor a, which describes the expansion history of the universe. However, in practice we often use the continuity equation

$$\dot{\rho} + 3H(\rho + p) = 0 \qquad (3)$$

instead of Eq. (2) because it is easier to solve. The two Friedmann equations and the continuity equation are not totally independent. It is easy to understand: from the Bianchi identities we know that the covariant derivative of the left side of the Einstein equation ($G_{\mu\nu}$) equals to 0, which implies that the covariant derivative of the right side ($T_{\mu\nu}$) should also be 0. And $\nabla_\mu T^{0\mu} = 0$ gives exactly the continuity equation. Thus the continuity equation is automatically ensured if the Einstein equation is satisfied.

(a) Show that from Eq. (1) and Eq. (3), we can obtain the second Friedmann equation Eq. (2). You can also try to derive any one of the three equations from the other two.

(b) For constant ω, solve the continuity equation to get the general expression of the energy density ρ in terms of a.

Problem 2: The age of the universe

From the observations of supernovae and cosmic microwave background (CMB), we know that about 30% of the contents of current universe is matter (baryonic matter+cold dark matter). The other 70% is the vacuum energy (or the cosmological constant). Unlike the matter and radiation, the vacuum energy has a negative pressure ($\omega = -1$), which leads to the late acceleration of the universe. It also makes the age of the universe different from what people firstly estimated by assuming a matter-only universe.

For convenience, we often write the current densities of different contents in terms of the critical density $\rho_{cri} = 3H_0^2/(8\pi G)$ and define the fractions as

$$\Omega_m = \frac{\rho_{m0}}{\rho_{cri}}, \quad \Omega_r = \frac{\rho_{r0}}{\rho_{cri}}, \quad \Omega_\Lambda = \frac{\Lambda}{8\pi G\rho_{cri}}, \quad \Omega_k = -\frac{3k}{8\pi G a_0^2 \rho_{cri}}. \tag{4}$$

Here the subscript "0" denotes the value at present time. We have treated the term $-k/a^2$ in the Friedmann equation Eq. (1) as an effective energy density, i.e. $-k/a^2 \equiv 8\pi G\rho_k/3$. The total energy density $\rho = \Sigma_i \rho_i$, with $i = m, r, \Lambda, k$. It is easy to see that $\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_k = 1$.

As we know, the light emitted from a distant object will be redshifted when we receive it. The redshift depends on the scale factor when the light is emitted a and the scale factor when it arrives at us a_0:

$$1 + z = \frac{a_0}{a}. \tag{5}$$

(a) Using the solution of Problem 1.(b), derive the energy density ρ_i in terms of a and Ω_i for different content i. Note that for dust $\omega = 0$, for radiation $\omega = 1/3$, for vacuum energy $\omega = -1$, and for spacial curvature $\omega = -1/3$. You can set $8\pi G = 1$ for simplicity. But be careful with the coefficient of each term.

(b) Using the solution of (a), solve the Friedmann equation Eq. (1) to derive the equation for the age of the universe t_0 in terms of H_0, Ω_m, Ω_r, Ω_Λ, Ω_k and z.

(c) Consider a matter-only universe, calculate the age of the universe. Use $H_0 = 67.8\text{km s}^{-1}\text{Mpc}^{-1}$ (Planck 2013). Compare it with the age of the oldest star found in the universe (about 14.46 ± 0.8Gyr). Is it possible that the universe contains only the dust matter? If the universe also contains some fraction of vacuum energy, how will the age of the universe change (decrease or increase)?

Problem 3: Cosmological redshift

When light travels in the universe, its wavelength will be stretched. So when the light from a distant object arrives at us, its color will be shifted towards the red end of the spectrum. This is called the cosmological redshift.

(a) What is the difference between the Doppler redshift and the cosmological redshift?

(b) The earliest photons we can observe in the universe is from the last scattering surface when the photons start to travel freely. They are redshifted during the traveling to the earth and observed by detectors as the cosmic microwave background (CMB). The redshift of CMB is about 1100. Normalizing the scale factor at present time to 1, what is the scale factor when these photons firstly start to travel freely? If the redshift is due to the Doppler effect, how fast must the source move away from us? Note that you should consider the relativistic Doppler effect.

Problem 4: Conformal time

Sometimes, it is convenient to use the conformal time in stead of the physical time. The conformal time η is defined as

$$\eta = \int \frac{dt}{a(t)}. \tag{6}$$

It is particularly useful when we consider the evolution of the cosmological perturbations.

(a) Assuming the spacial curvature $k = 0$, for different universe consisting of dust, radiation, vacuum energy, derive the expression of the conformal time η in terms of the physical time t, respectively.

(b) Express the scale factor a in terms of the conformal time η for different cases in (a).