Home exercise sheet 6

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Exercise 1: Solar convection zone

There exist two different mechanisms for energy transport inside stars: radiative transfer, which does not involve material motion and convection, which involves material motion. The energy transport inside the Sun is convective in the zone between $\mathrm{R}_{convec} = 0.7R_\odot$ and the solar surface at $R_\odot$. Find out how density, pressure and temperature vary as a function of radius r within this convection zone by making the following assumptions:

- The solar convection zone consists of an ideal gas.
- Consider for the convective region the following relation between its temperature T and pressure P

$$\frac{dT}{dr} = \left(1 - \frac{1}{\gamma}\right) \frac{T}{P} \frac{dP}{dr} \tag{1}$$

  where γ is the adiabatic exponent of the gas which makes up this stellar region and can be assumed constant.

- The convection zone only contains a small fraction of the Sun's total mass so that the gravitational force on a mass element dm located at a radius r inside the convection zone can be approximated by

$$F = -\frac{GM_\odot dm}{r^2} \tag{2}$$

  This assumption is quite accurate, since the convection zone contains only about 2% of the total solar mass.

Hint: The temperature, pressure and density in the solar convection zone are determined by a set of two differential equations. To solve these, we need two boundary conditions. As a first boundary condition, you can set $T(R_\odot) = 0$. Since the temperature and pressure are not independent, we need a second independent boundary condition, which can be fixed by the value of either the temperature or pressure at the inner boundary of the convection zone. Since we do not know this value, you can introduce an arbitrary normalisation constant c, which could in principle be adapted to this unknown temperature or pressure.

Exercise 2: Upper limit to radiative energy outflow

The Schwarzschild criterion defines the condition for stability against convection for a homogeneous medium (i.e. for a constant chemical composition). Consider this criterion and the Eulerian description of the temperature gradient for the diffusion of radiative energy from the lecture notes to show that in the case of an ideal, non-degenerate gas the luminosity (i.e. energy outflow) is limited in zones of radiative equilibrium to:

$$L(r) \leq 1.22 \times 10^{-18} \frac{\mu T^3}{\kappa\rho} M(r) \tag{3}$$

in cgs units.

Hints: consider hydrostatic equilibrium and remember that the adiabatic temperature gradient can be expressed as

$$\left(\frac{dT}{dr}\right)_{ad} = \left(1 - \frac{1}{\gamma}\right) \frac{T}{P} \frac{dP}{dr}$$

where γ is the adiabatic exponent of the gas.

Exercise 3: Neutrinos

Type II supernovae are the result of core collapse of massive stars and these explosions emit a large number of neutrinos. Depending on the initial mass of the star, such an explosion leads to the formation of a neutron star or a black hole. In 1987, a star underwent core collapse inside the Large Magellanic Cloud at a distance $D = 55$ kpc from Earth. Neutrinos from the so-called Supernova 1987A were detected on Earth and found to have energies in the range E = 6–39 MeV. If the spread of $\Delta t = 12s$ in their arrival times was caused by neutrinos of different energies travelling at different speeds, show that the neutrino mass cannot be larger than about $m \approx 20$ eV/c^2.

Exercise 4: White dwarfs

In this exercise we introduce the Fermi gas model for atomic nuclei, as done in the lectures and then apply it to the white dwarf stage of stellar evolution.

a) Consider here only the neutrons in the atom nuclei and assume that they are homogeneously distributed between the atoms. Determine how the nuclear volume depends on the mass number A under this assumption. Use $r_0 = 1.3$ fm. With the outcome calculate the Fermi energy for both a typical light nucleus (N = 0.5A) and a typical heavy nucleus (N = 0.6A) by using the formula:

$$E_F = \frac{\hbar^2}{m} \left(\frac{3\sqrt{2} N_F \pi^2}{4V}\right)^{\frac{2}{3}}$$

where N_F is the number of particles and V is the volume. Express the result in MeV.

Application to white dwarfs:

By using the Fermi gas model for atomic nuclei you can obtain a simple estimate of the radius of a white dwarf. For this purpose, assume that all electrons in the white dwarf can be described by a Fermi gas model, as these are fermions.

b) Determine first the equilibrium radius corresponding to the minimum total energy. The total energy consists of the kinetic energy – that is contributed almost exclusively by the electrons – and the gravitational potential energy. What is unusual in the resulting relation between radius and mass?

   Hint: It can be shown that the gravitational potential energy of a sphere is $E_G = -\frac{3}{5}G\frac{M^2}{R}$. Additionally, the average kinetic energy per electron is $< T >= \frac{3}{5}E_F$.

c) The mass of the Sun is 1.99×10^{30} kg. Also, for a typical white dwarf the atomic number is $Z \simeq \frac{A}{2}$. How large would the radius of the Sun be, if it turned into a white dwarf in about 6 billion years? Also, calculate its average density.