Homework exercise sheet 4

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Exercise 1: Star's inside

In a star of mass M, the density decreases from the center to the surface as a function of radial distance r, according to

$$\rho = \rho_c \; [1 - \left(\frac{r}{R}\right)^2],$$

where ρ_c is a given constant and R is the star's radius.

a) Find the mass $m(r)$ inside a sphere of radius r.

b) Derive the relation between M and R.

c) Show that the average density of the star (total mass divided by total volume) is $0.4 \; \rho_c$.

d) With the assumption of a hydrostatic equilibrium between the gravitational and pressure forces, we can write this balance as a differential equation

$$\frac{dP}{dr} = -\rho \frac{Gm}{r^2},$$

where P denotes the pressure and m the mass inside the sphere at radius r. Find the central pressure P_c in the case of a uniform density and a density profile like shown in the first equation. Show that the following inequality holds:

$$P_c > \frac{GM^2}{8\pi R^4} = 4.4 \times 10^{13} \left(\frac{M}{M_\odot}\right)^2 \left(\frac{R_\odot}{R}\right)^4 \mathrm{N\,m^{-2}}.$$

e) Assuming a density distribution as given by the first equation, derive a maximum estimate for the core temperature of the star that only depends on its mass. Consider that the star consists of an completely ionized ideal and *non*-degenerate gas. The mean atomic mass number of stellar material μ_I is defined as

$$\frac{1}{\mu_\mathrm{I}} = \sum_i \frac{X_i}{\mathcal{A}_i},$$

where X_i is the mass fraction of the element and $\mathcal{A}_i$ its atomic mass number. For stellar material this can be simplified by introducing the mass fractions X for Hydrogen and Y for Helium mass, which leaves $(1 - X - Y)$ as the fraction of higher order elements (*metals*). Therefore μ_I can be approximated as

$$\frac{1}{\mu_\mathrm{I}} \approx X + \frac{Y}{4} + \frac{1 - X - Y}{\langle \mathcal{A} \rangle},$$

which for the Sun with $X = 0.707$, $Y = 0.274$ and $\langle \mathcal{A} \rangle \approx 20$ yields $\mu_{\mathrm{I}} = 1.29$.
The pressure of a degenerate electron gas is given by

$$P_{\mathrm{e,deg}} = K_1' \left(\frac{\rho}{\mu_{\mathrm{e}}} \right)^{5/3} ,$$

where $K_1' = \frac{h^2}{20 m_{\mathrm{e}}} \left(\frac{3}{\pi}\right)^{2/3} \frac{1}{m_{\mathrm{H}}^{5/3}} = 1.00 \times 10^7 \mathrm{m}^4 \mathrm{kg}^{-2/3} \mathrm{s}^{-2} = 1.00 \times 10^{13} \mathrm{cm}^4 \mathrm{g}^{-2/3} \mathrm{s}^{-2}$ is a constant for electrons and $\frac{1}{\mu_{\mathrm{e}}} \approx \frac{1}{2}(1 + X)$ is the average number of free electrons per Hydrogen mass element M_{H} which again only depends on the mass fraction X of hydrogen in the star (for the Sun $X = 0.707$ and $\mu_{\mathrm{e}} = 1.17$).

Exercise 2: Derivation of the Free-Fall Time Scale

a) The free-fall time can be derived with the concept of pressure-less ($P = 0$) collapse. As initial configuration we take a static gas cloud (velocity $u(t = 0) = 0$) with initial radius $R(t = 0)$ and mass $M(t = 0) = \frac{4\pi}{3} \bar{\rho} R^3$ at a mean density $\bar{\rho}$.

Hints:

- As first step use the continuity equation in spherical symmetry, rewrite into the mass variable using the integrated mass

$$m(r) = \int_0^r 4\pi r'^2 \rho dr'$$

  and derive an expression for the mass in the form of a total derivative (often also called *Lagrange form*).
- Use the equation of motion in spherical symmetry in total derivative formulation (Lagrange form) and solve it for dt respectively for t in this collapse scenario. It is useful to make a variable substitution $\frac{r}{R} \equiv \xi$.
- Your result of τ_{ff} should only depend on constants and $\bar{\rho}$.

b) Compare the equation derived in a) to the one given in the lecture. Describe the differences.

c) If the Sun had no pressure support, what would be its free-fall time? (Use your result from a) and assume the mean density of the Sun is 1.41×10^3 kg m $^{-3}$.)

Exercise 3: Accretion to a hydrostatic core

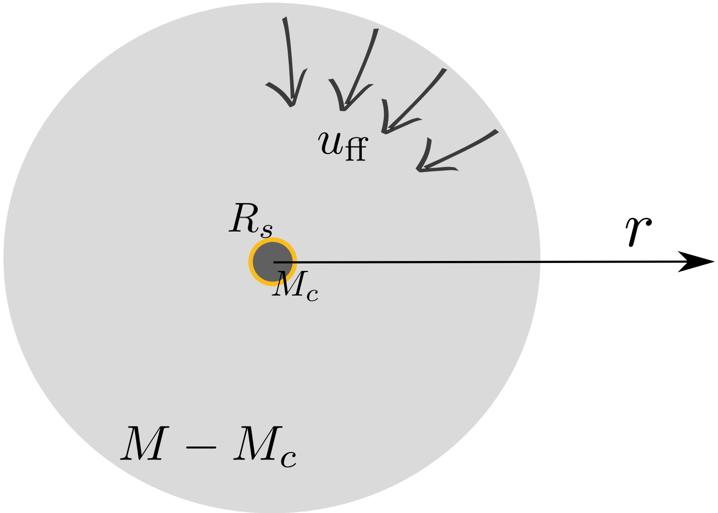

Figure 1: Accretion on hydrostatic core with accretion shock at R_s.

Suppose the accretion of an extended gas cloud onto an existing hydrostatic core with mass M_c, a scenario that represents the accretion phase of prestellar evolution. For the sake of simplicity we can assume that the gas from the cloud (shell) falls freely onto the core which is given by

$$u_{\mathrm{ff}}(r) = \sqrt{\frac{2GM_c}{r}}.$$

a) For a stationary configuration (i.e. not time dependent), the mass accretion rate can be written as

$$\frac{dM}{dt} = \dot{M} = 4\pi r^2 \rho u_{\mathrm{ff}}.$$

What is the functional dependence of the density in this case? Draw a plot of the function and think about a physically reasonable condition for $r \to 0$.

b) As depicted in upper figure, the infalling region is bordered by an accretion shock front at constant radius R_s and we assume a density ρ_s at this point. The density profile is then

$$\rho(r) = \rho_s \left(\frac{r}{R_s}\right)^{-3/2}.$$

What is the remaining shell mass $M - M_c$ in this configuration?

Hint: For the integration you can neglect the lower boundary since we assume that the core radius R_c is much smaller than the radius of the shell R.

c) Now assume again that the mass accretion is ideal in the sense that the velocity of the infalling gas at the point of the accretion shock R_s is given by the free-fall velocity at this radius, i.e. $u_{\text{ff}}(R_s) = u_s$. With the result from b) ($M - M_c \cong \frac{8\pi}{3}\rho_s R_s^{3/2} R^{3/2}$) derive a differential equation for the core mass accretion $\frac{dM_c}{dt}$ and solve it. Plot the function $\frac{M_c}{M}$ for $t \in [0, 3\tau_{\text{ff}}]$ and discuss its behavior .

Hints: After having constructed the differential equation it is useful to make a variable transformation for the time to a dimensionless parameter $\tau = \frac{t}{\tau_{\text{ff}}}$ and for the mass to $\mu = \frac{M_c}{M}$.

Exercise 4: Energy conservation

In the lecture, the conservation of energy has already been discussed. On way to describe energy conservation is the following:

$$\frac{dL_r}{dr} = 4\pi r^2 \rho(r)\varepsilon(r)$$

Here we will consider how one can derive the conservation of energy from the radiation transport equation. Starting point is

$$\frac{dI_\nu}{ds} = \vec{n} \cdot \nabla I_\nu = -\alpha_\nu I_\nu + \eta_\nu$$

where α_ν is the absorption coefficient and η_ν is the sum of thermal emission and energy production.

a) Write down the radiative transfer equation in spherical coordinates.

b) Integrate over the whole solid angle.
Hint: partial integration

c) Integrate over all frequenzies.
Hint: $\int d\nu\ \alpha_\nu < I_\nu > = \bar{\alpha} \cdot \int d\nu < I_\nu >$
$\int d\nu\ \eta_\nu = j + \frac{\varepsilon}{4\pi}$
In thermal equilibrium, the volume absorption $\bar{\alpha} \cdot \int d\nu < I_\nu >$ is equal to the volume emission j

d) Use $L_r = 4\pi r^2 F$, to get the upper formulation of the conservation of energy.