Report: MPC with rate limitation and switching frequency minimisation

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1 Quantisation

Let $w \in \mathbb{R}$ be the input, $\mathbf{Q}$ be the quantiser and $y \in \mathbb{U}$ the quantiser output, as shown in the additive model of quantisation in the Figure 1. Let us define the quantisation error as

$$q = \mathbf{Q}(w) - w = y - w. \tag{1}$$

The quantisation requires the signal to be mapped to a finite signal where each value of the output y is restriced to
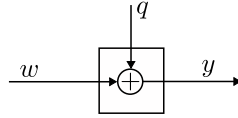

Figure 1: Quantiser additive model

belong to a finite set $\mathbb{U}$. The elements of the set $\mathbb{U}$ represent the quantiser levels and depends on the word-size of the quantiser.

2 Noise and Distortion in the DACs.

2.1 Slewing Distortion

Slew rate defined as the maximum rate of change of output voltage level. In other words, it referes to how fast the DAC is capable of changing its output voltage in response to a change in input. An ideal DAC creates an output voltage that is instantaneously proportional to the input number which is not possible in practice due to temporal limitations. If the input number to the DAC change from one value another at time t_1, the output voltage reaches new value at time later than time t_1. This is one of the most common distortions in DACs and is governed in large part by the slewing of the output amplifier and its compensation in physical DACs and shown in the following figure.
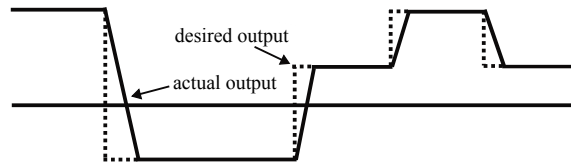

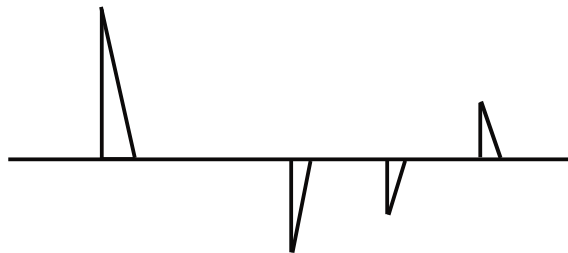
Figure 2: Effect of slew limiting on the output of the DAC

Figure 3: Slew limited error

3 Moving horizon optimal quantiser (MHOQ)

The design criteria for the MHOQ is the minimization of the perceived errors defined as follows:

$$e(t) = H(z)(w(t) - y(t)) \tag{2}$$

where $H(z)$ is a stable time-invariant linear low-pass filter with the following state-space

$$H(z) = 1 + C(zI - A)^{-1}B \tag{3}$$

The error e then can be written as the output of the following state-space representation of H

$$\begin{aligned}
x(t+1) &= Ax(t) + B(w(t) - y(t)) \\
e(t) &= Cx(t) + w(t) - y(t)
\end{aligned} \tag{4}$$

where $x \in \mathbb{R}^n$ is the state vector. The error e corresponds to the difference between the filtered quantised signal and the filtered input signal.

For moving horison implementation, the optimisation problem is defined as the problem of finding $y \in \mathbb{U}$ that minimises the cost function while satisfying the state equations as follows:

3.1 Mixed Integer Quadratic Optimisation Problem

$$y^*(t) = \arg\min_{y(t)} V_N = \sum_{t=k}^{k+N-1} e^2(t) \tag{5}$$

subject to

$$x(t+1) = Ax(t) + B(w(t) - y(t)) \tag{6}$$
$$e(t) = Cx(t) + w(t) - y(t) \tag{7}$$
$$y(t) \in \mathbb{U}. \tag{8}$$

3.2 Mixed Integer (Binary) Quadratic Optimisation Problem

The optimization problem (5)-(8) can be reformulated as an optimization problem with the binary variables. Let $\mathcal{B}$ be the number of bits. $b_i = \{0, 1\}$ and Q_i, $i = \{0, 1, \ldots, 2^{\mathcal{B}} - 1\}$, be the binary variables and quantisation levels, respectively.

$$y^*(t) = \arg\min_{y(t)} V_N = \sum_{t=k}^{k+N-1} e^2(t) \tag{9}$$

subject to

$$x(t+1) = Ax(t) + B(w(t) - y(t)) \tag{10}$$
$$e(t) = Cx(t) + (w(t) - y(t)) \tag{11}$$

$$y(t) = \sum_{i=0}^{2^{\mathcal{B}}-1} Q_i b_i, \quad \sum_{i=0}^{2^{\mathcal{B}}-1} b_i = 1, \quad b_i = \{0, 1\}. \tag{12}$$

4 MPC with Switching Frequency Minimization

The amount of the states that switches at each sampling time can be minimised by penalising the total number of switched with some weighting factor. Two control objective are achieved in the propose algorithm, optimal quantisation and switching frequency minimisation. The minimised switching frequency leads to the mitigation of the distortion in the output signal due to slewing effectin the DACs. The optimisation problem is as follows,

$$y^*(t) = \arg\min_{y(t)} V_N = \sum_{t=k}^{k+N-1} e^2(t) + \beta \mathcal{N}_s \tag{13}$$

subject to

$$x(t+1) = Ax(t) + B(w(t) - y(t)) \tag{14}$$
$$e(t) = Cx(t) + (w(t) - y(t)) \tag{15}$$

$$y(t) = \sum_{i=0}^{2^B-1} Q_i b_i(t), \quad \sum_{i=0}^{2^B-1} b_i(t) = 1, \quad b_i = \{0,1\}^N. \tag{16}$$

$$\mathcal{N}_s = \sum_{i=0}^{2^B-1} |b_i(t) - b_i(t-1)|^2 \tag{17}$$

where $b_i(t-1)$ is the state of the switch i at the previous state and $b_i(t)$ is the sate of switch i at the current state and β is the weighting factor. The average switching frequency of the coverters is defined as follows

$$f_{sw} := \frac{1}{MT_s} \sum_{i=1}^{M} \|\Delta b_i\|_1 \tag{18}$$

where M is the total number of samples, T_s is the sampling rate and $\Delta b_i = b_i - b_{i-1}$.
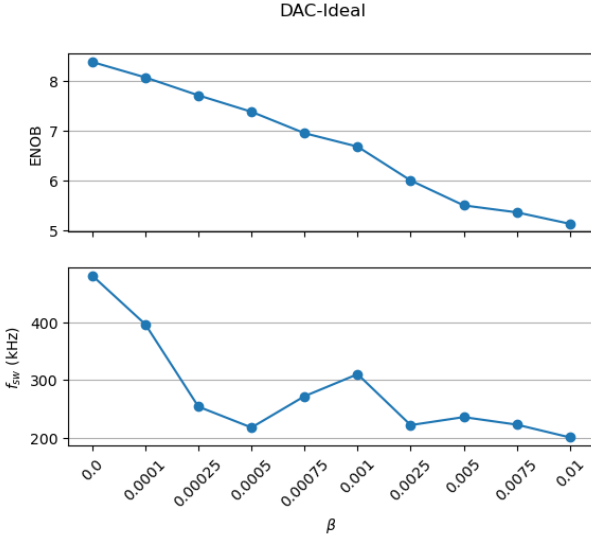

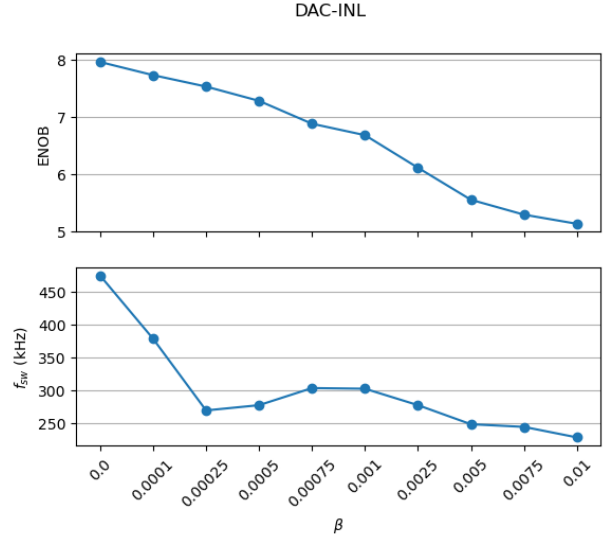Figure 4: DAC-Ideal: Effect of switching frequency β on average switching frequency f_{sw} and ENOB.

Figure 5: DAC-INL: Effect of switching frequency β on average switching frequency f_{sw} and ENOB.

5 MPC with switching rate limitation:

The MPC formulation with the rate limitation is as follows:

$$y^*(t) = \arg\min_{y(t)} V_N = \sum_{t=k}^{k+N-1} e^2(t) \tag{19}$$

subject to

$$x(t+1) = Ax(t) + B(w(t) - y(t)) \tag{20}$$
$$e(t) = Cx(t) + w(t) - y(t) \tag{21}$$
$$-L_r\Delta t \leq \Delta y(t) \leq L_r\Delta t \tag{22}$$
$$y(t) \in \mathbb{U}. \tag{23}$$

The rate limitation is imposed by the constraint (22) where $\Delta y(t) = y(t) - y(t - \Delta t)$ and the constraint (22) can be further simplifed as

$$\Delta y(t) \leq L_r\Delta t$$
$$-\Delta y(t) \leq L_r\Delta t$$

Here $\Delta t = T_s$ and the rate limitation L_r are assumed to constant for all sampling instants.

Simulations results:

Table 1: Simulation Results: Carrier Frequency $999Hz$, Total number of samples: 21021.

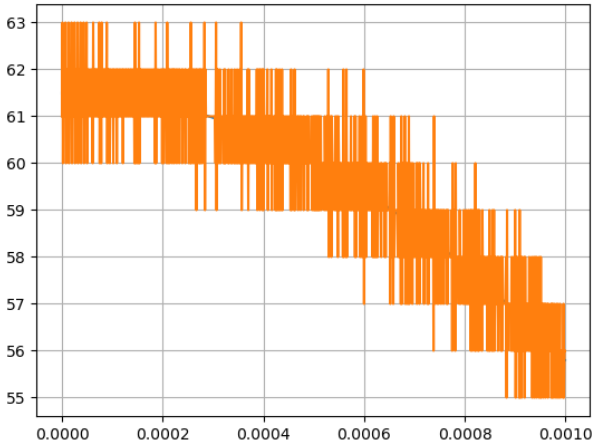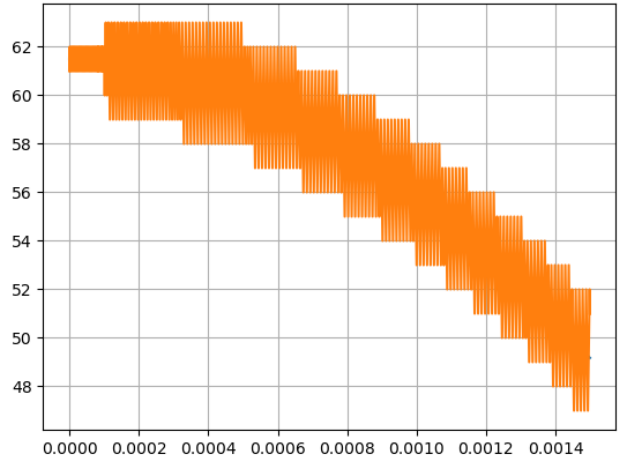| Rate limit $(L_r\Delta t)$	N/A	1e6	-	-	-	-	-
ENOB (99Hz)	9.444	4.18	-	-	-	-	-

Figure 6: No rate limit. Carrier frequency: $99Hz$.

Figure 7: Carrier frequency: $999Hz, \beta = 100$