Report: Synthesis of optimal filter for MHOQ

June 27, 2024

Contents

1 Quantisation

Let $w \in \mathbb{R}$ be the input, $\mathbf{Q}$ be the quantiser and $y \in \mathbb{U}$ the quantiser output, as shown in the additive model of quantisation in the Figure 1. Let us define the quantisation error as

$$q = \mathbf{Q}(w) - w = y - w. \tag{1}$$

The quantisation requires the signal to be mapped to a finite signal where each value of the output y is restriced to
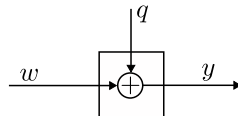

Figure 1: Quantiser additive model

belong to a finite set $\mathbb{U}$. The elements of the set $\mathbb{U}$ represent the quantiser levels and depends on the word-size of the quantiser.

2 Noise shaping quantiser

Noise-shaping quantisers can reduce the effective quantisation error by moving quantisation noise to higher frequencies through oversampling and feedback. The reconstruction filter is then used to attenuate the frequency-shaped quantisation noise. It operates by estimating the uniform quantisation error and employing a feedback filter to shape the noise power at the output of the DAC. A block diagram for a noise-shaping quantiser is shown in Fig. 3. The feedback filter $F(z)$ is designed such that the transfer function $y = (1 - F(z))\epsilon$ is a high-pass filter.
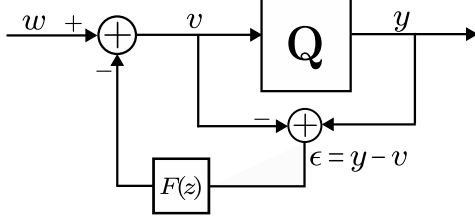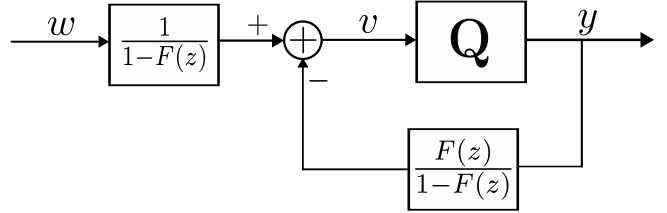

Figure 2: Noise shaping quantiser

Figure 3: Noise shaping quantiser

In linear analysis, the output is given by

$$Y(z) = \mathbf{STF}.W(z) + \mathbf{NTF}.E(Z) \tag{2}$$

where the signal transfer function $\mathbf{STF} = 1$, noise transfer function $\mathbf{NTF} = (1 - F)$ with F being a noise shaping filter. $F = z^{-1}$ is the special case known as the first-order delta-sigma modulator.

3 Moving horizon optimal quantiser (MHOQ)

The design criteria for the MHOQ is the minimization of the perceived errors defined as follows:

$$e(t) = H(z)(w(t) - y(t)) \tag{3}$$

where $H(z)$ is a stable time-invariant linear low-pass filter with the following state-space

$$H(z) = 1 + C(zI - A)^{-1}B \tag{4}$$

The error e then can be written as the output of the following state-space representation of H

$$\begin{aligned} x(t+1) &= Ax(t) + B(w(t) - y(t)) \\ e(t) &= Cx(t) + w(t) - y(t) \end{aligned} \tag{5}$$

where $x \in \mathbb{R}^n$ is the state vector. The error e corresponds to the difference between the filtered quantised signal and the filtered input signal.

For moving horison implementation, the optimisation problem is defined as the problem of finding $y \in \mathbb{U}$ that minimises the cost function while satisfying the state equations as follows:

$$y^*(t) = \arg\min_{y(t)} V_N = \sum_{t=k}^{k+N-1} e^2(t) \tag{6}$$

subject to

$$x(t+1) = Ax(t) + B(w(t) - y(t)) \tag{7}$$
$$e(t) = Cx(t) + w(t) - y(t) \tag{8}$$
$$y(t) \in \mathbb{U}. \tag{9}$$

3.1 Alternative binary formulation

The optimization problem (6)-(9) can be reformulated as an optimization problem with the binary variables. Let $\mathcal{B}$ be the number of bits. $b_i = \{0, 1\}$ and Q_i, $i = \{0, 1, \ldots, 2^{\mathcal{B}} - 1\}$, be the binary variables and quantisation levels, respectively.

$$y^*(t) = \arg\min_{y(t)} V_N = \sum_{t=k}^{k+N-1} e^2(t) \tag{10}$$

subject to

$$x(t+1) = Ax(t) + B(w(t) - y(t)) \tag{11}$$
$$e(t) = Cx(t) + (w(t) - y(t)) \tag{12}$$
$$y(t) = \sum_{i=0}^{2^{\mathcal{B}}-1} Q_i b_i, \quad \sum_{i=0}^{2^{\mathcal{B}}-1} b_i = 1, \quad b_i = \{0, 1\}. \tag{13}$$

4 Noise Transfer Function(NTF)

The frequency response of the noise transfer functions due to butterworth filters at different cutoff frequencies are shown in the figure Fig. 4. In the figure, we can see that the net area under the curve remain the same. In Fig. 5 the frequency reponse of the low pass filter is plotted along with that of the noise transefer function. This observation shows that the better performance can be achieved by increasing the cutoff frequency during MHOQ while keeping the cutoff frequency of the reconstruction as same. The simulation results in the following table confirm this observation.

Table 1: ENOB at different cutoff frequencies. Reconstruction filter: Butterworth LPF with $n = 2$, $Fc = 100$kHz and $Fs = 1$Mhz.

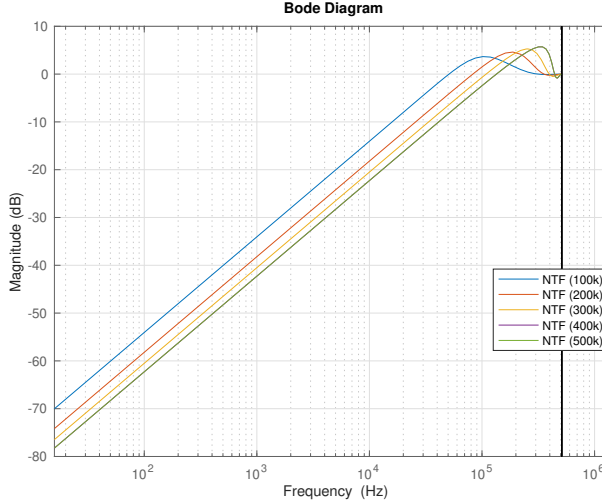| Fc	100 kHz	200 kHz	300 kHz	400 kHz	500 kHz
ENOB	3.981	5.307	7.817	10.481	10.936

Figure 4: Frequency response of NTF for different cutoff frequency
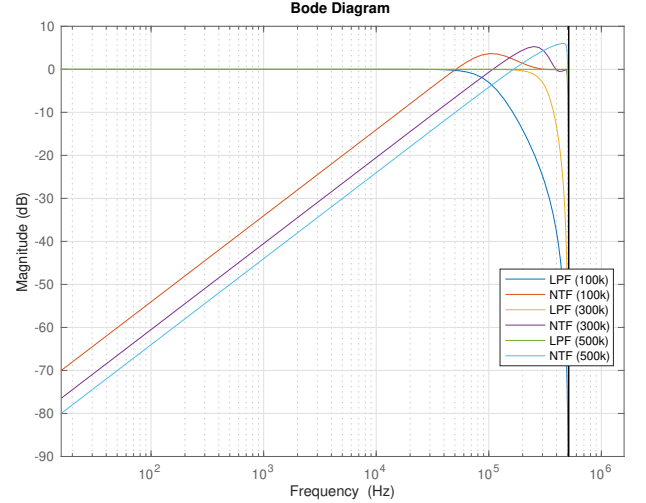
Figure 5: Frequency response of LPF and NTF for different cutoff frequency

5 Spectrum of quantisation noise

The sampling frequency effects the spectrum of the quantisation noise and consequently the ENOB as shown in the following figures.
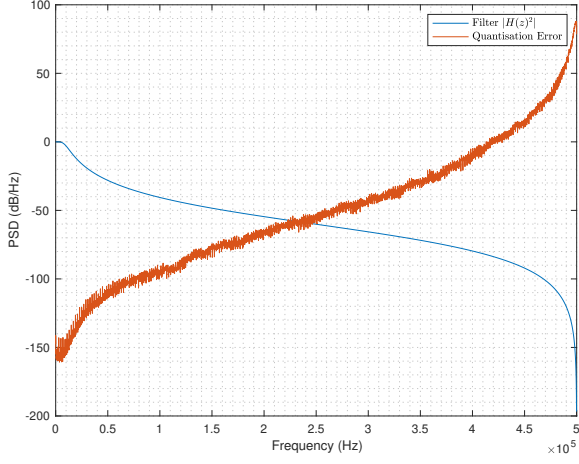
Figure 6: Butterworth Frequency response and frequency spectrum of quantisation noise: **Fc** = 10 kHz, **Fs** = 1 MHz, **ENOB** = 16.58, **INL**

Figure 7: Butterworth Frequency response and frequency spectrum of quantisation noise: **Fc** = 100 kHz, **Fs** = 1 MHz, **ENOB** =7.43, **INL**

Figure 8: Butterworth Frequency response and frequency spectrum of quantisation noise: **Fc** = 10 kHz, **Fs** = 10 MHz, **ENOB** = 25.12, **INL**

Figure 9: Butterworth Frequency response and frequency spectrum of quantisation noise: **Fc** = 100 kHz, **Fs** = 10 MHz, **ENOB** = 17.03, **INL**

6 Synthesis of optimal noise-shaping filter
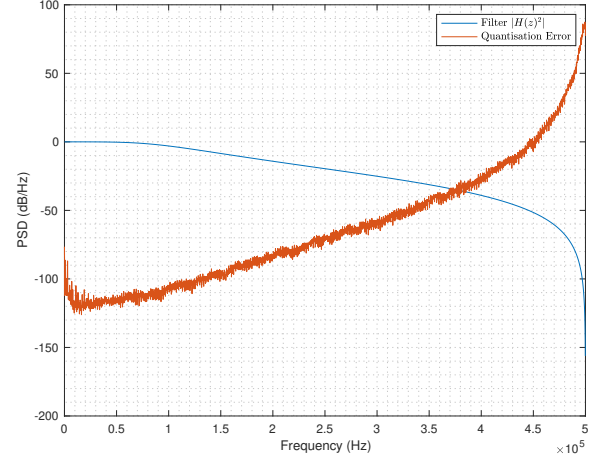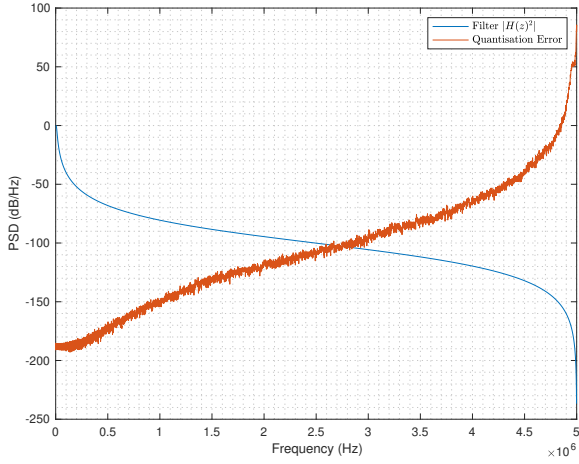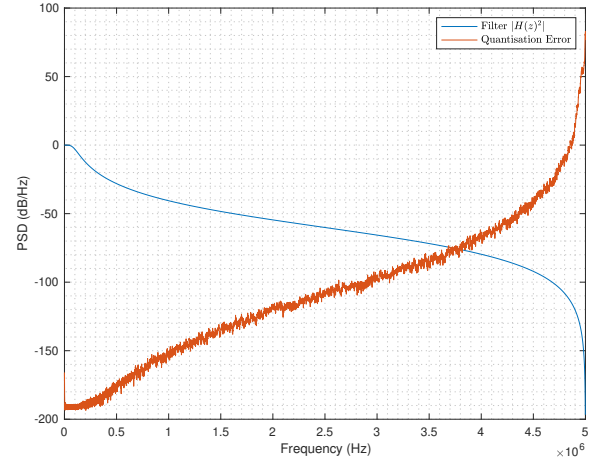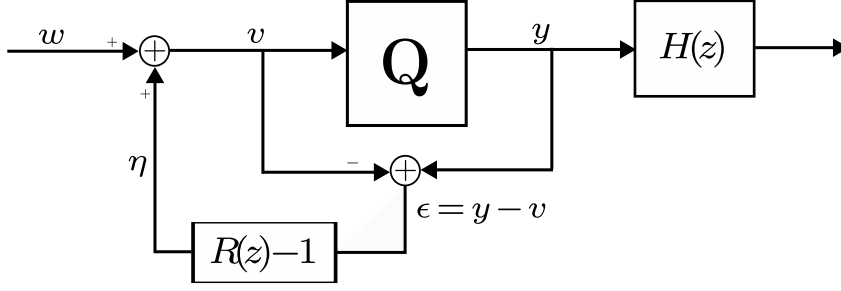

Figure 10: Noise shaping quantiser and a filter $H(z)$.

In noise shaping quantiser with error-feedback structure as shown in the Figure 10, the input to the quantiser is $v = w + \eta = w + (R(z) - 1)\epsilon$ and feedback error ϵ is

$$\epsilon = y - v = y - w - (R(z) - 1)\epsilon. \tag{14}$$

Then quantisaion noise defined as $q := y - w$ can be expressed as

$$q = y - w = R(z)\epsilon. \tag{15}$$

Then the effect of the quantisation error on the system $H(z)$ can be expressed as

$$e = H(z)R(z)\epsilon. \tag{16}$$

and it shows that we can reduce the error in the plant output by properly designing the noise shaping filter $R(z)$ with the knowledge of the plant $H(z)$.

The objective is to design stable noise-shaping filter such that it minimises the effect of the quantisation noise in the plant output. A constraint on the error feedback signal should be imposed to prevent the quantiser from overloading and achieve a stable noise-shaping quantiser as

$$\eta = (R(z) - 1)\epsilon.$$

- **Design Problem:** For a fixed pair (p,q) [1],

$$\min_{R(z) \in \mathbb{RH}_\infty} \|e\|_p$$

subject to.

$$R(\infty) = 1,$$
$$\|\eta\|_q < \gamma_\eta \tag{17}$$

where $\mathbb{RH}_\infty = \mathbb{R} \cap \mathbb{H}_\infty$ is the set of proper stable rational transfer functions.

6.1 Solution 1: Optimal noise shaping filter without constraint

The optimal noise shaping filter without constraint on the error feedback signal η is the scaled inverse of the system $H(z)$. If the quantisation error is assumed to be an i.i.d random variable with zero mean, then the variance of the error e at time k can be expressed as

$$E\{|e_k|^2\} = \|H(z)R(z)\|_2^2 \sigma_\epsilon^2 \tag{18}$$

where $E\{.\}$ is the expectation operator and σ_ϵ is variance of the ϵ and $\|H(z)R(z)\|_2^2$ is the $\mathbb{H}_2$-norm.

It is shown in [2], any casual and stable rational sytem function $H(z)$ can always be stated as the product of the minimum phase system $H_{min}(z)$ and all pass system $H_{ap}(z)$. Then it is shown that the optimal noise shaping filter is given by the scaled inverse of the plant, as follows,

$$R(z) = h_D H_{min}^{-1}(z) \tag{19}$$

where h_D is the first non-zero entry of the impulse response of the plant H_z.

6.2 Solution 2: Optimal noise shaping filter with constraint

The optimization problem is setup as the minimization of the upper bound of the $\|e\|_p$ and $\|\eta_q$ as follows:

$$\min_{R(z)\in\mathbb{RH}_\infty} \gamma_e \tag{20}$$

subject to $R(\infty) = 1$ and

$$\|H(z)R(z)\|_{ind.p} < \gamma_e \tag{21}$$
$$\|R(z) - 1\|_{ind.q} < \gamma_\eta \tag{22}$$

where $\|.\|_{ind.r}$ is the induced norm and γ_e and γ_η are the upper bound of $\|H(z)R(z)\|_{ind.p}$ and $\|R(z) - 1\|_{ind.q}$, respectively.

State space representation

6.2.1 System

Denoting the state-space representation of the $H(z)$ and $R(z)$ as (A_h, B_h, C_h, D_h) and $(A_r, B_r, C_r, 1)$ respectively, the state space realization of $H(z)R(z)$ is

$$x_{k+1} = Ax_k + B\epsilon_k \tag{23}$$
$$e_k = Cx_k + D\epsilon_k \tag{24}$$

where

$$A = \begin{bmatrix} A_h & B_hC_r \\ \mathbf{0} & A_r \end{bmatrix} \qquad B = \begin{bmatrix} B_h \\ B_r \end{bmatrix} \qquad C = \begin{bmatrix} C_h & D_hC_r \end{bmatrix} \qquad D = D_h. \tag{25}$$

Similary, the variance of the error e under the white noise assumption at time k in state-space form is given by $\mathbb{H}_2$-norm as

$$\|H(z)R(z)\|_2^2 = \sum_{k=0}^{\infty} \|CA^kB\|_2^2 + DD^\top. \tag{26}$$

Moreover, if A is Schur matrix, then there exists a positive semi-definite solution P of the discrete Lyapunov equation defined as

$$P = A^\top PA + BB^\top \tag{27}$$

and the squared $\mathbb{H}_2$ norm is given by

$$\|H(z)R(z)\|_2^2 = CPC^\top + DD^\top. \tag{28}$$

6.2.2 Constraint 1

Then $\|H(z)R(z)\|_2 < \gamma_e$ if and only if there exist a positive definite matrix P such that

$$(\textbf{BMI}) \quad \begin{bmatrix} P & PA & PB \\ A^\top & P & \mathbf{0} \\ B^\top & \mathbf{0} & 1 \end{bmatrix} \succ 0 \tag{29}$$

$$(\textbf{LMI}) \quad \begin{bmatrix} \mu_e & C & D \\ C^\top & P & \mathbf{0} \\ D^\top & \mathbf{0} & 1 \end{bmatrix} \succ 0 \tag{30}$$

$$\mu_e = \gamma_e^2. \tag{31}$$

6.2.3 Constraint 2

The variance of the noise shaping filter is given by

$$E\{|\eta_k|^2\} = \|R(z) - 1\|_2^2 = \sum_{k=1}^{\infty} \|\tilde{C}A^kB\|_2^2, \tag{32}$$

where $\tilde{C} = \begin{bmatrix} \mathbf{0} & C_r \end{bmatrix}$. Then $\|R(z) - 1\|_2^2 < \gamma_\eta$ if and only if there exist a positive definite matrix P that satisfies

$$\textbf{(BMI)} \qquad \begin{bmatrix} P & PA & PB \\ A^\top & P & \mathbf{0} \\ B^\top & \mathbf{0} & 1 \end{bmatrix} \succ 0 \tag{33}$$

$$\textbf{(LMI)} \qquad \begin{bmatrix} \mu_\eta & \tilde{C} \\ \tilde{C}^\top & P \end{bmatrix} \succ 0 \tag{34}$$

$$\mu_\eta = \gamma_\eta^2. \tag{35}$$

6.2.4 Optimization problem: State Space formulation

Thus the optimization problem can be written as follows,

$$\min_{R(z) \in \mathbb{RH}_\infty} \gamma_e \tag{36}$$

subject to $R(\infty) = 1$ and

$$\begin{bmatrix} P & PA & PB \\ A^\top & P & \mathbf{0} \\ B^\top & \mathbf{0} & 1 \end{bmatrix} \succ 0 \tag{37}$$

$$\begin{bmatrix} \mu_e & C & D \\ C^\top & P & \mathbf{0} \\ D^\top & \mathbf{0} & 1 \end{bmatrix} \succ 0 \tag{38}$$

$$\begin{bmatrix} \mu_\eta & \tilde{C} \\ \tilde{C}^\top & P \end{bmatrix} \succ 0 \tag{39}$$

$$\mu_e = \gamma_e^2, \mu_\eta = \gamma_\eta^2. \tag{40}$$

7 LMI Synthesis: Convert BMIs to convex LMIs

BMIs are not convex and NP hard to solve, but they can be converted to convex LMIs and can be solved numerically whereas the LMIs are convex. The non-convex BMIs can be converted to convex LMIs using change of variables [3].

Change of Variables:

Let the order of $H(z)$ is n and the set of $n \times n$ positive define matrices is denoted as $\text{PD}(n)$. Denote by $\mathcal{P}$ the set of variables $\mathbf{p} = \{P_f, P_g, W_f, W_g, W_h, L\}$ where $P_f \in \text{PD}(n)$, $P_g \in \text{PD}(n)$, $W_f \in \mathbb{R}^{1 \times n}$, $W_g \in \mathbb{R}^{n \times 1}$, $W_h \in \mathbb{R}$ and $L \in \mathbb{R}^{n \times n}$. Then define the following matrix values function on $\mathcal{P}$:

$$\begin{aligned} M_A &:= \begin{bmatrix} A_h P_f + B_h W_f & A_h \\ L & P_g A_h \end{bmatrix} \\ M_B &:= \begin{bmatrix} B_h \\ W_g \end{bmatrix} \\ M_C &:= \begin{bmatrix} C_h P_f + D_h W_f & C_h \end{bmatrix} \\ M_P &:= \begin{bmatrix} P_f & I_n \\ I_n & P_g \end{bmatrix}. \end{aligned} \tag{41}$$

Next, define

$$P^{-1} := \begin{bmatrix} P_f & S_f \\ S_f & S_f \end{bmatrix}, \qquad U := \begin{bmatrix} P_f & I_n \\ S_f & \mathbf{0} \end{bmatrix} \qquad \text{and} \tag{42}$$

$$S_f := P_f - P_g^{-1} (\succ 0) \tag{43}$$

then we have,

$$M_P = U^\top P U. \tag{44}$$

If the matrices (A_r, B_r, C_r) are given by

$$\begin{aligned} A_r &:= \begin{bmatrix} B_h W_f - P_g^{-1}(L - P_g A_h P_f) \end{bmatrix} S_f^{-1} \\ B_r &:= \begin{bmatrix} B_h - P_g^{-1} W_g \end{bmatrix} \\ C_r &:= W_f S_f^{-1} \end{aligned} \tag{45}$$

7

then (A, B, C) satisfy,

$$M_A = U^\top P A U \tag{46}$$

$$M_B = U^\top P B \tag{47}$$

$$M_C = C U \tag{48}$$

$$M_p = U^\top P U. \tag{49}$$

Multiplying with the transformation $\phi = \text{diag}(U, U, 1)$ form the RHS and $\phi^\top$ from the LHS the BMI condition 37 takes the following form:

$$\begin{bmatrix} M_P & M_A & M_B \\ M_A^\top & M_P & \mathbf{0} \\ M_B^\top & \mathbf{0} & \mathbf{I} \end{bmatrix} \succ 0. \tag{50}$$

Similarly, using the transformation $\text{diag}(1, U, 1)$, the LMI condition 38 is

$$\begin{bmatrix} \mu_e & M_C & D^\top \\ M_C^\top & M_P & \mathbf{0} \\ D & \mathbf{0} & \mathbf{I} \end{bmatrix} \succ 0 \tag{51}$$

and finally the constraint 39 using the transformation is

$$\begin{bmatrix} \gamma_\eta^2 & M_{\tilde{C}} \\ M_{\tilde{C}}^\top & M_P \end{bmatrix} \succ 0 \tag{52}$$

with $M_{\tilde{C}} := \tilde{C} U$. The LMI conditions 50, 51 and 52 are convex and the minimization of γ_e with these constraints is a convex optimization problem.

7.1 Optimization Problem:

With the change of the variables the BMI is converted to the LMI and since all the LMIs are convex, the optimization problem is a convex optimization problem. Also, since the objective is linear and the constraints are LMIs and with linearity constraints, the optimization problem is Semi-Definite Program (SDP). Such SDP can be solved numerically using CVX. The optimization problem takes the following form,

$$\min \mu_e = \gamma_e^2 \tag{53}$$

subject to,

$$\begin{bmatrix} M_P & M_A & M_B \\ M_A^\top & M_P & \mathbf{0} \\ M_B^\top & \mathbf{0} & \mathbf{I} \end{bmatrix} \succ 0 \tag{54}$$

$$\begin{bmatrix} \mu_e & M_C & D^\top \\ M_C^\top & M_P & \mathbf{0} \\ D & \mathbf{0} & \mathbf{I} \end{bmatrix} \succ 0. \tag{55}$$

$$\begin{bmatrix} \mu_\eta & M_{\tilde{C}} \\ M_{\tilde{C}}^\top & M_P \end{bmatrix} \succ 0 \tag{56}$$

where μ_e is the variance of the output error and μ_η the variance of the feedback error.

8 Simulation: Optimal noise shaping

Notations:

Let us denote the transfer function of a Butterworth low pass filter as $H(z)$. Then from Figure 10, the noise transfer function (NTF) is denoted by $R(z)$ and comparing Figure 2 and Figure 10, the noise shaping transfer function $F(z) = 1 - R(z)$. Moreover, it is shown in the moving horizon implementation of the quantisation [4], the low-pass filter and the noise shaping filter are related as follows,

$$F(z) = \frac{H(z) - 1}{H(z)} \qquad \Leftrightarrow \qquad H(z) = \frac{1}{1 - F(z)} = \frac{1}{R(z)}. \tag{57}$$

Next, let the optimal noise transfer function (NTF) for low-pass filter $H(z)$ obtained by solving the optimization problem (53)-(56) is denoted as $R_{opt}(z) = \dfrac{b_r}{a_r}$, where b_r and a_r are the numerator and denominator of the noise transfer function, respectively. Then the optimal noise-shaping transfer function (NSF) is $F_{opt}(z) = 1 - R_{opt}(z) = \dfrac{a_r - b_r}{a_r}$. Finally, the optimal low pass filter for MPC implementation is $H_{mpc} = \dfrac{1}{R_{opt}(z)} = \dfrac{a_r}{b_r}$.

8.1 Simulation 1:

Let us consider a third-order butter-worth low pass filter $H(z)$ with $F_c = 100kHz$ and $F_s = 1Mhz$. Then, from the expression in the equation 57 the corresponding noise shaping filter $F(z)$ and the noise transfer function $R(z)$ can be obtained. Next, the optimization problem (53)-(56) with the constraint $\gamma_\eta < 1.5^2$ (Lee's condition) is solved to obtain the optimal noise transfer function $R_{opt}(z)$ and consequently the $F_{opt}(z)$ and $H_{mpc}(z)$ are obtained. The frequency responses of $H(z)$, $F(z)$ and $R(z)$ are shown in the Figure. 11 and that of $H_{opt}(z)$, $F_{opt}(z)$ and $R_{opt}(z)$ are shown in Figure 12.
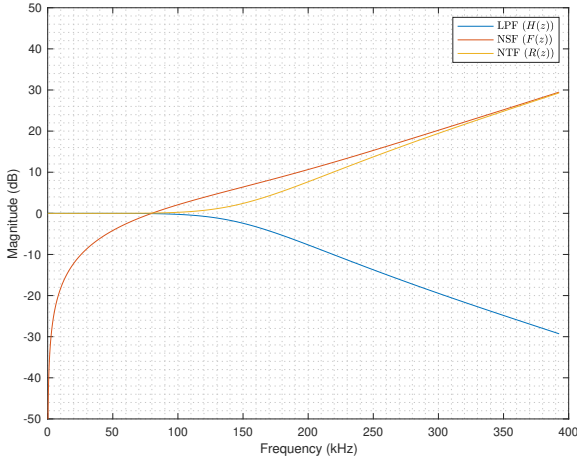

Figure 11: Frequency response: Butterworth filter $H(z)$ with $n = 3$ and $F_c = 100kHz$ and corresponding noise shaping filter $F(z)$ and noise transfer function $R(z)$.
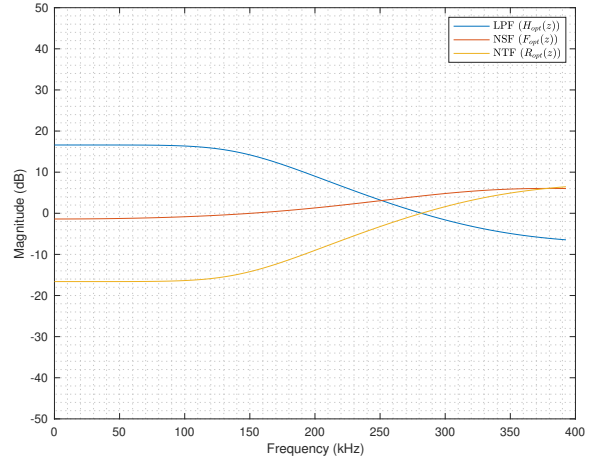
Figure 12: Frequency response: Optimal noise transfer function $R_{opt}(z)$ for $H(z)$ and corresponding noise shaping filter $F_{opt}(z)$ and low pass filter $H_{opt}(z)$.

Table 2: ENOB obtained for different methods with uniform quantisation models. Optimal NTF solved for second order butterworth filter with $Fc = 100kHz$ and $Fs = 1Mhz$.

ENOB/ Methods	Direct	DSM	NSD (Optimal)	MPC ($N=1$)	MPC ($N=2$)	MPC ($N=3$)
6-bit	6.936	8.291	9.30	9.30	9.303	9.361
8-bit	9.650	10.366	11.357	11.357	11.326	11.380
12-bit	13.285	14.365	15.372	15.372	15.290	15.281
16-bit	17.148	18.390	19.373	19.373	19.247	19.134

Table 3: ENOB obtained for different methods with nonlinear quantisation models. Optimal NTF solved for second order butterworth filter with $Fc = 100kHz$ and $Fs = 1Mhz$.

ENOB/ Methods	Direct	DSM	NSD (Optimal)	MPC ($N=1$)	MPC ($N=2$)	MPC ($N=3$)
6-bit	4.177	7.977	7.022	7.022	6.99	7.007
8-bit	6.526	9.968	9.228	9.228	9.251	9.241
12-bit	10.830	13.679	13.524	13.524	13.503	13.509
16-bit	13.577	15.77	16.156	16.156	16.157	16.145

8.2 Simulation 2:

Consider a third order Butterworth filter with $Fc = 100kHz$ and $Fs = 10Mhz$.
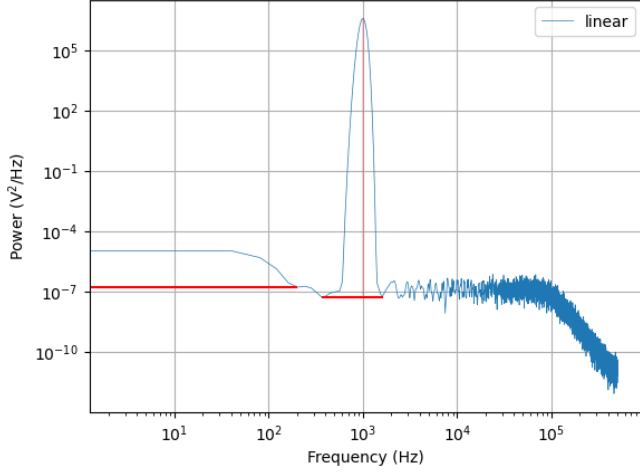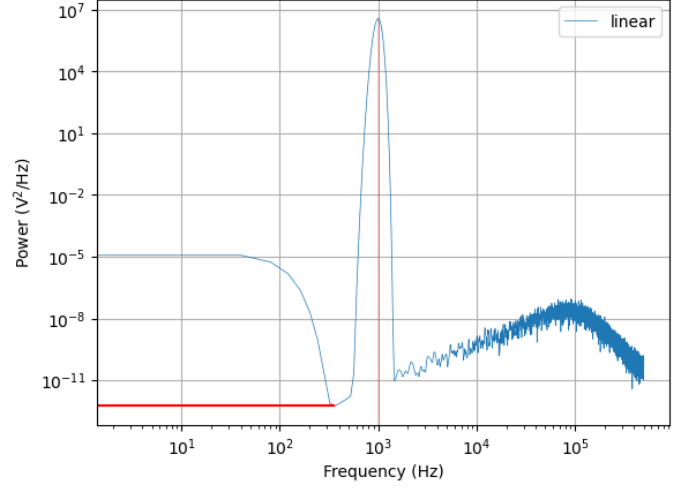
Figure 13: Direct Quantisation
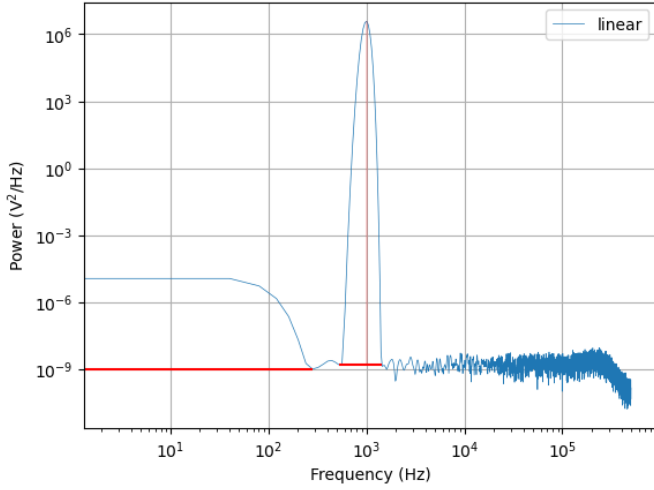

Figure 14: Delta sigma modulator
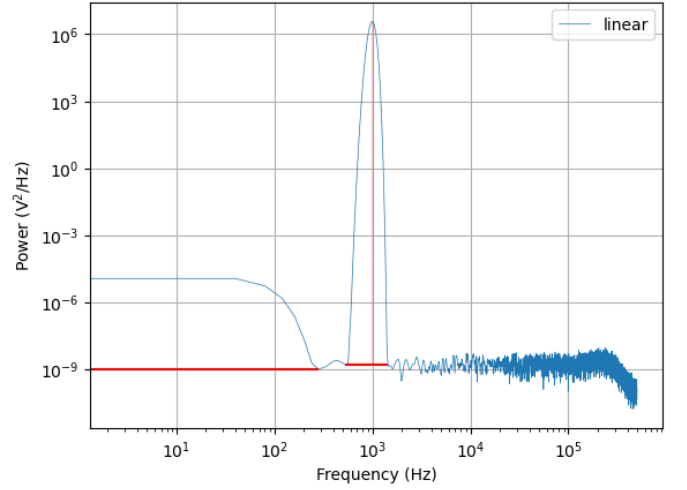

Figure 15: Noise shaping quantiser

Figure 16: MPC with optimal noise shaping

Table 4: ENOB obtained for different methods with nonlinear quantisation models. Optimal NTF solved for second order butterworth filter with $Fc = 100kHz$ and $Fs = 10Mhz$.

ENOB/ Methods	Direct	DSM	NSD (Optimal)	MPC ($N = 1$)	MPC ($N = 2$)	MPC ($N = 3$)
6-bit	5.041	12.355	15.576	15.576	15.548	15.577
8-bit	7.392	13.950	16.988	16.988	16.406	17.008
12-bit	11.210	17.368	20.952	20.952	20.950	20.931
16-bit	13.877	19.413	23.369	23.369	23.372	23.372

9 NSD performance: ENOB vs γ_η

Table 5: ENOB for different values of γ_η

Methods/γ_η	1.5	2	3	4	5	6	7	8	9	10	11	12
Direct	17.148	-	-	-	-	-	-	-	-	-	-	-
DSM	18.390	-	-	-	-	-	-	-	-	-	-	-
NSD	19.158	19.373	19.533	19.518	19.465	19.357	19.231	19.111	18.994	3.793	4.227	2.845
MPC(N=1)	19.158	19.373	19.533	19.518	19.465	19.357	19.231	19.111	18.994	3.793	4.227	2.845
MPC(N=2)	19.098	19.237	19.533	19.402	19.301	19.191	19.023	18.909	18.787	18.738	18.641	18.577
MPC(N=3)	18.955	19.133	19.266	19.180	19.040	18.868	18.746	18.677	18.638	18.560	18.456	18.379

10 Closed Form Solution:

Let N be the prediction horizon of the moving horizon implementation and $\mathbb{U}$ the quantisation levels. The Nth power cartesian product of the set $\mathbb{U}$ for N is defined as $\mathbb{U}^N := \mathbb{U} \ldots \times \ldots \mathbb{U}$. Let $n_{\mathbb{U}}$ represent the number of element in the set $\mathbb{U}$ i.e., number of quantisation levels, then the number of elements on the set $\mathbb{U}^N$ is $n_{\mathbb{U}}^N$. Also, let $v_i, i = \{1, 2, \ldots, n_{\mathbb{U}}^N\}$ represent the elements of the set $\mathbb{U}^N$. Then, the closed form solution of the moving horizon implementation of the optimisation problem (6)-(9) is

$$\mathbf{y}^*(k) = \Psi^{-1} q_{\tilde{\mathbb{U}}^N}\big[\Psi\mathbf{w}(k) + \Gamma x(k)\big] \tag{58}$$

where

$$\mathbf{y}^*(k) = \begin{bmatrix} y(k) \\ y(k+1) \\ \vdots \\ y(k+N-1) \end{bmatrix}, \quad \mathbf{w}(k) = \begin{bmatrix} w(k) \\ w(k+1) \\ \vdots \\ w(k+N-1) \end{bmatrix}, \quad \Gamma = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{N-1} \end{bmatrix}, \quad \Psi = \begin{bmatrix} h_0 & 0 & \cdots & 0 \\ h_1 & h_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ h_{N-1} & \cdots & h_1 & h_0 \end{bmatrix},$$

and $q_{\tilde{\mathbb{U}}^N}[.]$ represent the nearest neighbor vector quantisation (direct quantisation) and image of this mapping is the set

$$\tilde{\mathbb{U}}^N := \{\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_r\} \subset \mathbb{R}^N, \qquad r = n_{\mathbb{U}}^N,$$

with $\tilde{v}_i = \Psi v_i$ and $v_i \in \mathbb{U}^N = \{v_1, v_2, \ldots, v_{n_{\mathbb{U}}}\}$. The elements of the matrices Ψ, $\{h_0, h_1, \ldots, h_{N-1}\}$ are the impulse response of the filter $H(z) = 1 + C(zI - A)^{-1}B$ and $h_0 = 1$.

The moving horizon implementation of the above solution is

$$\tilde{y}(k) = [1, 0, \ldots, 0]\mathbf{y}^*(k) = [1, 0, \ldots, 0]\Psi^{-1} q_{\tilde{\mathbb{U}}^N}\big[\Psi\mathbf{w}(k) + \Gamma x(k)\big]. \tag{59}$$

Replacing $x(k) = (zI - A)^{-1}B(a(k) - u(k))$ and with further manipulation, we get,

$$\tilde{y}(k) = [1, 0, \ldots, 0]\Psi^{-1} q_{\tilde{\mathbb{U}}^N}\Big[\Psi\big\{\mathcal{H}(z)\mathbf{w}(k) - \big[\mathcal{H}(z) - I\big]\mathbf{y}^*(k)\big\}\Big]. \tag{60}$$

where

$$\mathcal{H}(z) = \begin{bmatrix} 1 + \mathcal{H}_1'(z) & 0 & \cdots & \cdots & 0 \\ \mathcal{H}_2'(z) & 1 & 0 & \cdots & 0 \\ \mathcal{H}_3'(z) & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathcal{H}_N'(z) & 0 & 0 & \cdots & 1 \end{bmatrix} \quad \text{and} \quad \mathcal{H}_i'(z) = \big[\Psi^{-1}\big]_{i,*}\Gamma(zI - A)^{-1}B \tag{61}$$

and $\big[\Psi^{-1}\big]_{i,*}$ denotes the i th row of Ψ^{-1}.

10.1 Prediction horizon $N = 1$:

When the prediction horizon is $N = 1$, $\Psi = 1$ since $h_0 = 1$ and

$$\mathcal{H}(z) = 1 + \mathcal{H}_1'(z) = 1 + C(zI - A)^{-1}B = H(z)$$

and the moving horizon implementation is

$$\tilde{y}(k) = q_{\tilde{\mathbb{U}}^N}\big[H(z)w(k) - \big[H(z) - I\big]y^*(k)\big]. \tag{62}$$

and $q_{\tilde{\mathbb{U}}^N}$ becomes scalar (direct) quantisation.

10.2 Prediction horizon $N = 2$:

When the prediction horizon is $N = 1$, $\Psi = \begin{bmatrix} h_0 & 0 \\ h_1 & h_0 \end{bmatrix}$ and thus $\Psi^{-1} = \frac{1}{h_0^2}\begin{bmatrix} h_0 & -h_1 \\ 0 & h_0 \end{bmatrix} = \begin{bmatrix} 1 & -h_1 \\ 0 & 1 \end{bmatrix}$ since $h_0 = 1$. We get,

$$\mathcal{H}_1'(z) = \begin{bmatrix} 1 & -h1 \end{bmatrix}\Gamma(zI - A)^{-1}B = (C - h_1CA)(zI - A)^{-1}B = H(z) - h_1CA(zI - A)^{-1}B - 1$$
$$\mathcal{H}_2'(z) = \begin{bmatrix} 0 & h_0 \end{bmatrix}\Gamma(zI - A)^{-1}B = CA(zI - A)^{-1}B$$

Then

$$\mathcal{H}(z) = \begin{bmatrix} 1 + \mathcal{H}_1'(z) & 0 \\ \mathcal{H}_2'(z) & 1 \end{bmatrix} = \begin{bmatrix} 1 + \mathcal{H}_1'(z) & 0 \\ \mathcal{H}_2'(z) & 1 \end{bmatrix}$$

and the moving horizon implementation is

$$
\begin{aligned}
\tilde{y}(k) &= [1,0] \Psi^{-1} q_{\tilde{\mathbb{U}}^N} \Big[\Psi \big\{ \mathcal{H}(z) \mathbf{w}(k) - \big[\mathcal{H}(z) - I \big] \mathbf{y}^*(k) \big\} \Big] \\
&= [1,0] \Psi^{-1} q_{\tilde{\mathbb{U}}^N} \Big[\Psi \mathcal{H}(z) \mathbf{w}(k) - \Psi \big[\mathcal{H}(z) - I \big] \mathbf{y}^*(k) \Big] \\
&= [1,0] \Psi^{-1} q_{\tilde{\mathbb{U}}^N} \Big[\Psi \mathcal{H}(z) \big(\mathbf{w}(k) - \mathbf{y}^*(k) \big) + \Psi \mathbf{y}^*(k) \Big] \\
&= [1,0] \Psi^{-1} q_{\mathbb{U}^N} \left[\begin{bmatrix} H(z) - h_1 CA(zI - A)^{-1}B & 0 \\ h_1 H(z) - (h_1^2 - 1)CA(zI - A)^{-1}B & 1 \end{bmatrix} \mathbf{w}(k) + \begin{bmatrix} 1 + h_1 CA(zI - A)^{-1}B - H(z) & 0 \\ (h_1^2 - 1)CA(zI - A)^{-1}B + h_1(1 - H(z)) & 0 \end{bmatrix} \mathbf{y}^*(k) \right]
\end{aligned}
$$

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11 Scribble

We want to minimise this error: $\bar{e} = Hy - Hw = Hv + H\epsilon - Hw$
$\bar{e} = Hy - Hw = Hv + H\epsilon - Hw$
What is $\bar{e}$ when

- the feedback filter is a (double) delay (standard delta-sigma)

- the feedback filter is optimal

- the feedback filter is implemented in the MPC formulation with horizon length 1, 2, 3, ...