Report: Synthesis of optimal filter for MHOQ

June 11, 2024

1 Quantisation

Let $w \in \mathbb{R}$ be the input, $\mathbf{Q}$ be the quantiser and $y \in \mathbb{U}$ the quantiser output. The quantiser output Let us define the quantisation error as

$$q = \mathbf{Q}(w) - w = y - w. \tag{1}$$

The quantisation requires the signal to be mapped to a finite signal where each value of the output y is restriced to belong to a finite set $\mathbb{U}$. The elements of the set $\mathbb{U}$ represent the quantiser levels and depends on the word-size of the quantiser.

2 Noise shaping quantiser

Noise-shaping quantisers can reduce the effective quantisation error by moving quantisation noise to higher frequencies through oversampling and feedback. The reconstruction filter is then used to attenuate the frequency-shaped quantisation noise. It operates by estimating the uniform quantisation error and employing a feedback filter to shape the noise power at the output of the DAC. A block diagram for a noise-shaping quantiser is shown in Fig. **??**. The feedback filter $F(z)$ is designed such that the transfer function $y = (1 - F(z))\epsilon$ is a high-pass filter.

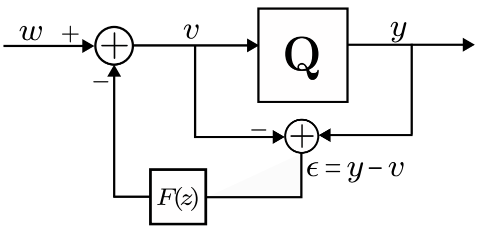

Figure 1: Noise shaping quantiser

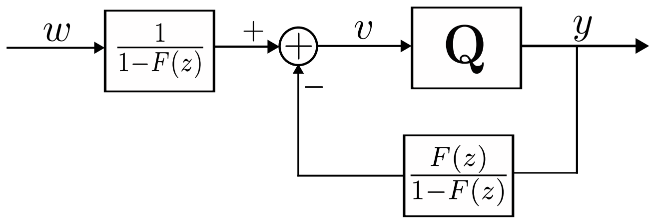

Figure 2: Noise shaping quantiser

In linear analysis, the output is given by

$$Y(z) = \mathbf{STF}.W(z) + \mathbf{NTF}.E(Z) \tag{2}$$

where the signal transfer function $\mathbf{STF} = 1$, noise transfer function $\mathbf{NTF} = (1 - F)$ and $F = z^{-1}$ for the first-order delta sigma modulator.

3 Moving horizon optimal quantiser (MHOQ)

The design criteria for the MHOQ is the minimization of the perceived errors defined as follows:

$$e(t) = H(z)(u(t) - y(t)) \tag{3}$$

where $H(z)$ is a stable time-invariant linear low-pass filter with the following state-space

$$H(z) = 1 + C(zI - A)^{-1}B \tag{4}$$

The error e then can be written as the output of the following state-space representation of H

$$\begin{aligned} x(t+1) &= Ax(t) + B(u(t) - y(t)) \\ e(t) &= Cx(t) + u(t) - y(t) \end{aligned} \tag{5}$$

where $x \in \mathbb{R}^n$ is the state vector. The error e corresponds to the difference between the filtered quantised signal and the filtered input signal.

For moving horison implementation, at time $t = k$ consider the quadratic cost is defined as

$$V_N = \sum_{t=k}^{k+N-1} e^2(t) \tag{6}$$

where $e(t)$ is the error defined in equation (3). Then, the optimisation problem can be defined as the problem of finding $y \in \mathbb{U}$ that minimises the cost function (6) while satisfying the state equations (5), that is,

$$y^*(t) = \arg\min_{y(t)} V_N \tag{7a}$$

subject to

$$x(t+1) = Ax(t) + B(y(t) - w(t)) \tag{7b}$$
$$e(t) = Cx(t) + y(t) - w(t) \tag{7c}$$
$$y(t) \in \mathbb{U}. \tag{7d}$$

4 MHOQ and Noise-shaping Quantiser

The moving horizon quantiser (MHOQ) with prediction horizon $N = 1$ is equivalent to the noise shaping quantiser. By choosing low-pass filter

$$H(z) = \frac{1}{1 - F(z)},$$

the MHOQ conicides with the noise shaping quantiser.

5 Frequency response and NTF

5.1 Perception filter

The frequency response and the STF and NTF of the perception filter used in [1] are shown in the figure below,

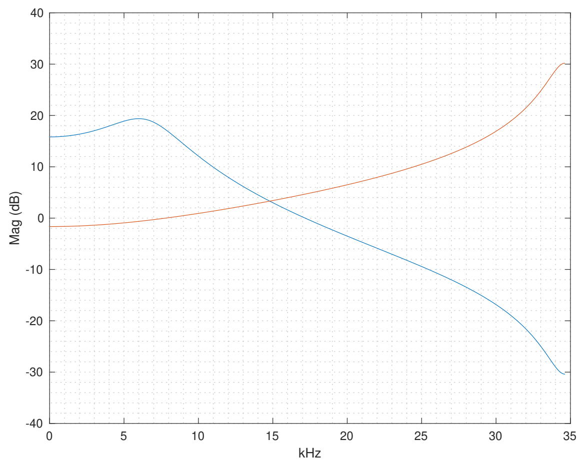

Figure 3: Perception filter Frequency response

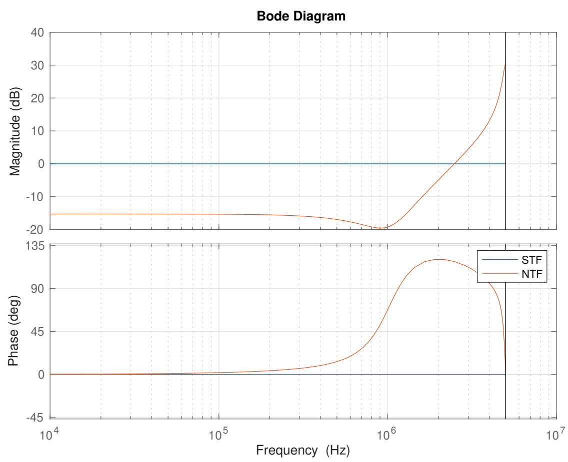

Figure 4: STF and NTF using perception filter

5.2 Frequency response and NTF: Butterworth Filter

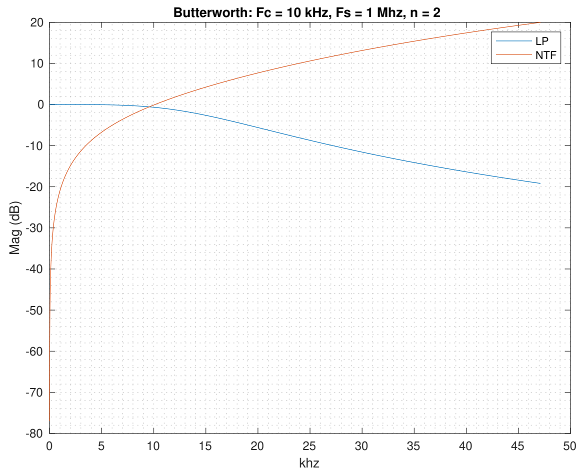

Figure 5: Frequency response of low-pass Butterworth and corresponding high-pass NTF

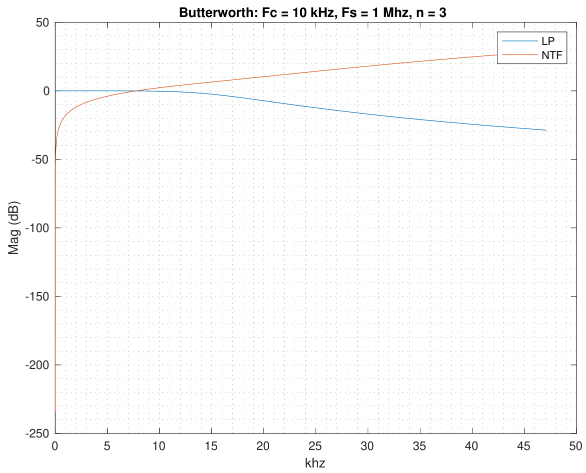

Figure 6: Frequency response of low-pass Butterworth and corresponding high-pass NTF

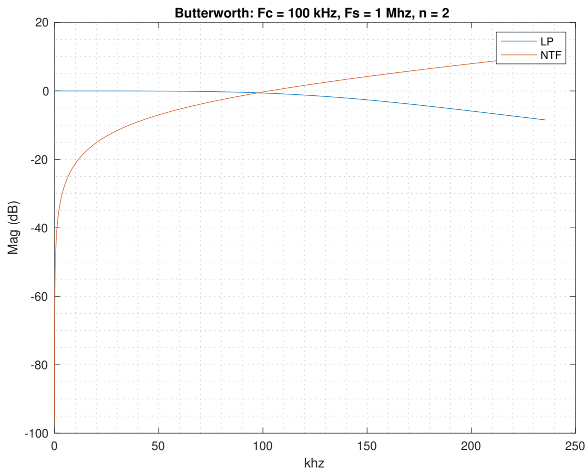

Figure 7: Frequency response of low-pass Butterworth and corresponding high-pass NTF

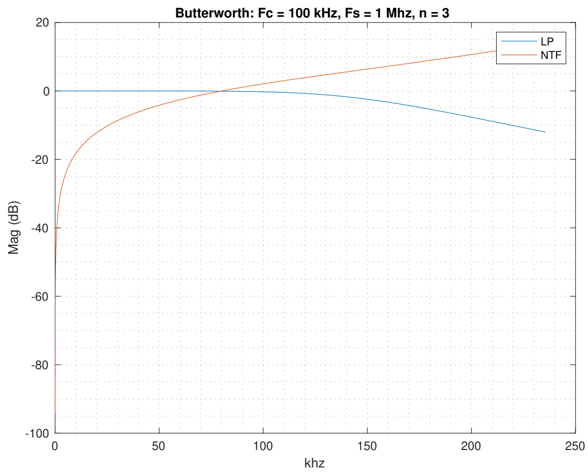

Figure 8: Frequency response of low-pass Butterworth and corresponding high-pass NTF

6 Noise Transfer Function(NTF)

The frequency response of the noise transfer functions due to butterworth filters at different cutoff frequencies are shown in the figure Fig. 9. In the figure, we can see that the net area under the curve remain the same. In Fig. reffig:LPFNTFvsfreq the frequency reponse of the low pass filter is plotted along with that of the noise transefer function. This observation shows that the better performance can be achieved by increasing the cutoff frequency during MHOQ while keeping the cutoff frequency of the reconstruction as same. The simulation results in the following table confirm this observation.

Table 1: ENOB at different cutoff frequencies. Reconstruction filter: Butterworth LPF with $n = 2$, $Fc = 100$kHz and $Fs = 1$Mhz.

Fc	100 kHz	200 kHz	300 kHz	400 kHz	500 kHz
ENOB	3.981	5.307	7.817	10.481	10.936

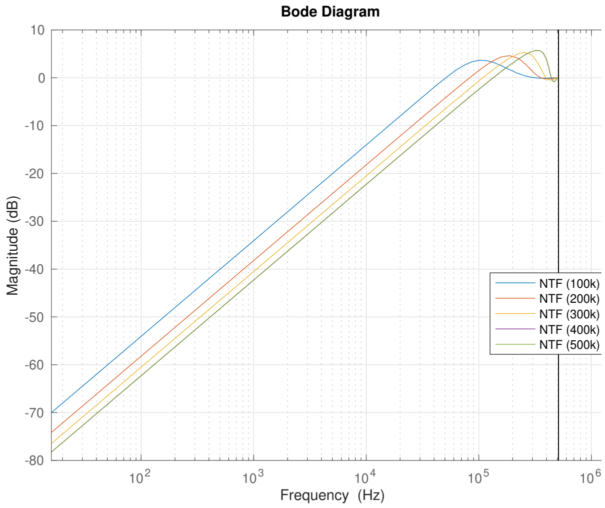

Figure 9: Frequency response of NTF for different cutoff frequency

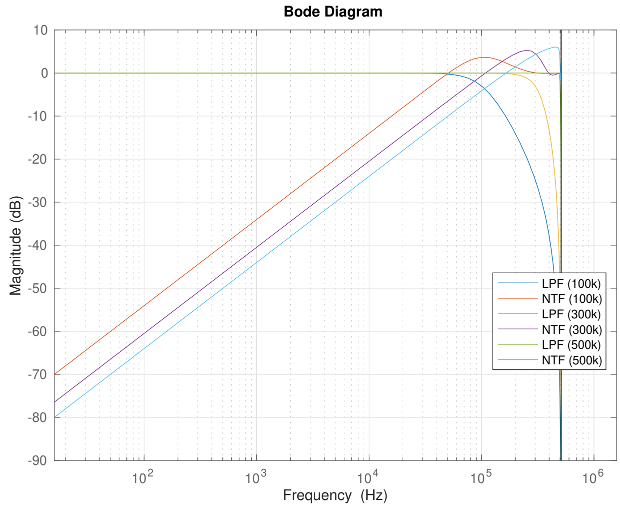

Figure 10: Frequency response of LPF and NTF for different cutoff frequency

References

[1] Graham C Goodwin, Daniel E Quevedo, and David McGrath. Moving-horizon optimal quantizer for audio signals. *Journal of the Audio Engineering Society*, 51(3):138–149, 2003.