Singular Perturbation Based Synchronization of Linear Heterogeneous Systems

1 Problem Formulation

1.1 System Model

Let us consider a network of N heterogeneous linear systems called *agents*. The state of each agent at any time t is characterized by the scalar $x_i(t) \in \mathbb{R}^{n_x}$ such that its dynamics is given by

$$\dot{x_i}(t) = A_i x_i(t) + u_i(t) \qquad \forall i \in \mathcal{N} = \{1, \cdots, N\}, \tag{1}$$

where $u_i \in \mathbb{R}^{n_x}$ is the control input and $A_i \in \mathbb{R}^{n_x \times n_x}$.

Remark 1 *We assume that the systems are heterogeneous i.e., the matrices $A_i, \forall i \in \mathcal{N}$ are different but of the same dimensions $\mathbb{R}^{n_x \times n_x}$.*

1.2 Network Model

Let $G = \left(V, E, A^d\right)$ be the weighted directed graph with the set of nodes $V = \{v_1, v_2, \cdots, v_N\}$, set of edges $E \subseteq V \times V$, and a weighted adjacency matrix $A^d_{ij} = [g_{ij}]_{i,j \in \mathcal{N}}$, where, $g_{ij} > 0$ for $i \neq j$ and there exists a directed link from j to i, while $g_{ij} = 0$ for $i = j$. We consider the network with a dynamic topology that is time-varying, also commonly known as *switching networks* [**?**]. These time-varying networks can be modelled using the dynamic graph $G^{\sigma_k(t)}$ parameterized with switching signal $\sigma_k(t) : \mathbb{R} \to \mathcal{I}, k \in \mathbb{N}$ that takes values in the index set $\mathcal{I}$ i.e., the network can switch among a finite number of graphs, $\Gamma_n = \{G_1, G_2, \cdots, G_n\}$. Also, we assume at any time instance, the network topology is directed and strongly connected. The network units are assumed to be connected via diffusive coupling, i.e., the i^{th} unit coupling, at any time instance, is given by

$$u_i(t) = -\gamma \sum_{j=1}^{N} g_{ij} \left(x_i - x_j\right), \tag{2}$$

where $\gamma \in \mathbb{R}$ corresponds to the coupling gain between the units.

Let us represent the state and control vectors in compact notation as follows:

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \in \mathbb{R}^{n_x N}, \qquad u = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} \in \mathbb{R}^{n_x N},$$

and also we represent the state matrix in block diagonal form as follows,

$$\mathcal{A} = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & A_N \end{bmatrix} \in \mathbb{R}^{n_x N \times n_x N}.$$

Hence the control input can be represented in the following compact notation,

$$u^{\sigma_k} = -\gamma (L^{\sigma_k} \otimes I_{n_x}) x, \tag{3}$$

where $\sigma_k \in \mathcal{I}$ is the topological index associated with the elements of Γ_n, $L^{\sigma_k} = L(G^{\sigma_k(t)})$ is the Laplacian associated with the graph $G^{\sigma_k(t)}$ and $\otimes$ is the Kronecker product. Then the resulting network dynamics is

$$\dot{x}(t) = (\mathcal{A} - \gamma (L^{\sigma_k} \otimes I_{n_x})) x(t). \tag{4}$$

2 Collective Behavior

In this section, we describe the collective behavior of the heterogeneous systems over the time-varying topology. We introduce the coordinate transformation to exhibit the collective behavior on the networked systems, namely the mean-field (x_s) and synchronization error (e). Also, we show that due to this transformation, jumps are introduced in the system dynamics leading us to the switching-impulsive system.

2.1 Graph Properties

We start by mentioning some graph properties that allow us the define the x_s and e. By definition of graph Laplacian, for every $\sigma_k \in \mathcal{I}$ the row sums of L^{σ_k} are zero, hence, L^{σ_k} has at least one zero eigenvalue, say $\lambda_1^{\sigma_k} = 0$ and assuming that the graph is strongly connected, the remaining eigenvalues of L^{σ_k} has non-negative real parts, i.e., $0 = \Re e\left\{\lambda_1^{\sigma_k}\right\} < \Re e\left\{\lambda_2^{\sigma_k}\right\} \leq \cdots \leq \Re e\left\{\lambda_N^{\sigma_k}\right\}$. Moreover, if the graph is directed and has a spanning tree, the eigenvalue $\lambda_1^{\sigma_k} = 0$ is simple and there exists left eigenvector ($v_{l1}^{\sigma_k}$) and right eigenvector ($v_{r1}^{\sigma_k}$) of L^{σ_k}, such that ${v_{l1}^{\sigma_k}}^T L^{\sigma_k} = 0$ and $L^{\sigma_k} v_{r1}^{\sigma_k} = 0$, for each $\sigma_k \in \mathcal{I}$. These left and right eigenvectors are given by

$$v_{l1}^{\sigma_k} = \begin{bmatrix} \vartheta_1^{\sigma_k} \\ \vartheta_2^{\sigma_k} \\ \vdots \\ \vartheta_N^{\sigma_k} \end{bmatrix} \in \mathbb{R}^N \qquad v_{r1}^{\sigma_k} = \mathbf{1}_N := \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^N. \tag{5}$$

Furthermore, there exists a Jordan decomposition of the Laplacian L^{σ_k}, of the form $L^{\sigma_k} = U^{\sigma_k} \Lambda^{\sigma_k} {U^{\sigma_k}}^{-1}$, where $U^{\sigma_k} \in \mathbb{R}^{N \times N}$ is nonsingular and $\Lambda^{\sigma_k} \in \mathbb{C}^{N \times N}$ is the block diagonal Jordan matrix of the following form

$$\Lambda = \begin{bmatrix} \Lambda_1 & & 0 \\ & \ddots & \\ 0 & & \Lambda_m \end{bmatrix}$$

where, for the networks with rooted-out branching, the algebraic multiplicity of the $\lambda_1 = 0$ is 1 and Λ_i are Jordan blocks of appropriate dimensions. Additionally, the matrix U^{σ_k} is composed of generalized right eigenvectors of the Laplacian matrix L^{σ_k} among which the first is $v_{r1}^{\sigma_k} = \mathbf{1}_N$. For further purpose, we decompose the matrix U^{σ_k} as

$$U^{\sigma_k} = [\mathbf{1}_N \quad V^{\sigma_k}] \tag{6}$$

where $V^{\sigma_k} \in \mathbb{R}^{N \times N-1}$. Similarly, the first row of the ${U^{\sigma_k}}^{-1}$ corresponds to the first left eigenvector of L^{σ_k}, $v_{l1}^{\sigma_k}$, which can be decomposed as follows,

$${U^{\sigma_k}}^{-1} = \begin{bmatrix} {v_{l1}^{\sigma_k}}^T \\ {V^{\sigma_k}}^{\dagger} \end{bmatrix} \tag{7}$$

and the following are satisfied,

$${v_{l1}^{\sigma_k}}^T V^{\sigma_k} = 0 \qquad {V^{\sigma_k}}^{\dagger} V^{\sigma_k} = I_{N-1}. \tag{8}$$

Let us define the *weighted average* [?] of the systems' states x_s as

$$x_s = \sum_{i=1}^{N} \vartheta_i^{\sigma_k} x_i, \qquad \sum_{i=1}^{N} \vartheta_i = 1. \tag{9}$$

2.2 Coordinate Transformation

Next, we introduce the transformation from [?] such that the existing model (4) can be expressed in terms of mean-field and synchronization error dynamics. The transformation matrix emerges from the Jordan decomposition of the Laplacian matrix and the transformation proceed as follows,

$$\bar{x} := {\mathcal{U}^{\sigma_k}}^{-1} x \tag{10}$$

where $\mathcal{U}^{\sigma_k} = (U^{\sigma_k} \otimes I_{n_x}) \in \mathbb{R}^{n_x N \times n_x N}$. Then, the partition is done using equation (7) to new coordinates, *i.e.*,

$$\bar{x} =: \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} := \begin{bmatrix} v_{l1}^{\sigma_k T} \otimes I_{n_x} \\ V^{\sigma_k \dagger} \otimes I_{n_x} \end{bmatrix} x.$$

By remarking that weighted average is defined using the left eigenvectors $v_{l1}^{\sigma_k}$, similarly to (9), the coordinate $\bar{x}_1$ represent the weighted average of the systems states as follows:

$$x_s(t) = (v_{l1}^{\sigma_k T} \otimes I_{n_x}) x(t), \qquad \forall t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}. \tag{11}$$

The synchronization error [**?**], [**?**], is the difference between the individual states and the state of the "average"(x_s) unit i.e.,

$$e(t) = x(t) - (\mathbf{1}_N \otimes I_{n_x}) x_s(t), \qquad \forall t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}. \tag{12}$$

The second coordinate $\bar{x}_2 = e_v \in \mathbb{R}^{n_x(N-1)}$ is the projection of the synchronization error (12) into the subspace orthogonal to that of vector $v_{l1}^{\sigma_k}$ and can be represented as follows,

$$\begin{aligned} e_v(t) &= (V^{\sigma_k \dagger} \otimes I_{n_x}) x(t), \qquad\qquad \forall t \in [t_k, t_{k+1}), \quad k \in \mathbb{N} \\ &= (V^{\sigma_k \dagger} \otimes I_{n_x})(e(t) - (\mathbf{1}_N \otimes I_{n_x}) x_s(t)) \\ &= (V^{\sigma_k \dagger} \otimes I_{n_x}) e(t), \end{aligned} \tag{13}$$

since $(V^{\sigma_k \dagger} \otimes I_{n_x})(\mathbf{1}_N \otimes I_{n_x}) x_s(t) = (V^{\sigma_k \dagger} \mathbf{1}_N \otimes I_{n_x}) x_s(t) = 0$. Moreover, $e(t) = (V^{\sigma_k} \otimes I_{n_x}) e_v(t)$.

Although the underlying topology in the original dynamics (4) is only switching, as a consequence of the coordinate transformation (10), impulses are introduced in the new variables x_s and e_v. The resulting impulse dynamics is as follows,

$$\begin{aligned} x_s(t_k) &= (v_{l1}^{\sigma_k T} \otimes I_{n_x}) x(t_k) \\ x_s(t_k^-) &= (v_{l1}^{\sigma_{k-1} T} \otimes I_{n_x}) x(t_k) \end{aligned} \tag{14}$$

and

$$\begin{aligned} e_v(t_k) &= (V^{\sigma_k \dagger} \otimes I_{n_x}) e(t_k) \\ e_v(t_k^-) &= (V^{\sigma_{k-1} \dagger} \otimes I_{n_x}) e(t_k). \end{aligned} \tag{15}$$

Hence, the impulse in the x_s and e_v, for each k is,

$$\begin{aligned} x_s(t_k) &= x_s(t_k^-) + \big((v_{l1}^{\sigma_k T} - v_{l1}^{\sigma_{k-1} T}) \otimes I_{n_x}\big) x(t_k) \\ &= x_s(t_k^-) + \big((v_{l1}^{\sigma_k T} - v_{l1}^{\sigma_{k-1} T}) \otimes I_{n_x}\big)\big((V^{\sigma_{k-1}} \otimes I_{n_x}) e_v(t_k^-) + (\mathbf{1}_N \otimes I_{n_x}) x_s(t_k^-)\big) \\ &= x_s(t_k^-) + \big((v_{l1}^{\sigma_k T} - v_{l1}^{\sigma_{k-1} T}) V^{\sigma_{k-1}} \otimes I_{n_x}\big) e_v(t_k^-) + \big((v_{l1}^{\sigma_k T} - v_{l1}^{\sigma_{k-1} T}) \mathbf{1}_N \otimes I_{n_x}\big) x_s(t_k^-) \\ &= x_s(t_k^-) + (v_{l1}^{\sigma_k T} V^{\sigma_{k-1}} \otimes I_{n_x}) e_v(t_k^-). \end{aligned} \tag{16}$$

$$\begin{aligned} e_v(t_k) &= e_v(t_k^-) + \left((V^{\sigma_k \dagger} - V^{\sigma_{k-1} \dagger}) \otimes I_{n_x}\right) e(t_k) \\ &= e_v(t_k^-) + \left((V^{\sigma_k \dagger} - V^{\sigma_{k-1} \dagger}) \otimes I_{n_x}\right)(V^{\sigma_{k-1}} \otimes I_{n_x}) e_v(t_k^-) \\ &= e_v(t_k^-) + \left((V^{\sigma_k \dagger} - V^{\sigma_{k-1} \dagger}) V^{\sigma_{k-1}} \otimes I_{n_x}\right) e_v(t) = \big(V^{\sigma_k \dagger} V^{\sigma_{k-1}} \otimes I_{n_x}\big) e_v(t_k^-). \end{aligned} \tag{17}$$

Remark 2 *As the topology changes, there is a change in the transformation matrix (eigenvectors of the Laplacian), this change introduces the jumps in the original system* (4) *and transforms the switching system to hybrid (switching-impulsive) system.*

3 Network Dynamics

Now that we have defined the mean-field states and the synchronization error, we proceed with the derivation of their dynamics over time-varying topology. Furthermore, we show that the obtained model is in singular perturbation form.

3.1 Mean-Field Dynamics

Since we have defined the x_s and e_v, we continue with the derivation of the mean-field dynamics x_s, defined in (11). In the new coordinate system, differentiating equation (11) both sides and using the network dynamics (4) , as well as substituting x from (12), we obtain $\forall t \in [t_k, t_{k+1})\,, k \in \mathbb{N}$,

$$\begin{aligned}
\dot{x}_s &= ({v_{l1}^{\sigma_k}}^T \otimes I_{n_x})\dot{x} = ({v_{l1}^{\sigma_k}}^T \otimes I_{n_x})\left(\mathcal{A} - \gamma(L^{\sigma_k} \otimes I_{n_x})\right)x(t) \\
\dot{x}_s &= ({v_{l1}^{\sigma_k}}^T \otimes I_{n_x})\left(\mathcal{A} - \gamma(L^{\sigma_k} \otimes I_{n_x})\right)\left(e + (\mathbf{1}_N \otimes I_{n_x})x_s\right) \\
\dot{x}_s &= \Bigg(({v_{l1}^{\sigma_k}}^T \otimes I_{n_x})\mathcal{A}(\mathbf{1}_N \otimes I_{n_x})\Bigg)x_s + ({v_{l1}^{\sigma_k}}^T \otimes I_{n_x})\mathcal{A}e
\end{aligned} \tag{18}$$

also, equivalently remarking that $e(t) = (V^{\sigma_k} \otimes I_{n_x})e_v(t)$, we get

$$\dot{x}_s = \Big(({v_{l1}^{\sigma_k}}^T \otimes I_{n_x})\mathcal{A}(\mathbf{1}_N \otimes I_{n_x})\Big)\,x_s + ({v_{l1}^{\sigma_k}}^T \otimes I_{n_x})\mathcal{A}(V^{\sigma_k} \otimes I_{n_x})e_v. \tag{19}$$

The equation (19) represents the *mean-field dynamics*. Also, it is important to note that the mean-field dynamics is independent of the coupling strength γ due to the properties of the Laplacian matrix. However, the dynamics of x_s is affected by γ through the error e_v.

3.2 Error Dynamics

Similarly to obtain the dynamics of the error we start by differentiating on both sides of equation (13) and substituting $\dot{x}_s$ from (18) $\forall t \in [t_k, t_{k+1})\,, k \in \mathbb{N}$ it follows:

$$\begin{aligned}
\dot{e}_v &= ({V^{\sigma_k}}^\dagger \otimes I_{n_x})\dot{e}(t) = ({V^{\sigma_k}}^\dagger \otimes I_{n_x})\big(\dot{x}(t) - (\mathbf{1}_N \otimes I_{n_x})\dot{x}_s(t)\big) \\
&= ({V^{\sigma_k}}^\dagger \otimes I_{n_x})\Bigg\{\big(\mathcal{A} - \gamma(L^{\sigma_k} \otimes I_{n_x})\big)x(t) - (\mathbf{1}_N \otimes I_{n_x})\big(({v_{l1}^{\sigma_k}}^T \otimes I_{n_x})\mathcal{A}(\mathbf{1}_N \otimes I_{n_x})\big)x_s \\
&\qquad + (\mathbf{1}_N \otimes I_{n_x})({v_{l1}^{\sigma_k}}^T \otimes I_{n_x})\mathcal{A}(V^{\sigma_k} \otimes I_{n_x})e_v\Bigg\} \\
&= ({V^{\sigma_k}}^\dagger \otimes I_{n_x})\Bigg\{\big(\mathcal{A} - \gamma(L^{\sigma_k} \otimes I_{n_x})\big)e(t) + \mathcal{A}(\mathbf{1}_N \otimes I_{n_x})x_s(t) - (\mathbf{1}_N {v_{l1}^{\sigma_k}}^T \otimes I_{n_x})\mathcal{A}(\mathbf{1}_N \otimes I_{n_x})x_s \\
&\qquad + (\mathbf{1}_N {v_{l1}^{\sigma_k}}^T \otimes I_{n_x})\mathcal{A}(V^{\sigma_k} \otimes I_{n_x})e_v\Bigg\} \\
&= ({V^{\sigma_k}}^\dagger \otimes I_{n_x})\Bigg\{\big(\mathcal{A} - \gamma(L^{\sigma_k} \otimes I_{n_x})\big)(V^{\sigma_k} \otimes I_{n_x})e_v(t) + \mathcal{A}(\mathbf{1}_N \otimes I_{n_x})x_s(t) - (\mathbf{1}_N {v_{l1}^{\sigma_k}}^T \otimes I_{n_x})\mathcal{A}(\mathbf{1}_N \otimes I_{n_x})x_s \\
&\qquad + (\mathbf{1}_N {v_{l1}^{\sigma_k}}^T \otimes I_{n_x})\mathcal{A}(V^{\sigma_k} \otimes I_{n_x})e_v\Bigg\}
\end{aligned} \tag{20}$$

From equations (6) and (7), we know ${V^{\sigma_k}}^\dagger \mathbf{1}_N = 0$, hence from equation (20), it follows,

$$\dot{e}_v = -\gamma({V^{\sigma_k}}^\dagger L^{\sigma_k} V^{\sigma_k} \otimes I_{n_x})e_v + ({V^{\sigma_k}}^\dagger \otimes I_{n_x})\mathcal{A}(\mathbf{1}_N \otimes I_{n_x})x_s + ({V^{\sigma_k}}^\dagger \otimes I_{n_x})\mathcal{A}(V^{\sigma_k} \otimes I_{n_x})e_v$$

and finally, the error dynamics is obtained as follows,

$$\dot{e}_v = -\gamma(\Lambda^{\sigma_k'} \otimes I_{n_x})e_v + ({V^{\sigma_k}}^\dagger \otimes I_{n_x})\mathcal{A}(\mathbf{1}_N \otimes I_{n_x})x_s + ({V^{\sigma_k}}^\dagger \otimes I_{n_x})\mathcal{A}(V^{\sigma_k} \otimes I_{n_x})e_v \tag{21}$$

here, $\Lambda^{\sigma_k'} = \mathrm{diag}\,(\lambda_2^{\sigma_k} \cdots \lambda_N^{\sigma_k})$.

3.3 Switching and Impulsive Dynamics of Networked System

In this section, we show that the resulting model (19) and (21), which represent the dynamics of the mean-field and error is in singular perturbation form with the variables x_s and e_v, where x_s is the slow variable and e_v is the fast one. The coupling strength, which is commonly assumed to be large in the study of the network distributed systems acts a perturbation parameter.

Consider the equation (19) and (21) as follows,

$$\begin{pmatrix} \dot{x_s}(t) \\ \dot{e_v}(t) \end{pmatrix} = \begin{pmatrix} A_0^{\sigma_k} & B_1^{\sigma_k} \\ B_2^{\sigma_k} & A_{22}^{\sigma_k} + B_3^{\sigma_k} \end{pmatrix} \begin{pmatrix} x_s(t) \\ e_v(t) \end{pmatrix}, \qquad \forall t \in [t_k, t_{k+1}), k \in \mathbb{N} \tag{22}$$

where for all $\sigma_k \in \mathcal{I}$ one has

$$\begin{aligned} A_0^{\sigma_k} &= ({v_{l1}^{\sigma_k}}^T \otimes I_{n_x})\mathcal{A}(\mathbf{1}_N \otimes I_{n_x}) = \sum_{i=1}^{N} v_{l1_i}^{\sigma_k} A_i, \qquad B_1^{\sigma_k} = ({v_{l1}^{\sigma_k}}^T \otimes I_{n_x})\mathcal{A}(V^{\sigma_k} \otimes I_{n_x}) \\ B_2^{\sigma_k} &= ({V^{\sigma_k}}^\dagger \otimes I_{n_x})\mathcal{A}(\mathbf{1}_N \otimes I_{n_x}), \qquad B_3^{\sigma_k} = ({V^{\sigma_k}}^\dagger \otimes I_{n_x})\mathcal{A}(V^{\sigma_k} \otimes I_{n_x}) \\ A_{22}^{\sigma_k} &= -\gamma(\Lambda^{\sigma_k'} \otimes I_{n_x}). \end{aligned} \tag{23}$$

We assume that the coupling parameter is γ *large* and dividing both sides of equation (21) by γ, we obtain the dynamics of the system in standard singular perturbation form as follows:

$$\begin{pmatrix} \dot{x_s}(t) \\ \epsilon\dot{e_v}(t) \end{pmatrix} = \begin{pmatrix} A_0^{\sigma_k} & B_1^{\sigma_k} \\ \epsilon B_2^{\sigma_k} & A_{22}^{\sigma_k} + \epsilon B_3^{\sigma_k} \end{pmatrix} \begin{pmatrix} x_s(t) \\ e_v(t) \end{pmatrix}, \qquad \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}. \tag{24}$$

We set $\epsilon = \frac{1}{\gamma} \ll 1$ as the perturbation parameter and hence it can be seen that the matrix $A_0^{\sigma_k}$, $B_1^{\sigma_k}$ $B_2^{\sigma_k}$ and $B_3^{\sigma_k}$ are of the order $\mathcal{O}(1)$, while $A_{22}^{\sigma_k}$ is of the order $\mathcal{O}(\frac{1}{\epsilon})$.

The impulsive dynamics is as follows,

$$\begin{pmatrix} x_s(t_k) \\ e_v(t_k) \end{pmatrix} = \begin{pmatrix} J_{11}^{\sigma_{k-1}\overset{v_k}{\to}\sigma_k} & J_{12}^{\sigma_{k-1}\overset{v_k}{\to}\sigma_k} \\ \mathbf{0}_{n_x(N-1),n_xN} & J_{22}^{\sigma_{k-1}\overset{v_k}{\to}\sigma_k} \end{pmatrix} \begin{pmatrix} x_s(t_k^-) \\ e_v(t_k^-) \end{pmatrix}. \tag{25}$$

where $J_{11}^{\sigma_{k-1}\overset{v_k}{\to}\sigma_k} = I_{n_xN}$, $J_{12}^{\sigma_{k-1}\overset{v_k}{\to}\sigma_k} = ({v_{l1}^{\sigma_k}}^T V^{\sigma_{k-1}} \otimes I_{n_x})$ and $J_{22}^{\sigma_{k-1}\overset{v_k}{\to}\sigma_k} = \left({V^{\sigma_k}}^\dagger V^{\sigma_{k-1}} \otimes I_{n_x}\right)$. Hence, the system has standard singular perturbations form with reduced (slow) variable x_s and boundary layer (fast) variable e_v. This time scale separation on the network of linear heterogeneous systems is due to the large coupling parameter γ and the coordinate transformation (10) which are the intrinsic properties of the network.

Remark 3 *The matrix $A_{22}^{\sigma_k}$ are Hurwitz for all $\sigma_k \in \mathcal{I}$ since the diagonal matrix $\Lambda^{\sigma_k'} = diag\ (\lambda_2^{\sigma_k}, \cdots \lambda_N^{\sigma_k})$ has all positive eigenvalues i.e., $\lambda_i^{\sigma_k} > 0, \forall i = 2, \cdots, N$ and $\forall \sigma_k \in \mathcal{I}$.*

3.4 From Linear switching to Singularly-perturbed hybrid systems

The linear switching multi-agent system, with the individual dynamics as follows,

$$\dot{x_i}(t) = A_i x_i(t) + u_i(t) \qquad \forall i \in \mathcal{N} = \{1, \cdots, N\},$$

has the following network dynamics

$$\dot{x}(t) = (\mathcal{A} - \gamma(L^{\sigma_k} \otimes I_{n_x}))x(t). \tag{26}$$

Introducing the transformation from [**?**], we define the mean-field and synchronization error, which are the intrinsic properties of the network distributed systems. The mean-field x_s is the weighted average of the systems state and the error e_v is the projection of the synchronization errors e, defined as the differences between the individual states and the state of the averaged units ($e(t) = x(t) - (\mathbf{1}_N \otimes I_{n_x})x_s(t)$), onto the subspace orthogonal to that of vector $v_{l1}^{\sigma_k}, k \in \mathbb{N}$, represented respectively $\forall t \in [t_k, t_{k+1}), k \in \mathbb{N}$ as

$$\begin{aligned} x_s(t) &= ({v_{l1}^{\sigma_k}}^T \otimes I_{n_x})x(t), \\ e(t) &= x(t) - (\mathbf{1}_N \otimes I_{n_x})x_s(t). \end{aligned}$$

The linear multi-agent system over a time-varying topology has been transformed into the hybrid system which is in the standard singular-perturbation. The perturbation parameter $\epsilon = \frac{1}{\gamma}$ introduces the time scale separation in our transformed model, where γ is the coupling strength that is commonly assumed to be large. In singular perturbation form, the mean-field dynamics restricted to the synchronization manifold is the slow dynamics (also called emergent dynamics) and the dynamics of the error behaves as the fast dynamics. The resulting singularly-perturbed hybrid system, as a result of the transformation of the linear systems, is as follows,

$$\begin{pmatrix} \dot{x_s}(t) \\ \epsilon \dot{e_v}(t) \end{pmatrix} = \begin{pmatrix} A_0^{\sigma_k} & B_1^{\sigma_k} \\ \epsilon B_2^{\sigma_k} & A_{22}^{\sigma_k} + \epsilon B_3^{\sigma_k} \end{pmatrix} \begin{pmatrix} x_s(t) \\ e_v(t) \end{pmatrix}, \qquad \forall t \in [t_k, t_{k+1}), k \in \mathbb{N} \tag{27}$$

and

$$\begin{pmatrix} x_s(t_k) \\ e_v(t_k) \end{pmatrix} = \begin{pmatrix} I_{n_x N} & (v_{l1}^{\sigma_k T} V^{\sigma_{k-1}} \otimes I_{n_x}) \\ \mathbf{0}_{n_x(N-1), n_x N} & (V^{\sigma_k \dagger} V^{\sigma_{k-1}} \otimes I_{n_x}) \end{pmatrix} \begin{pmatrix} x_s(t_k^-) \\ e_v(t_k^-) \end{pmatrix}. \tag{28}$$

4 Approximate Models

As we know that for the continuous-time systems [?], the original system in the singular-perturbation form can be approximated by the reduced ordered systems that evolve in different time scale. If the fast subsystem is asymptotically stable then the original system is approximated by the reduced ordered systems for the finite time $t \geq t_0$. Moreover, if the slow dynamics is asymptotically stable then the reduced ordered systems approximate the original system for time $t \in [t_0, \infty)$.

In this paper, we wish to develop similar approximation results for the singularly perturbed hybrid systems. While the fast system is stable in each mode, we consider the slow subsystem (emergent dynamics) to be bounded. Hence we establish the condition for the boundedness of the slow system that depends on the ratio of the time the system remains in the stable mode to that of the unstable one. Finally, we prove that the similar approximation results, like the one provided in the [?] for continuous-time system exist for the hybrid systems.

Assumption 1 *For the family of the state matrices $\left\{A_0^i\right\}_{i\in\mathcal{I}}$, both stable and unstable mode exists.*

We introduce the following definition of the *measure* and upper bound on the exponential of the matrices from [?].

Definition 1 *The measure of a square matrix M is defined as $\nu(M) = \frac{1}{2}\lambda_{max}(M + M^T)$, where λ_{max} is the largest eigenvalue of the symmetric matrix $M + M^T$.*

Lemma 4.1 *For any square matrix $\mathbb{A}$, $\|e^{\mathbb{A}t}\| \leq e^{\nu(\mathbb{A})t}$ holds.*

Then from the definition of the measure, it follows

$$\nu(A_0^{\sigma_k}) = \frac{1}{2}\lambda_{max}^{\sigma_k}(A_0^{\sigma_k} + A_0^{\sigma_k T}), \qquad \forall \sigma_k \in \mathcal{I}. \tag{29}$$

Based on Assumption 1 and the definition of the measure, the measure $\nu(A_0^{\sigma_k})$ may be positive or negative. Let $\mathcal{S}$ and $\mathcal{U}$ be the set of stable and unstable modes, respectively. In other words, $\sigma_k \in \mathcal{S}$ iff $\nu(A_0^{\sigma_k}) < 0$, $\sigma_k \in \mathcal{U}$ iff $\nu(A_0^{\sigma_k}) \geq 0$ and $\mathcal{S} \bigcup \mathcal{U} = \mathcal{I}$. It is important to note that the measure is defined for the symmetric matrix $(M + M^T)$, hence the eigenvalues are always real.

4.1 Emergent Dynamics (Slow-Subsystem)

To obtain the slow subsystem of the original system (27)-(28), we set $\epsilon = 0$. Then, we have the slow subsystem as follows,

$$\dot{x_e}(t) = A_0^{\sigma_k} x_e(t) \qquad \forall t \in [t_k, t_{k+1}), k \in \mathbb{N} \tag{30}$$

$$e_s(t) = 0. \tag{31}$$

with the impulsive dynamics as follows:

$$x_e(t_k) = x_e(t_k^-), \qquad \forall k \geq 1. \tag{32}$$

The dynamics (30)-(32) is also referred to as *emergent dynamics*, which is the dynamics of the mean-field unit restricted to the synchronization manifold, where $e_v = 0$. Due to the existence of the both stable and unstable modes, as remarked in assumption 1, we wish to bound the trajectories of the slow subsystems. Hence, in the following lemma, we provide the necessary conditions to ensure that the trajectories of the slow subsystems are bounded. Eventually, we will prove that the mean-field dynamics can be approximated by the emergent dynamics with the order of approximation O(ϵ).

Let t_s and t_u be the total amount of time the system remains in the stable and unstable modes, respectively. Also, we define the following,

$$\begin{aligned} \lambda_s &= \max_{\sigma_k \in \mathcal{S}} \nu(A_0^{\sigma_k}) \\ \lambda_u &= \max_{\sigma_k \in \mathcal{U}} \nu(A_0^{\sigma_k}). \end{aligned} \tag{33}$$

Lemma 4.2 *The trajectories of the slow subsystems* (30)-(32) *are bounded if*

$$\frac{t_s}{t_u} \geq \frac{\lambda_u}{\lambda_s} \tag{34}$$

is satisfied for $t_s > 0$ and $t_u > 0$.

Proof: Let $\{t_k\}_{k\in\mathbb{N}}$ be the switching time for the switching event σ_k. Then for any t such that $t_k < t < t_{k+1}$, from equations (30)-(32) we have,

$$\begin{aligned}
x_e(t) &= e^{A_0^{\sigma_k}(t-t_k)}x_e(t_k) \qquad \forall t \in [t_k, t_{k+1}), k \in \mathbb{N} \\
x_e(t_k) &= x_e(t_k^-), \qquad \forall k \in \mathbb{N} \\
x_e(t_k^-) &= e^{A_0^{\sigma_{k-1}}(t_k-t_{k-1})}x_e(t_{k-1}), \qquad \forall t \in [t_{k-1}, t_k), k \in \mathbb{N}.
\end{aligned}$$

and hence,

$$x_e(t) = e^{A_0^{\sigma_k}(t-t_k)}e^{A_0^{\sigma_{k-1}}(t_k-t_{k-1})}x_e(t_{k-1})$$

Now, starting from t_0, for every k, we have,

$$x_e(t) = e^{A_0^{\sigma_k}(t-t_k)}e^{A_0^{\sigma_{k-1}}(t_k-t_{k-1})}\cdots e^{A_0^{\sigma_0}(t_1-t_0)}x_e(t_0). \tag{35}$$

Then following from Lemma (4.1) and equation (86), we get,

$$\begin{aligned}
\|x_e(t)\| &= \|e^{A_0^{\sigma_k}(t-t_k)}e^{A_0^{\sigma_{k-1}}(t_k-t_{k-1})}\cdots e^{A_0^{\sigma_0}(t_1-t_0)}x_e(t_0)\| \\
&\leq \|e^{A_0^{\sigma_k}(t-t_k)}\|\|e^{A_0^{\sigma_{k-1}}(t_k-t_{k-1})}\|\cdots\|e^{A_0^{\sigma_0}(t_1-t_0)}\|\|x_e(t_0)\| \\
&\leq e^{\nu(A_0^{\sigma_k})(t-t_k)+\nu(A_0^{\sigma_{k-1}})(t_k-t_{k-1})+\cdots+\nu(A_0^{\sigma_0})(t_1-t_0)}\|x_e(t_0)\| \\
&\leq e^{\lambda_u t_u+\lambda_s t_s}\|x_e(t_0)\|.
\end{aligned} \tag{36}$$

We can conclude from equation (88), the trajectories of the emergent dynamics will be bounded, while k is increasing, iff

$$\lambda_u t_u + \lambda_s t_s \leq 0 \tag{37}$$

and remarking that $\lambda_s < 0$ we get the required condition (34). □

4.2 Fast Subsystem

The representation of the fast subsystems assumes that the emergent dynamics is constant during the fast transients. As a result, we obtain the following fast subsystem as follows,

$$\frac{de_f(\tau)}{d\tau} = A_{22}^{\sigma_k}e_f(\tau), \qquad \forall \tau \in \left[\frac{t_k}{\epsilon}, \frac{t_{k+1}}{\epsilon}\right), \forall k \in \mathbb{N} \tag{38}$$

$$e_f\left(\frac{t_k}{\epsilon}\right) = J_{22}^{\sigma_{k-1}\xrightarrow{v_k}\sigma_k}e_f\left(\frac{t_k^-}{\epsilon}\right). \tag{39}$$

where, $J_{22}^{\sigma_{k-1}\xrightarrow{v_k}\sigma_k} = \left(V^{\sigma_k\dagger}V^{\sigma_{k-1}} \otimes I_{n_x}\right)$ and $\tau = \frac{t-t_k}{\epsilon}$. We know from Remark 3, for the continuous dynamics between two events, the matrix $A_{22}^{\sigma_k}, \forall k \in \mathbb{N}$ is Hurwitz. However, only this stability property does not ensure the stability of the hybrid systems due to the presence of the impulse effects. Hence to ensure the stability of the fast dynamics with jumps, we establish the dwell-time conditions.

Lemma 4.3 *If all the fast modes are stable i.e.,$\forall k \in \mathbb{N}, A_{22}^{\sigma_k}$ are Hurwitz, then there exists $\tau^* > 0$ such that for all sequences $(t_k)_{k\geq 0}$ of event times satisfying the dwell time condition $\tau_k \geq \tau^*$, $\forall k \in \mathbb{N}$ with*

$$\tau^* > \frac{\epsilon k \ln(\gamma_{22})}{(k+1)\lambda_f} \tag{40}$$

the overall fast dynamics (90)-(91) *is asymptotically stable.*

Proof: From equations (90) and (91) we have the following,

$$e_f(\tau) = e^{A_{22}^{\sigma_k}(\frac{t-t_k}{\epsilon})} J_{22}^{\sigma_{k-1}\overset{v_k}{\to}\sigma_k} e^{A_{22}^{\sigma_{k-1}}(\frac{t_k-t_{k-1}}{\epsilon})} e_f(\frac{t_{k-1}}{\epsilon}), \qquad \forall \tau \in \left[\frac{t_k}{\epsilon}, \frac{t_{k+1}}{\epsilon}\right), k \in \mathbb{N} \tag{41}$$

and eventually, we have,

$$e_f(\tau) = e^{A_0^{\sigma_k}(\frac{t-t_k}{\epsilon})} J_{22}^{\sigma_{k-1}\overset{v_k}{\to}\sigma_k} e^{A_0^{\sigma_{k-1}}(\frac{t_k-t_{k-1}}{\epsilon})} \dots J_{22}^{\sigma_0\overset{v_1}{\to}\sigma_1} e^{A_0^{\sigma_0}(\frac{t_1-t_0}{\epsilon})} e_f(0). \tag{42}$$

Since we know that all the fast modes are stable i.e., the measure $\nu(A_{22}^{\sigma_k}) < 0, \forall k \in \mathbb{N}$, let us define the following,

$$\max_{\sigma_k \in \mathcal{I}} \nu(A_{22}^{\sigma_k}) = -\lambda_f \tag{43}$$

and

$$\gamma_{22} = \max_{\sigma_k, \sigma_{k-1} \in \mathcal{I}, v_k \in \mathcal{J}} \|J_{22}^{\sigma_{k-1}\overset{v_k}{\to}\sigma_k}\|. \tag{44}$$

Let $\tau^* > 0$ such that $(t_k - t_{k-1}) \geq \tau^*, \forall k \in \mathbb{N}$ then we have the following

$$\begin{aligned}
\|e_f(\tau)\| &= \|e^{A_0^{\sigma_k}(\frac{t-t_k}{\epsilon})} J_{22}^{\sigma_{k-1}\overset{v_k}{\to}\sigma_k} e^{A_0^{\sigma_{k-1}}(\frac{t_k-t_{k-1}}{\epsilon})} \dots J_{22}^{\sigma_0\overset{v_1}{\to}\sigma_1} e^{A_0^{\sigma_0}(\frac{t_1-t_0}{\epsilon})} e_f(0)\| \\
&\leq \|e^{A_0^{\sigma_k}(\frac{t-t_k}{\epsilon})}\| \|J_{22}^{\sigma_{k-1}\overset{v_k}{\to}\sigma_k}\| \|e^{A_0^{\sigma_{k-1}}(\frac{t_k-t_{k-1}}{\epsilon})}\| \cdots \|J_{22}^{\sigma_0\overset{v_1}{\to}\sigma_1}\| \|e^{A_0^{\sigma_0}(\frac{t_1-t_0}{\epsilon})}\| \|e_f(0)\| \\
&\leq \gamma_{22}^k e^{\nu(A_{22}^{\sigma_k})\frac{\tau^*}{\epsilon} + \nu(A_{22}^{\sigma_{k-1}})\frac{\tau^*}{\epsilon} + \cdots + \nu(A_{22}^{\sigma_0})\frac{\tau^*}{\epsilon}} \|e_f(0)\| \\
&\leq \gamma_{22}^k e^{-\lambda_f (k+1)\frac{\tau^*}{\epsilon}} \|e_f(0)\|.
\end{aligned} \tag{45}$$

Then from equation (97) we can conclude, for sufficiently large k, $e_f(\tau)$ tends to 0 when

$$\gamma_{22}^k e^{-\lambda_f (k+1)\frac{\tau^*}{\epsilon}} < 1 \tag{46}$$

is satisfied and the dwell-time condition (92) follows from (98). Hence we can conclude that the fast subsystem is asymptotically stable when every mode is asymptotically stable and provided the dwell-time condition (92) is satisfied. □

4.3 Validation of the Approximate Models

Now that we have the necessary conditions for the boundedness of the emergent dynamics and the asymptotic stability of the fast dynamics, which are pivotal in proving that the original dynamics is approximated by the slow and fast subsystems. We develop our result for the hybrid systems, based on the result from [?], for the continuous dynamics between two events, the original dynamics is approximated by the reduced ordered dynamics. We propose the following proposition that provide the approximation of synchronization error dynamics e_v by the fast-variable e_f.

Proposition 1 *The error dynamics of the original system* (27)-(28) *is approximated for time* $t \geq t_0$ *by* $e_v(t) = e_f(\tau) + O(\epsilon)$.

Proof: The error dynamics of the singularly perturbed switching-impulsive system (27)-(28) is

$$\dot{e}_v(t) = B_2^{\sigma_k} x_s + (B_3^{\sigma_k} + \frac{A_{22}^{\sigma_k}}{\epsilon}) e_v(t), \qquad \forall t \in [t_k, t_{k+1}), k \in \mathbb{N} \tag{47}$$

$$e_v(t_k) = J_{22}^{\sigma_{k-1}\overset{v_k}{\to}\sigma_k} e_v(t_k^-). \tag{48}$$

Solving this differential equation (112) and substituting the equation (113), while ignoring the higher order epsilon term ($n \geq 2$) we have,

$$
\begin{aligned}
e_v(t) &= e^{(A_{22}^{\sigma_k}+\epsilon B_3^{\sigma_k})(\frac{t-t_k}{\epsilon})}e_v(t_k) + O(\epsilon) = (e^{A_{22}^{\sigma_k}(\frac{t-t_k}{\epsilon})} + O(\epsilon))e_v(t_k) + O(\epsilon) \\
&= e^{A_{22}^{\sigma_k}(\frac{t-t_k}{\epsilon})}e_v(t_k) + O(\epsilon) \\
e_v(t_k) &= J_{22}^{\sigma_{k-1}\overset{v_k}{\rightarrow}\sigma_k}e_v(t_k^-) \\
e_v(t_k^-) &= e^{(A_{22}^{\sigma_{k-1}}+\epsilon B_3^{\sigma_{k-1}})(\frac{t_k-t_{k-1}}{\epsilon})}e_v(t_{k-1}) + O(\epsilon) = (e^{A_{22}^{\sigma_{k-1}}(\frac{t_k-t_{k-1}}{\epsilon})} + O(\epsilon))e_v(t_{k-1}) + O(\epsilon) \\
&= e^{A_{22}^{\sigma_{k-1}}(\frac{t_k-t_{k-1}}{\epsilon})}e_v(t_{k-1}) + O(\epsilon)
\end{aligned}
$$

which implies,

$$
\begin{aligned}
e_v(t) &= (e^{A_{22}^{\sigma_k}(\frac{t-t_k}{\epsilon})}J_{22}^{\sigma_{k-1}\overset{v_k}{\rightarrow}\sigma_k})(e^{A_{22}^{\sigma_{k-1}}(\frac{t_k-t_{k-1}}{\epsilon})}e_v(t_{k-1}) + O(\epsilon)) + O(\epsilon) \\
e_v(t) &= e^{A_{22}^{\sigma_k}(\frac{t-t_k}{\epsilon})}J_{22}^{\sigma_{k-1}\overset{v_k}{\rightarrow}\sigma_k}e^{A_{22}^{\sigma_{k-1}}(\frac{t_k-t_{k-1}}{\epsilon})}e_v(t_{k-1}) + O(\epsilon). \qquad (49)
\end{aligned}
$$

From [?], we know that for the continuous dynamics between the two events, i.e., $\forall t \in [t_k, t_{k+1})$, the error e_v can be approximated as $e_v(t) = e_f + O(\epsilon)$, hence from (114) it follows,

$$
\begin{aligned}
e_v(t) &= e^{A_{22}^{\sigma_k}(\frac{t-t_k}{\epsilon})}J_{22}^{\sigma_{k-1}\overset{v_k}{\rightarrow}\sigma_k}e^{A_{22}^{\sigma_{k-1}}(\frac{t_k-t_{k-1}}{\epsilon})}(e_f(\tau) + O(\epsilon)) + O(\epsilon) \\
e_v(t) &= e^{A_{22}^{\sigma_k}(\frac{t-t_k}{\epsilon})}J_{22}^{\sigma_{k-1}\overset{v_k}{\rightarrow}\sigma_k}e^{A_{22}^{\sigma_{k-1}}(\frac{t_k-t_{k-1}}{\epsilon})}e_f(\tau) + O(\epsilon). \qquad (50)
\end{aligned}
$$

It follows,

$$
\begin{aligned}
e_v(t) &= e^{A_{22}^{\sigma_k}(\frac{t-t_k}{\epsilon})}J_{22}^{\sigma_{k-1}\overset{v_k}{\rightarrow}\sigma_k}e^{A_{22}^{\sigma_{k-1}}(\frac{t_k-t_{k-1}}{\epsilon})}\dots J_{22}^{\sigma_0\overset{v_1}{\rightarrow}\sigma_1}e^{A_{22}^{\sigma_0}(\frac{t_1-t_0}{\epsilon})}e_f(t_0) + O(\epsilon) \\
\|e_v(t)\| &\leq e^{\nu(A_{22}^{\sigma_k})(\frac{t-t_k}{\epsilon})}\|J_{22}^{\sigma_{k-1}\overset{v_k}{\rightarrow}\sigma_k}\|e^{\nu(A_{22}^{\sigma_{k-1}})(\frac{t_k-t_{k-1}}{\epsilon})}\dots\|J_{22}^{\sigma_0\overset{v_1}{\rightarrow}\sigma_1}\|e^{\nu(A_{22}^{\sigma_0})(\frac{t_1-t_0}{\epsilon})}\|e_f(t_0)\| + O(\epsilon). \qquad (51)
\end{aligned}
$$

Now from equation (95) and (96) we obtain,

$$
\|e_v(t)\| \leq \gamma_2^k e^{-\lambda_f(\frac{t-t_0}{\epsilon})}\|e_f(t_0)\| + O(\epsilon) \qquad (52)
$$

and from (32) and Lemma (4.2) the following can be concluded,

$$
\|e_v(t)\| \leq \|e_f(\tau)\| + O(\epsilon), \qquad (53)
$$

provided that the dwell-condition (92) is satisfied. □

It can be seen that the the synchronization error is approximated by the error in fast dynamics only. This is significant only during the initial short period which is of the order $O(\epsilon \ln \epsilon)$ [?].

Proposition 2 *Under assumption 1, if the trajectories of the emergent dynamics are bounded i.e., when the inequality* (34) *is satisfied, then trajectories of the mean-field dynamics are approximated for time* $t \geq t_0$ *by the trajectories of the emergent dynamics with the order of approximation* $O(\epsilon)$*, i.e.,*

$$
x_s = x_e + O(\epsilon). \qquad (54)
$$

Proof: The mean-field dynamics of the singularly perturbed switching-impulsive system (28)-(29) is

$$
\dot{x}_s(t) = A_0^{\sigma_k}x_s(t) + B_1^{\sigma_k}e_v(t) \qquad \forall t \in [t_k, t_{k+1}), \forall k \in \mathbb{N} \qquad (55)
$$

$$
x_s(t_k) = x_s(t_k^-) + J_{12}^{\sigma_{k-1}\overset{v_k}{\rightarrow}\sigma_k}e_v(t_k^-) = x_s(t_k^-) + (v_{l1}^{\sigma_k\,T}V^{\sigma_{k-1}} \otimes I_{n_x})e_v(t_k^-) \qquad (56)
$$

$$
x_s(t_k^-) = e^{A_0^{\sigma_{k-1}}(t_k-t_{k-1})}x_s(t_{k-1}). \qquad (57)
$$

Now from equations (120), (121) and (122) we have,

$$
\begin{aligned}
x_s(t) &= e^{A_0^{\sigma_k}(t-t_k)} x_s(t_k) = e^{A_0^{\sigma_k}(t-t_k)} \big(x_s(t_k^-) + J_{12}^{\sigma_{k-1}\overset{v_k}{\to}\sigma_k} e_v(t_k^-)\big) \\
&= e^{A_0^{\sigma_k}(t-t_k)} x_s(t_k^-) + e^{A_0^{\sigma_k}(t-t_k)} J_{12}^{\sigma_{k-1}\overset{v_k}{\to}\sigma_k} e_v(t_k^-) \\
&= e^{A_0^{\sigma_k}(t-t_k)} e^{A_0^{\sigma_{k-1}}(t_k-t_{k-1})} x_s(t_{k-1}) + e^{A_0^{\sigma_k}(t-t_k)} J_{12}^{\sigma_{k-1}\overset{v_k}{\to}\sigma_k} e_v(t_k^-)
\end{aligned} \tag{58}
$$

The continuous dynamics for $t \in [t_k, t_{k+1})$, $\forall k \in \mathbb{N}$, can be approximated as follows [**?**],

$$
x_s(t) = x_e(t) + O(\epsilon) \tag{59}
$$

$$
e_v(t) = e_f(\tau) + O(\epsilon) \tag{60}
$$

Hence from equations (123) and (124) we have,

$$
\begin{aligned}
x_s(t_k) &= e^{A_0^{\sigma_k}(t-t_k)+A_0^{\sigma_{k-1}}(t_k-t_{k-1})} (x_e(t_{k-1}) + O(\epsilon)) + e^{A_0^{\sigma_k}(t-t_k)} J_{12}^{\sigma_{k-1}\overset{v_k}{\to}\sigma_k} e_v(t_k^-) \\
&= e^{A_0^{\sigma_k}(t-t_k)+A_0^{\sigma_{k-1}}(t_k-t_{k-1})} x_e(t_{k-1}) + e^{A_0^{\sigma_k}(t-t_k)} J_{12}^{\sigma_{k-1}\overset{v_k}{\to}\sigma_k} e_v(t_k^-) + O(\epsilon)
\end{aligned} \tag{61}
$$

Following equation (126), we obtain the following:

$$
\begin{aligned}
x_s(t) &= e^{A_0^{\sigma_k}(t-t_k)} e^{A_0^{\sigma_{k-1}}(t_k-t_{k-1})} \cdots e^{A_0^{\sigma_0}(t_1-t_0)} x_e(t_0) + e^{A_0^{\sigma_k}(t-t_k)} J_{12}^{\sigma_{k-1}\overset{v_k}{\to}\sigma_k} e_v(t_k^-) \\
&\quad + e^{A_0^{\sigma_k}(t-t_k)+A_0^{\sigma_{k-1}}(t_k-t_{k-1})} J_{12}^{\sigma_{k-2}\overset{v_k}{\to}\sigma_{k-1}} e_v(t_{k-1}^-) + \cdots \\
&\quad + e^{A_0^{\sigma_k}(t-t_k)+A_0^{\sigma_{k-1}}(t_k-t_{k-1})+\cdots+A_0^{\sigma_0}(t_1-t_0)} J_{12}^{\sigma_0\overset{v_k}{\to}\sigma_1} e_v(0) + O(\epsilon) \\
\|x_s(t)\| &\le e^{\nu(A_0^{\sigma_k})(t-t_k)} e^{\nu(A_0^{\sigma_{k-1}})(t_k-t_{k-1})} \cdots e^{\nu(A_0^{\sigma_0})(t_1-t_0)} \|x_e(t_0)\| + e^{\nu(A_0^{\sigma_k})(t-t_k)} J_{12}^{\sigma_{k-1}\overset{v_k}{\to}\sigma_k} \|e_v(t_k^-)\| \\
&\quad + e^{\nu(A_0^{\sigma_k})(t-t_k)+\nu(A_0^{\sigma_{k-1}})(t_k-t_{k-1})} J_{12}^{\sigma_{k-2}\overset{v_k}{\to}\sigma_{k-1}} \|e_v(t_{k-1}^-)\| + \cdots \\
&\quad + e^{\nu(A_0^{\sigma_k})(t-t_k)+\nu(A_0^{\sigma_{k-1}})(t_k-t_{k-1})+\cdots+\nu(A_0^{\sigma_0})(t_1-t_0)} J_{12}^{\sigma_0\overset{v_k}{\to}\sigma_1} \|e_v(0)\| + O(\epsilon) \\
\|x_s(t)\| &\le e^{\nu(A_0^{\sigma_k})(t-t_k)} e^{\nu(A_0^{\sigma_{k-1}})(t_k-t_{k-1})} \cdots e^{\nu(A_0^{\sigma_0})(t_1-t_0)} \|x_e(t_0)\| + \mathcal{P} + O(\epsilon)
\end{aligned} \tag{62}
$$

where, $\mathcal{P}$ is as follows,

$$
\begin{aligned}
\mathcal{P} &= e^{\nu(A_0^{\sigma_k})(t-t_k)} J_{12}^{\sigma_{k-1}\overset{v_k}{\to}\sigma_k} \|e_v(t_k^-)\| + e^{\nu(A_0^{\sigma_k})(t-t_k)+\nu(A_0^{\sigma_{k-1}})(t_k-t_{k-1})} J_{12}^{\sigma_{k-2}\overset{v_k}{\to}\sigma_{k-1}} \|e_v(t_{k-1}^-)\| + \cdots \\
&\quad + e^{\nu(A_0^{\sigma_k})(t-t_k)+\nu(A_0^{\sigma_{k-1}})(t_k-t_{k-1})+\cdots+\nu(A_0^{\sigma_0})(t_1-t_0)} J_{12}^{\sigma_0\overset{v_k}{\to}\sigma_1} \|e_v(0)\| \\
&\le e^{\nu(A_0^{\sigma_k})(t-t_k)} J_{12}^{\sigma_{k-1}\overset{v_k}{\to}\sigma_k} \|e_f(t_k^-)\| + e^{\nu(A_0^{\sigma_k})(t-t_k)+\nu(A_0^{\sigma_{k-1}})(t_k-t_{k-1})} J_{12}^{\sigma_{k-2}\overset{v_k}{\to}\sigma_{k-1}} \|e_f(t_{k-1}^-)\| + \cdots \\
&\quad + e^{\nu(A_0^{\sigma_k})(t-t_k)+\nu(A_0^{\sigma_{k-1}})(t_k-t_{k-1})+\cdots+\nu(A_0^{\sigma_0})(t_1-t_0)} J_{12}^{\sigma_0\overset{v_k}{\to}\sigma_1} \|e_f(0)\| + O(\epsilon) \\
&\le \gamma_{12} \big(e^{\nu(A_0^{\sigma_k})(t-t_k)} \|e_f(t_k^-)\| + e^{\nu(A_0^{\sigma_k})(t-t_k)+\nu(A_0^{\sigma_{k-1}})(t_k-t_{k-1})} \|e_f(t_{k-1}^-)\| + \cdots \\
&\quad + e^{\nu(A_0^{\sigma_k})(t-t_k)+\nu(A_0^{\sigma_{k-1}})(t_k-t_{k-1})+\cdots+\nu(A_0^{\sigma_0})(t_1-t_0)} \|e_f(0)\|\big) + O(\epsilon).
\end{aligned}
$$

where, $\gamma_{12} = \max\limits_{\sigma_k, \sigma_{k-1} \in \mathcal{I}, v_k \in \mathcal{J}} \|J_{12}^{\sigma_{k-1}\overset{v_k}{\to}\sigma_k}\|$. Since, we know that the trajectories of the slow subsystems are bounded and the fast subsystems are asymptotically stable between the two switching events, we have,

$$
\mathcal{P} \le O(\epsilon). \tag{63}
$$

From equation (62), (63) and (34) it follows,

$$
\begin{aligned}
\|x_s(t)\| &\le e^{\nu_s t_s + \nu_u t_u} \|x_e(t_0)\| + \mathcal{P} + O(\epsilon) \\
\|x_s(t)\| &\le \|x_e(t)\| + O(\epsilon)
\end{aligned} \tag{64}
$$

and (119) can be concluded. This proves that the emergent dynamics, which is the weighted average of the systems states, approximate the mean-field dynamics with O(ϵ) order of approximation. Also, it is important to note that the asymptotic stability of the fast dynamics is necessary for the $O(\epsilon)$ approximation of the original variables. □

5 Stability Analysis

As stated in Remark 3, the matrix $A_{22}^{\sigma_k}$ is Hurwitz hence there exists symmetric positive definite matrices $Q_f^{\sigma_k} \geq I_{n_x(N-1)}$ and $\lambda_f^{\sigma_k} > 0$, $\forall \sigma_k \in \mathcal{I}$. However, for the matrix $A_0^{\sigma_k}$ with stable and unstable modes, we have $Q_s^{\sigma_k} \geq I_{n_x}, \forall \sigma_k \in \mathcal{S}$ and $Q_u^{\sigma_k} \geq I_{n_x}$, $\forall \sigma_k \in \mathcal{U}$ with positive numbers $\lambda_s^{\sigma_k}$ and $\lambda_u^{\sigma_k}$, such that,

$$
\begin{aligned}
{A_0^{\sigma_k}}^T Q_s^{\sigma_k} + Q_s^{\sigma_k} A_0^{\sigma_k} &\leq -2\lambda_s^{\sigma_k} Q_s^{\sigma_k} \qquad \forall \sigma_k \in \mathcal{S} \\
{A_0^{\sigma_k}}^T Q_s^{\sigma_k} + Q_s^{\sigma_k} A_0^{\sigma_k} &\leq 2\lambda_u^{\sigma_k} Q_s^{\sigma_k} \qquad \forall \sigma_k \in \mathcal{U} \\
{A_{22}^{\sigma_k}}^T Q_f^{\sigma_k} + Q_f^{\sigma_k} A_{22}^{\sigma_k} &\leq -2\lambda_f^{\sigma_k} Q_f^{\sigma_k} \qquad \forall \sigma_k \in \mathcal{I}
\end{aligned}
$$

Let us denote, $\lambda_s = \min_{\sigma_k \in \mathcal{S}} \lambda_s^{\sigma_k}$, $\lambda_u = \max_{\sigma_k \in \mathcal{S}} \lambda_u^{\sigma_k}$ and $\lambda_f = \min_{\sigma_k \in \mathcal{I}} \lambda_f^{\sigma_k}$.

Assumption 2 *The trajectories of the slow subsystems are bounded.*

Let us consider the composite Lyapunov functions as follows for the slow and fast subsystems $W_s(t) = x_s^T \left[\gamma Q_s^i + (1-\gamma) Q_u^i\right] x_s$ and $W_f(t) = e_v^T Q_f^i e_v$, where $i \in \mathcal{I}$ and $\gamma \in \{0, 1\}$ where $\gamma = 1$ if $i \in \mathcal{S}$ and $\gamma = 0$ if $i \in \mathcal{U}$.
The time derivative of the Lyapunov function $W_s(t)$ along the trajectories of the slow system is

$$
\dot{W}_s(t) = \tag{65}
$$

Proposition 3 *The function $V_i(x_s, e_v) = x_s^T \left[\gamma Q_s^i + (1-\gamma) Q_u^i\right] x_s + e_v^T Q_f^i e_v$, $\forall i \in \mathcal{I}$, is the composite Lyapunov function for the system (21) for all $\epsilon \in (0, \epsilon^*)$ and $\gamma = 1$ if $i \in \mathcal{S}$ and $\gamma = 0$ if $i \in \mathcal{U}$.*

Proof: The time derivative of the Lyapunov function along the trajectories of the system (21) is

$$
\begin{aligned}
\dot{V}_i &= 2x_s^T \left[\gamma Q_s^i + (1-\gamma) Q_u^i\right] \dot{x_s} + 2e_v^T Q_f^i \dot{e_v} \\
&= 2x_s^T \left[\gamma Q_s^i + (1-\gamma) Q_u^i\right] (A_0^i x_s + B_1^i e_v) + 2e_v^T Q_f^i (B_2^i x_s + B_3^i e_v + \frac{1}{\epsilon} A_{22}^i e_v) \\
&= 2x_s^T \left[\gamma Q_s^i + (1-\gamma) Q_u^i\right] A_0^i x_s + 2x_s^T \left[\gamma Q_s^i + (1-\gamma) Q_u^i\right] B_1^i e_v + 2e_v^T Q_f^i B_2^i x_s + 2e_v^T Q_f^i B_3^i e_v + \frac{2}{\epsilon} e_v^T Q_f^i A_{22}^i e_v \\
&= 2\gamma x_s^T Q_s^i A_0^i x_s + 2(1-\gamma) x_s^T Q_u^i A_0^i x_s + 2\gamma x_s^T Q_s^i B_1^i e_v + 2(1-\gamma) x_s^T Q_u^i B_1^i e_v + 2e_v^T Q_f^i B_2^i x_s + \\
&\qquad 2e_v^T Q_f^i B_3^i e_v + \frac{2}{\epsilon} e_v^T Q_f^i A_{22}^i e_v \\
&\leq -2\lambda_s \gamma x_s^T Q_s^i x_s + 2\lambda_u (1-\gamma) x_s^T Q_u^i x_s + 2\gamma x_s^T Q_s^i B_1^i e_v + 2(1-\gamma) x_s^T Q_u^i B_1^i e_v + 2e_v^T Q_f^i B_2^i x_s + \\
&\qquad 2e_v^T Q_f^i B_3^i e_v - \frac{2}{\epsilon} \lambda_f e_v^T Q_f^i e_v
\end{aligned} \tag{66}
$$

Let $b_1^i = \|{Q_s^i}^{\frac{1}{2}} B_1^i {Q_f^i}^{-\frac{1}{2}}\|$, $b_2^i = \|{Q_u^i}^{\frac{1}{2}} B_1^i {Q_f^i}^{-\frac{1}{2}}\|$, $b_3^i = \|{Q_f^i}^{\frac{1}{2}} B_2^i \big(\gamma Q_s^i + (1-\gamma) Q_u^i\big)^{-\frac{1}{2}}\|$, $b_4^i = \|{Q_f^i}^{\frac{1}{2}} B_3^i {Q_f^i}^{-\frac{1}{2}}\|$ and $b_j = \max_{i \in \mathcal{I}} b_j^i, j \in \{1, 2, 3, 4\}$.

$$
x_s^T Q_s^i B_1^i e_v = x_s^T {Q_s^i}^{\frac{1}{2}} {Q_s^i}^{\frac{1}{2}} B_1^i {Q_f^i}^{-\frac{1}{2}} {Q_f^i}^{\frac{1}{2}} e_v \leq b_1^i \|x_s^T {Q_s^i}^{\frac{1}{2}}\| \|{Q_f^i}^{\frac{1}{2}} e_v\| = b_1^i \sqrt{x_s^T Q_s^i x_s} \sqrt{e_v^T Q_f^i e_v} \tag{67}
$$

$$
x_s^T Q_u^i B_1^i e_v = x_s^T {Q_u^i}^{\frac{1}{2}} {Q_u^i}^{\frac{1}{2}} B_1^i {Q_f^i}^{-\frac{1}{2}} {Q_f^i}^{\frac{1}{2}} e_v \leq b_2^i \|x_s^T {Q_u^i}^{\frac{1}{2}}\| \|{Q_f^i}^{\frac{1}{2}} e_v\| = b_2^i \sqrt{x_s^T Q_u^i x_s} \sqrt{e_v^T Q_f^i e_v} \tag{68}
$$

$$
\begin{aligned}
e_v^T Q_f^i B_2^i x_s &= e_v^T {Q_f^i}^{\frac{1}{2}} {Q_f^i}^{\frac{1}{2}} B_2^i \big(\gamma Q_s^i + (1-\gamma) Q_u^i\big)^{-\frac{1}{2}} \big(\gamma Q_s^i + (1-\gamma) Q_u^i\big)^{\frac{1}{2}} x_s \\
&\leq b_3^i \|e_v^T {Q_f^i}^{\frac{1}{2}}\| \|\big(\gamma Q_s^i + (1-\gamma) Q_u^i\big)^{\frac{1}{2}} x_s\| = b_3^i \sqrt{e_v^T {Q_f^i}^{\frac{1}{2}} e_v} \sqrt{x_s^T (\gamma Q_s^i + (1-\gamma) Q_u^i) x_s}
\end{aligned} \tag{69}
$$

$$
e_v^T Q_f^i B_3^i e_v = e_v^T {Q_f^i}^{\frac{1}{2}} {Q_f^i}^{\frac{1}{2}} B_3^i {Q_f^i}^{-\frac{1}{2}} {Q_f^i}^{\frac{1}{2}} e_v \leq b_4^i e_v^T Q_f^i e_v \tag{70}
$$

Now, substiuting the value from equations (85), (86), (87) and (88) into equation (34), we have,

$$\begin{aligned}
\dot{V}_i &\leq -2\lambda_s\gamma x_s^T Q_s^i x_s + 2\lambda_u(1-\gamma)x_s^T Q_u^i x_s + 2\gamma b_1\sqrt{x_s^T Q_s^i x_s}\sqrt{e_v^T Q_f^i e_v} + 2(1-\gamma)b_2\sqrt{x_s^T Q_u^i x_s}\sqrt{e_v^T Q_f^i e_v} \\
&\quad + 2b_3\sqrt{e_v^T Q_f^i e_v}\sqrt{x_s^T(\gamma Q_s^i + (1-\gamma)Q_u^i)x_s} + 2b_4 e_v^T Q_f^i e_v - \frac{2}{\epsilon}\lambda_f e_v^T Q_f^i e_v \\
&\leq -2\lambda_s\gamma x_s^T Q_s^i x_s + 2\lambda_u(1-\gamma)x_s^T Q_u^i x_s + \gamma b_1 x_s^T Q_s^i x_s + \gamma b_1 e_v^T Q_f^i e_v + (1-\gamma)b_2 x_s^T Q_u^i x_s + (1-\gamma)b_2 e_v^T Q_f^i e_v \\
&\quad + b_3 e_v^T Q_f^i e_v + b_3 x_s^T(\gamma Q_s^i + (1-\gamma)Q_u^i)x_s + 2b_4 e_v^T Q_f^i e_v - \frac{2}{\epsilon}\lambda_f e_v^T Q_f^i e_v \\
&= \gamma\bigg(-2\lambda_s + b_1 + b_3\bigg)x_s^T Q_s^i x_s + (1-\gamma)\bigg(2\lambda_u + b_2 + b_3\bigg)x_s^T Q_u^i x_s + \\
&\quad \bigg(\gamma b_1 + (1-\gamma)b_2 + b_3 + 2b_4 - \frac{2}{\epsilon}\lambda_f\bigg)e_v^T Q_f^i e_v
\end{aligned}$$

Case 1: For $i \in \mathcal{S}$, $\gamma = 1$, we have,

$$\dot{V}_i \leq \bigg(-2\lambda_s + b_1 + b_3\bigg)x_s^T Q_s^i x_s + \bigg(b_1 + b_3 + 2b_4 - \frac{2}{\epsilon}\lambda_f\bigg)e_v^T Q_f^i e_v \tag{71}$$

$$\dot{V}_i \leq -\sqrt{(2\lambda_s - b_1 - b_3)^2 + (\frac{2}{\epsilon}\lambda_f - b_1 - b_3 - 2b_4)}\sqrt{(x_s^T Q_s^i x_s)^2 + (e_v^T Q_f^i e_v)^2} \tag{72}$$

$$\dot{V}_i \leq -\sqrt{(2\lambda_s - b_1 - b_3)^2 + (\frac{2}{\epsilon}\lambda_f - b_1 - b_3 - 2b_4)}\sqrt{(x_s^T Q_s^i x_s + e_v^T Q_f^i e_v)^2} \tag{73}$$

$$\dot{V}_i \leq -\sqrt{(2\lambda_s - b_1 - b_3)^2 + (\frac{2}{\epsilon}\lambda_f - b_1 - b_3 - 2b_4)}V_i \tag{74}$$

$$V_i(t) = \exp\left\{-\sqrt{(2\lambda_s - b_1 - b_3)^2 + (\frac{2}{\epsilon}\lambda_f - b_1 - b_3 - 2b_4)^2}(t - t_k)\right\}V_i(t_k) \tag{75}$$

Hence, $\dot{V}_i < 0$ iff $\lambda_s > \frac{b_1+b_3}{2}$ and for all $\epsilon \in (0, \epsilon_1)$, where, $\epsilon_1 = \frac{2\lambda_f}{b_1+b_3+2b_4}$.

Case 2: For $i \in \mathcal{U}$, $\gamma = 0$, then we have,

$$\dot{V}_i \leq \bigg(2\lambda_u + b_2 + b_3\bigg)x_s^T Q_u^i x_s + \bigg(b_2 + b_3 + 2b_4 - \frac{2}{\epsilon}\lambda_f\bigg)e_v^T Q_f^i e_v \tag{76}$$

$$\dot{V}_i \leq \sqrt{(2\lambda_u + b_2 + b_3)^2 + (b_2 + b_3 + 2b_4 - \frac{2}{\epsilon}\lambda_f)^2}\sqrt{(x_s^T Q_u^i x_s)^2 + (e_v^T Q_f^i e_v)}^2 \tag{77}$$

$$\dot{V}_i \leq \sqrt{(2\lambda_u + b_2 + b_3)^2 + (b_2 + b_3 + 2b_4 - \frac{2}{\epsilon}\lambda_f)^2}\sqrt{(x_s^T Q_u^i x_s + e_v^T Q_f^i e_v)}^2 \tag{78}$$

$$\dot{V}_i \leq \sqrt{(2\lambda_u + b_2 + b_3)^2 + (b_2 + b_3 + 2b_4 - \frac{2}{\epsilon}\lambda_f)^2}V_i \tag{79}$$

$$V_i \leq \exp\left\{\sqrt{(2\lambda_u + b_2 + b_3)^2 + (b_2 + b_3 + 2b_4 - \frac{2}{\epsilon}\lambda_f)^2}(t - t_k)\right\}V_i(t_k) \tag{80}$$

Then, for the system switching between the stable and unstable modes, let $\mu > 0$, then we have the following,

$$V(t_{k+1}) \leq \mu V(t_k), \qquad \forall k \in \mathbb{N}. \tag{81}$$

If $t_{k+1} - t_k \geq \tau^* > 0$ is the average dwell time then by induction we have,

$$V(t) \leq \mu^{|\mathcal{I}|}\exp\left\{\sqrt{(2\lambda_u + b_2 + b_3)^2 + (b_2 + b_3 + 2b_4 - \frac{2}{\epsilon}\lambda_f)^2}\tau_u - \sqrt{(2\lambda_s - b_1 - b_3)^2 + (\frac{2}{\epsilon}\lambda_f - b_1 - b_3 - 2b_4)^2}\tau_s\right\}V(0) \tag{82}$$

Hence, the system is stable when

$$\sqrt{(2\lambda_u + b_2 + b_3)^2 + (b_2 + b_3 + 2b_4 - \frac{2}{\epsilon}\lambda_f)^2}\tau_u - \sqrt{(2\lambda_s - b_1 - b_3)^2 + (\frac{2}{\epsilon}\lambda_f - b_1 - b_3 - 2b_4)^2}\tau_s < 0$$

$$\frac{\tau_s}{\tau_u} > \frac{\sqrt{(2\lambda_u + b_2 + b_3)^2 + (b_2 + b_3 + 2b_4 - \frac{2}{\epsilon}\lambda_f)^2}}{\sqrt{(2\lambda_s - b_1 - b_3)^2 + (\frac{2}{\epsilon}\lambda_f - b_1 - b_3 - 2b_4)^2}}. \tag{83}$$

Lemma 5.1 *If all the fast modes are stable i.e.,* $A_{22}^{\sigma_k}, \forall k \in \mathbb{N}$ *are Hurwitz, then there exists* $\tau^* > 0$ *such that the overall fast dynamics is asymptotically stable iff* $\forall k \in \mathbb{N}$, $t_{k+1} - t_k > \tau^*$, *where* $\tau^* = \frac{\epsilon k \ln(\gamma_2)}{(k+1)\lambda_f}$.

Proof: From equations (23) and (24), we obtain the fast subsystem with the impulsive dynamics as follows:

$$\frac{de_f}{d\tau} = A_{22}^{\sigma_k} e_f(\tau) \qquad \forall \tau \in \left[\frac{t_k}{\epsilon}, \frac{t_{k+1}}{\epsilon}\right), \forall k \in \mathbb{N}$$
$$e_f\left(\frac{t_k}{\epsilon}\right) = J_{22}^{\sigma_{k-1} \overset{v_k}{\rightarrow} \sigma_k} e_f\left(\frac{t_k^-}{\epsilon}\right),$$

where $J_{22}^{\sigma_{k-1} \overset{v_k}{\rightarrow} \sigma_k} = ((V^{\sigma_k \dagger} - V^{\sigma_{k-1} \dagger}) V^{\sigma_{k-1}} \otimes I_{n_x})$. Then from equations (88) and (89) it follows,

$$e_f(\tau) = e^{A_{22}^{\sigma_k} (\frac{t - t_k}{\epsilon})} J_{22}^{\sigma_{k-1} \overset{v_k}{\rightarrow} \sigma_k} e^{A_{22}^{\sigma_{k-1}} (\frac{t_k - t_{k-1}}{\epsilon})} e_f(\frac{t_{k-1}}{\epsilon}), \qquad \forall \tau \in \left[\frac{t_k}{\epsilon}, \frac{t_{k+1}}{\epsilon}\right), k \in \mathbb{N}$$

and eventually we have,

$$e_f(t) = e^{A_0^{\sigma_k} (\frac{t - t_k}{\epsilon})} J_{22}^{\sigma_{k-1} \overset{v_k}{\rightarrow} \sigma_k} e^{A_0^{\sigma_{k-1}} (\frac{t_k - t_{k-1}}{\epsilon})} \cdots J_{22}^{\sigma_0 \overset{v_1}{\rightarrow} \sigma_1} e^{A_0^{\sigma_0} (\frac{t_1 - t_0}{\epsilon})} e_f(0).$$

We know from remark 3, that the matrices $A_{22}^{\sigma_k}, \forall \sigma_k \in \mathcal{I}$ are Hurwitz i.e., the measure $\nu(A_{22}^{\sigma_k}) < 0, \forall \sigma_k \in \mathcal{I}$. Let us define the following,

$$\max_{\sigma_k \in \mathcal{I}} \nu(A_{22}^{\sigma_k}) = -\lambda_f$$

and

$$\gamma_2 = \max_{\sigma_k, \sigma_{k-1} \in \mathcal{I}, v_k \in \mathcal{J}} \| J_{22}^{\sigma_{k-1} \overset{v_k}{\rightarrow} \sigma_k} \|.$$

Let $\tau^* > 0$ such that $t_k - t_{k-1} \geq \tau^*, \forall k \in \mathbb{N}$ then we have the following

$$\begin{aligned}
\|e_f(t)\| &= \|e^{A_0^{\sigma_k} (\frac{t - t_k}{\epsilon})} J_{22}^{\sigma_{k-1} \overset{v_k}{\rightarrow} \sigma_k} e^{A_0^{\sigma_{k-1}} (\frac{t_k - t_{k-1}}{\epsilon})} \cdots J_{22}^{\sigma_0 \overset{v_1}{\rightarrow} \sigma_1} e^{A_0^{\sigma_0} (\frac{t_1 - t_0}{\epsilon})} e_f(0)\| \\
&\leq \|e^{A_0^{\sigma_k} (\frac{t - t_k}{\epsilon})}\| \|J_{22}^{\sigma_{k-1} \overset{v_k}{\rightarrow} \sigma_k}\| \|e^{A_0^{\sigma_{k-1}} (\frac{t_k - t_{k-1}}{\epsilon})}\| \cdots \|J_{22}^{\sigma_0 \overset{v_1}{\rightarrow} \sigma_1}\| \|e^{A_0^{\sigma_0} (\frac{t_1 - t_0}{\epsilon})}\| \|e_f(0)\| \\
&\leq \gamma_2^k e^{\nu(A_{22}^{\sigma_k}) \frac{\tau^*}{\epsilon} + \nu(A_{22}^{\sigma_{k-1}}) \frac{\tau^*}{\epsilon} + \cdots + \nu(A_{22}^{\sigma_0}) \frac{\tau^*}{\epsilon}} \|e_f(0)\| \\
&\leq \gamma_2^k e^{-\lambda_f (k+1) \frac{\tau^*}{\epsilon}} \|e_f(0)\|.
\end{aligned}$$

Then from equation (94) we can conclude, for sufficiently large k, $e_f(t)$ tends to 0 when the dwell-time condition

$$\tau^* > \frac{\epsilon k \ln(\gamma_2)}{(k+1)\lambda_f}$$

is satisfied. Hence we can conclude that the fast subsystem is asymptotically stable when every modes are asymptotically stable and provided the dwell-time condition (95) is satisfied. □

6 Stability Analysis

As stated in Remark 3, the matrix $A_{22}^{\sigma_k}$ is Hurwitz hence there exists symmetric positive definite matrices $Q_f^{\sigma_k} \geq I_{n_x(N-1)}$ and $\lambda_f^{\sigma_k} > 0$, $\forall \sigma_k \in \mathcal{I}$. However, for the matrix $A_0^{\sigma_k}$ with stable and unstable modes, we have $Q_s^{\sigma_k} \geq I_{n_x}, \forall \sigma_k \in \mathcal{S}$ and $Q_u^{\sigma_k} \geq I_{n_x}$, $\forall \sigma_k \in \mathcal{U}$ with positive numbers $\lambda_s^{\sigma_k}$ and $\lambda_u^{\sigma_k}$, such that,

$$\begin{aligned}
{A_0^{\sigma_k}}^T Q_s^{\sigma_k} + Q_s^{\sigma_k} A_0^{\sigma_k} &\leq 2\nu(A_0^{\sigma_k}) Q_s^{\sigma_k} \qquad \forall \sigma_k \in \mathcal{I} \\
{A_{22}^{\sigma_k}}^T Q_f^{\sigma_k} + Q_f^{\sigma_k} A_{22}^{\sigma_k} &\leq -2\lambda_f^{\sigma_k} Q_f^{\sigma_k} \qquad \forall \sigma_k \in \mathcal{I}
\end{aligned}$$

Let us denote, $\nu(A_0^{\sigma_k}) = \frac{1}{2}\lambda_{max}(A_0^{\sigma_k} + {A_0^{\sigma_k}}^T)$ and $\lambda_f = \min\limits_{\sigma_k \in \mathcal{I}} \lambda_f^{\sigma_k}$ where $\lambda_f^{\sigma_k} = \frac{1}{2}(A_{22}^{\sigma_k} + {A_{22}^{\sigma_k}}^T)$.

Assumption 3 *The trajectories of the slow subsystems are bounded i.e.,* $\frac{t_u}{t_s} \leq -\frac{\nu_s}{\nu_u}$.

Proposition 4 *The function* $V_i(x_s, e_v) = x_s^T Q_s^i x_s + e_v^T Q_f^i e_v$, $\forall i \in \mathcal{I}$, *is the composite Lyapunov function for the system (21) for all* $\epsilon \in (0, \epsilon^*)$.

Proof: The time derivative of the Lyapunov function along the trajectories of the system (21) is

$$\begin{aligned}
\dot{V}_i &= 2x_s^T Q_s^i \dot{x_s} + 2e_v^T Q_f^i \dot{e_v} \\
&= 2x_s^T Q_s^i (A_0^i x_s + B_1^i e_v) + 2e_v^T Q_f^i (B_2^i x_s + B_3^i e_v + \frac{1}{\epsilon} A_{22}^i e_v) \\
&= 2x_s^T Q_s^i A_0^i x_s + 2x_s^T Q_s^i B_1^i e_v + 2e_v^T Q_f^i B_2^i x_s + 2e_v^T Q_f^i B_3^i e_v + \frac{2}{\epsilon} e_v^T Q_f^i A_{22}^i e_v \\
&\leq 2\nu(A_0^{\sigma_k}) x_s^T Q_s^i x_s + 2x_s^T Q_s^i B_1^i e_v + 2e_v^T Q_f^i B_2^i x_s + 2e_v^T Q_f^i B_3^i e_v - \frac{2}{\epsilon}\lambda_f e_v^T Q_f^i e_v
\end{aligned} \tag{84}$$

Let $b_1^i = \|{Q_s^i}^{\frac{1}{2}} B_1^i {Q_f^i}^{-\frac{1}{2}}\|$, $b_2^i = \|{Q_s^i}^{\frac{1}{2}} B_1^i {Q_f^i}^{-\frac{1}{2}}\|$, $b_3^i = \|{Q_f^i}^{\frac{1}{2}} B_3^i {Q_f^i}^{-\frac{1}{2}}\|$ and $b_j = \max\limits_{i \in \mathcal{I}} b_j^i, j \in \{1, 2, 3\}$.

$$x_s^T Q_s^i B_1^i e_v = x_s^T {Q_s^i}^{\frac{1}{2}} {Q_s^i}^{\frac{1}{2}} B_1^i {Q_f^i}^{-\frac{1}{2}} {Q_f^i}^{\frac{1}{2}} e_v \leq b_1^i \|x_s^T {Q_s^i}^{\frac{1}{2}}\| \|{Q_f^i}^{\frac{1}{2}} e_v\| = b_1^i \sqrt{x_s^T Q_s^i x_s} \sqrt{e_v^T Q_f^i e_v} \tag{85}$$

$$e_v^T Q_f^i B_1^i x_s = e_v^T {Q_f^i}^{\frac{1}{2}} {Q_f^i}^{\frac{1}{2}} B_1^i {Q_s^i}^{-\frac{1}{2}} {Q_s^i}^{\frac{1}{2}} x_s \leq b_2^i \|e_v^T {Q_f^i}^{\frac{1}{2}}\| \|{Q_s^i}^{\frac{1}{2}} x_s\| = b_2^i \sqrt{x_s^T Q_s^i x_s} \sqrt{e_v^T Q_f^i e_v} \tag{86}$$

$$e_v^T Q_f^i B_3^i e_v = e_v^T {Q_f^i}^{\frac{1}{2}} {Q_f^i}^{\frac{1}{2}} B_3^i {Q_f^i}^{-\frac{1}{2}} {Q_f^i}^{\frac{1}{2}} e_v \leq b_3^i e_v^T Q_f^i e_v \tag{87}$$

Now, substiuting the value from equations (85), (86) and (87) into equation (84), we have,

$$\begin{aligned}
\dot{V}_i &\leq 2\nu(A_0^i) x_s^T Q_s^i x_s + (2b_1 + 2b_2)\sqrt{x_s^T Q_s^i x_s}\sqrt{e_v^T Q_f^i e_v} + 2b_3 e_v^T Q_f^i e_v - \frac{2}{\epsilon}\lambda_f e_v^T Q_f^i e_v \\
\dot{V}_i &\leq 2\nu(A_0^i) x_s^T Q_s^i x_s + (2b_1 + 2b_2)\sqrt{x_s^T Q_s^i x_s}\sqrt{e_v^T Q_f^i e_v} + (2b_3 - \frac{2}{\epsilon}\lambda_f) e_v^T Q_f^i e_v
\end{aligned}$$

Case 1: For $i \in \mathcal{S}$, $\nu(A_0^i) < 0$ and let $\nu(A_0^i) = -\lambda_s$, then, we have,

$$\dot{V}_i \leq -2\lambda_s x_s^T Q_s^i x_s + 2(b_1 + b_2)\sqrt{x_s^T Q_s^i x_s}\sqrt{e_v^T Q_f^i e_v} + (2b_3 - \frac{2}{\epsilon}\lambda_f) e_v^T Q_f^i e_v \tag{88}$$

Case 2: For $i \in \mathcal{U}$,$\nu(A_0^i) > 0$ and let $\nu(A_0^i) = \lambda_u$, then, we have,

$$\dot{V}_i \leq 2\lambda_u x_s^T Q_s^i x_s + 2(b_1 + b_2)\sqrt{x_s^T Q_s^i x_s}\sqrt{e_v^T Q_f^i e_v} + (2b_3 - \frac{2}{\epsilon}\lambda_f) e_v^T Q_f^i e_v \tag{89}$$

Then for the events t_k where $\sigma_k \in \mathcal{S}$ the stability can be guaranteed by choosing $\epsilon \in (0, \epsilon_1]$, where $\epsilon_1 = \frac{\lambda_f}{\frac{(b_1+b_2)^2}{4\lambda_s} + b_3}$, while for $\sigma_k \in \mathcal{U}$, $\dot{V}_i$ may or maynot be negative. ■

The stability analysis of the switching-impulsive system with unstable subsystems is done using the following mode dependent functions.

$$W_s(t) = \sqrt{x_s^T Q_s^{\sigma_k} x_s}$$
$$W_f(t) = \sqrt{e_v^T Q_f^{\sigma_k} e_v} \qquad \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}$$

we have, $\forall \epsilon \in (0, \epsilon_2]$

$$W_s(t) \leq W_s(0) e^{\nu(A_0^{\sigma_k})t} + \epsilon\beta_2 W_f(0) + \epsilon\beta_3 \sqrt{V(0)} \tag{90}$$
$$W_f(t) \leq W_f(0) e^{-\frac{\lambda_f}{\epsilon}t} + \epsilon\beta_1 \sqrt{V(0)} \tag{91}$$

where, $\beta_1 = \frac{\sqrt{b_2^2 + b_3^2}}{\lambda_f}$, $\beta_2 = \frac{b_1}{\lambda_f - \epsilon_2 \lambda_s}$, $\beta_3 = \frac{b_1 \beta_1}{\lambda_s}$ and $\epsilon \in (0, \epsilon_1] \cap (0, \frac{\lambda_f}{\lambda_s})$

Lemma 6.1 *Let $\tau_k = t_{k+1} - t_k$ and let $\epsilon \in (0, \epsilon_2]$ for the sequence of the time events $(t_k)_{k \geq 0}$ between two switching. Then for all $k \in \mathbb{N}$ and for all $\sigma_k \in \mathcal{I}$,*

$$W_s(t_{k+1}^-) \leq W_s(t_k)(e^{\nu(A_0^{\sigma_k})\tau_k} + \epsilon\beta_3) + W_f(t_k)(\epsilon\beta_2 + \epsilon\beta_3) \tag{92}$$
$$W_f(t_{k+1}^-) \leq W_s(t_k)\epsilon\beta_1 + W_f(t_k)(e^{-\frac{\lambda_f}{\epsilon}\tau_k} + \epsilon\beta_1) \tag{93}$$

Lemma 6.2 *Let a sequence $(t_k)_{k \geq 0}$, of event times, then for all $k \geq 1$, we have,*

$$W_s(t_k) \leq \gamma_{11} W_s(t_k^-) + \gamma_{12} W_f(t_k^-) \tag{94}$$
$$W_f(t_k) \leq \gamma_{22} W_f(t_k^-) \tag{95}$$

*where, γ_{11}, γ_{12} and γ_{22} are defined in a similar way to the one in [**?**].*

Now, from equations (92), (93), (94) and (95) we have the following,

$$\begin{pmatrix} W_s(t_k) \\ W_f(t_k) \end{pmatrix} \leq \Gamma M_\tau \begin{pmatrix} W_s(t_{k-1}) \\ W_f(t_{k-1}) \end{pmatrix} \tag{96}$$

where, $\Gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ 0 & \gamma_{22} \end{pmatrix}$ and $M_\tau = \begin{pmatrix} (e^{\nu(A_0^{\sigma_k})\tau_k} + \epsilon\beta_3) & (\epsilon\beta_2 + \epsilon\beta_3) \\ \epsilon\beta_1 & (e^{-\frac{\lambda_f}{\epsilon}\tau_k} + \epsilon\beta_1) \end{pmatrix}$.

Lemma 6.3 *Let $\epsilon \in (0, \epsilon_2]$ and let $\tau^* \geq 0$ such that the matrix ΓM_τ, $\sigma_k \in \mathcal{S}$ is Schur and assumption 2 is satisfied, then for all $(t_k)_{k \geq 0}$ and $\tau^k \geq \tau^*$ the system is asymptotically stable.*

Proof: From Lemma 3.3 and 3.4, $\forall k \in \mathbb{N}$, we have the following,

$$\begin{pmatrix} W_s(t_k) \\ W_f(t_k) \end{pmatrix} \leq \Gamma M_{\tau_k} \cdots \Gamma M_{\tau_1} \Gamma M_{\tau_0} \begin{pmatrix} W_s(t_0) \\ W_f(t_0) \end{pmatrix} \tag{97}$$

and it follows,

$$\begin{pmatrix} W_s(t_k) \\ W_f(t_k) \end{pmatrix} \leq (\Gamma M_{\tau^*})^k \begin{pmatrix} W_s(t_{k-1}) \\ W_f(t_{k-1}) \end{pmatrix} \tag{98}$$

However, we have both stable and unstable slow subsystems, hence from equation (30), we have,

$$M_\tau^s = \begin{pmatrix} (e^{\nu_s \tau^*} + \epsilon\beta_3) & (\epsilon\beta_2 + \epsilon\beta_3) \\ \epsilon\beta_1 & (e^{-\frac{\lambda_f}{\epsilon}\tau^*} + \epsilon\beta_1) \end{pmatrix} \qquad \forall \sigma_k \in \mathcal{S} \tag{99}$$

and

$$M_\tau^u = \begin{pmatrix} (e^{\nu_u \tau^*} + \epsilon\beta_3) & (\epsilon\beta_2 + \epsilon\beta_3) \\ \epsilon\beta_1 & (e^{-\frac{\lambda_f}{\epsilon}\tau^*} + \epsilon\beta_1) \end{pmatrix} \qquad \forall \sigma_k \in \mathcal{U}. \tag{100}$$

Without loss of generality, we assume that slow systems $A_0^1, \cdots A_0^r$ are unstable and the remaining modes are stable then for n switching, we have,

$$\begin{pmatrix} W_s(t_k) \\ W_f(t_k) \end{pmatrix} \leq (\Gamma M_{\tau^*})^{k-r} (\Gamma M_{\tau^*})^r \begin{pmatrix} W_s(t_0) \\ W_f(t_0) \end{pmatrix} \tag{101}$$

$$\begin{pmatrix} W_s(t_k) \\ W_f(t_k) \end{pmatrix} \leq \begin{pmatrix} \gamma_{11}^k e^{\nu_s(k-r)\tau^* + \nu_u r\tau^*} + O(\epsilon) & O(\epsilon) \\ O(\epsilon) & \gamma_{22}^k e^{-r\frac{\lambda_f}{\epsilon}\tau^*} + O(\epsilon) \end{pmatrix} \begin{pmatrix} W_s(t_0) \\ W_f(t_0) \end{pmatrix} \tag{102}$$

Let, $\tau_s = (k-r)\tau^*$ and $\tau_u = r\tau^*$ be the total amount of time spent in stable and unstable mode, respectively, then we have,

$$\begin{pmatrix} W_s(t_k) \\ W_f(t_k) \end{pmatrix} \leq \begin{pmatrix} \gamma_{11}^k e^{\nu_s\tau_s + \nu_u\tau_u} & 0 \\ 0 & \gamma_{22}^k e^{-(\tau_s+\tau_u)\frac{\lambda_f}{\epsilon}} \end{pmatrix} \begin{pmatrix} W_s(t_0) \\ W_f(t_0) \end{pmatrix} + O(\epsilon) \tag{103}$$

and by Assumption 2, $(W_s(t_k))_{k\geq 0}$ and $(W_f(t_k))_{k\geq 0}$ goes to 0 as $k \to \infty$.

Remark 4 *The asymptotic stability of the system (23)-(24) is guaranteed by the Schur stability property of the stable modes when the Assumption 2 is satisfied. In other words, it can be understood as the stable modes drive the system towards stability provided the proportion of the stable and unstable modes satisfy the certain boundedness condition i.e.,* $\nu_s\tau_s + \nu_u\tau_u < 0$*. Hence, the dwell time condition can be derived by using the Schur stability criterion.*

Theorem 6.4 *Dwell Time Conditions:*

Proof:

Case 1: $\gamma_{11} > 1$,

A positive matrix is Schur stable iff $A^T\xi < \xi$ where $\xi \in \mathbb{R}^n$ and $\xi > 0$ [**?**]. Let $\xi = (1, a\epsilon)^T$, then $(\Gamma M_{\tau_s^*})^T\xi < \xi$ leads to the following,

$$\gamma_{11}e^{\nu_s\tau^*} + \epsilon(\gamma_{11}\beta_3 + \gamma_{12}\beta_1 + \epsilon a\gamma_{22}\beta_1) < 1$$

$$\gamma_{12}e^{-\frac{\lambda_f}{\epsilon}\tau^*} + \epsilon a\gamma_{22}e^{-\frac{\lambda_f}{\epsilon}\tau^*} + \epsilon(\gamma_{11}(\beta_2 + \beta_3) + (\gamma_{12} + \epsilon a\gamma_{22})\beta_1) < a\epsilon.$$

Let $\delta_1 = \gamma_{11}\beta_3 + \gamma_{12}\beta_1$,$\delta_2 = \gamma_{11}(\beta_2 + \beta_3) + \gamma_{12}\beta_1$ and $\delta_3 = \gamma_{22}\beta_1$, then we have,

$$\gamma_{11}e^{\nu_s\tau^*} + \epsilon\delta_1 + a\epsilon^2\delta_3 < 1 \tag{104}$$

$$\gamma_{12}e^{-\frac{\lambda_f}{\epsilon}\tau^*} + \epsilon a\gamma_{22}e^{-\frac{\lambda_f}{\epsilon}\tau^*} + \epsilon\delta_2 + a\epsilon^2\delta_3 < a\epsilon. \tag{105}$$

Then from equation (104) we have,

$$\tau^* > \frac{1}{\lambda_s}\ln\left(\frac{\gamma_{11}}{1 - \epsilon\delta_1 - a\epsilon^2\delta_3}\right) = \frac{1}{\lambda_s}\ln(\gamma_{11}) + \eta_1(\epsilon) \tag{106}$$

where, $\eta_1(\epsilon) = \ln\left(\frac{1}{1-\epsilon\delta_1 - a\epsilon^2\delta_3}\right) = O(\epsilon)$ and $\eta_1(\epsilon)$ is only defined if $\epsilon < \epsilon_3$ where,

$$\epsilon_3 = \frac{-\delta_1 + \sqrt{\delta_1^2 + 4a\delta_3}}{2a\delta_3}. \tag{107}$$

From equation (105), we have,

$$\tau^* > \frac{\epsilon}{\lambda_f}\ln\left(\frac{\gamma_{12} + \epsilon a\gamma_{22}}{a\epsilon - \epsilon\delta_2 - a\epsilon^2\delta_3}\right) = \frac{\epsilon}{\lambda_f}\left(\ln\left(\frac{\gamma_{12} + \epsilon a\gamma_{22}}{a - \delta_2 - a\epsilon\delta_3}\right) - \ln(\epsilon)\right). \tag{108}$$

Since, $\tau^* > \frac{\ln(\gamma_{11})}{\lambda_s} + \eta_1(\epsilon) > \frac{\ln(\gamma_{11})}{\lambda_s}$, we choose $\epsilon_4 > 0$, such that for all $\epsilon \in (0, \epsilon_4)$, $\frac{\epsilon}{\lambda_f}\left(\ln\left(\frac{\gamma_{12}+\epsilon a\gamma_{22}}{a-\delta_2-a\epsilon\delta_3}\right) - \ln(\epsilon)\right)$ holds. Then we set $\epsilon_1^* = \min\{\epsilon_3, \epsilon_4\}$ to prove the theorem.

Case 2: When $\gamma_{11} = 1$, from (119), $\tau^* > \eta_1(\epsilon)$ satisfy the inequality (104) and inequality (105) is satisfied when $\tau^* > -\frac{\epsilon}{\lambda_f}\ln(\epsilon) + \eta_2(\epsilon)$ where, $\eta_2(\epsilon) = \frac{\epsilon}{\lambda_f}\ln\left(\frac{\gamma_{12}+\epsilon a\gamma_{22}}{a-\delta_2-a\epsilon\delta_3}\right) = O(\epsilon)$. Moreover, $\eta_2(\epsilon)$ is defined iff $\epsilon \in (0, \epsilon)$

$$
\begin{aligned}
W_s &= \sqrt{x_s^T e^{\nu(A_0^{\sigma_k})(t-t_k)} x_s} \\
W_f &= \sqrt{e_v^T e^{\nu(A_{22}^{\sigma_k})(t-t_k)} e_v}
\end{aligned} \tag{109}
$$

Let $V = x_s^T e^{\nu(A_0^{\sigma_k})(t-t_k)} x_s + e_v^T e^{\nu(A_{22}^{\sigma_k})(t-t_k)} e_v$ be the Lyapunov function. Then the time derivative of the Lyapunov function is given by ,

$$
\dot{V}_i = 2x_s^T e^{\nu(A_0^i)(t-t_k)} \dot{x_s} + e_v^T e^{\nu(A_{22}^i)(t-t_k)} \dot{e_v} \tag{110}
$$

$$
\dot{V}_i = 2x_s^T e^{\nu(A_0^i)(t-t_k)} (A_0^i x_s + B_2^i e_v) + e_v^T e^{\nu(A_{22}^i)(t-t_k)} \dot{e_v} \tag{111}
$$

7 Validation of the Approximation Models

Following up from the previous section, we have the necessary conditions for the boundedness of the emergent dynamics and the asymptotic stability of the fast dynamics. In this section, these conditions are used in proving that the original dynamics is approximated by the slow and fast subsystems.
We develop our result for the hybrid systems, based on the result from [**?**], for the continuous dynamics between two events, the original dynamics is approximated by the reduced ordered dynamics. We propose the following proposition that provide the approximation of e_v by the fast-variable e_f.

Proposition 5 *The error dynamics of the original system* (28)-(29) *is approximated for time* $t \geq t_0$ *by* $e_v(t) = e_s(t) + e_f(\tau) + O(\epsilon)$.

Proof: The error dynamics of the singularly perturbed switching-impulsive system (28)-(29) is

$$\dot{e}_v(t) = B_2^{\sigma_k} x_s + (B_3^{\sigma_k} + \frac{A_{22}^{\sigma_k}}{\epsilon}) e_v(t), \qquad \forall t \in [t_k, t_{k+1}), k \in \mathbb{N} \tag{112}$$

$$e_v(t_k) = J_{22}^{\sigma_{k-1} \overset{v_k}{\to} \sigma_k} e_v(t_k^-). \tag{113}$$

Solving this differential equation (112) and substituting the equation (113), while ignoring the higher order epsilon term ($n \geq 2$) we have,

$$\begin{aligned}
e_v(t) &= e^{(A_{22}^{\sigma_k} + \epsilon B_3^{\sigma_k})(\frac{t-t_k}{\epsilon})} e_v(t_k) + O(\epsilon) = (e^{A_{22}^{\sigma_k}(\frac{t-t_k}{\epsilon})} + O(\epsilon)) e_v(t_k) + O(\epsilon) \\
&= e^{A_{22}^{\sigma_k}(\frac{t-t_k}{\epsilon})} e_v(t_k) + O(\epsilon) \\
e_v(t_k) &= J_{22}^{\sigma_{k-1} \overset{v_k}{\to} \sigma_k} e_v(t_k^-) \\
e_v(t_k^-) &= e^{(A_{22}^{\sigma_{k-1}} + \epsilon B_3^{\sigma_{k-1}})(\frac{t_k - t_{k-1}}{\epsilon})} e_v(t_{k-1}) + O(\epsilon) = (e^{A_{22}^{\sigma_{k-1}}(\frac{t_k - t_{k-1}}{\epsilon})} + O(\epsilon)) e_v(t_{k-1}) + O(\epsilon) \\
&= e^{A_{22}^{\sigma_{k-1}}(\frac{t_k - t_{k-1}}{\epsilon})} e_v(t_{k-1}) + O(\epsilon)
\end{aligned}$$

which implies,

$$e_v(t) = (e^{A_{22}^{\sigma_k}(\frac{t-t_k}{\epsilon})} J_{22}^{\sigma_{k-1} \overset{v_k}{\to} \sigma_k})(e^{A_{22}^{\sigma_{k-1}}(\frac{t_k - t_{k-1}}{\epsilon})} e_v(t_{k-1}) + O(\epsilon)) + O(\epsilon)$$

$$e_v(t) = e^{A_{22}^{\sigma_k}(\frac{t-t_k}{\epsilon})} J_{22}^{\sigma_{k-1} \overset{v_k}{\to} \sigma_k} e^{A_{22}^{\sigma_{k-1}}(\frac{t_k - t_{k-1}}{\epsilon})} e_v(t_{k-1}) + O(\epsilon) \tag{114}$$

From [**?**], we know that for the continuous dynamics between the two events, i.e., $\forall t \in [t_k, t_{k+1})$, the error e_v can be approximated as $e_v(t) = e_f + O(\epsilon)$, hence from (114) it follows,

$$e_v(t) = e^{A_{22}^{\sigma_k}(\frac{t-t_k}{\epsilon})} J_{22}^{\sigma_{k-1} \overset{v_k}{\to} \sigma_k} e^{A_{22}^{\sigma_{k-1}}(\frac{t_k - t_{k-1}}{\epsilon})} (e_f(\tau) + O(\epsilon)) + O(\epsilon)$$

$$e_v(t) = e^{A_{22}^{\sigma_k}(\frac{t-t_k}{\epsilon})} J_{22}^{\sigma_{k-1} \overset{v_k}{\to} \sigma_k} e^{A_{22}^{\sigma_{k-1}}(\frac{t_k - t_{k-1}}{\epsilon})} e_f(\tau) + O(\epsilon) \tag{115}$$

It follows,

$$e_v(t) = e^{A_{22}^{\sigma_k}(\frac{t-t_k}{\epsilon})} J_{22}^{\sigma_{k-1} \overset{v_k}{\to} \sigma_k} e^{A_{22}^{\sigma_{k-1}}(\frac{t_k - t_{k-1}}{\epsilon})} \dots J_{22}^{\sigma_0 \overset{v_1}{\to} \sigma_1} e^{A_{22}^{\sigma_0}(\frac{t_1 - t_0}{\epsilon})} e_f(t_0) + O(\epsilon)$$

$$\|e_v(t)\| \leq e^{\nu(A_{22}^{\sigma_k})(\frac{t-t_k}{\epsilon})} \|J_{22}^{\sigma_{k-1} \overset{v_k}{\to} \sigma_k}\| e^{\nu(A_{22}^{\sigma_{k-1}})(\frac{t_k - t_{k-1}}{\epsilon})} \dots \|J_{22}^{\sigma_0 \overset{v_1}{\to} \sigma_1}\| e^{\nu(A_{22}^{\sigma_0})(\frac{t_1 - t_0}{\epsilon})} \|e_f(t_0)\| + O(\epsilon). \tag{116}$$

Now from equation (95) and (96) we obtain,

$$\|e_v(t)\| \leq \gamma_2^k e^{-\lambda_f(\frac{t-t_0}{\epsilon})} \|e_f(t_0)\| + O(\epsilon) \tag{117}$$

and from (32) and Lemma (4.2) the following can be concluded,

$$\|e_v(t)\| \leq \|e_s(t)\| + \|e_f(\tau)\| + O(\epsilon), \tag{118}$$

provided that the dwell-condition (92) is satisfied.

Proposition 6 *Under assumption 1, if the trajectories of the emergent dynamics are bounded i.e., when the inequality* (84) *is satisfied, then trajectories of the mean field dynamics are approximated for time* $t \geq t_0$ *by the trajectories of the emergent dynamics with the order of approximation* $O(\epsilon)$*, i.e.,*

$$x_s = x_e + O(\epsilon). \tag{119}$$

Proof: The mean field dynamics of the singularly perturbed switching-impulsive system (28)-(29) is

$$\dot{x}_s(t) = A_0^{\sigma_k} x_s(t) + B_1^{\sigma_k} e_v(t) \qquad \forall t \in [t_k, t_{k+1}), \forall k \in \mathbb{N} \tag{120}$$

$$x_s(t_k) = J_{11}^{\sigma_{k-1} \overset{v_k}{\to} \sigma_k} x_s(t_k^-). \tag{121}$$

The continuous dynamics for $t \in [t_k, t_{k+1})$, $\forall k \in \mathbb{N}$, can be approximated as follows [**?**],

$$x_s(t) = x_e(t) + O(\epsilon) = e^{A_0^{\sigma_k}(t-t_k)} x_e(t_k) + O(\epsilon) \tag{122}$$

also,

$$x_s(t_k^-) = e^{A_0^{\sigma_{k-1}}(t_k - t_{k-1})} x_e(t_{k-1}) + O(\epsilon). \tag{123}$$

Now from equations (121), (122) and (123) we have,

$$x_s(t) = e^{A_0^{\sigma_k}(t-t_k)} J_{11}^{\sigma_{k-1} \overset{v_k}{\to} \sigma_k} e^{A_0^{\sigma_{k-1}}(t_k - t_{k-1})} x_e(t_k) + O(\epsilon) \tag{124}$$

Following from equation (124), we obtain the following:

$$\begin{aligned} x_s(t) &= e^{A_0^{\sigma_k}(t-t_k)} J_{11}^{\sigma_{k-1} \overset{v_k}{\to} \sigma_k} e^{A_0^{\sigma_{k-1}}(t_k - t_{k-1})} \dots J_{11}^{\sigma_0 \overset{v_1}{\to} \sigma_1} e^{A_0^{\sigma_0}(t_1 - t_0)} x_e(t_0) + O(\epsilon) \\ \|x_s(t)\| &\leq e^{\nu(A_0^{\sigma_k})(t-t_k)} \|J_{11}^{\sigma_{k-1} \overset{v_k}{\to} \sigma_k}\| e^{\nu(A_0^{\sigma_{k-1}})(t_k - t_{k-1})} \dots \\ &+ \|J_{11}^{\sigma_0 \overset{v_1}{\to} \sigma_1}\| e^{\nu(A_0^{\sigma_0})(t_1 - t_0)} \|x_e(t_0)\| + O(\epsilon). \end{aligned} \tag{125}$$

From equation (34) and (87), it follows,

$$\begin{aligned} \|x_s(t)\| &\leq \gamma_1^k e^{\nu_s t_s + \nu_u t_u} \|x_s(t_0)\| + O(\epsilon) \\ \|x_s(t)\| &\leq \|x_e(t)\| + O(\epsilon) \end{aligned} \tag{126}$$

and (119) can be concluded. This proves that the emergent dynamics, which is the weighted average of the systems states, approximate the mean-field dynamics with O(ϵ) order of approximation. Also, it is important to note that the asymptotic stability of the fast dynamics is necessary for the $O(\epsilon)$ approximation of the original variables. Otherwise the $O(\epsilon)$ approximation we derived will not hold. □

References