Singular Perturbation Based Stability Analysis of Linear Systems

1 Problem Formulation

Let us consider a network of N identical linear systems called *agents*. The state of each agent at any time t is characterized by the scalar $x_i(t) \in \mathbb{R}$ such that its dynamics is given by

$$\dot{x_i}(t) = a_i x_i(t) + b_i u_i(t) \qquad \forall i \in \mathcal{N} = \{1, \cdots, N\}, \tag{1}$$

where $u_i \in \mathbb{R}$ is the control input.

Let $G = (V, E, A)$ be the weighted directed graph with the set of nodes $V = \{v_1, v_2, \cdots, v_N\}$, set of edges $E \subseteq V \times V$, and a weighted adjacency matrix $A_{ij} = [g_{ij}]$ where $g_{ij} > 0$ for $i \neq j$ and there exists a directed link from j to i, while $g_{ij} = 0$ for $i = j$. We consider the network with a dynamic topology that is time-varying, also commonly known as *switching networks* [3]. These time-varying networks can be modelled using the dynamic graph $G^{\sigma_k(t)}$ parameterized with switching signal $\sigma_k(t) : \mathbb{R} \to \mathcal{I}, k \in \mathbb{N}$ that takes values in the finite index set $\mathcal{I} = \{1, 2, \cdots, m\}$ i.e. the network can switch among a finite number of graphs, $\Gamma_m = \{G_1, G_2, \cdots, G_m\}$. Also, we assume at any time instance, the network topology is strongly connected and balanced. The network units are assumed to be connected via diffusive coupling, i.e., the i^{th} unit coupling, at any time instance, is given by

$$u_i(t) = -\gamma \sum_{j=1}^{N} g_{ij} (x_i - x_j), \tag{2}$$

where γ corresponds to the coupling gain between the units. Let $u(t) = (u_1(t), u_2(t), \cdots, u_N(t))^T \in \mathbb{R}^N$, then the feedback mechanism in a network at any time instance t with time varying topology is

$$u^{\sigma_k(t)} = -\gamma L^{\sigma_k} x, \tag{3}$$

where $\sigma_k(t) \in \mathcal{I}$ is the topological index associated with the elements of Γ_m and $L^{\sigma_k} = L(G^{\sigma_k(t)})$ is the Laplacian associated with the graph $G^{\sigma_k(t)}$. Also, let $x(t) = (x_1(t), x_2(t), \cdots x_N(t))^T \in \mathbb{R}^N$ be the collective state of the agents in the network, then the resulting network dynamics is

$$\dot{x}(t) = (\mathcal{A} - \gamma \mathcal{B} L^{\sigma_k}) x(t), \tag{4}$$

where $\mathcal{A} = \mathrm{diag}(a_1, \cdots a_n)$ and $\mathcal{B} = \mathrm{diag}(b_1 \cdots b_n)$.

2 Coordinate Transformation and Model Reformulation

By definition of graph Laplacian, the row sums of L are zero, hence, L has at least one zero eigenvalue, say $\lambda_1 = 0$ and the remaining eigenvalues of L^{σ_k} has non-negative real parts, i.e., $0 = \lambda_1^{\sigma_k} < \lambda_2^{\sigma_k} \leq \cdots \leq \lambda_N^{\sigma_k}$. Then corresponding to $\lambda_1^{\sigma_k} = 0$, there exists left eigenvector $(v_{l1}^{\sigma_k})$ and right eigenvector $(v_{r1}^{\sigma_k})$ of L^{σ_k}, such that ${v_{l1}^{\sigma_k}}^T L^{\sigma_k} = 0$ and $L^{\sigma_k} v_{r1}^{\sigma_k} = 0$, for each $\sigma_k \in \mathcal{I}$. These left and right eigenvectors are given by

$$v_{l1}^{\sigma_k} = \begin{bmatrix} \nu_1^{\sigma_k} \\ \nu_2^{\sigma_k} \\ \vdots \\ \nu_N^{\sigma_k} \end{bmatrix} \in \mathbb{R}^N \qquad v_{r1}^{\sigma_k} = \mathbf{1}_N := \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^N. \tag{5}$$

Moreover, since L is symmetric there exists a Jordan decomposition of the Laplacian L^{σ_k}, of the form $L^{\sigma_k} = U^{\sigma_k} \Lambda^{\sigma_k} {U^{\sigma_k}}^T$, where $U^{\sigma_k} \in \mathbb{R}^{N \times N}$ is nonsingular and $\Lambda^{\sigma_k} = \mathrm{diag}(\lambda_1^{\sigma_k}, \cdots \lambda_N^{\sigma_k})$. Furthermore, the matrix U^{σ_k} is composed of generalized right eigenvectors of the Laplacian matrix L^{σ_k} among which the first is $v_{r1}^{\sigma_k} = \mathbf{1}_N$. For further development, we decompose the matrix U^{σ_k} as

$$U^{\sigma_k} = [\mathbf{1}_N \quad V^{\sigma_k}] \tag{6}$$

where $V^{\sigma_k} \in \mathbb{R}^{N \times N-1}$. Similarly, the first row of the $U^{\sigma_k T}$ corresponds to the first left eigenvector of L^{σ_k}, $v_{l1}^{\sigma_k}$, which can be decomposed as follows,

$$U^{\sigma_k T} = \begin{bmatrix} v_{l1}^{\sigma_k T} \\ V^{\sigma_k T} \end{bmatrix} \tag{7}$$

and necessarily,

$$v_{l1}^{\sigma_k T} V^{\sigma_k} = 0 \qquad\qquad V^{\sigma_k T} V^{\sigma_k} = I_{N-1}.$$

The *weighted average* [4] of the systems' states x_s is

$$x_s = \sum_{i=1}^{N} \nu_i^{\sigma_k} x_i \tag{8}$$

and we introduce the transformation from [4], that leads to,

$$\bar{x} := U^{\sigma_k T} x. \tag{9}$$

Then, the partition is done using equation (7) to new coordinates, *i.e.,*

$$\bar{x} =: \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} := \begin{bmatrix} v_{l1}^{\sigma_k T} \\ V^{\sigma_k T} \end{bmatrix} x.$$

Remark 1 *It should be noted that due to the coordinate transformation (9) the switching system has changed into the* ***switching impulsive system*** *and the weighted average and the error terms experiences the jumps when topology changes. In other words, when there is a change in topology, the eigenvectors (transformation vectors) changes and this introduces, in addition to switching, jumps in the system dynamics.*

The coordinate $\bar{x}_1$ is equivalent to the weighted average (8)

$$x_s(t) = v_{l1}^{\sigma_k T} x(t), \qquad \forall t \in [t_k, t_{k+1}), \quad k \in \mathbb{N} \tag{10}$$

with the impulse,

$$x_s(t_k) = J_{11}^{\sigma_{k-1} \overset{v_k}{\to} \sigma_k} x(t_k^-), \qquad \forall k \geq 1 \tag{11}$$

where $J_{11}^{\sigma_{k-1} \overset{v_k}{\to} \sigma_k} x(t_k^-) = (v_{l1}^{\sigma_k} - v_{l1}^{\sigma_{k-1}})$, $x(t^-) = \lim_{\delta \to 0, \delta > 0} x(t - \delta)$, $0 = t_0 < t_1 < \cdots$ is the monotonically increasing and unbounded sequence of instants of discrete time events and $\sigma_k \in \mathcal{I}$ and $v_k \in \mathcal{J}$, where $\mathcal{J}$ is a finite set of indices.

The second coordinate $\bar{x}_2 = e_v \in \mathbb{R}^{N-1}$ is the projection of the *synchronization error* [2] into the subspace orthogonal to that of vector $v_{l1}^{\sigma_k}$. The synchronization error is the difference between the individual states and the state of the "average" unit, i.e.,

$$e^{\sigma_k}(t) = x(t) - \mathbf{1}_N x_s(t), \qquad \forall t \in [t_k, t_{k+1}), k \in \mathbb{N} \tag{12}$$

then,

$$e_v^{\sigma_k}(t) = V^{\sigma_k T} e^{\sigma_k}(t). \tag{13}$$

Similarly, the impulsive dynamics of the error is,

$$e_v^{\sigma_k}(t_k) = J_{22}^{\sigma_{k-1} \overset{v_k}{\to} \sigma_k} e(t_k^-), \tag{14}$$

with $J_{22}^{\sigma_{k-1} \overset{v_k}{\to} \sigma_k} = (V^{\sigma_k T} - V^{\sigma_{k-1} T})$.

In the new coordinate systems, differentiating equation (10) both sides and using the network dynamics (4) , as well as x from (11), we obtain,

$$
\begin{aligned}
\dot{x}_s = {v_{l1}^{\sigma_k}}^T \dot{x} = {v_{l1}^{\sigma_k}}^T \left(\mathcal{A} - \gamma \mathcal{B} L^{\sigma_k}\right) x(t) \\
\dot{x}_s &= {v_{l1}^{\sigma_k}}^T \left(\mathcal{A} - \gamma \mathcal{B} L^{\sigma_k}\right) \left(e^{\sigma_k} + \mathbf{1}_N x_s\right) \\
\dot{x}_s &= {v_{l1}^{\sigma_k}}^T \mathcal{A} \mathbf{1}_N x_s + {v_{l1}^{\sigma_k}}^T \left(\mathcal{A} - \gamma \mathcal{B} L^{\sigma_k}\right) e^{\sigma_k}
\end{aligned}
\tag{15}
$$

or equivalently,

$$
\dot{x}_s = {v_{l1}^{\sigma_k}}^T \mathcal{A} \mathbf{1}_N x_s + {v_{l1}^{\sigma_k}}^T \left(\mathcal{A} - \gamma \mathcal{B} L^{\sigma_k}\right) V^{\sigma_k} e_v^{\sigma_k}. \tag{16}
$$

The equation (13) and (14) represents the *mean-field dynamics*. Differentiating on both sides of equation (11) and substituting $\dot{x}_s$ from (13), we get,

$$
\begin{aligned}
\dot{e}^{\sigma_k} &= \dot{x} - \mathbf{1}_N \dot{x}_s \\
&= \left(\mathcal{A} - \gamma \mathcal{B} L^{\sigma_k}\right) \left(e^{\sigma_k} + \mathbf{1}_N x_s\right) - \mathbf{1}_N \left({v_{l1}^{\sigma_k}}^T \mathcal{A} \mathbf{1}_N x_s + {v_{l1}^{\sigma_k}}^T \left(\mathcal{A} - \gamma \mathcal{B} L^{\sigma_k}\right) e^{\sigma_k}\right) \\
&= \mathcal{A} e^{\sigma_k} + \mathcal{A} \mathbf{1}_N x_s - \gamma \mathcal{B} L^{\sigma_k} e^{\sigma_k} - \mathbf{1}_N {v_{l1}^{\sigma_k}}^T \mathcal{A} \mathbf{1}_N x_s - \mathbf{1}_N {v_{l1}^{\sigma_k}}^T \mathcal{A} e^{\sigma_k} + \gamma \mathbf{1}_N {v_{l1}^{\sigma_k}}^T \mathcal{B} L^{\sigma_k} e^{\sigma_k} \\
\dot{e}^{\sigma_k} &= -\gamma \mathcal{B} L^{\sigma_k} e^{\sigma_k} + \left(I_N - \mathbf{1}_N {v_{l1}^{\sigma_k}}^T\right) \mathcal{A} e^{\sigma_k} + \left(I_N - \mathbf{1}_N {v_{l1}^{\sigma_k}}^T\right) \mathcal{A} \mathbf{1}_N x_s + \gamma \mathbf{1}_N {v_{l1}^{\sigma_k}}^T \mathcal{B} L^{\sigma_k} e^{\sigma_k}.
\end{aligned}
\tag{17}
$$

Since the error $e_v^{\sigma_k}$ is the projection of the synchronization error dynamics e^{σ_k} onto the subspace orthogonal to that of the vector $v_{l1}^{\sigma_k}$, we have from equation (12),

$$
\dot{e}_v^{\sigma_k} = {V^{\sigma_k}}^T \left(-\gamma \mathcal{B} L^{\sigma_k} e^{\sigma_k} + \left(I_N - \mathbf{1}_N {v_{l1}^{\sigma_k}}^T\right) \mathcal{A} e^{\sigma_k} + \left(I_N - \mathbf{1}_N {v_{l1}^{\sigma_k}}^T\right) \mathcal{A} \mathbf{1}_N x_s + \gamma \mathbf{1}_N {v_{l1}^{\sigma_k}}^T \mathcal{B} L^{\sigma_k} e^{\sigma_k}\right).
$$

We can cancel out the terms, ${V^{\sigma_k}}^T \mathbf{1}_N {v_{l1}^{\sigma_k}}^T \mathcal{A} e^{\sigma_k}$, ${V^{\sigma_k}}^T \mathbf{1}_N {v_{l1}^{\sigma_k}}^T \mathcal{A} \mathbf{1}_N x_s$ and $\gamma {V^{\sigma_k}}^T \mathbf{1}_N {v_{l1}^{\sigma_k}}^T \mathcal{B} L^{\sigma_k} e^{\sigma_k}$ because ${V^{\sigma_k}}^T \mathbf{1}_N = (\mathbf{1}_N^T V^{\sigma_k})^T = 0$. Then we have,

$$
\dot{e}_v^{\sigma_k} = {V^{\sigma_k}}^T \mathcal{A} \mathbf{1}_N x_s + {V^{\sigma_k}}^T \mathcal{A} V^{\sigma_k} e_v^{\sigma_k} - \gamma {V^{\sigma_k}}^T \mathcal{B} L^{\sigma_k} V^{\sigma_k} e_v^{\sigma_k}. \tag{18}
$$

Now, dividing both sides of equation (16) by γ, we obtain the *error dynamics*,

$$
\epsilon \dot{e_v}^{\sigma_k} = \epsilon ({V^{\sigma_k}}^T \mathcal{A} V^{\sigma_k} e_v^{\sigma_k} + {V^{\sigma_k}}^T \mathcal{A} \mathbf{1}_N x_s) - {V^{\sigma_k}}^T \mathcal{B} L^{\sigma_k} V^{\sigma_k} e_v^{\sigma_k}. \tag{19}
$$

We assume that the coupling parameter is γ *large*. Then, $\epsilon = \frac{1}{\gamma} \ll 1$, is the perturbation parameter characterizing the time scale separation between the fast and slow dynamics in singularly perturbed systems. In other words, considering the equation (14) and (17) as follows,

$$
\begin{pmatrix} \dot{x_s}(t) \\ \epsilon \dot{e_v}(t) \end{pmatrix} = \begin{pmatrix} A_0^{\sigma_k} & B_1^{\sigma_k} \\ \epsilon B_2^{\sigma_k} & A_{22}^{\sigma_k} + \epsilon B_3^{\sigma_k} \end{pmatrix} \begin{pmatrix} x_s(t) \\ e_v(t) \end{pmatrix}, \qquad \forall t \in (t_k, t_{k+1}], k \in \mathbb{N} \tag{20}
$$

where for all $\sigma_k \in \mathcal{I}$ one has

$$
\begin{aligned}
&A_0^{\sigma_k} = {v_{l1}^{\sigma_k}}^T \mathcal{A} \mathbf{1}_N, && B_1^{\sigma_k} = {v_{l1}^{\sigma_k}}^T \left(\mathcal{A} - \gamma \mathcal{B} L^{\sigma_k}\right) V^{\sigma_k}, \\
&B_2^{\sigma_k} = {V^{\sigma_k}}^T \mathcal{A} \mathbf{1}_N, && B_3^{\sigma_k} = {V^{\sigma_k}}^T \mathcal{A} V^{\sigma_k}, \qquad A_{22}^{\sigma_k} = -{V^{\sigma_k}}^T \mathcal{B} L^{\sigma_k} V^{\sigma_k},
\end{aligned}
$$

and it can be clearly seen that the matrix $A_0^{\sigma_k}$, $B_1^{\sigma_k}$, $B_2^{\sigma_k}$ and $B_3^{\sigma_k}$ are of the order $\mathcal{O}(1)$, while $A_{22}^{\sigma_k}$ is of the order $\mathcal{O}(\frac{1}{\epsilon})$. The impulsive dynamics is as follows,

$$
\begin{pmatrix} x_s(t_k) \\ e_v(t_k) \end{pmatrix} = J^{\sigma_{k-1} \xrightarrow{v_k} \sigma_k} \begin{pmatrix} x(t_k^-) \\ e(t_k^-) \end{pmatrix} \tag{21}
$$

with the jump matrix,

$$
J^{\sigma_{k-1} \xrightarrow{v_k} \sigma_k} = \begin{pmatrix} {v_{l1}^{\sigma_k}}^T - {v_{l1}^{\sigma_{k-1}}}^T & \mathbf{0}_{1,N} \\ \mathbf{0}_{N-1,N} & {V^{\sigma_k}}^T - {V^{\sigma_{k-1}}}^T \end{pmatrix}. \tag{22}
$$

Hence, the system has standard singular perturbations form with reduced (slow) variable x_s and boundary layer (fast) variable $e_v^{\sigma_k}$. This time scale separation on the network of linear homogeneous systems is due to the large coupling parameter γ and the coordinate transformation (9).

Reduced Ordered Systems

In order to obtain the reduced order model in single time scale from the equation (18), we set $\epsilon = 0$. Then, we have the slow subsystem as follows,

$$\dot{x_s} = A_0^{\sigma_k} x_s(t) \qquad \forall t \in [t_k, t_{k+1}), k \in \mathbb{N} \tag{23}$$

with the impulsive dynamics as follows:

$$x(t_k) = ({v_{l1}^{\sigma_k}}^T - {v_{l1}^{\sigma_{k-1}}}^T) x(t_k^-), \qquad \forall k \geq 1 \tag{24}$$

The dynamics (23) is also referred as *emergent dynamics*, which is the dynamics of the mean field unit restricted to the synchronization manifold.

3 Stability

In this section, we analyze the stability of the system (20) and (21) and we impose the following assumption on the dynamics of the original system, related to the stability of the slow and fast subsystems.

Assumption 1: $\sum_{i=1}^{N} a_i < 0$ and $\mathcal{B}$ is a positive definite matrix.

Remark 2 *The assumption 1 is imposed in the original system so that the matrix $A_0^{\sigma_k}$ and $A_{22}^{\sigma_k}$ are Hurwitz, in other words, the slow and fast dynamics are stable. It is important to note, as a consequence of the assumption 1, we can have a stable network dynamics even if we have unstable individual agents in the network.*

Since the matrix $A_0^{\sigma_k}$ and $A_{22}^{\sigma_k}$ are Hurwitz, there exists symmetric positive definite matrices $Q_s^{\sigma_k} \geq I_1$ and $Q_f^{\sigma_k} \geq I_{N-1}$, $\forall \sigma_k \in \mathcal{I}$ and positive numbers $\lambda_s^{\sigma_k}$ and $\lambda_f^{\sigma_k}$, $\forall \sigma_k \in \mathcal{I}$, such that,

$$\begin{aligned} {A_0^{\sigma_k}}^T Q_s^{\sigma_k} + Q_s^{\sigma_k} A_0^{\sigma_k} &\leq -2\lambda_s^{\sigma_k} Q_s^{\sigma_k} \\ {A_{22}^{\sigma_k}}^T Q_f^{\sigma_k} + Q_f^{\sigma_k} A_{22}^{\sigma_k} &\leq -2\lambda_f^{\sigma_k} Q_f^{\sigma_k} \end{aligned}$$

Let us denote, $\lambda_s = \min_{\sigma_k \in \mathcal{I}} \lambda_s^{\sigma_k}$ and $\lambda_f = \min_{\sigma_k \in \mathcal{I}} \lambda_f^{\sigma_k}$. Then, from [1] following proposition follows which states the existence of composite Lyapunov function.

Proposition 1 *Under Assumption 1, there exists $\epsilon_1 \geq 0$, such that ,*

$$V(x_s, e_v) = x_s^T Q_s^{\sigma_k} x_s + e_v^T Q_f^{\sigma_k} e_v \tag{25}$$

is a Lyapunov function for the system (20), for all $\epsilon \in (0, \epsilon_1]$, where, $\epsilon_1 = \frac{\lambda_f}{\frac{(b_1+b_2)^2}{4\lambda_s} + b_3}$ and $\forall$ $\sigma_k \in \mathcal{I}$, $b_1^{\sigma_k} = \|(Q_s^{\sigma_k})^{\frac{1}{2}} B_1^{\sigma_k} (Q_f^{\sigma_k})^{-\frac{1}{2}}\|$, $b_2^{\sigma_k} = \|(Q_f^{\sigma_k})^{\frac{1}{2}} B_2^{\sigma_k} (Q_s^{\sigma_k})^{-\frac{1}{2}}\|$, $b_3^{\sigma_k} = \|(Q_f^{\sigma_k})^{\frac{1}{2}} B_3^{\sigma_k} (Q_f^{\sigma_k})^{-\frac{1}{2}}\|$ and $b_j = \max_{\sigma_k \in \mathcal{I}} b_j^{\sigma_k}, j \in \{1, 2, 3\}$.

Also, denoting the Lyapunov function for decoupled dynamics as, $W_s = \sqrt{x_s^T Q_s^{\sigma_k} x_s}$ and $W_f = \sqrt{e_v^T Q_f^{\sigma_k} e_v}$, for all $t \in [t_k, t_{k+1}), k \in \mathbb{N}$ and assuming that after each switching the system remains in the current topology for time 'τ_k' i.e., $t_{k+1} - t_k = \tau_k$ $\forall k \in \mathbb{N}$, the Lemma 4 from [1], provides us the following results.

$$W_s(t_{k+1}^-) \leq W_s(t_k)(e^{-\lambda_s \tau_k} + \epsilon\beta_3) + W_f(t_k)\epsilon(\beta_2 + \beta_3) \tag{26}$$

$$W_f(t_{k+1}^-) \leq W_s(t_k)\epsilon\beta_1 + W_f(t_k)(e^{\frac{-\lambda_f}{\epsilon}\tau_k} + \epsilon\beta_1), \tag{27}$$

where, $\beta_1 = \frac{\sqrt{b_2^2 + b_3^2}}{\lambda_f}$, $\beta_2 = \frac{b_1}{\lambda_f - \epsilon_2 \lambda_s}$ and $\beta_3 = \frac{b_1 \beta_1}{\lambda_s}$, where, $\epsilon_2 \in (0, \epsilon_1] \cap (0, \frac{\lambda_f}{\lambda_s})$. The equations (26) and (27) characterize the variation of the decoupled Lyapunov function W_s and W_f during the continuous dynamics between two events.

Then during the jump, variations of W_s and W_f can be characterized as follows:

Lemma 3.1 *Let us consider a sequence $(t_k)_{k\geq 0}$ of even times, then for all $k \geq 1$, we have,*

$$W_s(t_k) \leq \rho_1 W_s(t_k^-) \tag{28}$$

$$W_f(t_k) \leq \rho_2 W_f(t_k^-) \tag{29}$$

where $\rho_1 = \max\limits_{i,i^{'}\in\mathcal{I},j\in\mathcal{J}} \|(Q_s^{i^{'}})^{\frac{1}{2}}(v_{l1}^{i} - v_{l1}^{i^{'}})(Q_s^i)^{-\frac{1}{2}}\|$ *and* $\rho_2 = \max\limits_{i,i^{'}\in\mathcal{I},j\in\mathcal{J}} \|(Q_f^{i^{'}})^{\frac{1}{2}}(V^i - V^{i^{'}})(Q_f^i)^{-\frac{1}{2}}\|$.

Proof:

$$\begin{aligned} W_s(t_k) &= \sqrt{x_s(t_k)^T Q_s^{\sigma_k} x_s(t_k)} = \|(Q_s^{\sigma_k})^{\frac{1}{2}} x_s(t_k)\| \\ &\leq \|(Q_s^{\sigma_k})^{\frac{1}{2}}(v_{l1}^{\sigma_k} - v_{l1}^{\sigma_{k-1}})x(t_k^-)\| \\ &\leq \|(Q_s^{\sigma_k})^{\frac{1}{2}}(v_{l1}^{\sigma_k} - v_{l1}^{\sigma_{k-1}})(Q_s^{\sigma_k})^{-\frac{1}{2}}\| W_s(t_k^-) \leq \rho_1 W_s(t_k^-). \end{aligned}$$

The inequality (29) is obtained similarly.

Then multiplying bothsides of (26) and (27) by ρ_1 and ρ_2, respectively and from (28) and (29), we have

$$W_s(t_{k+1}) \leq \rho_1 W_s(t_{k+1}^-) \leq \rho_1(e^{-\lambda_s\tau_k} + \epsilon\beta_3)W_s(t_k) + \epsilon\rho_1(\beta_2+\beta_3)W_f(t_k) \tag{30}$$

$$W_f(t_{k+1}) \leq \rho_2 W_f(t_{k+1}^-) \leq \epsilon\rho_2\beta_1 W_s(t_k) + \rho_2(e^{\frac{-\lambda_f}{\epsilon}\tau_k} + \epsilon\beta_1)W_f(t_k), \tag{31}$$

Then it follows, for any $\tau \geq 0$,

$$\begin{pmatrix} W_s(t_{k+1}) \\ W_f(t_{k+1}) \end{pmatrix} \leq M_\tau \begin{pmatrix} W_s(t_k) \\ W_f(t_k) \end{pmatrix}, \tag{32}$$

where, $M_\tau = \begin{pmatrix} \rho_1(e^{-\lambda_s\tau} + \epsilon\beta_3) & \epsilon\rho_1(\beta_2+\beta_3) \\ \epsilon\rho_2\beta_1 & \rho_2(e^{-\frac{\lambda_f}{\epsilon}\tau} + \epsilon\beta_1) \end{pmatrix}$. Then from the equation (32) we can express the relation between $W_s(t_k)$ and $W_f(t_k)$ with $W_s(t_0)$ and $W_{s(t_0)}$, respectively, as follows:

$$\begin{pmatrix} W_s(t_k) \\ W_f(t_k) \end{pmatrix} \leq M_{\tau_{k-1}} \cdots M_\tau \begin{pmatrix} W_s(t_0) \\ W_f(t_0) \end{pmatrix} \iff \begin{pmatrix} W_s(t_k) \\ W_f(t_k) \end{pmatrix} \leq (M_{\tau^*})^k \begin{pmatrix} W_s(t_0) \\ W_f(t_0) \end{pmatrix}, \forall k \in \mathbb{N}.$$

Hence from (25), we have

$$\begin{aligned} V(t_{k+1}) &= W_s^2(t_{k+1}) + W_f^2(t_{k+1}) \\ &\leq \|(M_{\tau^*})\|_\infty^2 (W_s(t_k) + W_f(t_k))^2 \\ &\leq 2\|(M_{\tau^*})\|_\infty^2 V(t_k) \end{aligned}$$

This shows that the V is decreasing. This implies that if the positive matrix M_{τ^*} is Schur, then both the sequences, $(W_s(t_k))_{k\geq 0}$ and $(W_f(t_k)_{k\geq})0$ go to 0, and the system (20) and (21) are globally asymptotically stable. Clearly, a necessary condition for (asymptotic) stability under arbitrary switching is that all of the individual subsystems are (asymptotically) stable.

It is clear that the stability of the (20) and (21) can be investigating by studying the spectral properties of the matrix M_{τ^*}. Hence, the bounds on the value of τ^* so that the matrix M_{τ^*} is Schur ensure that the singularly perturbed system is stable. By definition, the positive matrix M_{τ^*} is Schur iff there exists $p \in \mathbb{R}_+^2$, such that $M_{\tau^*}p < p$ [5]. Taking the form of p as $(1, a\epsilon)^T$ with $a > \rho_1(\beta_2+\beta_3)$, then $M_{\tau^*}p < p$ is equivalent to

$$\rho_1(e^{-\lambda_s\tau^*} + \epsilon\beta_3) + a\epsilon^2\rho_2\beta_1 < 1 \tag{33}$$

$$\epsilon\rho_1(\beta_2+\beta_3) + a\rho_2\epsilon(e^{-\frac{\lambda_f}{\epsilon}\tau^*} + \epsilon\beta_1) < a\epsilon \tag{34}$$

Case 1:
From equation (33) we have,

$$\tau^* > -\frac{1}{\lambda_s}\ln\left(\frac{1 - \epsilon\rho_1\beta_3 - a\epsilon^2\rho_2\beta_1}{\rho_1}\right) = \frac{\ln(\rho_1)}{\lambda_s} + \eta_1(\epsilon) \tag{35}$$

where $\eta_1(\epsilon) = -\frac{\ln(1-\epsilon\rho_1\beta_3 - a\epsilon^2\rho_2\beta_1)}{\lambda_s} = \mathcal{O}(\epsilon)$ and it is defined only for $\epsilon < \epsilon_3$ such that $\epsilon_3 = \frac{-\rho_1\beta_3+\sqrt{\rho_1^2\beta_3^2+4a\rho_2\beta_1}}{2a\rho_2\beta_1}$.

Similarly from equation (34),

$$\tau^* > \frac{\epsilon}{\lambda_f}\ln\left(\frac{a\rho_2}{a-\rho_1(\beta_2+\beta_3)-a\epsilon\rho_2\beta_1}\right) \tag{36}$$

As ϵ goes to 0, it is clear that the right hand side of equation (36) also goes to 0. Since, from equation (35), we have $\tau^* > \frac{\ln(\rho_1)}{\lambda_s} + \eta_1(\epsilon) > \frac{\ln(\rho_1)}{\lambda_s}$, we can find $\epsilon_4 > 0$ such that for all $\epsilon \in (0, \epsilon_4)$ following holds

$$\frac{\ln(\rho_1)}{\lambda_s} > \frac{\epsilon}{\lambda_f}\ln\left(\frac{a\rho_2}{a-\rho_1(\beta_2+\beta_3)-a\epsilon\rho_2\beta_1}\right).$$

Hence setting $\epsilon_1^* = \min(\epsilon_2, \epsilon_3, \epsilon_4)$, we have for all $\epsilon \in (0, \epsilon_1^*)$,

$$\tau^* > \ln(\frac{\rho_1}{\lambda_s}) + \eta_1(\epsilon). \tag{37}$$

Case 2:
When $\rho_1 = 1$, it is sufficient to find the conditions such that the equation (33) and (34) holds and we have,

$$\tau^* > \eta_1(\epsilon) \qquad \text{and} \qquad \tau^* > \frac{\epsilon}{\lambda_f}\ln\left(\frac{a\rho_2}{a-(\beta_2+\beta_3)-a\epsilon\rho_2\beta_1}\right).$$

Hence, for all $\epsilon \in (0, \epsilon_2^*)$ where $\epsilon_2^* = \epsilon_3$, we can have the condition as follows:

$$\tau^* > \eta_2(\epsilon), \qquad \text{where} \qquad \eta_2 = \max\left(\eta_1(\epsilon), \frac{\epsilon}{\lambda_f}\ln\left(\frac{a\rho_2}{a-(\beta_2+\beta_3)-a\epsilon\rho_2\beta_1}\right)\right). \tag{38}$$

Case 3:
To derive the necessary condition for a matrix M_{τ^*} to be Schur when $\rho_1 < 1$, we look for $p = (1,1)^T$. Then $M_{\tau^*}p < p$ is equivalent to

$$\rho_1(e^{-\lambda_s\tau} + \epsilon\beta_3) + \epsilon\rho_2\beta_1 < 1 \tag{39}$$

$$\epsilon\rho_1(\beta_2+\beta_3) + \rho_2(e^{-\frac{\lambda_f}{\epsilon}\tau} + \epsilon\beta_1) < 1 \tag{40}$$

which in turn is equivalent to

$$\tau^* > \frac{1}{\lambda_s}\ln\left(\frac{\rho_1}{1-\epsilon\rho_2\beta_1-\rho_1\epsilon\beta_3}\right) \tag{41}$$

$$\tau^* > \frac{\epsilon}{\lambda_f}\ln\left(\frac{\rho_2}{1-\epsilon\rho_1(\beta_2+\beta_3)-\rho_2\epsilon\beta_1}\right). \tag{42}$$

Since $\rho_1 < 1$, the inequality (41) holds for all $\epsilon \in (0, \epsilon_5)$, where $\epsilon_5 = \frac{1-\rho_1}{\rho_2\beta_1+\rho_1\beta_2}$. Then the second inequality is equivalent to $\tau > \eta_3(\epsilon)$ where

$$\eta_3(\epsilon) = \frac{\epsilon}{\lambda_f}\ln\left(\frac{\rho_2}{1-\epsilon\rho_1(\beta_2+\beta_3)-\rho_2\epsilon\beta_1}\right). \tag{43}$$

It is clear that $\eta_3(\epsilon) = \mathcal{O}(\epsilon)$ and is well defined for $\epsilon < \epsilon_6$ given by

$$\epsilon_6 = \tag{44}$$

Stability of Reduced order system

The reduced order system (slow dynamics) with single time scale is

$$\dot{x_s}(t) = {v_{l1}^{\sigma_k}}^T \mathcal{A}\mathbf{1}_N x_s(t) = A_0^{\sigma_k} x_s(t) \tag{45}$$

Similarly, as above let $V(x_s) = x_s^T Q_s^{\sigma_k} x_s$ be the Lyapuov function. Let us define, $W_s = \sqrt{x_s(t)^T Q_s^{\sigma_k} x_s}$ for all $t \in [t_k, t_{k+1}), k \in \mathbb{N}$. Also, it follows that $W_s(t_{k+1}) \leq W_s(t_k)e^{-\lambda_s t}$, which implies, $W_s(t_k) \leq W_s(t_0)e^{-\lambda_s k t}$. Then for all $k \in \mathbb{N}$, $W_s(t_k) \to 0$ as $k \to \infty$. Hence, the reduced order system (24) is globally asymptotically stable.

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