Singular Perturbation Based Stability Analysis of Linear Heterogeneous Systems

1 Problem Formulation

System Model

Let us consider a network of N heterogeneous linear systems called *agents*. The state of each agent at any time t is characterized by the scalar $x_i(t) \in \mathbb{R}^{n_x}$ such that its dynamics is given by

$$\dot{x_i}(t) = A_i x_i(t) + u_i(t) \qquad \forall i \in \mathcal{N} = \{1, \cdots, N\}, \tag{1}$$

where $u_i \in \mathbb{R}^{n_x}$ is the control input and $A_i \in \mathbb{R}^{n_x \times n_x}$.

Remark 1 *We assume that the systems are heterogeneous, that is, the matrices A_i $\forall i \in \mathcal{N}$ are different but of the same dimensions $\mathbb{R}^{n_x \times n_x}$.*

Network Model

Let $G = \left(V, E, A^d\right)$ be the weighted directed graph with the set of nodes $V = \{v_1, v_2, \cdots, v_N\}$, set of edges $E \subseteq V \times V$, and a weighted adjacency matrix $A^d_{ij} = [g_{ij}]_{i,j\in\mathcal{N}}$, where, $g_{ij} > 0$ for $i \neq j$ and there exists a directed link from j to i, while $g_{ij} = 0$ for $i = j$. We consider the network with a dynamic topology that is time-varying, also commonly known as *switching networks* [5]. These time-varying networks can be modelled using the dynamic graph $G^{\sigma_k(t)}$ parameterized with switching signal $\sigma_k(t) : \mathbb{R} \to \mathcal{I}, k \in \mathbb{N}$ that takes values in the index set $\mathcal{I} = \{1, 2, \cdots, n\}$ i.e. the network can switch among a finite number of graphs, $\Gamma_n = \{G_1, G_2, \cdots, G_n\}$. Also, we assume at any time instance, the network topology is strongly connected. The network units are assumed to be connected via diffusive coupling, i.e., the i^{th} unit coupling, at any time instance, is given by

$$u_i(t) = -\gamma \sum_{j=1}^{N} g_{ij} \left(x_i - x_j\right), \tag{2}$$

where $\gamma \in \mathbb{R}$ corresponds to the coupling gain between the units.

Let us represent the state and control vectors in compact notation as follows:

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \in \mathbb{R}^{n_x N}, \qquad u = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} \in \mathbb{R}^{mN},$$

and also we represent the state matrix in block diagonal form as follows,

$$\mathcal{A} = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & A_N \end{bmatrix} \in \mathbb{R}^{n_x N \times n_x N}.$$

Hence the control input can be represented in the following compact notation,

$$u^{\sigma_k} = -\gamma (L^{\sigma_k} \otimes I_{n_x}) x, \tag{3}$$

where $\sigma_k \in \mathcal{I}$ is the topological index associated with the elements of Γ_n, $L^{\sigma_k} = L(G^{\sigma_k(t)})$ is the Laplacian associated with the graph $G^{\sigma_k(t)}$. Then the resulting network dynamics is

$$\dot{x}(t) = (\mathcal{A} - \gamma(L^{\sigma_k} \otimes I_{n_x}))x(t). \tag{4}$$

2 Network Dynamics

Coordinate Transformation

By definition of graph Laplacian, for every $\sigma_k \in \mathcal{I}$ the row sums of L^{σ_k} are zero, hence, L^{σ_k} has at least one zero eigenvalue, say $\lambda_1^{\sigma_k} = 0$ and assuming that the graph is strongly connected, the remaining eigenvalues of L^{σ_k} has non-negative real parts, i.e., $0 = \Re e\left\{\lambda_1^{\sigma_k}\right\} < \Re e\left\{\lambda_2^{\sigma_k}\right\} \leq \cdots \leq \Re e\left\{\lambda_N^{\sigma_k}\right\}$. Moreover, if the graph is directed and has a spanning tree, the eigenvalue $\lambda_1^{\sigma_k} = 0$ is simple and there exists left eigenvector $(v_{l1}^{\sigma_k})$ and right eigenvector $(v_{r1}^{\sigma_k})$ of L^{σ_k}, such that ${v_{l1}^{\sigma_k}}^T L^{\sigma_k} = 0$ and $L^{\sigma_k} v_{r1}^{\sigma_k} = 0$, for each $\sigma_k \in \mathcal{I}$. These left and right eigenvectors are given by

$$v_{l1}^{\sigma_k} = \begin{bmatrix} \nu_1^{\sigma_k} \\ \nu_2^{\sigma_k} \\ \vdots \\ \nu_N^{\sigma_k} \end{bmatrix} \in \mathbb{R}^N \qquad v_{r1}^{\sigma_k} = \mathbf{1}_N := \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^N. \tag{5}$$

Furthermore, there exists a Jordan decomposition of the Laplacian L^{σ_k}, of the form $L^{\sigma_k} = U^{\sigma_k} \Lambda^{\sigma_k} {U^{\sigma_k}}^{-1}$, where $U^{\sigma_k} \in \mathbb{R}^{N \times N}$ is nonsingular and $\Lambda^{\sigma_k} \in \mathbb{C}^{N \times N}$ is the block diagonal Jordan matrix of the following form

$$\Lambda = \begin{bmatrix} \Lambda_1 & & 0 \\ & \ddots & \\ 0 & & \Lambda_m \end{bmatrix}$$

where, for the networks with rooted-out branching, the algebraic multiplicity of the $\lambda_1 = 0$ is 1 and Λ_i are Jordan blocks of appropriate dimensions. In addition, the matrix U^{σ_k} is composed of generalized right eigenvectors of the Laplacian matrix L^{σ_k} among which the first is $v_{r1}^{\sigma_k} = \mathbf{1}_N$. For further development, we decompose the matrix U^{σ_k} as

$$U^{\sigma_k} = [\mathbf{1}_N \quad V^{\sigma_k}] \tag{6}$$

where $V^{\sigma_k} \in \mathbb{R}^{N \times N-1}$. Similarly, the first row of the ${U^{\sigma_k}}^{-1}$ corresponds to the first left eigenvector of L^{σ_k}, $v_{l1}^{\sigma_k}$, which can be decomposed as follows,

$${U^{\sigma_k}}^{-1} = \begin{bmatrix} {v_{l1}^{\sigma_k}}^T \\ {V^{\sigma_k}}^\dagger \end{bmatrix} \tag{7}$$

and necessarily,

$${v_{l1}^{\sigma_k}}^T V^{\sigma_k} = 0 \qquad {V^{\sigma_k}}^\dagger V^{\sigma_k} = I_{N-1}. \tag{8}$$

The *weighted average* [6] of the systems' states x_s is

$$x_s = \sum_{i=1}^{N} \nu_i^{\sigma_k} x_i \tag{9}$$

and we introduce the transformation from [6], that leads to,

$$\bar{x} := {\mathcal{U}^{\sigma_k}}^{-1} x \tag{10}$$

where $\mathcal{U}^{\sigma_k} = (U^{\sigma_k} \otimes I_{n_x}) \in \mathbb{R}^{n_x N \times n_x N}$. Then, the partition is done using equation (7) to new coordinates, *i.e.*,

$$\bar{x} =: \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} := \begin{bmatrix} {v_{l1}^{\sigma_k}}^T \otimes I_{n_x} \\ {V^{\sigma_k}}^\dagger \otimes I_{n_x} \end{bmatrix} x.$$

Remark 2 *It should be noted that due to the coordinate transformation (9) the switching system has changed into the* ***switching impulsive system*** *and the weighted average and the error terms experiences the jumps when topology changes. In other words, when there is a switching in topology, the eigenvectors (transformation vectors) changes and this introduces, in addition to switching, jumps in the system dynamics.*

The coordinates $\bar{x}_1$ is equivalent to the weighted average (9)

$$x_s(t) = (v_{l1}^{\sigma_k T} \otimes I_{n_x})x(t), \qquad \forall t \in [t_k, t_{k+1}), \quad k \in \mathbb{N} \tag{11}$$

with the impulsive dynamics,

$$x_s(t_k) = J_{11}^{\sigma_{k-1} \overset{v_k}{\rightarrow} \sigma_k} x(t_k^-), \qquad \forall k \geq 1 \tag{12}$$

where $J_{11}^{\sigma_{k-1} \overset{v_k}{\rightarrow} \sigma_k} = \left((v_{l1}^{\sigma_k T} - v_{l1}^{\sigma_{k-1} T}) \otimes I_{n_x} \right)$, $x(t^-) = \lim_{\delta \to 0, \delta > 0} x(t-\delta)$, $0 = t_0 < t_1 < \cdots$ is the monotonically increasing and unbounded sequence of instants of discrete time events and $\sigma_k \in \mathcal{I}$ and $v_k \in \mathcal{J}$, where $\mathcal{J}$ is a finite set of indices.

The second coordinate $\bar{x}_2 = e_v \in \mathbb{R}^{n_x(N-1)}$ is the projection of the *synchronization error* [4] into the subspace orthogonal to that of vector $v_{l1}^{\sigma_k}$. The synchronization error is the difference between the individual states and the state of the "average" unit, i.e.,

$$e(t) = x(t) - (\mathbf{1}_N \otimes I_{n_x})x_s(t), \qquad \forall t \in [t_k, t_{k+1}), \quad k \in \mathbb{N} \tag{13}$$

then,

$$e_v(t) = (V^{\sigma_k \dagger} \otimes I_{n_x})e(t) \iff e(t) = (V^{\sigma_k} \otimes I_{n_x})e_v(t). \tag{14}$$

Similarly, the impulsive dynamics of the error is,

$$e_v(t_k) = J_{22}^{\sigma_{k-1} \overset{v_k}{\rightarrow} \sigma_k} e(t_k^-), \tag{15}$$

with $J_{22}^{\sigma_{k-1} \overset{v_k}{\rightarrow} \sigma_k} = \left((V^{\sigma_k \dagger} - V^{\sigma_{k-1} \dagger}) \otimes I_{n_x} \right)$.

Mean-Field Dynamics

In the new coordinate system, differentiating equation (11) both sides and using the network dynamics (4) , as well as substituting x from (13), we obtain $\forall t \in [t_k, t_{k+1}), k \in \mathbb{N}$,

$$\begin{aligned}
\dot{x}_s &= (v_{l1}^{\sigma_k T} \otimes I_{n_x})\dot{x} = (v_{l1}^{\sigma_k T} \otimes I_{n_x}) \left(\mathcal{A} - \gamma(L^{\sigma_k} \otimes I_{n_x})\right) x(t) \\
\dot{x}_s &= (v_{l1}^{\sigma_k T} \otimes I_{n_x}) \left(\mathcal{A} - \gamma(L^{\sigma_k} \otimes I_{n_x})\right) (e + (\mathbf{1}_N \otimes I_{n_x})x_s) \\
\dot{x}_s &= \left((v_{l1}^{\sigma_k T} \otimes I_{n_x})\mathcal{A}(\mathbf{1}_N \otimes I_{n_x}) \right) x_s + (v_{l1}^{\sigma_k T} \otimes I_{n_x})\mathcal{A}e
\end{aligned} \tag{16}$$

also, equivalently substituting the value of e from equation (14), we get

$$\dot{x}_s = \left((v_{l1}^{\sigma_k T} \otimes I_{n_x})\mathcal{A}(\mathbf{1}_N \otimes I_{n_x}) \right) x_s + (v_{l1}^{\sigma_k T} \otimes I_{n_x})\mathcal{A}(V^{\sigma_k} \otimes I_{n_x})e_v. \tag{17}$$

The equation (16) and (17) represents the *mean-field dynamics*.

Synchronization Error Dynamics

Differentiating on both sides of equation (13) and substituting $\dot{x}_s$ from (16) $\forall t \in [t_k, t_{k+1}), k \in \mathbb{N}$ it follows:

$$\begin{aligned}
\dot{e} &= \dot{x} - (\mathbf{1}_N \otimes I_{n_x})\dot{x}_s \\
&= (\mathcal{A} - \gamma(L^{\sigma_k} \otimes I_{n_x}))(e + (\mathbf{1}_N \otimes I_{n_x})x_s) - \\
&\quad (\mathbf{1}_N \otimes I_{n_x}) \left[((v_{l1}^{\sigma_k T} \otimes I_{n_x})\mathcal{A}(\mathbf{1}_N \otimes I_{n_x}))x_s + (v_{l1}^{\sigma_k T} \otimes I_{n_x})\mathcal{A}e \right] \\
&= \left(\mathcal{A}(\mathbf{1}_N \otimes I_{n_x}) - (\mathbf{1}_N v_{l1}^{\sigma_k T} \otimes I_{n_x})\mathcal{A}(\mathbf{1}_N \otimes I_{n_x}) \right) x_s + \left(\mathcal{A} - (\mathbf{1}_N v_{l1}^{\sigma_k T} \otimes I_{n_x})\mathcal{A} \right) e \\
&\quad - \gamma(L^{\sigma_k} \otimes I_{n_x})e.
\end{aligned} \tag{18}$$

Since the error e_v is the projection of the synchronization error e onto the subspace orthogonal to that of the vector $v_{l1}^{\sigma_k}$, hence from equations (14) and (18), we have,

$$\dot{e}_v = ({V^{\sigma_k}}^{\dagger} \otimes I_{n_x})\Big\{\Big(\mathcal{A}(\mathbf{1}_N \otimes I_{n_x}) - (\mathbf{1}_N {v_{l1}^{\sigma_k}}^T \otimes I_{n_x})\mathcal{A}(\mathbf{1}_N \otimes I_{n_x})\Big)x_s + \Big(\mathcal{A} - (\mathbf{1}_N {v_{l1}^{\sigma_k}}^T \otimes I_{n_x})\mathcal{A}\Big)e + \\ - \gamma(L^{\sigma_k} \otimes I_{n_x})e\Big\}$$

Also, we know that ${V^{\sigma_k}}^{\dagger}\mathbf{1}_N = 0$ and substituting the value of e from equation (14), it follows

$$\begin{aligned}
\dot{e}_v &= -\gamma({V^{\sigma_k}}^{\dagger} L^{\sigma_k} V^{\sigma_k} \otimes I_{n_x})e_v + ({V^{\sigma_k}}^{\dagger} \otimes I_{n_x})\mathcal{A}(\mathbf{1}_N \otimes I_{n_x})x_s + \\
&\quad ({V^{\sigma_k}}^{\dagger} \otimes I_{n_x})\mathcal{A}(V^{\sigma_k} \otimes I_{n_x})e_v \\
\dot{e}_v &= -\gamma(\Lambda^{\sigma_k'} \otimes I_{n_x})e_v + ({V^{\sigma_k}}^{\dagger} \otimes I_{n_x})\mathcal{A}(\mathbf{1}_N \otimes I_{n_x})x_s + \\
&\quad ({V^{\sigma_k}}^{\dagger} \otimes I_{n_x})\mathcal{A}(V^{\sigma_k} \otimes I_{n_x})e_v
\end{aligned} \tag{19}$$

where, $\Lambda^{\sigma_k'} = \operatorname{diag}\left(\lambda_2^{\sigma_k} \cdots \lambda_N^{\sigma_k}\right)$.

Switching and Impulsive Dynamics of Networked System

Consider the equation (17) and (19) as follows,

$$\begin{pmatrix} \dot{x_s}(t) \\ \dot{e_v}(t) \end{pmatrix} = \begin{pmatrix} A_0^{\sigma_k} & B_1^{\sigma_k} \\ B_2^{\sigma_k} & A_{22}^{\sigma_k} + B_3^{\sigma_k} \end{pmatrix} \begin{pmatrix} x_s(t) \\ e_v(t) \end{pmatrix}, \qquad \forall t \in (t_k, t_{k+1}]\,, k \in \mathbb{N} \tag{20}$$

where for all $\sigma_k \in \mathcal{I}$ one has

$$\begin{aligned}
A_0^{\sigma_k} &= ({v_{l1}^{\sigma_k}}^T \otimes I_{n_x})\mathcal{A}(\mathbf{1}_N \otimes I_{n_x}) = \sum_{i=1}^{N} v_{l1_i}^{\sigma_k} A_i, \qquad B_1^{\sigma_k} = ({v_{l1}^{\sigma_k}}^T \otimes I_{n_x})\mathcal{A}(V^{\sigma_k} \otimes I_{n_x}) \\
B_2^{\sigma_k} &= ({V^{\sigma_k}}^{\dagger} \otimes I_{n_x})\mathcal{A}(\mathbf{1}_N \otimes I_{n_x}), \qquad B_3^{\sigma_k} = ({V^{\sigma_k}}^{\dagger} \otimes I_{n_x})\mathcal{A}(V^{\sigma_k} \otimes I_{n_x}) \\
A_{22}^{\sigma_k} &= -\gamma(\Lambda^{\sigma_k'} \otimes I_{n_x}).
\end{aligned}$$

We assume that the coupling parameter is γ *large* and dividing both sides of equation (19) by γ, we obtain the dynamics of the system in standard singular perturbation form as follows:

$$\begin{pmatrix} \dot{x_s}(t) \\ \epsilon\dot{e_v}(t) \end{pmatrix} = \begin{pmatrix} A_0^{\sigma_k} & B_1^{\sigma_k} \\ \epsilon B_2^{\sigma_k} & A_{22}^{\sigma_k} + \epsilon B_3^{\sigma_k} \end{pmatrix} \begin{pmatrix} x_s(t) \\ e_v(t) \end{pmatrix}, \qquad \forall t \in (t_k, t_{k+1}]\,, k \in \mathbb{N}. \tag{21}$$

We set $\epsilon = \frac{1}{\gamma} \ll 1$ as the perturbation parameter and hence it can be clearly seen that the matrix $A_0^{\sigma_k}$, $B_1^{\sigma_k}$ $B_2^{\sigma_k}$ and $B_3^{\sigma_k}$ are of the order $\mathcal{O}(1)$, while $A_{22}^{\sigma_k}$ is of the order $\mathcal{O}(\frac{1}{\epsilon})$. The impulsive dynamics is as follows,

$$\begin{pmatrix} x_s(t_k) \\ e_v(t_k) \end{pmatrix} = J^{\sigma_{k-1} \xrightarrow{v_k} \sigma_k} \begin{pmatrix} x(t_k^-) \\ e(t_k^-) \end{pmatrix} \tag{22}$$

with the jump matrix,

$$J^{\sigma_{k-1} \xrightarrow{v_k} \sigma_k} = \begin{pmatrix} \left({v_{l1}^{\sigma_k}}^T - {v_{l1}^{\sigma_{k-1}}}^T\right) \otimes I_{n_x} & \mathbf{0}_{n_x, n_x N} \\ \mathbf{0}_{n_x(N-1), n_x N} & \left({V^{\sigma_k}}^T - {V^{\sigma_{k-1}}}^T\right) \otimes I_{n_x} \end{pmatrix}. \tag{23}$$

Hence, the system has standard singular perturbations form with reduced (slow) variable x_s and boundary layer (fast) variable e_v. This time scale separation on the network of linear heterogeneous systems is due to the large coupling parameter γ and the coordinate transformation (10) which are the intrinsic properties of the network.

Remark 3 *After simple calculation, we can see that* $A_0^{\sigma_k} = \sum_{i=1}^{N} v_{l1_i}^{\sigma_k} A_i, \forall \sigma_k \in \mathcal{I}$ *and it is also important to note that the matrix* $A_0^{\sigma_k}$ *is the convex combination of the matrices* A_i, $\forall i \in \mathcal{N}$*. This is due to the property of the left eigenvector of the Laplacian of the directed graphs[1] i.e.,* $v_{l1_i}^{\sigma_k} \in (0,1), \forall i \in \mathcal{N}$ *and* $\sum_{i=1}^{N} v_{l1_i}^{\sigma_k} = 1$, $\forall \sigma_k \in \mathcal{I}$ *. Additionally, the matrix* $A_{22}^{\sigma_k}$ *is Hurwitz because the diagonal matrix* $\Lambda^{\sigma_k'} = diag$ $(\lambda_2^{\sigma_k}, \cdots \lambda_N^{\sigma_k})$ *has all positive eigenvalues i.e.,* $\lambda_i^{\sigma_k} > 0, \forall i = 2, \cdots, N$ *and* $\forall \sigma_k \in \mathcal{I}$.

Reduced Ordered System (Emergent Dynamics)

In order to obtain the reduced order model in single time scale, of the system (21) and (22), we set $\epsilon = 0$. Then, we have the slow subsystem as follows,

$$\dot{x_e}(t) = A_0^{\sigma_k} x_e(t) \qquad \forall t \in [t_k, t_{k+1}), k \in \mathbb{N} \tag{24}$$

$$e_v(t) = 0. \tag{25}$$

with the impulsive dynamics as follows:

$$x_e(t_k) = ((v_{l1}^{\sigma_k T} - v_{l1}^{\sigma_{k-1} T}) \otimes I_{n_x}) x(t_k^-), \qquad \forall k \geq 1. \tag{26}$$

The dynamics (24) and (25) is also referred as *emergent dynamics*, which is the dynamics of the mean field unit restricted to the synchronization manifold.

3 Stability

In this section, we analyze the stability of the system (21) and (22) using the Lyapunov function arguments. In the analysis of the singularly perturbed system we consider that the slow subsystems are rather bounded than stable, hence we make the following assumption.

Assumption 1 *For the family of the state matrices* $\{A_0^i\}_{i \in \mathcal{I}}$*, both stable and unstable mode exists.*

We introduce the following definition of the *measure* and a result from [2] that provide the upper bound on the exponential of the matrices.

Definition 1 *The measure of a square matrix* M *is defined as* $\nu(M) = \frac{1}{2}\lambda_{max}(M + M^T)$*, where* λ_{max} *is the largest eigenvalue of the symmetric matrix* $M + M^T$.

Lemma 3.1 *For any square matrix* $\mathbb{A}$, $\|e^{\mathbb{A}t}\| \leq e^{\nu(\mathbb{A})t}$ *holds.*

Then from the definition of the measure it follows

$$\nu(A_0^{\sigma_k}) = \frac{1}{2}\lambda_{max}^{\sigma_k}(A_0^{\sigma_k} + A_0^{\sigma_k T}), \qquad \forall \sigma_k \in \mathcal{I}. \tag{27}$$

Based on the Assumption 1 and the definition of the measure, the measure $\nu(A_0^{\sigma_k})$ may be positive or negative. Let $\mathcal{S}$ and $\mathcal{U}$ be the set of stable and unstable modes, respectively. In other words, $\sigma_k \in \mathcal{S}$ iff $\nu(A_0^{\sigma_k}) < 0$, $\sigma_k \in \mathcal{U}$ iff $\nu(A_0^{\sigma_k}) \geq 0$ and $\mathcal{S} \bigcup \mathcal{U} = \mathcal{I}$. It is important to note that the measure is defined for the symmetric matrix $(M + M^T)$, hence the eigenvalues are always real. Furthermore, we define the following maximum for the positive and negative measure

$$\begin{aligned} \nu_u &= \max_{\nu^k \in \mathcal{U}} \quad \nu^k \\ \nu_s &= \max_{\nu^k \in \mathcal{S}} \quad \nu^k. \end{aligned} \tag{28}$$

Let $\tau^* > 0$ is the time between two consecutive switching i.e., mimimum dwell time and t_u and t_s be the total amount of time in the unstable and stable mode respectively then we define the following *switching law* for the boundedness of the slow subsystem,

$$\nu_u t_u + \nu_s t_s \leq 0 \iff \frac{t_u}{t_s} \leq -\frac{\nu_s}{\nu_u}, \tag{29}$$

which gives the proportion of the time the systems should be in the respective mode for the trajectories of the slow systems to be bounded.

Lemma 3.2 *If $\frac{t_u}{t_s} \leq -\frac{\nu_s}{\nu_u}$, the trajectories of the slow subsystems are bounded.*

Proof: Let $\{t_k\}_{k\in\mathbb{N}}$ be the switching time for the switching event σ_k. Then for any t such that $t_k < t < t_{k+1}$,we have

$$x_e(t) = e^{A_0^{\sigma_{k+1}}(t-t_k)} x_e(t_k) \qquad \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}, \forall \sigma_k \in \mathcal{I}. \tag{30}$$

Similarly for every k,we have $x_e(t_k) = e^{A_0^{\sigma_k}\tau_k} x_e(t_{k-1})$, where $t_k - t_{k-1} = \tau_k \geq \tau^*, \forall k \in \mathbb{N}$ which leads to the following,

$$x_e(t) = e^{A_0^{\sigma_{k+1}}(t-t_k)} e^{A_0^{\sigma_k}\tau_k} \cdots e^{A_0^1\tau_k} x_e(t_0). \tag{31}$$

Then we have,

$$\begin{aligned} \|x_e(t)\| &= \|e^{A_0^{\sigma_{k+1}}(t-t_k)} e^{A_0^{\sigma_k}\tau_k} \cdots e^{A_0^1\tau_k} x_e(t_0)\|. \\ &\leq \|e^{A_0^{\sigma_{k+1}}(t-t_k)}\| \|e^{A_0^{\sigma_k}\tau_k}\| \cdots \|e^{A_0^{\sigma_1}\tau_k}\| \|x_e(t_0)\|. \\ &\leq \|e^{A_0^{\sigma_{k+1}}(t-t_k) + A_0^{\sigma_k}\tau_k + \cdots + A_0^{\sigma_1}\tau_k}\| \|x_e(t_0)\|. \\ &\leq e^{\nu(A_0^{\sigma_{k+1}})(t-t_k) + \nu(A_0^{\sigma_k})\tau_k + \cdots + \nu(A_0^{\sigma_1})\tau_k} \|x_e(t_0)\|. \\ &\leq e^{\nu_u t_u + \nu_s t_s} \|x_e(t_0)\| \end{aligned} \tag{32}$$

Now, it is evident that the trajectories of slow subsystem are bounded iff $e^{\nu_u t_u + \nu_s t_s}$ is non increasing and it is true when $\nu_u t_u + \nu_s t_s \leq 0$. ■

Remark 4 *If the measure of the matrix $A_0^{\sigma_k}$ is negative, i.e., $\bar{\nu} = 0$, for all $\sigma_k \in \mathcal{I}$, then the trajectories of slow systems are bounded for all $t_s > 0$.*

Theorem 3.3 *If Re $\lambda(A_{22}^{\sigma_k}) < 0$, there exists an $\epsilon^* > 0$ such that for all $\epsilon \in (0, \epsilon^*)$, the states of the original system (21), for any bounded initial conditions are approximated for all finite time $t \geq t_0$ by*

$$x_s(t) = x_e(t) + \Theta \tag{33}$$

$$e_v(t) = \tag{34}$$

If the trajectories of the slow systems are bounded then 33 and 34 holds for $t \in [t_0, \infty)$

Proof: We use the following similarity transformation from [3] to decouple the fast and slow states of the system (21), as follows,

$$\begin{pmatrix} x_s \\ e_v \end{pmatrix} = T \begin{pmatrix} \xi \\ \eta \end{pmatrix}, T = \begin{bmatrix} I_n & \epsilon H \\ -L & I_m - \epsilon LH \end{bmatrix} \tag{35}$$

where, L and H will be defined later. Then, by the similarity transformation, the system (21), is as follows,

$$\begin{pmatrix} \dot{\xi} \\ \epsilon\dot{\eta} \end{pmatrix} = T^{-1} \begin{pmatrix} \dot{x_s} \\ \epsilon\dot{e_v} \end{pmatrix} = T^{-1} \begin{pmatrix} A_0^{\sigma_k} & B_1^{\sigma_k} \\ \epsilon B_2^{\sigma_k} & \epsilon B_2^{\sigma_k} + A_{22}^{\sigma_k} \end{pmatrix} T \begin{pmatrix} \xi \\ \eta \end{pmatrix} \tag{36}$$

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} \begin{array}{l} (I_n - \epsilon HL)(A_0^{\sigma_k} - B_1^{\sigma_k} L) \\ -\epsilon H(B_2^{\sigma_k} - L(B_3 + \frac{A_{22}}{\epsilon})) \end{array} & S(H, \epsilon) \\ R(L, \epsilon) & \begin{array}{r} L(\epsilon A_0 H + B_1^{\sigma_k}(I_n - \epsilon LH)) \\ +\epsilon B_2 H + (B_3 + \frac{A_{22}}{\epsilon})(I_n - \epsilon LH) \end{array} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \tag{37}$$

where, we choose L and H such that

$$R(L,\epsilon) = L(A_0 - B_1 L) + B_2 - L(B_3 + \frac{A_{22}}{\epsilon}) = 0 \tag{38}$$

$$S(H,\epsilon) = (I_n - \epsilon HL)(\epsilon A_0 H + B_1 - \epsilon B_1 LH) - \epsilon H(\epsilon B_2 H + B_3 + \frac{A_{22}}{\epsilon} - \epsilon B_3 LH - A_{22}H) = 0. \tag{39}$$

The unique solution for above equations exists, as follows,

$$L(\epsilon) = -\epsilon A_{22}^{-1} B_2 + O(\epsilon^2) \tag{40}$$

$$H(\epsilon) = B_1 A_{22}^{-1} + O(\epsilon). \tag{41}$$

Then for $O(\epsilon)$ approximation of L and H, we have,

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} A_0 & 0 \\ 0 & B_3 + \frac{A_{22}}{\epsilon} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}. \tag{42}$$

Then, solving the decoupled slow and fast subsystems are transforming back to the original variables we have,

$$\begin{pmatrix} x(t) \\ e_v(\tau) \end{pmatrix} = \begin{pmatrix} I_n & \epsilon B_1 A_{22}^{-1} \\ 0 & I_m \end{pmatrix} \begin{pmatrix} \exp\{A_0(t-t_0)\}\,\xi_0 \\ \exp\left\{(\epsilon B_3 + A_{22})(\frac{t-t_0}{\epsilon})\right\}\eta_0 \end{pmatrix} \tag{43}$$

and $\xi_0 = x_s^0 - \epsilon H \eta_0$ and $\eta_0 = e_v^0$. Then, we have,

$$x_s(t) = \exp\{A_0(t-t_0)\}\, x_s^0 + O(\epsilon) \tag{44}$$

$$e_v(\tau) = \exp\left\{(\epsilon B_3 + A_{22})(\frac{t-t_0}{\epsilon})\right\} e_v^0 \tag{45}$$

■.

As stated in Remark 3, the matrix $A_{22}^{\sigma_k}$ is Hurwitz hence there exists symmetric positive definite matrices $Q_f^{\sigma_k} \geq I_{n_x N-1}$ and $\lambda_f^{\sigma_k} > 0$, $\forall \sigma_k \in \mathcal{I}$. However, for the matrix $A_0^{\sigma_k}$ with stable and unstable modes, we have $Q_s^{\sigma_k} \geq I_{n_x}, \forall \sigma_k \in \mathcal{S}$ and $Q_u^{\sigma_k} \geq I_{n_x}$, $\forall \sigma_k \in \mathcal{U}$ with positive numbers $\lambda_s^{\sigma_k}$ and $\lambda_u^{\sigma_k}$, such that,

$$\begin{aligned} {A_0^{\sigma_k}}^T Q_s^{\sigma_k} + Q_s^{\sigma_k} A_0^{\sigma_k} &\leq -2\lambda_s^{\sigma_k} Q_s^{\sigma_k} \qquad \forall \sigma_k \in \mathcal{S} \\ {A_0^{\sigma_k}}^T Q_u^{\sigma_k} + Q_u^{\sigma_k} A_0^{\sigma_k} &\leq 2\lambda_u^{\sigma_k} Q_u^{\sigma_k} \qquad \forall \sigma_k \in \mathcal{U} \\ {A_{22}^{\sigma_k}}^T Q_f^{\sigma_k} + Q_f^{\sigma_k} A_{22}^{\sigma_k} &\leq -2\lambda_f^{\sigma_k} Q_f^{\sigma_k} \qquad \forall \sigma_k \in \mathcal{I} \end{aligned}$$

Let us denote, $\lambda_s = \min\limits_{\sigma_k \in \mathcal{S}} \lambda_s^{\sigma_k}$, $\lambda_u = \max\limits_{\sigma_k \in \mathcal{U}} \lambda_u^{\sigma_k}$ and $\lambda_f = \min\limits_{\sigma_k \in \mathcal{I}} \lambda_f^{\sigma_k}$.

Assumption 2 *The trajectories of the slow subsystems are bounded i.e.,* $\frac{t_u}{t_s} \leq -\frac{\nu_s}{\nu_u}$.

Let us consider the composite Lyapunov functions as follows for the slow and fast subsystems $W_s(t) = x_s^T \left[\gamma Q_s^i + (1-\gamma) Q_u^i\right] x_s$ and $W_f(t) = e_v^T Q_f^i e_v$, where $i \in \mathcal{I}$ and $\gamma \in \{0,1\}$ where $\gamma = 1$ if $i \in \mathcal{S}$ and $\gamma = 0$ if $i \in \mathcal{U}$.

Proposition 1 *The function* $V_i(x_s, e_v) = x_s^T \left[\gamma Q_s^i + (1-\gamma) Q_u^i\right] x_s + e_v^T Q_f^i e_v$, $\forall i \in \mathcal{I}$, *is the composite Lyapunov function for the system (21) for all* $\epsilon \in (0, \epsilon^*)$ *and* $\gamma = 1$ *if* $i \in \mathcal{S}$ *and* $\gamma = 0$ *if* $i \in \mathcal{U}$.

Proof: The time derivative of the Lyapunov function along the trajectories of the system (21) is

$$\begin{aligned} \dot{V}_i &= 2x_s^T \left[\gamma Q_s^i + (1-\gamma) Q_u^i\right] \dot{x_s} + 2e_v^T Q_f^i \dot{e_v} \\ &= 2x_s^T \left[\gamma Q_s^i + (1-\gamma) Q_u^i\right] (A_0^i x_s + B_1^i e_v) + 2e_v^T Q_f^i (B_2^i x_s + B_3^i e_v + \frac{1}{\epsilon} A_{22}^i e_v) \\ &= 2x_s^T \left[\gamma Q_s^i + (1-\gamma) Q_u^i\right] A_0^i x_s + 2x_s^T \left[\gamma Q_s^i + (1-\gamma) Q_u^i\right] B_1^i e_v + 2e_v^T Q_f^i B_2^i x_s + 2e_v^T Q_f^i B_3^i e_v + \frac{2}{\epsilon} e_v^T Q_f^i A_{22}^i e_v \\ &= 2\gamma x_s^T Q_s^i A_0^i x_s + 2(1-\gamma) x_s^T Q_u^i A_0^i x_s + 2\gamma x_s^T Q_s^i B_1^i e_v + 2(1-\gamma) x_s^T Q_u^i B_1^i e_v + 2e_v^T Q_f^i B_2^i x_s + \\ &\quad 2e_v^T Q_f^i B_3^i e_v + \frac{2}{\epsilon} e_v^T Q_f^i A_{22}^i e_v \\ &\leq -2\lambda_s \gamma x_s^T Q_s^i x_s + 2\lambda_u (1-\gamma) x_s^T Q_u^i x_s + 2\gamma x_s^T Q_s^i B_1^i e_v + 2(1-\gamma) x_s^T Q_u^i B_1^i e_v + 2e_v^T Q_f^i B_2^i x_s + \\ &\quad 2e_v^T Q_f^i B_3^i e_v - \frac{2}{\epsilon} \lambda_f e_v^T Q_f^i e_v \end{aligned} \tag{46}$$

Let $b_1^i = \|{Q_s^i}^{\frac{1}{2}} B_1^i {Q_f^i}^{-\frac{1}{2}}\|$, $b_2^i = \|{Q_u^i}^{\frac{1}{2}} B_1^i {Q_f^i}^{-\frac{1}{2}}\|$, $b_3^i = \|{Q_f^i}^{\frac{1}{2}} B_2^i \big(\gamma Q_s^i + (1-\gamma) Q_u^i\big)^{-\frac{1}{2}}\|$, $b_4^i = \|{Q_f^i}^{\frac{1}{2}} B_3^i {Q_f^i}^{-\frac{1}{2}}\|$ and $b_j = \max_{i\in\mathcal{I}} b_j^i, j \in \{1,2,3,4\}$.

$$x_s^T Q_s^i B_1^i e_v = x_s^T {Q_s^i}^{\frac{1}{2}} {Q_s^i}^{\frac{1}{2}} B_1^i {Q_f^i}^{-\frac{1}{2}} {Q_f^i}^{\frac{1}{2}} e_v \le b_1^i \|x_s^T {Q_s^i}^{\frac{1}{2}}\| \|{Q_f^i}^{\frac{1}{2}} e_v\| = b_1^i \sqrt{x_s^T Q_s^i x_s} \sqrt{e_v^T Q_f^i e_v} \tag{47}$$

$$x_s^T Q_u^i B_1^i e_v = x_s^T {Q_u^i}^{\frac{1}{2}} {Q_u^i}^{\frac{1}{2}} B_1^i {Q_f^i}^{-\frac{1}{2}} {Q_f^i}^{\frac{1}{2}} e_v \le b_2^i \|x_s^T {Q_u^i}^{\frac{1}{2}}\| \|{Q_f^i}^{\frac{1}{2}} e_v\| = b_2^i \sqrt{x_s^T Q_u^i x_s} \sqrt{e_v^T Q_f^i e_v} \tag{48}$$

$$\begin{aligned} e_v^T Q_f^i B_2^i x_s &= e_v^T {Q_f^i}^{\frac{1}{2}} {Q_f^i}^{\frac{1}{2}} B_2^i \big(\gamma Q_s^i + (1-\gamma) Q_u^i\big)^{-\frac{1}{2}} \big(\gamma Q_s^i + (1-\gamma) Q_u^i\big)^{\frac{1}{2}} x_s \\ &\le b_3^i \|e_v^T {Q_f^i}^{\frac{1}{2}}\| \|\big(\gamma Q_s^i + (1-\gamma) Q_u^i\big)^{\frac{1}{2}} x_s\| = b_3^i \sqrt{e_v^T {Q_f^i}^{\frac{1}{2}} e_v} \sqrt{x_s^T (\gamma Q_s^i + (1-\gamma) Q_u^i) x_s} \end{aligned} \tag{49}$$

$$e_v^T Q_f^i B_3^i e_v = e_v^T {Q_f^i}^{\frac{1}{2}} {Q_f^i}^{\frac{1}{2}} B_3^i {Q_f^i}^{-\frac{1}{2}} {Q_f^i}^{\frac{1}{2}} e_v \le b_4^i e_v^T Q_f^i e_v \tag{50}$$

Now, substiuting the value from equations (47), (48), (49) and (50) into equation (34), we have,

$$\begin{aligned} \dot{V}_i \le& -2\lambda_s \gamma x_s^T Q_s^i x_s + 2\lambda_u (1-\gamma) x_s^T Q_u^i x_s + 2\gamma b_1 \sqrt{x_s^T Q_s^i x_s} \sqrt{e_v^T Q_f^i e_v} + 2(1-\gamma) b_2 \sqrt{x_s^T Q_u^i x_s} \sqrt{e_v^T Q_f^i e_v} \\ &+ 2b_3 \sqrt{e_v^T Q_f^i e_v} \sqrt{x_s^T (\gamma Q_s^i + (1-\gamma) Q_u^i) x_s} + 2b_4 e_v^T Q_f^i e_v - \frac{2}{\epsilon} \lambda_f e_v^T Q_f^i e_v \\ \le& -2\lambda_s \gamma x_s^T Q_s^i x_s + 2\lambda_u (1-\gamma) x_s^T Q_u^i x_s + \gamma b_1 x_s^T Q_s^i x_s + \gamma b_1 e_v^T Q_f^i e_v + (1-\gamma) b_2 x_s^T Q_u^i x_s + (1-\gamma) b_2 e_v^T Q_f^i e_v \\ &+ b_3 e_v^T Q_f^i e_v + b_3 x_s^T (\gamma Q_s^i + (1-\gamma) Q_u^i) x_s + 2b_4 e_v^T Q_f^i e_v - \frac{2}{\epsilon} \lambda_f e_v^T Q_f^i e_v \\ =& \gamma \Big(-2\lambda_s + b_1 + b_3 \Big) x_s^T Q_s^i x_s + (1-\gamma) \Big(2\lambda_u + b_2 + b_3 \Big) x_s^T Q_u^i x_s + \\ & \Big(\gamma b_1 + (1-\gamma) b_2 + b_3 + 2b_4 - \frac{2}{\epsilon} \lambda_f \Big) e_v^T Q_f^i e_v \end{aligned}$$

Case 1: For $i \in \mathcal{S}$, $\gamma = 1$, we have,

$$\dot{V}_i \le \Big(-2\lambda_s + b_1 + b_3 \Big) x_s^T Q_s^i x_s + \Big(b_1 + b_3 + 2b_4 - \frac{2}{\epsilon} \lambda_f \Big) e_v^T Q_f^i e_v. \tag{51}$$

Hence, $\dot{V}_i < 0$ iff $\lambda_s > \frac{b_1+b_3}{2}$ and for all $\epsilon \in (0, \epsilon_1)$, where, $\epsilon_1 = \frac{2\lambda_f}{b_1+b_3+2b_4}$.

Case 2: For $i \in \mathcal{U}$, $\gamma = 0$, then we have,

$$\dot{V}_i \le \Big(2\lambda_u + b_2 + b_3 \Big) x_s^T Q_u^i x_s + \Big(b_2 + b_3 + 2b_4 - \frac{2}{\epsilon} \lambda_f \Big) e_v^T Q_f^i e_v \tag{52}$$

where, $\epsilon_2 = \frac{2\lambda_f}{b_2+b_3+2b_4}$ and let $\epsilon^* = \min\{\epsilon_1, \epsilon_2\}$. From equation 39 and 40 we can write the following,

$$V(t) \le \begin{cases} e^{\left\{(-2\lambda_s+b_1+b_3) x_s^T Q_s^i x_s + (b_1+b_3+2b_4-\frac{2}{\epsilon}\lambda_f) e_v^T Q_f^i e_v\right\}(t-t_k)} V(t_k) \\ e^{\left\{(2\lambda_u+b_2+b_3) x_s^T Q_u^i x_s + (b_2+b_3+2b_4-\frac{2}{\epsilon}\lambda_f) e_v^T Q_f^i e_v\right\}(t-t_k)} V(t_k), \end{cases} \forall t \in [t_k, t_{k+1}), k \in \mathbb{N} \tag{53}$$

Then, for the system switching between the stable and unstable modes, let $0 < \rho < 1$ and $\mu \ge 1$, then we have the following,

$$V(t_{k+1}) \le \rho V(t_k), \qquad \forall \sigma_k \in \mathcal{S} \tag{54}$$
$$V(t_{k+1}) \le \mu V(t_k), \qquad \forall \sigma_k \in \mathcal{U}. \tag{55}$$

If $t_{k+1} - t_k = \tau^* > 0$ is the average dwell time then

$$V(t_{k+1}) \le \begin{cases} \rho \tau^* r V(t_0) \\ \mu \tau^* (k+1-r) V(t_0) \end{cases} \tag{56}$$

where, r is the number of stable modes and $(k+1-r)$ is the number of unstable modes until time t_{k+1} i.e., $t_{k+1} - t_0 = \tau^* r + \tau^*(k+1-r)$.

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