Singular Perturbation Based Stability Analysis of Linear Heterogeneous Systems

1 Problem Formulation

System Model

Let us consider a network of N heterogeneous linear systems called *agents*. The state of each agent at any time t is characterized by the scalar $x_i(t) \in \mathbb{R}^{n_x}$ such that its dynamics is given by

$$\dot{x_i}(t) = A_i x_i(t) + u_i(t) \qquad \forall i \in \mathcal{N} = \{1, \cdots, N\}, \tag{1}$$

where $u_i \in \mathbb{R}^{n_x}$ is the control input and $A_i \in \mathbb{R}^{n_x \times n_x}$.

Remark 1 *We assume that the systems are heterogeneous, that is, the matrices A_i $\forall i \in \mathcal{N}$ are different but of the same dimensions $\mathbb{R}^{n_x \times n_x}$.*

Network Model

Let $G = \left(V, E, A^d\right)$ be the weighted directed graph with the set of nodes $V = \{v_1, v_2, \cdots, v_N\}$, set of edges $E \subseteq V \times V$, and a weighted adjacency matrix $A^d_{ij} = [g_{ij}]_{i,j \in \mathcal{N}}$, where, $g_{ij} > 0$ for $i \neq j$ and there exists a directed link from j to i, while $g_{ij} = 0$ for $i = j$. We consider the network with a dynamic topology that is time-varying, also commonly known as *switching networks* [5]. These time-varying networks can be modelled using the dynamic graph $G^{\sigma_k(t)}$ parameterized with switching signal $\sigma_k(t) : \mathbb{R} \to \mathcal{I}, k \in \mathbb{N}$ that takes values in the index set $\mathcal{I} = \{1, 2, \cdots, n\}$ i.e. the network can switch among a finite number of graphs, $\Gamma_n = \{G_1, G_2, \cdots, G_n\}$. Also, we assume at any time instance, the network topology is strongly connected. The network units are assumed to be connected via diffusive coupling, i.e., the i^{th} unit coupling, at any time instance, is given by

$$u_i(t) = -\gamma \sum_{j=1}^{N} g_{ij} \left(x_i - x_j\right), \tag{2}$$

where $\gamma \in \mathbb{R}$ corresponds to the coupling gain between the units.

Let us represent the state and control vectors in compact notation as follows:

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \in \mathbb{R}^{n_x N}, \qquad u = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} \in \mathbb{R}^{mN},$$

and also we represent the state matrix in block diagonal form as follows,

$$\mathcal{A} = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & A_N \end{bmatrix} \in \mathbb{R}^{n_x N \times n_x N}.$$

Hence the control input can be represented in the following compact notation,

$$u^{\sigma_k} = -\gamma (L^{\sigma_k} \otimes I_{n_x}) x, \tag{3}$$

where $\sigma_k \in \mathcal{I}$ is the topological index associated with the elements of Γ_n, $L^{\sigma_k} = L(G^{\sigma_k(t)})$ is the Laplacian associated with the graph $G^{\sigma_k(t)}$. Then the resulting network dynamics is

$$\dot{x}(t) = (\mathcal{A} - \gamma (L^{\sigma_k} \otimes I_{n_x})) x(t). \tag{4}$$

2 Network Dynamics

Coordinate Transformation

By definition of graph Laplacian, for every $\sigma_k \in \mathcal{I}$ the row sums of L^{σ_k} are zero, hence, L^{σ_k} has at least one zero eigenvalue, say $\lambda_1^{\sigma_k} = 0$ and assuming that the graph is strongly connected, the remaining eigenvalues of L^{σ_k} has non-negative real parts, i.e., $0 = \Re e\{\lambda_1^{\sigma_k}\} < \Re e\{\lambda_2^{\sigma_k}\} \leq \cdots \leq \Re e\{\lambda_N^{\sigma_k}\}$. Moreover, if the graph is directed and has a spanning tree, the eigenvalue $\lambda_1^{\sigma_k} = 0$ is simple and there exists left eigenvector $(v_{l1}^{\sigma_k})$ and right eigenvector $(v_{r1}^{\sigma_k})$ of L^{σ_k}, such that ${v_{l1}^{\sigma_k}}^T L^{\sigma_k} = 0$ and $L^{\sigma_k} v_{r1}^{\sigma_k} = 0$, for each $\sigma_k \in \mathcal{I}$. These left and right eigenvectors are given by

$$v_{l1}^{\sigma_k} = \begin{bmatrix} \nu_1^{\sigma_k} \\ \nu_2^{\sigma_k} \\ \vdots \\ \nu_N^{\sigma_k} \end{bmatrix} \in \mathbb{R}^N \qquad v_{r1}^{\sigma_k} = \mathbf{1}_N := \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^N. \tag{5}$$

Furthermore, there exists a Jordan decomposition of the Laplacian L^{σ_k}, of the form $L^{\sigma_k} = U^{\sigma_k} \Lambda^{\sigma_k} {U^{\sigma_k}}^{-1}$, where $U^{\sigma_k} \in \mathbb{R}^{N\times N}$ is nonsingular and $\Lambda^{\sigma_k} \in \mathbb{C}^{N\times N}$ is the block diagonal Jordan matrix of the following form

$$\Lambda = \begin{bmatrix} \Lambda_1 & & 0 \\ & \ddots & \\ 0 & & \Lambda_m \end{bmatrix}$$

where, for the networks with rooted-out branching, the algebraic multiplicity of the $\lambda_1 = 0$ is 1 and Λ_i are Jordan blocks of appropriate dimensions. In addition, the matrix U^{σ_k} is composed of generalized right eigenvectors of the Laplacian matrix L^{σ_k} among which the first is $v_{r1}^{\sigma_k} = \mathbf{1}_N$. For further development, we decompose the matrix U^{σ_k} as

$$U^{\sigma_k} = [\mathbf{1}_N \quad V^{\sigma_k}] \tag{6}$$

where $V^{\sigma_k} \in \mathbb{R}^{N\times N-1}$. Similarly, the first row of the ${U^{\sigma_k}}^{-1}$ corresponds to the first left eigenvector of L^{σ_k}, $v_{l1}^{\sigma_k}$, which can be decomposed as follows,

$${U^{\sigma_k}}^{-1} = \begin{bmatrix} {v_{l1}^{\sigma_k}}^T \\ {V^{\sigma_k}}^{\dagger} \end{bmatrix} \tag{7}$$

and necessarily,

$${v_{l1}^{\sigma_k}}^T V^{\sigma_k} = 0 \qquad {V^{\sigma_k}}^{\dagger} V^{\sigma_k} = I_{N-1}. \tag{8}$$

The *weighted average* [6] of the systems' states x_s is

$$x_s = \sum_{i=1}^{N} \nu_i^{\sigma_k} x_i \tag{9}$$

and we introduce the transformation from [6], that leads to,

$$\bar{x} := {\mathcal{U}^{\sigma_k}}^{-1} x \tag{10}$$

where $\mathcal{U}^{\sigma_k} = (U^{\sigma_k} \otimes I_{n_x}) \in \mathbb{R}^{n_x N \times n_x N}$. Then, the partition is done using equation (7) to new coordinates, *i.e.*,

$$\bar{x} =: \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} := \begin{bmatrix} {v_{l1}^{\sigma_k}}^T \otimes I_{n_x} \\ {V^{\sigma_k}}^{\dagger} \otimes I_{n_x} \end{bmatrix} x.$$

Remark 2 *It should be noted that due to the coordinate transformation (9) the switching system has changed into the* ***switching impulsive system*** *and the weighted average and the error terms experiences the jumps when topology changes. In other words, when there is a switching in topology, the eigenvectors (transformation vectors) changes and this introduces, in addition to switching, jumps in the system dynamics.*

The coordinates $\bar{x}_1$ is equivalent to the weighted average (9)

$$x_s(t) = (v_{l1}^{\sigma_k T} \otimes I_{n_x})x(t), \qquad \forall t \in [t_k, t_{k+1}), \quad k \in \mathbb{N} \tag{11}$$

with the impulsive dynamics,

$$x_s(t_k) = J_{11}^{\sigma_{k-1} \overset{v_k}{\rightarrow} \sigma_k} x(t_k^-), \qquad \forall k \geq 1 \tag{12}$$

where $J_{11}^{\sigma_{k-1} \overset{v_k}{\rightarrow} \sigma_k} = \Big((v_{l1}^{\sigma_k T} - v_{l1}^{\sigma_{k-1} T}) \otimes I_{n_x} \Big)$, $x(t^-) = \lim_{\delta \to 0, \delta > 0} x(t - \delta)$, $0 = t_0 < t_1 < \cdots$ is the monotonically increasing and unbounded sequence of instants of discrete time events and $\sigma_k \in \mathcal{I}$ and $v_k \in \mathcal{J}$, where $\mathcal{J}$ is a finite set of indices.

The second coordinate $\bar{x}_2 = e_v \in \mathbb{R}^{n_x(N-1)}$ is the projection of the *synchronization error* [4] into the subspace orthogonal to that of vector $v_{l1}^{\sigma_k}$. The synchronization error is the difference between the individual states and the state of the "average" unit, i.e.,

$$e(t) = x(t) - (\mathbf{1}_N \otimes I_{n_x})x_s(t), \qquad \forall t \in [t_k, t_{k+1}), \quad k \in \mathbb{N} \tag{13}$$

then,

$$e_v(t) = (V^{\sigma_k \dagger} \otimes I_{n_x})e(t) \iff e(t) = (V^{\sigma_k} \otimes I_{n_x})e_v(t). \tag{14}$$

Similarly, the impulsive dynamics of the error is,

$$e_v(t_k) = J_{22}^{\sigma_{k-1} \overset{v_k}{\rightarrow} \sigma_k} e(t_k^-), \tag{15}$$

with $J_{22}^{\sigma_{k-1} \overset{v_k}{\rightarrow} \sigma_k} = \Big((V^{\sigma_k \dagger} - V^{\sigma_{k-1} \dagger}) \otimes I_{n_x} \Big)$.

Mean-Field Dynamics

In the new coordinate system, differentiating equation (11) both sides and using the network dynamics (4) , as well as substituting x from (13), we obtain $\forall t \in [t_k, t_{k+1}), k \in \mathbb{N}$,

$$\begin{aligned}
\dot{x}_s &= (v_{l1}^{\sigma_k T} \otimes I_{n_x})\dot{x} = (v_{l1}^{\sigma_k T} \otimes I_{n_x}) \left(\mathcal{A} - \gamma (L^{\sigma_k} \otimes I_{n_x}) \right) x(t) \\
\dot{x}_s &= (v_{l1}^{\sigma_k T} \otimes I_{n_x}) \left(\mathcal{A} - \gamma (L^{\sigma_k} \otimes I_{n_x}) \right) (e + (\mathbf{1}_N \otimes I_{n_x})x_s) \\
\dot{x}_s &= \Big((v_{l1}^{\sigma_k T} \otimes I_{n_x}) \mathcal{A} (\mathbf{1}_N \otimes I_{n_x}) \Big) x_s + (v_{l1}^{\sigma_k T} \otimes I_{n_x}) \mathcal{A} e
\end{aligned} \tag{16}$$

also, equivalently substituting the value of e from equation (14), we get

$$\dot{x}_s = \Big((v_{l1}^{\sigma_k T} \otimes I_{n_x}) \mathcal{A} (\mathbf{1}_N \otimes I_{n_x}) \Big) x_s + (v_{l1}^{\sigma_k T} \otimes I_{n_x}) \mathcal{A} (V^{\sigma_k} \otimes I_{n_x}) e_v. \tag{17}$$

The equation (16) and (17) represents the *mean-field dynamics.*

Synchronization Error Dynamics

Differentiating on both sides of equation (13) and substituting $\dot{x}_s$ from (16) $\forall t \in [t_k, t_{k+1}), k \in \mathbb{N}$ it follows:

$$\begin{aligned}
\dot{e} &= \dot{x} - (\mathbf{1}_N \otimes I_{n_x})\dot{x}_s \\
&= (\mathcal{A} - \gamma(L^{\sigma_k} \otimes I_{n_x}))(e + (\mathbf{1}_N \otimes I_{n_x})x_s) - \\
&\quad (\mathbf{1}_N \otimes I_{n_x}) \Big[((v_{l1}^{\sigma_k T} \otimes I_{n_x}) \mathcal{A} (\mathbf{1}_N \otimes I_{n_x}))x_s + (v_{l1}^{\sigma_k T} \otimes I_{n_x}) \mathcal{A} e \Big] \\
&= \Big(\mathcal{A}(\mathbf{1}_N \otimes I_{n_x}) - (\mathbf{1}_N v_{l1}^{\sigma_k T} \otimes I_{n_x}) \mathcal{A} (\mathbf{1}_N \otimes I_{n_x}) \Big) x_s + \Big(\mathcal{A} - (\mathbf{1}_N v_{l1}^{\sigma_k T} \otimes I_{n_x}) \mathcal{A} \Big) e \\
&\quad - \gamma (L^{\sigma_k} \otimes I_{n_x}) e.
\end{aligned} \tag{18}$$

Since the error e_v is the projection of the synchronization error e onto the subspace orthogonal to that of the vector $v_{l1}^{\sigma_k}$, hence from equations (14) and (18), we have,

$$\dot{e}_v = (V^{\sigma_k \dagger} \otimes I_{n_x})\Big\{\Big(\mathcal{A}(\mathbf{1}_N \otimes I_{n_x}) - (\mathbf{1}_N v_{l1}^{\sigma_k T} \otimes I_{n_x})\mathcal{A}(\mathbf{1}_N \otimes I_{n_x})\Big)x_s + \Big(\mathcal{A} - (\mathbf{1}_N v_{l1}^{\sigma_k T} \otimes I_{n_x})\mathcal{A}\Big)e + \\ - \gamma(L^{\sigma_k} \otimes I_{n_x})e\Big\}$$

Also, we know that $V^{\sigma_k \dagger}\mathbf{1}_N = 0$ and substituting the value of e from equation (14), it follows

$$\begin{aligned}
\dot{e}_v &= -\gamma(V^{\sigma_k \dagger} L^{\sigma_k} V^{\sigma_k} \otimes I_{n_x})e_v + (V^{\sigma_k \dagger} \otimes I_{n_x})\mathcal{A}(\mathbf{1}_N \otimes I_{n_x})x_s + \\
&\quad (V^{\sigma_k \dagger} \otimes I_{n_x})\mathcal{A}(V^{\sigma_k} \otimes I_{n_x})e_v \\
\dot{e}_v &= -\gamma(\Lambda^{\sigma_k'} \otimes I_{n_x})e_v + (V^{\sigma_k \dagger} \otimes I_{n_x})\mathcal{A}(\mathbf{1}_N \otimes I_{n_x})x_s + \\
&\quad (V^{\sigma_k \dagger} \otimes I_{n_x})\mathcal{A}(V^{\sigma_k} \otimes I_{n_x})e_v
\end{aligned} \tag{19}$$

where, $\Lambda^{\sigma_k'} = \operatorname{diag}(\lambda_2^{\sigma_k} \cdots \lambda_N^{\sigma_k})$.

Switching and Impulsive Dynamics of Networked System

Consider the equation (17) and (19) as follows,

$$\begin{pmatrix} \dot{x_s}(t) \\ \dot{e_v}(t) \end{pmatrix} = \begin{pmatrix} A_0^{\sigma_k} & B_1^{\sigma_k} \\ B_2^{\sigma_k} & A_{22}^{\sigma_k} + B_3^{\sigma_k} \end{pmatrix} \begin{pmatrix} x_s(t) \\ e_v(t) \end{pmatrix}, \qquad \forall t \in (t_k, t_{k+1}], k \in \mathbb{N} \tag{20}$$

where for all $\sigma_k \in \mathcal{I}$ one has

$$\begin{aligned}
A_0^{\sigma_k} &= (v_{l1}^{\sigma_k T} \otimes I_{n_x})\mathcal{A}(\mathbf{1}_N \otimes I_{n_x}) = \sum_{i=1}^{N} v_{l1_i}^{\sigma_k} A_i, \qquad B_1^{\sigma_k} = (v_{l1}^{\sigma_k T} \otimes I_{n_x})\mathcal{A}(V^{\sigma_k} \otimes I_{n_x}) \\
B_2^{\sigma_k} &= (V^{\sigma_k \dagger} \otimes I_{n_x})\mathcal{A}(\mathbf{1}_N \otimes I_{n_x}), \qquad B_3^{\sigma_k} = (V^{\sigma_k \dagger} \otimes I_{n_x})\mathcal{A}(V^{\sigma_k} \otimes I_{n_x}) \\
A_{22}^{\sigma_k} &= -\gamma(\Lambda^{\sigma_k'} \otimes I_{n_x}).
\end{aligned}$$

We assume that the coupling parameter is γ *large* and dividing both sides of equation (19) by γ, we obtain the dynamics of the system in standard singular perturbation form as follows:

$$\begin{pmatrix} \dot{x_s}(t) \\ \epsilon\dot{e_v}(t) \end{pmatrix} = \begin{pmatrix} A_0^{\sigma_k} & B_1^{\sigma_k} \\ \epsilon B_2^{\sigma_k} & A_{22}^{\sigma_k} + \epsilon B_3^{\sigma_k} \end{pmatrix} \begin{pmatrix} x_s(t) \\ e_v(t) \end{pmatrix}, \qquad \forall t \in (t_k, t_{k+1}], k \in \mathbb{N}. \tag{21}$$

We set $\epsilon = \frac{1}{\gamma} \ll 1$ as the perturbation parameter and hence it can be clearly seen that the matrix $A_0^{\sigma_k}$, $B_1^{\sigma_k}$ $B_2^{\sigma_k}$ and $B_3^{\sigma_k}$ are of the order $\mathcal{O}(1)$, while $A_{22}^{\sigma_k}$ is of the order $\mathcal{O}(\frac{1}{\epsilon})$. The impulsive dynamics is as follows,

$$\begin{pmatrix} x_s(t_k) \\ e_v(t_k) \end{pmatrix} = J^{\sigma_{k-1} \xrightarrow{v_k} \sigma_k} \begin{pmatrix} x(t_k^-) \\ e(t_k^-) \end{pmatrix} \tag{22}$$

with the jump matrix,

$$J^{\sigma_{k-1} \xrightarrow{v_k} \sigma_k} = \begin{pmatrix} \Big(v_{l1}^{\sigma_k T} - v_{l1}^{\sigma_{k-1} T}\Big) \otimes I_{n_x} & \mathbf{0}_{n_x, n_x N} \\ \mathbf{0}_{n_x(N-1), n_x N} & \Big(V^{\sigma_k T} - V^{\sigma_{k-1} T}\Big) \otimes I_{n_x} \end{pmatrix}. \tag{23}$$

Hence, the system has standard singular perturbations form with reduced (slow) variable x_s and boundary layer (fast) variable e_v. This time scale separation on the network of linear heterogeneous systems is due to the large coupling parameter γ and the coordinate transformation (10) which are the intrinsic properties of the network.

Remark 3 *After simple calculation, we can see that* $A_0^{\sigma_k} = \sum_{i=1}^{N} v_{l1_i}^{\sigma_k} A_i, \forall \sigma_k \in \mathcal{I}$ *and it is also important to note that the matrix* $A_0^{\sigma_k}$ *is the convex combination of the matrices* A_i, $\forall i \in \mathcal{N}$. *This is due to the property of the left eigenvector of the Laplacian of the directed graphs[1] i.e.,* $v_{l1_i}^{\sigma_k} \in (0,1), \forall i \in \mathcal{N}$ *and* $\sum_{i=1}^{N} v_{l1_i}^{\sigma_k} = 1$, $\forall \sigma_k \in \mathcal{I}$ *. Additionally, the matrix* $A_{22}^{\sigma_k}$ *is Hurwitz because the diagonal matrix* $\Lambda^{\sigma_k'} =$ *diag* $(\lambda_2^{\sigma_k}, \cdots \lambda_N^{\sigma_k})$ *has all positive eigenvalues i.e.,* $\lambda_i^{\sigma_k} > 0, \forall i = 2, \cdots, N$ *and* $\forall \sigma_k \in \mathcal{I}$.

Reduced Ordered System

In order to obtain the reduced order model in single time scale, of the system (21) and (22), we set $\epsilon = 0$. Then, we have the slow subsystem as follows,

$$\begin{aligned} \dot{x_s}(t) &= A_0^{\sigma_k} x_s(t) \qquad \forall t \in [t_k, t_{k+1}), k \in \mathbb{N} \\ e_v(t) &= 0. \end{aligned} \tag{24}$$

with the impulsive dynamics as follows:

$$x_s(t_k) = ((v_{l1}^{\sigma_k T} - v_{l1}^{\sigma_{k-1} T}) \otimes I_{n_x}) x(t_k^-), \qquad \forall k \geq 1. \tag{25}$$

The dynamics (24) is also referred as *emergent dynamics*, which is the dynamics of the mean field unit restricted to the synchronization manifold.

3 Stability

In this section, we analyze the stability of the system (21) and (22) using the Lyapunov function arguments. The following result from [3], Theorem 1 provides us the conditions for the Lyapunov stability of hybrid systems.

Definition: The hybrid system (21) and (22) with the equilibrium point $(\bar{x_s}, \bar{e_v}) = (0,0)$ is globally asymptotically stable, for $\epsilon \in (0, \epsilon_1)$, $\epsilon_1 > 0$ if there exists a function $V(x_s, e_v)$ such that $V(0,0) = 0$, $V(x,y) > 0$, $\forall (x,y) \neq 0$ and the following conditions holds for any $(x(0), y(0)) \neq (0,0)$
(a) $\forall \epsilon \in [0, \epsilon_1)$ and $t \in [t_k, t_{k+1})$ we have $\dot{V}(x(t), y(t)) < 0$ and
(b) $\forall\ t_k,\ k \in \mathbb{N}$, $V(x(t_{k+1}), y(t_{k+1})) < V(x(t_k), y(t_k))$.

We impose the following assumption on the dynamics of the original system, related to the stability of the slow and fast subsystems at each mode.

Assumption 1: The matrices $\sum_{i=1}^{N} v_{l1_i}^{\sigma_k} A_i, \forall \sigma_k \in \mathcal{I}$ are strictly diagonally dominant and the diagonal entries are negative.

Remark 4 *We impose the Assumption 1 so that the matrix* $A_0^{\sigma_k}$ *is Hurwitz, in other words, the slow systems are stable. Also it is important to note, as a consequence of the Assumption 1, we can have a stable network dynamics even if we have unstable individual agents in the network.*

Since the matrix $A_0^{\sigma_k}$ and $A_{22}^{\sigma_k}$ are Hurwitz, there exists symmetric positive definite matrices $Q_s^{\sigma_k} \geq I_{n_x}$ and $Q_f^{\sigma_k} \geq I_{n_x N-1}$ and positive numbers $\lambda_s^{\sigma_k}$ and $\lambda_f^{\sigma_k}$, $\forall \sigma_k \in \mathcal{I}$, such that,

$$\begin{aligned} A_0^{\sigma_k T} Q_s^{\sigma_k} + Q_s^{\sigma_k} A_0^{\sigma_k} &\leq -2\lambda_s^{\sigma_k} Q_s^{\sigma_k} \\ A_{22}^{\sigma_k T} Q_f^{\sigma_k} + Q_f^{\sigma_k} A_{22}^{\sigma_k} &\leq -2\lambda_f^{\sigma_k} Q_f^{\sigma_k} \end{aligned}$$

Let us denote, $\lambda_s = \min_{\sigma_k \in \mathcal{I}} \lambda_s^{\sigma_k}$ and $\lambda_f = \min_{\sigma_k \in \mathcal{I}} \lambda_f^{\sigma_k}$. Then, from [2] following proposition follows, which states the existence of composite Lyapunov function.

Proposition 1 *Under Assumption 1, there exists* $\epsilon_1 \geq 0$*, such that ,*

$$V(x_s, e_v) = x_s^T Q_s^{\sigma_k} x_s + e_v^T Q_f^{\sigma_k} e_v \tag{26}$$

is a Lyapunov function for the system (21), for all $\epsilon \in (0, \epsilon_1]$*, where,* $\epsilon_1 = \frac{\lambda_f}{\frac{(b_1+b_2)^2}{4\lambda_s}+b_3}$ *and* $\forall$ $\sigma_k \in \mathcal{I}$*,* $b_1^{\sigma_k} = \|(Q_s^{\sigma_k})^{\frac{1}{2}} B_1^{\sigma_k} (Q_f^{\sigma_k})^{-\frac{1}{2}}\|$*,* $b_2^{\sigma_k} = \|(Q_f^{\sigma_k})^{\frac{1}{2}} B_2^{\sigma_k} (Q_s^{\sigma_k})^{-\frac{1}{2}}\|$*,* $b_3^{\sigma_k} = \|(Q_f^{\sigma_k})^{\frac{1}{2}} B_3^{\sigma_k} (Q_f^{\sigma_k})^{-\frac{1}{2}}\|$ *and* $b_j = \max_{\sigma_k \in \mathcal{I}} b_j^{\sigma_k}, j \in \{1, 2, 3\}$*.*

Hence, it follows from Proposition 1 that the linear dynamics (21) are Lyapunov stable for all $\epsilon \in (0, \epsilon_1)$.

Now, the stablility analysis of the system (21) and (22) is done using the mode dependent function, $W_s = \sqrt{x_s^T Q_s^{\sigma_k} x_s}$ and $W_f = \sqrt{e_v^T Q_f^{\sigma_k} e_v}$, for all $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$ and assuming that after each switching the system remains in the current topology for time 'τ_k' i.e., $t_{k+1}^- - t_k = \tau_k$ $\forall k \in \mathbb{N}$. Then from [2], Lemma 4 we have,

$$W_s(t_{k+1}^-) \leq W_s(t_k)(e^{-\lambda_s \tau_k} + \epsilon\beta_3) + W_f(t_k)\epsilon(\beta_2 + \beta_3) \tag{27}$$

$$W_f(t_{k+1}^-) \leq W_s(t_k)\epsilon\beta_1 + W_f(t_k)(e^{\frac{-\lambda_f}{\epsilon}\tau_k} + \epsilon\beta_1), \tag{28}$$

where, $\beta_1 = \frac{\sqrt{b_2^2+b_3^2}}{\lambda_f}$, $\beta_2 = \frac{b_1}{\lambda_f - \epsilon_2\lambda_s}$ and $\beta_3 = \frac{b_1\beta_1}{\lambda_s}$, where, $\epsilon_2 \in (0, \epsilon_1] \cap (0, \frac{\lambda_f}{\lambda_s})$. The equations (27) and (28) characterize the variation of the decoupled Lyapunov function W_s and W_f during the continuous dynamics between two events i.e, immediately after the jump and before the next jump. Then during the jump, variations of W_s and W_f can be characterized as follows:

Lemma 3.1 *Let us consider a sequence* $(t_k)_{k\geq 0}$ *of event times, then for all* $k \geq 1$*, we have,*

$$W_s(t_k) \leq \rho_1 W_s(t_k^-) \tag{29}$$

$$W_f(t_k) \leq \rho_2 W_f(t_k^-) \tag{30}$$

where $\rho_1 = \max_{i,i' \in \mathcal{I}, j \in \mathcal{J}} \|(Q_s^{i'})^{\frac{1}{2}}((v_{l1}^{i\,T} - v_{l1}^{i'\,T})\boldsymbol{1}_N \otimes I_{n_x})(Q_s^i)^{-\frac{1}{2}}\|$ *and*

$\rho_2 = \max_{i,i' \in \mathcal{I}, j \in \mathcal{J}} \|(Q_f^{i'})^{\frac{1}{2}}((V^{i^T} - V^{i'^T})V^{i'} \otimes I_{n_x})(Q_f^i)^{-\frac{1}{2}}\|$.

Proof:

$$\begin{aligned}
W_s(t_k) &= \sqrt{x_s(t_k)^T Q_s^{\sigma_k} x_s(t_k)} = \|(Q_s^{\sigma_k})^{\frac{1}{2}} x_s(t_k)\| \\
&\leq \|(Q_s^{\sigma_k})^{\frac{1}{2}}((v_{l1}^{\sigma_k T} - v_{l1}^{\sigma_{k-1} T}) \otimes I_{n_x})x(t_k^-)\| \\
&= \|(Q_s^{\sigma_k})^{\frac{1}{2}}((v_{l1}^{\sigma_k T} - v_{l1}^{\sigma_{k-1} T}) \otimes I_{n_x})(e(t_k^-) + (\mathbf{1}_N \otimes I_{n_x})x_s(t_k^-))\| \\
&\leq \|(Q_s^{\sigma_k})^{\frac{1}{2}}((v_{l1}^{\sigma_k T} - v_{l1}^{\sigma_{k-1} T})V^{\sigma_k} \otimes I_{n_x})e_v(t_k^-)\| + \|(Q_s^{\sigma_k})^{\frac{1}{2}}((v_{l1}^{\sigma_k T} - v_{l1}^{\sigma_{k-1} T})\mathbf{1}_N \otimes I_{n_x})x_s(t_k^-))\| \\
&\leq \|(Q_s^{\sigma_k})^{\frac{1}{2}}((v_{l1}^{\sigma_k T} - v_{l1}^{\sigma_{k-1} T})\mathbf{1}_N \otimes I_{n_x})(Q_s^{\sigma_{k-1}})^{-\frac{1}{2}}\|W_s(t_k^-) \leq \rho_1 W_s(t_k^-).
\end{aligned}$$

The inequality (30) is obtained similarly.

Then multiplying bothsides of (27) and (28) by ρ_1 and ρ_2, respectively and from (29) and (30), we have

$$W_s(t_{k+1}) \leq \rho_1 W_s(t_{k+1}^-) \leq \rho_1(e^{-\lambda_s\tau_k} + \epsilon\beta_3)W_s(t_k) + \epsilon\rho_1(\beta_2 + \beta_3)W_f(t_k) \tag{31}$$

$$W_f(t_{k+1}) \leq \rho_2 W_f(t_{k+1}^-) \leq \epsilon\rho_2\beta_1 W_s(t_k) + \rho_2(e^{\frac{-\lambda_f}{\epsilon}\tau_k} + \epsilon\beta_1)W_f(t_k). \tag{32}$$

Then it follows, for any $\tau \geq 0$,

$$\begin{pmatrix} W_s(t_{k+1}) \\ W_f(t_{k+1}) \end{pmatrix} \leq M_\tau \begin{pmatrix} W_s(t_k) \\ W_f(t_k) \end{pmatrix}, \tag{33}$$

where, $M_\tau = \begin{pmatrix} \rho_1(e^{-\lambda_s \tau} + \epsilon\beta_3) & \epsilon\rho_1(\beta_2+\beta_3) \\ \epsilon\rho_2\beta_1 & \rho_2(e^{-\frac{\lambda_f}{\epsilon}\tau} + \epsilon\beta_1) \end{pmatrix}$. Then from the equation (33) we can express the relation between $W_s(t_k)$ and $W_f(t_k)$ with $W_s(t_0)$ and $W_f(t_0)$, respectively, as follows:

$$\begin{pmatrix} W_s(t_k) \\ W_f(t_k) \end{pmatrix} \leq M_{\tau_{k-1}} \cdots M_{\tau_0} \begin{pmatrix} W_s(t_0) \\ W_f(t_0) \end{pmatrix} \iff \begin{pmatrix} W_s(t_k) \\ W_f(t_k) \end{pmatrix} \leq (M_{\tau^*})^k \begin{pmatrix} W_s(t_0) \\ W_f(t_0) \end{pmatrix}, \forall k \in \mathbb{N}. \tag{34}$$

This implies that if the positive matrix M_{τ^*} is Schur, then both the sequences, $(W_s(t_k))_{k\geq 0}$ and $(W_f(t_k)_{k\geq 0}$ go to 0, and the system (21) and (22) are globally asymptotically stable as follows.

$$\begin{aligned} V(t_{k+1}) &= W_s^2(t_{k+1}) + W_f^2(t_{k+1}) \\ &\leq \|(M_{\tau^*})\|_\infty^2 (W_s(t_k) + W_f(t_k))^2 \\ &\leq 2\|(M_{\tau^*})\|_\infty^2 V(t_k). \end{aligned}$$

It is clear from (34) and above inequality that the necessary condition for (asymptotic) stability of system (21) and (22), under arbitrary switching, is that all of the individual subsystems are (asymptotically) stable. Hence, ensuring that the spectral radius of the matrix $M_{\tau^*} < 1$, *i.e.*, the matrix is *schur*, provides us the necessary condition for the stability. Furthermore, it can be seen from the structure of the matrix M_{τ^*} that the bounds on the value of τ^* is the required necessary condition for ensuring the matrix to be Schur. Thus, we proceed with establishing the sufficient conditions for deriving such bounds for τ^*.

By definition, the positive matrix M_{τ^*} is Schur iff there exists $p \in \mathbb{R}_+^2$, such that $M_{\tau^*}^T p < p$ [7]. Taking the form of p as $(1, a\epsilon)^T$ with $a > \rho_1(\beta_2+\beta_3)$, then $M_{\tau^*}^T p < p$ is equivalent to

$$\rho_1(e^{-\lambda_s \tau^*} + \epsilon\beta_3) + a\epsilon^2\rho_2\beta_1 < 1 \tag{35}$$

$$\epsilon\rho_1(\beta_2+\beta_3) + a\rho_2\epsilon(e^{-\frac{\lambda_f}{\epsilon}\tau^*} + \epsilon\beta_1) < a\epsilon \tag{36}$$

Case 1:
From equation (34) we have, for $\rho_1 > 1$,

$$\tau^* > -\frac{1}{\lambda_s}\ln\left(\frac{1-\epsilon\rho_1\beta_3 - a\epsilon^2\rho_2\beta_1}{\rho_1}\right) = \frac{\ln(\rho_1)}{\lambda_s} + \eta_1(\epsilon) \tag{37}$$

where $\eta_1(\epsilon) = -\frac{\ln(1-\epsilon\rho_1\beta_3 - a\epsilon^2\rho_2\beta_1)}{\lambda_s} = \mathcal{O}(\epsilon)$ and it is defined only for $\epsilon < \epsilon_3$ such that $\epsilon_3 = \frac{-\rho_1\beta_3 + \sqrt{\rho_1^2\beta_3^2 + 4a\rho_2\beta_1}}{2a\rho_2\beta_1}$.
Similarly from equation (35), we have,

$$\tau^* > \frac{\epsilon}{\lambda_f}\ln\left(\frac{a\rho_2}{a-\rho_1(\beta_2+\beta_3) - a\epsilon\rho_2\beta_1}\right) \tag{38}$$

As ϵ goes to 0, it is clear that the right hand side of equation (37) also goes to 0. Since, from equation (36), we have $\tau^* > \frac{\ln(\rho_1)}{\lambda_s} + \eta_1(\epsilon) > \frac{\ln(\rho_1)}{\lambda_s}$, we can find $\epsilon_4 > 0$ such that for all $\epsilon \in (0, \epsilon_4)$ following holds

$$\frac{\ln(\rho_1)}{\lambda_s} > \frac{\epsilon}{\lambda_f}\ln\left(\frac{a\rho_2}{a-\rho_1(\beta_2+\beta_3) - a\epsilon\rho_2\beta_1}\right).$$

Hence setting $\epsilon_1^* = \min(\epsilon_2, \epsilon_3, \epsilon_4)$, we have for all $\epsilon \in (0, \epsilon_1^*)$,

$$\tau^* > \ln(\frac{\rho_1}{\lambda_s}) + \eta_1(\epsilon). \tag{39}$$

Case 2:
When $\rho_1 = 1$, it is sufficient to find the conditions such that the equation (34) and (35) holds and we have,

$$\tau^* > \eta_1(\epsilon) \qquad \text{and} \qquad \tau^* > \frac{\epsilon}{\lambda_f}\ln\left(\frac{a\rho_2}{a-(\beta_2+\beta_3) - a\epsilon\rho_2\beta_1}\right).$$

Hence, for all $\epsilon \in (0, \epsilon_2^*)$ where $\epsilon_2^* = \epsilon_3$, we can have the condition as follows:

$$\tau^* > \eta_2(\epsilon), \qquad \text{where} \qquad \eta_2 = \max\left(\eta_1(\epsilon), \frac{\epsilon}{\lambda_f}\ln\left(\frac{a\rho_2}{a - (\beta_2+\beta_3) - a\epsilon\rho_2\beta_1}\right)\right). \tag{40}$$

Case 3:
To derive the necessary condition for a matrix M_{τ^*} to be Schur when $\rho_1 < 1$, we look for $p = (1,1)^T$. Then $M_{\tau^*}p < p$ is equivalent to

$$\rho_1(e^{-\lambda_s\tau} + \epsilon\beta_3) + \epsilon\rho_2\beta_1 < 1 \tag{41}$$

$$\epsilon\rho_1(\beta_2+\beta_3) + \rho_2(e^{-\frac{\lambda_f}{\epsilon}\tau} + \epsilon\beta_1) < 1 \tag{42}$$

which in turn is equivalent to

$$\tau^* > \frac{1}{\lambda_s}\ln\left(\frac{\rho_1}{1 - \epsilon\rho_2\beta_1 - \rho_1\epsilon\beta_3}\right) \tag{43}$$

$$\tau^* > \frac{\epsilon}{\lambda_f}\ln\left(\frac{\rho_2}{1 - \epsilon\rho_1(\beta_2+\beta_3) - \rho_2\epsilon\beta_1}\right). \tag{44}$$

Since $\rho_1 < 1$, the inequality (43) holds for all $\epsilon \in (0, \epsilon_5)$, where $\epsilon_5 = \frac{1-\rho_1}{\rho_2\beta_1+\rho_1\beta_2}$. Then the second inequality is equivalent to $\tau^* > \eta_3(\epsilon)$ where

$$\eta_3(\epsilon) = \frac{\epsilon}{\lambda_f}\ln\left(\frac{\rho_2}{1 - \epsilon\rho_1(\beta_2+\beta_3) - \rho_2\epsilon\beta_1}\right). \tag{45}$$

It is clear that $\eta_3(\epsilon) = \mathcal{O}(\epsilon)$ and is well defined for $\epsilon < \epsilon_6$ given by

$$\epsilon_6 = \frac{1}{\rho_1(\beta_2+\beta_3) + \rho_2\beta_1}. \tag{46}$$

4 Example

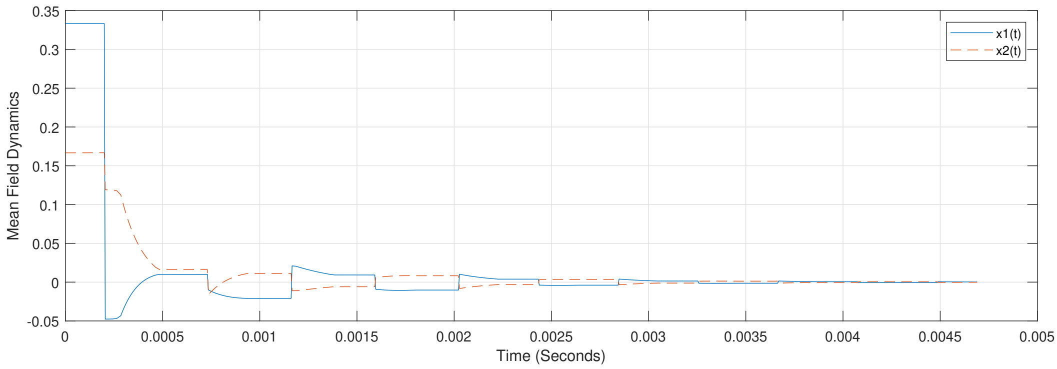
To perform the numerical simulation, let us consider the system (21) and (22), with four agents ($N = 4$) and periodic switching ($m = 2$). Let t_k, $k \geq 0$ be the sequence of discrete events where the switching takes place. We consider the following dynamics of the agents,

$$A_1 = \begin{pmatrix} -1 & 0.5 \\ -1 & -1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} -2 & -2 \\ 2 & 1 \end{pmatrix} \qquad A_3 = \begin{pmatrix} -1 & 0.5 \\ -2 & -2 \end{pmatrix} \qquad A_4 = \begin{pmatrix} 1 & -1 \\ 1 & 0.5 \end{pmatrix}$$

and the Laplacian of the graph G1 and G2 are as follows:

$$L_1 \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 2 \end{pmatrix} \qquad L_2 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}.$$

Let $\gamma = 1000$, i.e. $\epsilon = 10^{-3}$ and the initial condition $x_0 = [0, 2, 1, -1, -2, 3, 1, 0]$. It is verified that the assumption 1 is satisfied and hence the reduced (slow) subsystem and boundary layer (fast) subsystems are stable.



(a)

Figure 1: State trajectories of the mean field dynamics with $\tau_k = 0.0005$ sec.

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