Student Information

Full Name :
Id Number :

Answer 1

a.

Reflexive: Assume $a \in R$. Then, $|a - a| = 0$ which satisfies $0 \in R$, $0 < 4$ and so aAa. Hence, the relation is reflexive.

b.

Symmetric: Assume $x, y \in R$. Suppose xAy. Let $|x - y| = m$ where $m < 4$ and $m \in R$. Then $|y - x| = m$ (since the absolute value measures the distance between 2 numbers) which satisfies $m < 4$ and $m \in R$ and yAx. Thus, the relation is symmetric.

c.

Transitive: Let $x, y, z \in R$. Suppose xAy and yAz. Then $|x - y| < 4$ and $|y - z| < 4$ where $|x - y|, |y - z| \in R$. We can write the inequalities as follows:

$$-4 < x - y < 4 \tag{1}$$

$$-4 < y - z < 4 \tag{2}$$

When we sum the inequalities we get:

$$-8 < x - z < 8 \tag{3}$$

which is equivalent to $|x - z| < 8$. We can find a counter example where $|x - y| < 4$ and $|y - z| < 4$ is satisfied but $|x - z| < 4$ is not satisfied: If $x = 6$, $y = 3$ and $z = 1$, then $|6 - 3| < 4$ and $|3 - 1| < 4$ but $|6 - 1| \nless 4$. Thus, we conclude that the relation is not reflexive.

Answer 2

a.

T must be reflexive, symmetric and transitive in order to be an equivalence relation.

Reflexive: Given $a \in S$, we clearly have $a/a = 1$ which is a rational number; hence a T a and the relation is reflexive.

Symmetric: Assume $a, b \in S$ are given and a T b. By definition of the relation a/b is a rational number. Because $a \neq 0$ and $b \neq 0$, this implies that a/b is a nonzero rational number and therefore the inverse $1/(a/b) = b/a$ is also a rational number. Then again by the definition of T, this implies b T a and the relation is symmetric.

Transitive: Assume $a, b, c \in S$ are given with a T b and b T c. By the definition of T, equivalently a/b and b/c are rational numbers. The product of rational numbers is rational, so $(a/b).(b/c) = a/c$ is also rational. By the definition of T, a T c and the relation is transitive.

b.

We provide distinct equivalence classes by providing a unique representative from each of them. Consider the subset

$$B = \{r - \sqrt{5} : r \in Q\} \cup \{1\} \tag{4}$$

of S, where Q represents the set of rational numbers. We claim every element of S is equivalent to exactly one of the elements of S and therefore elements of B represent all distinct equivalence classes of T. Given $a = x - y\sqrt{5} \in S$ for $x, y \in Q$. If $y = 0$ then $a = x$ and therefore $a/1 = x \in Q$; hence a T 1. If $y \neq 0$, then $r = x/y \in Q$ and we can see that

$$\frac{a}{r - \sqrt{5}} = \frac{x - y\sqrt{5}}{\frac{x}{y} - \sqrt{5}} = y \in Q \tag{5}$$

Therefore a T $(r - \sqrt{5})$.

It only remains to show that distinct elements of B are not equivalent. Suppose $r - \sqrt{5}$ and $s - \sqrt{5}$ are distinct elements of B. Let

$$K = \frac{r - \sqrt{5}}{s - \sqrt{5}} = \frac{r - \sqrt{5}}{s - \sqrt{5}} \times \frac{s + \sqrt{5}}{s + \sqrt{5}} = \frac{rs + \sqrt{5}r - \sqrt{5}s - 5}{s^2 - 5} \tag{6}$$

From this we conclude

$$\sqrt{5} = \frac{K(s^2 - 5) - rs + 5}{r - s} \tag{7}$$

is rational which is not the case. Hence $r - \sqrt{5}$ and $s - \sqrt{5}$ are not equivalent. Also we can see that for every element $r - \sqrt{5}$ of B, $(r - \sqrt{5})/1 = r - \sqrt{5}$ is also irrational and therefore $r - \sqrt{5}$ is not equivalent to 1. Hence every two distinct elements of B are not equivalent and represent different equivalence classes.

Answer 3

Note: You are expected to explain your solution. In the below solution, the explanations of the application of the algorithm is taken from the student with ID=2094415.

The pseudocode of the Warshall algorithm is given below.

MR : n x n zero–one matrix W : = MR
for k : = 1 to n
    for i : = 1 to n
        for j : = 1 to n
            $wij := w_{ij} \vee (w_{ik} \wedge w_{kj})$
return W $\{W = [w_{ij}]$ is MR $\}$

We need adjacency matrix as n x n zero-one matrix. Let's assume $v_1 = 1$, $v_2 = 2$, $v_3 = 3$ and $v_4 = 4$. Then, we can define $A_R = W_0$ as the matrix of the relation, where the rows go from 1 to 4 starting from the top, and the columns go from 1 to 4 starting from the left. We have paths from 1 to 2, 2 to 4, 4 to 1 and 4 to 3. So:

$$\mathbf{W_0} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Now in W_1 we will add the paths that include v_1 as an interior vertex. This means that now we have a new path:
From 4 to 2, $(4,1) \to (1,2)$. So:

$$\mathbf{W_1} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & \mathbf{1} & 1 & 0 \end{pmatrix} \end{array}$$

In this next matrix W_2, we will add paths that also include v_2, meaning they can include both v_1 and v_2. Now we have two new paths:
From 1 to 4, $(1,2) \to (2,4)$
From 4 to 4, $(4,1) \to (1,2) \to (2,4)$. So:

$$\mathbf{W_2} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & \mathbf{1} \end{pmatrix} \end{array}$$

In W_3, we will have paths that can also use v_3 as their interior vertex. As there is no edge from 3 to any other vertex, this does not add any new paths and $W_3 = W_2$. So:

$$\mathbf{W_3} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

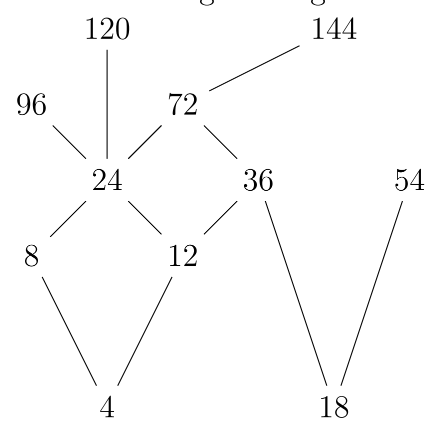
In W_4, we will have all possible paths, as now we can use all vertices (i.e. 1, 2, 3 and 4) as interior. This means we have five new paths:
From 1 to 1, $(1,2) \to (2,4) \to (4,1)$
From 1 to 3, $(1,2) \to (2,4) \to (4,3)$
From 2 to 1, $(2,4) \to (4,1)$
From 2 to 2, $(2,4) \to (4,1) \to (1,2)$
From 2 to 3, $(2,4) \to (4,3)$. So:

$$\mathbf{W_4} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} \mathbf{1} & 1 & \mathbf{1} & 1 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

Notice that the 1's in bold in each matrix is a result of the previous step. W_4 is the matrix of the transitive closure. The final transitive closure relation is following:
$R^* = \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),(4,1),(4,2),(4,3),(4,4)\}$

Answer 4

The Hasse diagram is given below:



a.

54,96,120,144

b.

4,18

c.

No.

d.

No.

e.

36,72,144

f.

36

g.

4,8,12,24

h.

24

Answer 5

In both graphs, there are 6 vertices, 8 edges and the degree sequence is 2,2,2,3,3,4. So we can expect that there exists an isomorphism. **However, when these invariants are the same, it does not necessarily mean that the two graphs are isomorphic. There are no useful sets of invariants currently known that can be used to determine whether simple graphs are isomorphic.**

We now will define a function f and then determine whether it is an isomorphism. Because $deg(b) = deg(q) = 4$, we can set f(b)=q. There are 2 vertices having degree 3, so there are two possible mappings either:

- f(a)=m and f(e)=n or
- f(a)=n and f(e)=m

In any of the above possibilities , we see that vertice d connects vertices a and e and vertice r connects vertices m and n. Hence we can set f(d)=r. Then, we check the first possibility and assume f(a)=m and f(e)=n. We see that vertice c connects vertices a and b and vertice p connects vertices m and q. Hence, we set f(c)=p. The only vertices remaining in both of the graphs are f and o; so we set f(f)=o. We now have a one-to-one correspondence between the vertex set of G and the vertex set of H, namely, f(a)=m, f(b)=q, f(c)=p, f(d)=r, f(e)=n, f(f)=o. To see f preserves edges, we examine the adjacency matrix of G and the adjacency matrix of H with the rows and columns labeled by the images of the corresponding vertices in G.

$$\mathbf{A_G} = \begin{array}{c|cccccc} & a & b & c & d & e & f \\ \hline a & 0 & 1 & 1 & 1 & 0 & 0 \\ b & 1 & 0 & 1 & 0 & 1 & 1 \\ c & 1 & 1 & 0 & 0 & 0 & 0 \\ d & 1 & 0 & 0 & 0 & 1 & 0 \\ e & 0 & 1 & 0 & 1 & 0 & 1 \\ f & 0 & 1 & 0 & 0 & 1 & 0 \end{array} \qquad \mathbf{A_H} = \begin{array}{c|cccccc} & m & q & p & r & n & o \\ \hline m & 0 & 1 & 1 & 1 & 0 & 0 \\ q & 1 & 0 & 1 & 0 & 1 & 1 \\ p & 1 & 1 & 0 & 0 & 0 & 0 \\ r & 1 & 0 & 0 & 0 & 1 & 0 \\ n & 0 & 1 & 0 & 1 & 0 & 1 \\ o & 0 & 1 & 0 & 0 & 1 & 0 \end{array}$$

Since $A_G = A_H$, it follows that f preserves edges. We conclude that f is an isomorphism, so G and H are isomorphic.

Answer 6

a.

There are possible ways to answer this question. Here, two of those are given:

1.(From the answer sheet of the student with ID=2098952)
Since no vertices in V_1 are adjacent, no edge is counted twice in the sum of degrees of vertices in V_1. And no vertices are adjacent in V_2, i.e. they all are adjacent with a vertice in V_1. So, the number of edges in G is the sum of the degrees of vertices in V_1. This is also true for V_2.

The sum of the degrees of vertices in V_1 is $|V_1|n$ because the degree of all vertices in V_1 is n. For the same reason, the sum of the degrees of vertices in V_2 is $|V_2|n$.

Since both sums are equal to the number of vertices in G, $|V_1|n = |V_2|n$.

Hence, $|V_1| = |V_2|$.

2.
Since all vertices in G has exactly the same degree $n > 0$, G is an n-regular graph. Since G is n-regular, the total number of edges is both $n|V_1|$ and $n|V_2|$. Hence, $n|V_1| = n|V_2|$ which simplifies to $|V_1| = |V_2|$

b.

Recall Hall's theorem says the bipartite graph G with bipartition (V_1, V_2) has a complete matching from V_1 to V_2 if and only if $|N(A)| \geq |A|$ for all subsets A of V_1, where N(A) denotes the set of neighbors of vertices in A. We prove by contradiction that this condition holds for any n-regular bipartite graph. Suppose not. Then there is a non-empty subset $C \subseteq V_1$, such that $|C| > |N(C)|$. This means that the $n|C|$ edges incident on vertices in C are all incident on vertices in N(C). But then, by the generalized pigeonhole principle, this means that some vertex in N(C) has degree greater than n. But this contradicts the fact that G is n-regular. Thus, Hall's condition holds and any n-regular bipartite graph has a perfect matching.