Answer 1

The A_k given in the question is a hypercube (a.k.a. n-cube with n-dimensions). We will use mathematical induction to prove the claim.

- **Basis Step:** As seen in figure 3, A_2 has a hamiltonian cycle with the following traversal: $00 \to 01 \to 11 \to 10 \to 00$

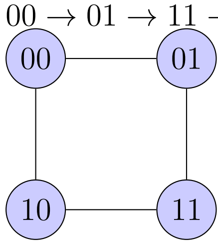
  

  Fig. 3

- **Inductive Step:** We will show that the conditional statement $P(n) \to P(n+1)$ is true $\forall n > 1$ where $P(n)$ represents A_n has a hamiltonian cycle. Suppose $P(n)$ holds for an arbitrary positive integer $n > 1$. The (n+1)-cube consists of two copies of an n-cube, where we join each vertex of one copy to its corresponding sister. Now we will show that a Hamiltonian cycle is possible for the (n+1)-cube. Start at vertex v_{a_1} in the first n-cube. By the inductive hypothesis, there is a Hamiltonian cycle for this n-cube whose last vertex (before returning to v_{a_1}) is v_{b_1}. Likewise, the second cube has a corresponding Hamiltonian path from v_{a_2} to v_{b_2}. Now consider the path that starts at v_{a_1}, traverses all vertices of the first copy ending with vertex v_{b_1}, then travels to the corresponding vertex v_{b_2} of the second n-cube, then travels backwards along the Hamiltonian path in the second cube from v_{b_2} to v_{a_2}, and then finally travels back to the first cube's vertex at v_{a_1}. This is a Hamiltonian cycle of the (n+1)-cube.

We have completed the basis step and the inductive step so the proof is completed.

Answer 2

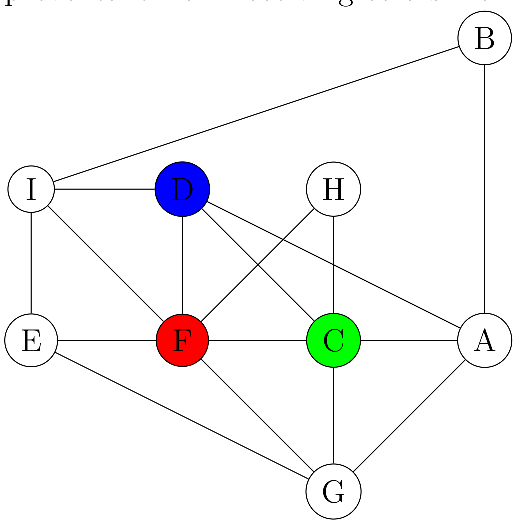
At first sight, we can observe that the chromatic number can not be less than 3; if we consider nodes I, E and F which are adjacent to each other pairwise, they all should have different colors. Also, after a careful analysis, we also arrive the result that chromatic number must be greater than 3. To show that, consider three nodes D, C and F which are colored as D blue, C green and F red. It is clear that I should be green. Then E is blue because it is adjacent to I and F. Then, since G is adjacent to E, F and C which have 3 different colors, we need one more color; hence chromatic number should be at least 4.
In order to color vertices, we can use a greedy algorithm to color vertices. It does not always succeed in finding the minimum number (the chromatic number), but at least provides some proper coloring and gives an upper bound for the chromatic number. The procedure requires us to number consecutively the colors that we use, so each time we introduce a new color, we number it also. Here is the procedure:

1. Color a vertex with color 1.

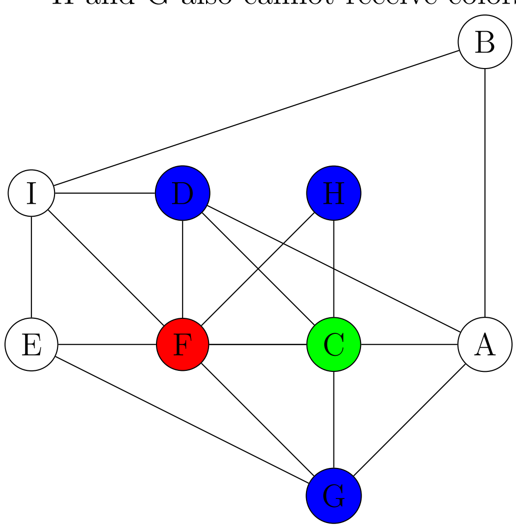
2. Pick an uncolored vertex v. Color it with the lowest-numbered color that has not been used on any previously-colored vertices adjacent to v. (If all previously-used colors appear on vertices adjacent to v, this means that we must introduce a new color and number it.)

3. Repeat the previous step until all vertices are colored.

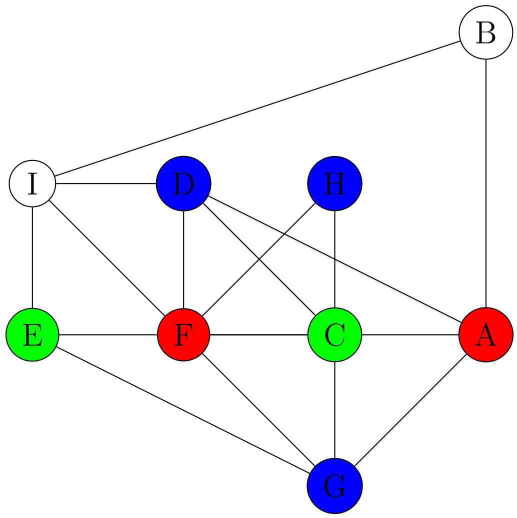
Clearly, this produces a proper coloring, since we are careful to avoid conflicts each time we color a new vertex. Suppose we decide to color the vertices in order C, F, D, H, G, A, E, I, B. Then we would color C with color 1 (green), F with color 2 (red) since adjacency with C prevents it from receiving color 1 (green), and we color D with color 3 (blue) since adjacency with C and F prevents it from receiving colors 1 and 2 (green and red). So we have:



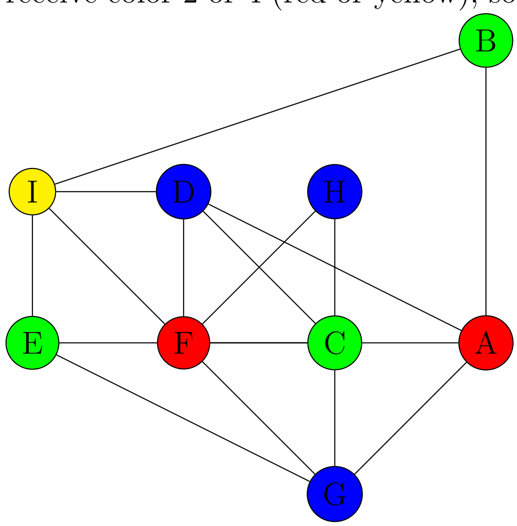
H and G also cannot receive colors 1 and 2 (green and red), so they are given color 3 (blue):



Then A cannot receive colors 1 and 3 (green and blue), so we give it color 2 (red), while E cannot receive colors 2 and 3 (red and blue), so we give it color 1 (green).



Vertex I cannot receive color 1, 2, or 3, and so we give it color 4 (say, yellow). Vertex B cannot receive color 2 or 4 (red or yellow), so we give it color 1 (green). The final coloring is given below:



As a result, the chromatic number of the graph is 4.

Answer 3

For T_1,

1. The root node of T_1 has 1 child.
2. In the first level, there are one node and it has 2 children.
3. In the second level, there are 2 nodes and each node has 3 children.
4. In the third level, there are $2 \cdot 3$ nodes and each node has 4 children.

So in a level l, there are $l!$ nodes in T_1. The total number of nodes in T_1 is,

$$\sum_{l=0}^{h} l!$$

For T_2,

1. The root node of T_2 has h child.
2. In the first level, there are h nodes and each node has $(h-1)$ children.
3. In the second level, there are $h \cdot (h-1)$ nodes and each node has $h-2$ children.
4. In the third level, there are $h \cdot (h-1) \cdot (h-2)$ nodes and each node has $h-3$ children.

So in a level l, there are $\frac{h!}{(h-l)!}$ nodes in T_2. The total number of nodes in T_2 is,

$$\sum_{l=0}^{h} \frac{h!}{(h-l)!}$$

We need to check if

$$\sum_{l=0}^{h} l! \leq \sum_{l=0}^{h} \frac{h!}{(h-l)!}$$

By taking each term, we get

$$\begin{aligned} l! &\leq \frac{h!}{(h-l)!} \\ 1 &\leq \frac{h!}{(h-l)!l!} \\ 1 &\leq C(h, l) \end{aligned}$$

Since $0 \leq l \leq h$, by the definition of combinations, this inequality holds. Hence, for each level l, the number of nodes in T_2 is greater than or equal to the number of nodes in T_1 in that level and therefore, the total number of nodes in in T_2 is greater than or equal to the total number of nodes in T_1.

Answer 4

A tree with n nodes has $n-1$ edges. So, the complement of T must have

$$\binom{n}{2} - (n-1) = \frac{n \cdot (n-1)}{2} - (n-1)$$

edges. If its complement is also a tree, then it should also have $(n-1)$ edges. So,

$$\frac{n \cdot (n-1)}{2} - (n-1) = (n-1)$$

$$n \cdot (n-1) = 4 \cdot (n-1)$$

Since $n \geq 2$, n must be equal to 4.