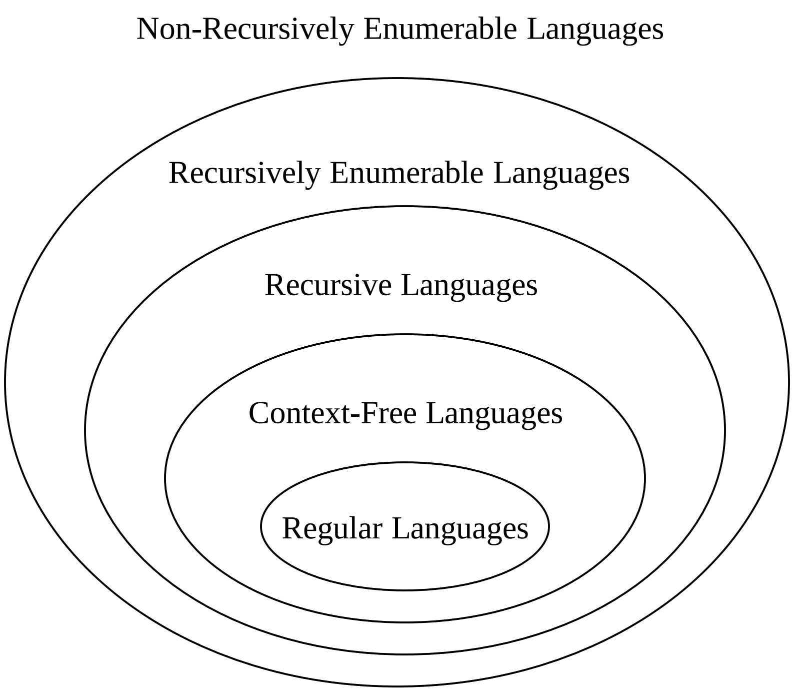
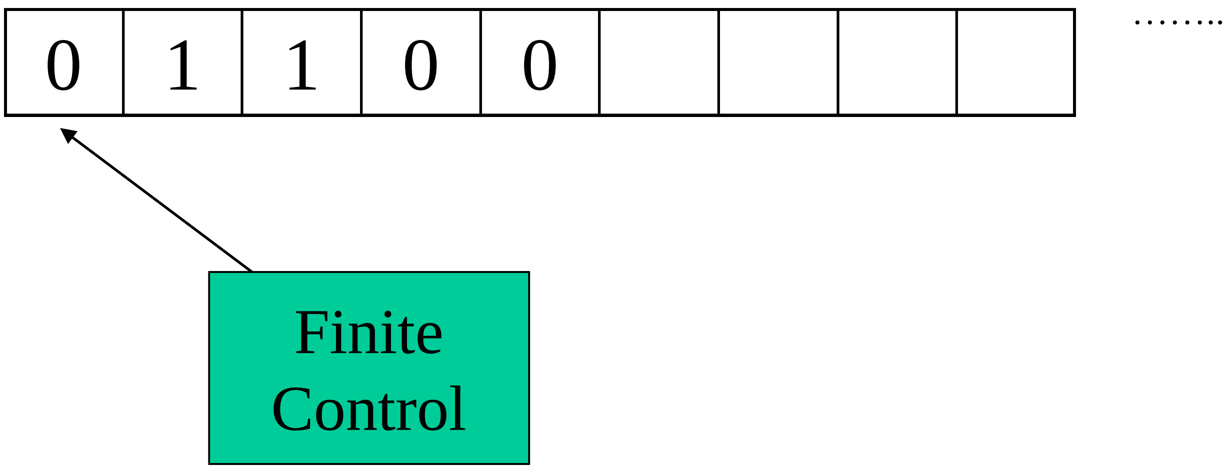
Hierarchy of languages



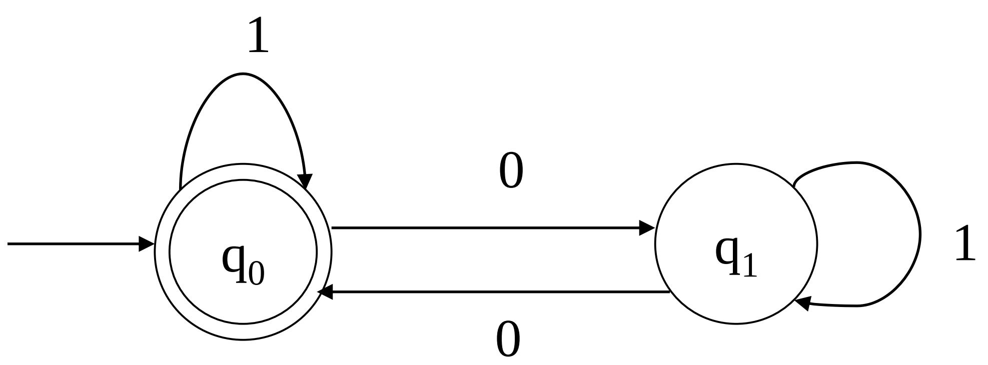
Deterministic Finite State Automata (DFA)



- One-way, infinite tape, broken into cells
- One-way, read-only tape head.
- Finite control, i.e.,
  - finite number of states, and
  - transition rules between them, i.e.,
  - a program, containing the position of the read head, current symbol being scanned, and the current "state."
- A string is placed on the tape, read head is positioned at the left end, and the DFA will read the string one symbol at a time until all symbols have been read. The DFA will then either *accept* or *reject* the string.

- The finite control can be described by a transition diagram or table:

- Example #1:



$$\begin{array}{cccccccccccc} & 1 & & 0 & & 0 & & 1 & & 1 & \\ q_0 & & q_0 & & q_1 & & q_0 & & q_0 & & q_0 \end{array}$$

- One state is final/accepting, all others are rejecting.
- The above DFA accepts those strings that contain an even number of 0's, including the *null* string, over *Sigma* = {0,1}

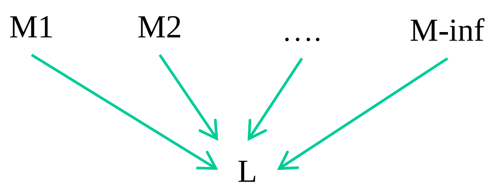
    L = {all strings with zero or more 0's}

- Note, the DFA must reject all other strings

Note:

- *Machine is for accepting a language, language is the purpose!*

- *Many equivalent machines may accept the same language, but a machine cannot accept multiple languages!*

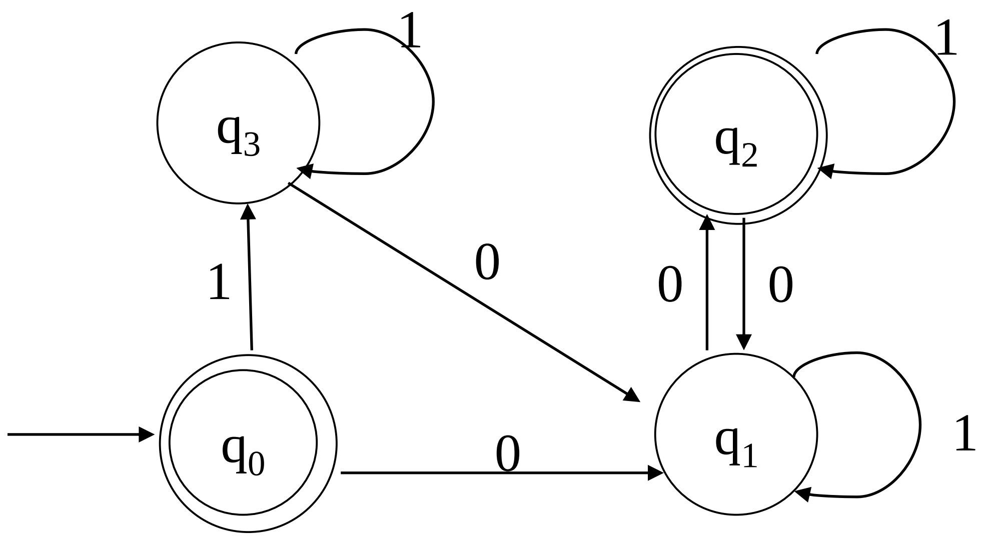


- *Id's of the characters or states are irrelevant, you can call them by any names!*

Sigma = {0, 1} ≡ {a, b}

States = {q0, q1} ≡ {u, v}, as long as they have identical (isomorphic) transition table

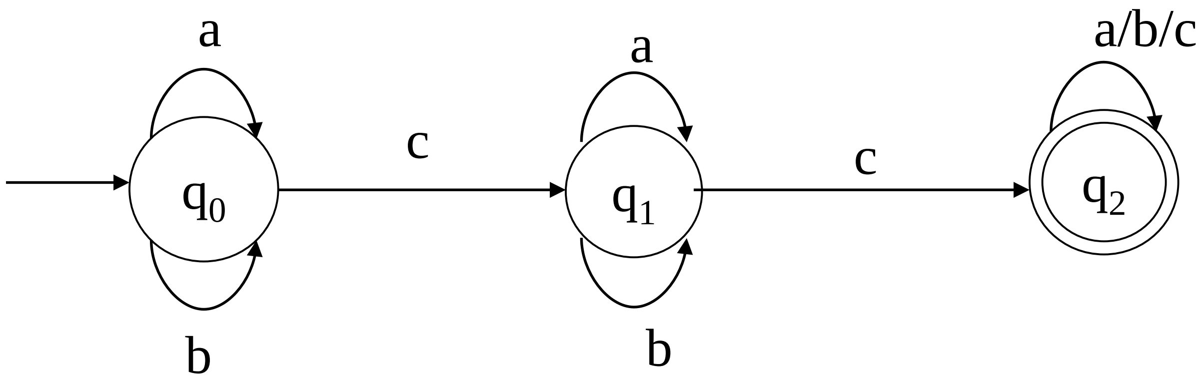
- An equivalent machine to the previous example (DFA for even number of 0's):



	1		0		0		1		1		
q_0		q_3		q_1		q_2		q_2		q_2	*accept string*

- One state is final/accepting, all others are rejecting.
- The above DFA accepts those strings that contain an even number of 0's, including null string, over *Sigma* = {0,1}
- *Can you draw a machine for a language by excluding the null string from the language?* L = {all strings with 2 or more *0*'s}

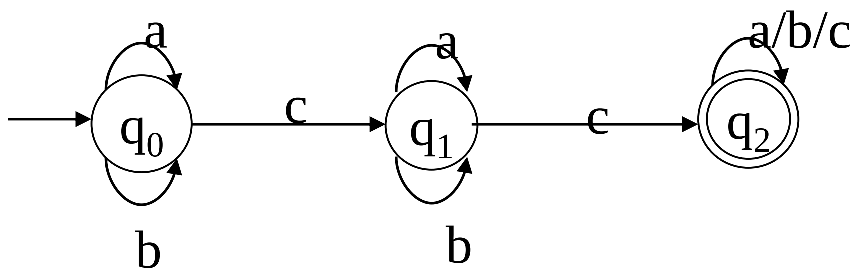
- Example #2:



	a	c	c	c	b	<u>accepted</u>
q_0	q_0	q_1	q_2	q_2	q_2	

	a	a	c	<u>rejected</u>
q_0	q_0	q_0	q_1	

- Accepts those strings that contain <u>at least</u> two *c*'s



Inductive Proof (sketch): that the machine correctly accepts strings with at least two *c*'s
Proof goes over the length of the string.

Base: x a string with |x|=0. state will be q0 => rejected.
Inductive hypothesis: |x|= integer *k*, & string *x* is *rejected* - in state q0 (x must have *zero* c),
OR, *rejected* – in state q1 (x must have *one* c),
OR, *accepted* – in state q2 (x has already with *two* c's)

Inductive steps: Each case for symbol *p*, for string *xp* (|*xp*| = *k*+1), the last symbol *p* = *a*, *b* or *c*

	xa	**xb**	**xc**
x ends in q0	q0 =>reject *(still zero c => should reject)*	q0 =>reject *(still zero c => should reject)*	q1 =>reject *(still zero c => should reject)*
x ends in q1	q1 =>reject *(still one c => should reject)*	q1 =>reject *(still one c => should reject)*	q2 =>accept *(two c now=> should accept)*
x ends in q2	q2 =>accept (two c already => should accept)	q2 =>accept (two c already => should accept)	q2 =>accept (two c already => should accept)

Formal Definition of a DFA

- A DFA is a five-tuple:

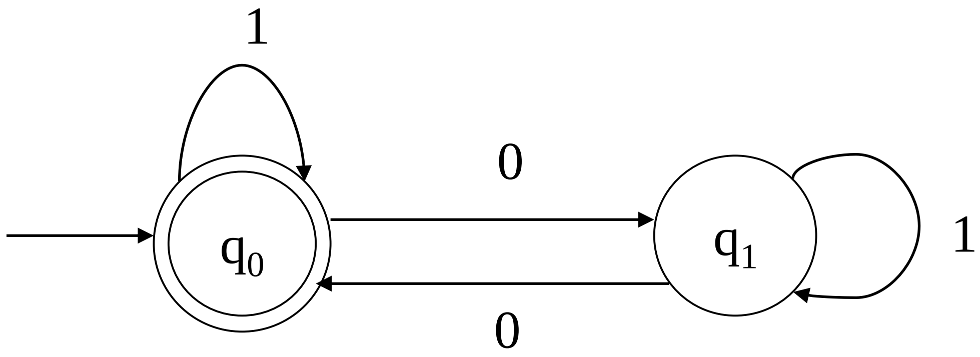
  $M = (Q, \Sigma, \delta, q_0, F)$

Q	A finite set of states
Σ	A finite input alphabet
q_0	The initial/starting state, q_0 is in Q
F	A set of final/accepting states, which is a subset of Q
δ	A transition function, which is a total function from $Q \times \Sigma$ to Q

$\delta: (Q \times \Sigma) \rightarrow Q$	δ is defined for any q in Q and s in Σ, and
$\delta(q,s) = q'$	is equal to some state q' in Q, could be q'=q

  Intuitively, $\delta(q,s)$ is the state entered by M after reading symbol s while in state q.

- Revisit example #1:



$Q = \{q_0, q_1\}$

$\Sigma = \{0, 1\}$

Start state is q_0

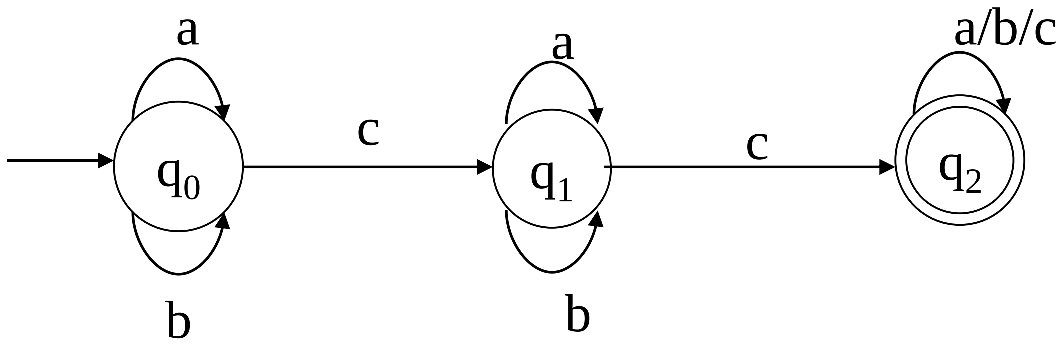
$F = \{q_0\}$

δ:

	0	1
q_0	q_1	q_0
q_1	q_0	q_1

- Revisit example #2:

$Q = \{q_0, q_1, q_2\}$
$\Sigma = \{a, b, c\}$
Start state is q_0
$F = \{q_2\}$



δ:

	a	b	c
q_0	q_0	q_0	q_1
q_1	q_1	q_1	q_2
q_2	q_2	q_2	q_2

- Since δ is a function, at each step M has exactly one option.
- It follows that for a given string, there is exactly one computation.

Extension of δ to Strings

$\delta^\wedge : (Q \times \Sigma^*) \rightarrow Q$

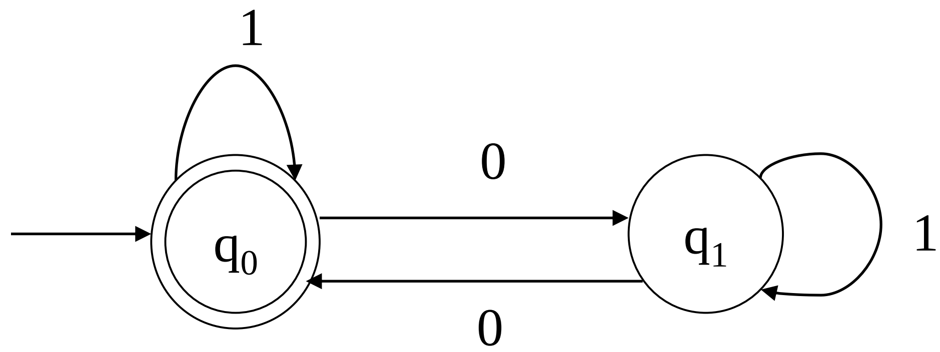
$\delta^\wedge(q,w)$ – The state entered after reading string *w* having started in state *q*.

Formally:

1) $\delta^\wedge(q, \varepsilon) = q$, and
2) For all w in Σ^* and a in Σ

$$\delta^\wedge(q,wa) = \delta\,(\delta^\wedge(q,w), a)$$

- Recall Example #1:



- What is $\delta^\wedge(q_0, 011)$? Informally, it is the state entered by M after processing 011 having started in state $q_{0.}$
- Formally:

$\delta^\wedge(q_0, 011)$	$= \delta\ (\delta^\wedge(q_0,01), 1)$	by rule #2
	$= \delta\ (\delta\ (\ \delta^\wedge(q_0,0), 1), 1)$	by rule #2
	$= \delta\ (\delta\ (\delta\ (\delta^\wedge(q_0, \lambda), 0), 1), 1)$	by rule #2
	$= \delta\ (\delta\ (\delta(q_0,0), 1), 1)$	by rule #1
	$= \delta\ (\delta\ (q_1, 1), 1)$	by definition of δ
	$= \delta\ (q_1, 1)$	by definition of δ
	$= q_1$	by definition of δ

- Is 011 accepted? No, since $\delta^\wedge(q_0, 011) = q_1$ is not a final state.

- Note that:

$$\delta^{\wedge}(q,a) = \delta(\delta^{\wedge}(q, \varepsilon), a) \qquad \text{by definition of } \delta^{\wedge}\text{, rule \#2}$$
$$= \delta(q, a) \qquad \text{by definition of } \delta^{\wedge}\text{, rule \#1}$$

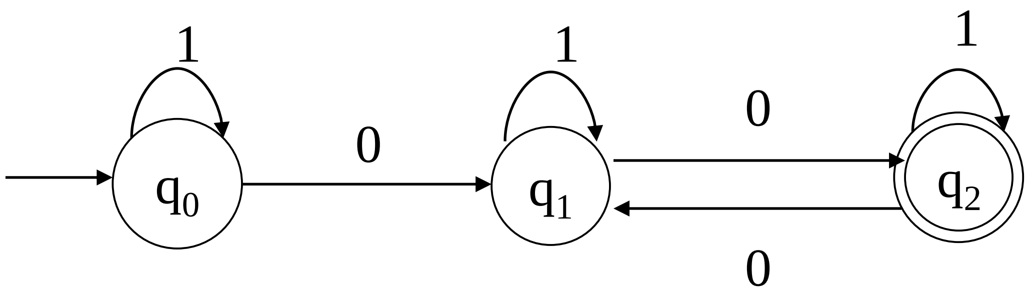
- Therefore:

$$\delta^{\wedge}(q, a_1a_2\ldots a_n) = \delta(\delta(\ldots\delta(\delta(q, a1), a2)\ldots), a_n)$$

- However, we will abuse notations, and use δ in place of $\delta^{\wedge}$:

$$\delta^{\wedge}(q, a_1a_2\ldots a_n) = \delta(q, a_1a_2\ldots a_n)$$

- Example #3:

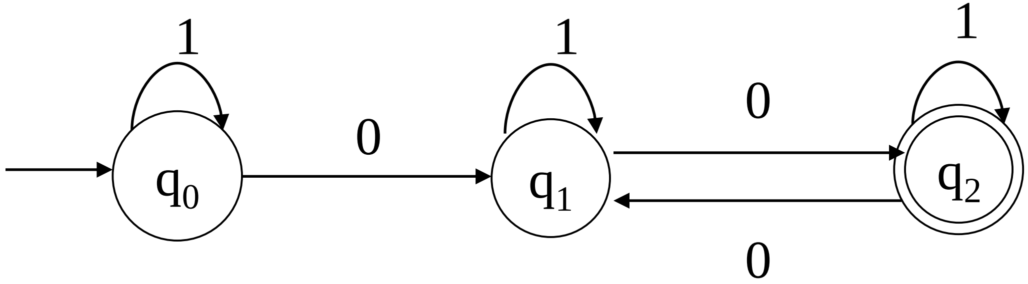


- What is $\delta(q_0, 011)$? Informally, it is the state entered by M after processing 011 having started in state q_0.
- Formally:

$\delta(q_0, 011)$	$= \delta(\delta(q_0,01), 1)$	by rule #2
	$= \delta(\delta(\delta(q_0,0), 1), 1)$	by rule #2
	$= \delta(\delta(q_1, 1), 1)$	by definition of δ
	$= \delta(q_1, 1)$	by definition of δ
	$= q_1$	by definition of δ

- Is 011 accepted? No, since $\delta(q_0, 011) = q_1$ is not a final state.
- *Language?*
- L ={ all strings over {0,1} that has 2 or more *0* symbols}

- Recall Example #3:



- What is $\delta(q_1, 10)$?

$$
\begin{aligned}
\delta(q_1, 10) &= \delta(\delta(q_1,1), 0) && \text{by rule \#2} \\
&= \delta(q_1, 0) && \text{by definition of } \delta \\
&= q_2 && \text{by definition of } \delta
\end{aligned}
$$

- Is 10 accepted? No, since $\delta(q_0, 10) = q_1$ is not a final state. The fact that $\delta(q_1, 10) = q_2$ is irrelevant, q1 is not the start state!

Definitions related to DFAs

- Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA and let w be in Σ^*. Then w is *accepted* by M iff $\delta(q_0, w) = p$ for some state p in F.

- Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA. Then the *language accepted* by M is the set:

$$L(M) = \{w \mid w \text{ is in } \Sigma^* \text{ and } \delta(q_0, w) \text{ is in } F\}$$

- Another equivalent definition:

$$L(M) = \{w \mid w \text{ is in } \Sigma^* \text{ and } w \text{ is accepted by } M\}$$

- Let L be a language. Then L is a ***regular language*** iff there exists a DFA M such that $L = L(M)$.

- Let $M_1 = (Q_1, \Sigma_1, \delta_1, q_0, F_1)$ and $M_2 = (Q_2, \Sigma_2, \delta_2, p_0, F_2)$ be DFAs. Then M_1 and M_2 are *equivalent* iff $L(M_1) = L(M_2)$.

- Notes:
  - A DFA $M = (Q, \Sigma, \delta, q_0, F)$ partitions the set Σ^* into two sets: L(M) and Σ^* - L(M).
  - If L = L(M) then L is a subset of L(M) and L(M) is a subset of L (def. of set equality).
  - Similarly, if $L(M_1) = L(M_2)$ then $L(M_1)$ is a subset of $L(M_2)$ and $L(M_2)$ is a subset of $L(M_1)$.
  - Some languages are regular, others are not. For example, if

    Regular: L_1 = {x | x is a string of 0's and 1's containing an even number of 1's} and

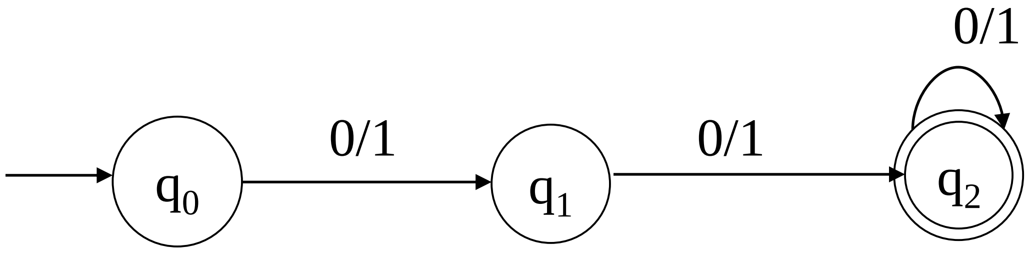
    Not-regular: $L_2 = \{x \mid x = 0^n1^n \text{ for some } n >= 0\}$

- *Can you write a program to "simulate" a given DFA, or any arbitrary input DFA?*

- Question we will address later:
  - How do we determine whether or not a given language is regular?

- Give a DFA M such that:

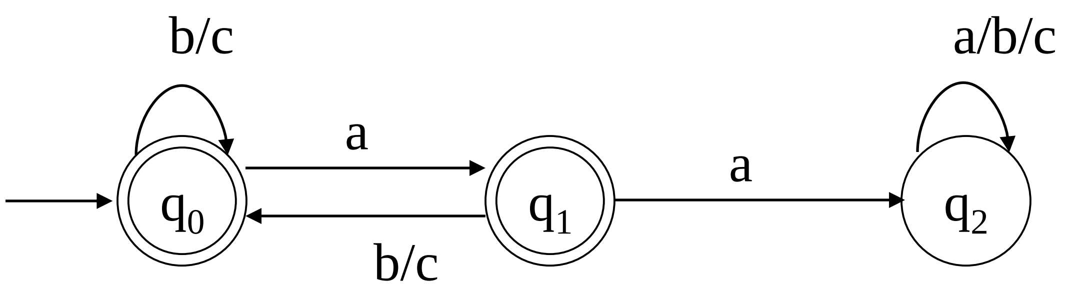
$$L(M) = \{x \mid x \text{ is a string of 0's and 1's and } |x| >= 2\}$$



Prove this by induction

- Give a DFA M such that:

L(M) = {x | x is a string of (zero or more) a's, b's and c's such that x does *not* contain the substring *aa*}



Logic:

In Start state (q0): b's and c's: ignore – stay in same state

q0 is also "accept" state

First 'a' appears: get ready (q1) to reject

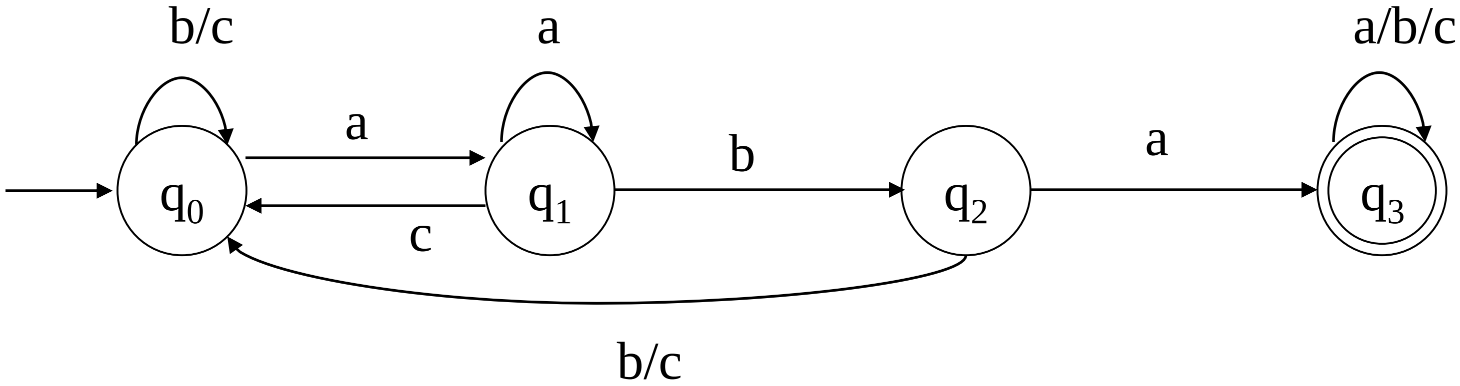
But followed by a 'b' or 'c': go back to start state q0

When second 'a' appears after the "ready" state: go to reject state q2

Ignore everything after getting to the "reject" state q2

- Give a DFA M such that:

L(M) = {x | x is a string of a's, b's and c's such that x contains the substring *aba*}



Logic: acceptance is straight forward, progressing on each expected symbol

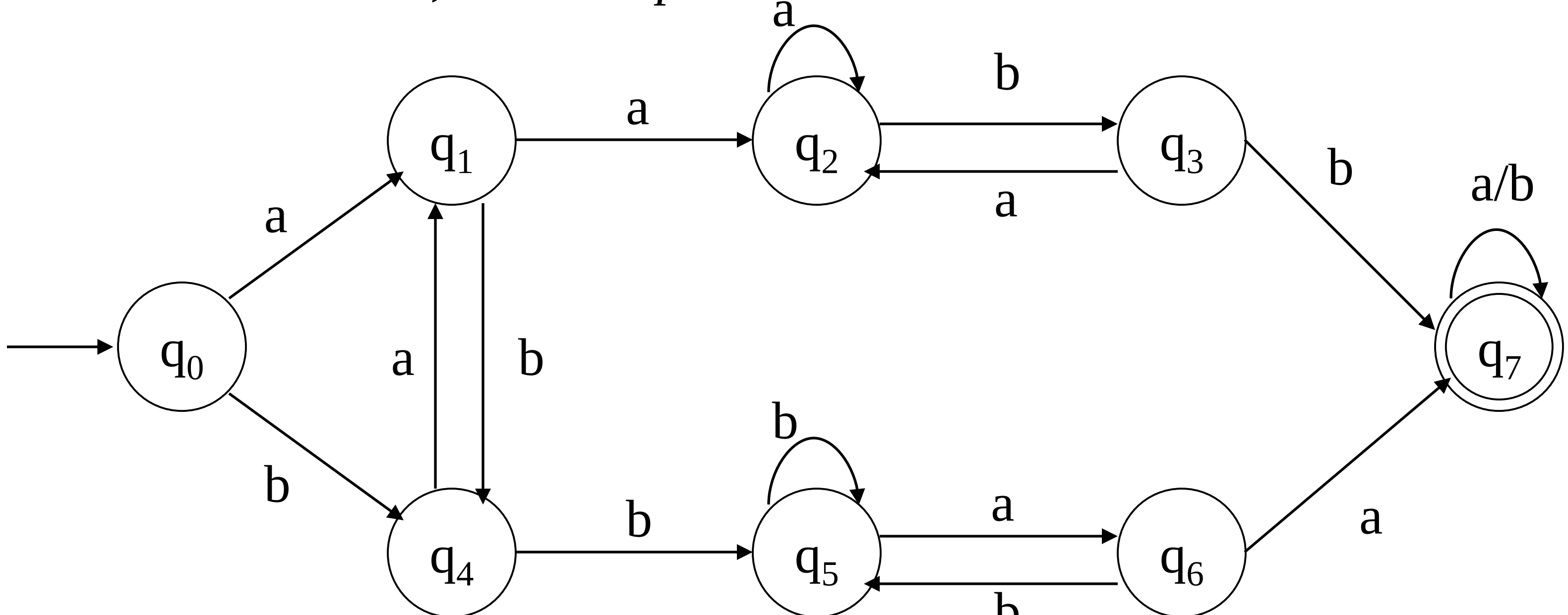
However, rejection needs special care, in each state (for DFA, we will see this becomes easier in NFA, non-deterministic machine)

- Give a DFA M such that:

L(M) = {x | x is a string of a's and b's such that x contains both *aa* and *bb*}

First do, for a language where 'aa' comes before 'bb'

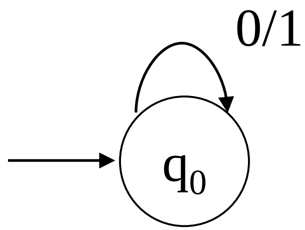
Then do its reverse; and then parallelize them.



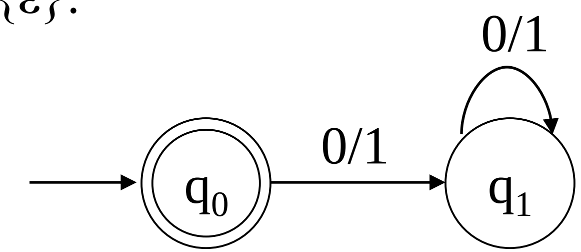
Remember, you may have multiple "final" states, but only one "start" state

- Let $\Sigma = \{0, 1\}$. Give DFAs for $\{\}$, $\{\varepsilon\}$, Σ^*, and Σ^+.

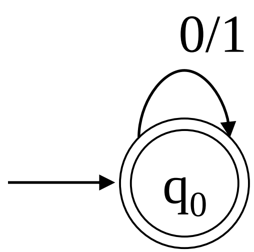
For $\{\}$:



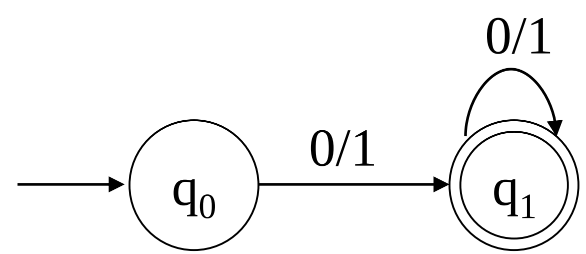
For $\{\varepsilon\}$:



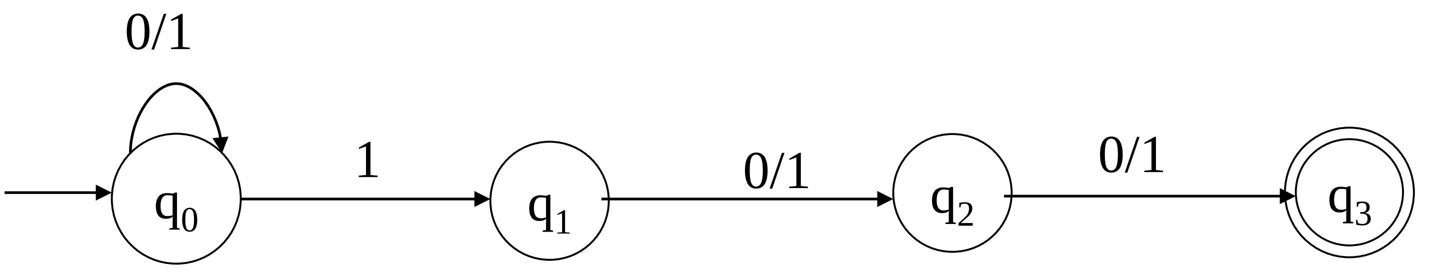
For Σ^*:



For Σ^+:



- Problem: Third symbol from last is *1*



Is this a DFA?

No, but it is a *Non-deterministic Finite Automaton*

Nondeterministic Finite State Automata (NFA)

- An NFA is a five-tuple:

  $M = (Q, \Sigma, \delta, q_0, F)$

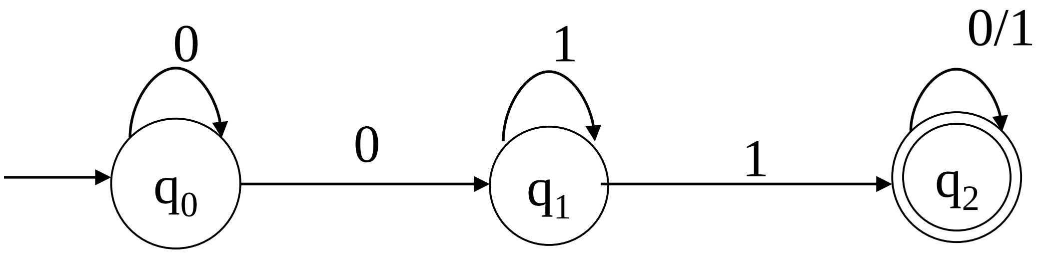
Q	A <u>finite</u> set of states
Σ	A <u>finite</u> input alphabet
q_0	The initial/starting state, q_0 is in Q
F	A set of final/accepting states, which is a subset of Q
δ	A transition function, which is a total function from $Q \times \Sigma$ to 2^Q

$\delta: (Q \times \Sigma) \rightarrow$ **2^Q**	:2^Q is the power set of Q, the set of *all subsets* of Q
$\delta(q,s)$	:The set of all states p such that there is a transition labeled s from q to p

$\delta(q,s)$ is a function from $Q \times S$ to 2^Q (but not only to Q)

- Example #1: one or more 0's followed by one or more 1's

$Q = \{q_0, q_1, q_2\}$
$\Sigma = \{0, 1\}$
Start state is q_0
$F = \{q_2\}$



δ:

	0	1
q_0	$\{q_0, q_1\}$	$\{\}$
q_1	$\{\}$	$\{q_1, q_2\}$
q_2	$\{q_2\}$	$\{q_2\}$

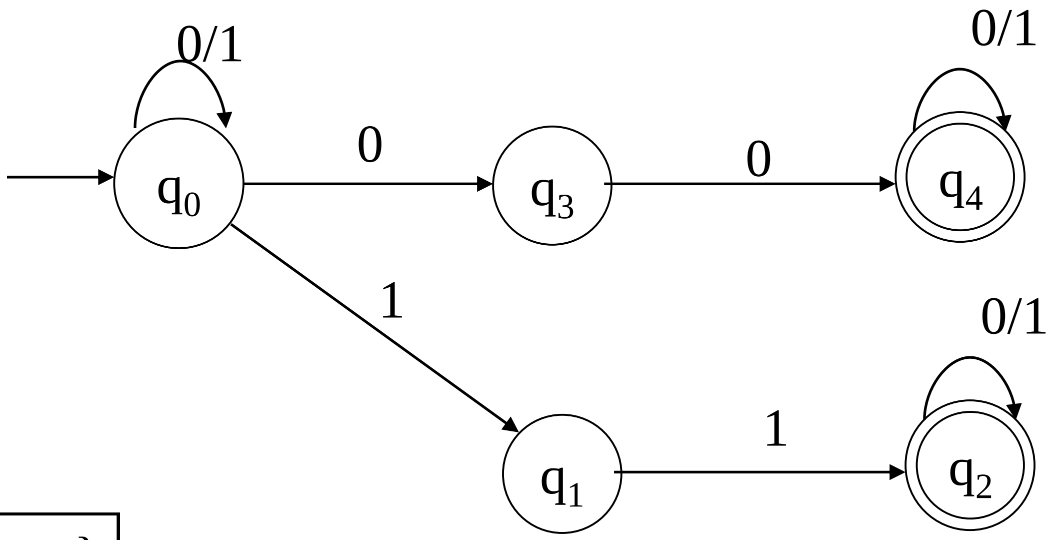
- Example #2: pair of 0's *or* pair of 1's as substring

$Q = \{q_0, q_1, q_2, q_3, q_4\}$

$\Sigma = \{0, 1\}$

Start state is q_0

$F = \{q_2, q_4\}$



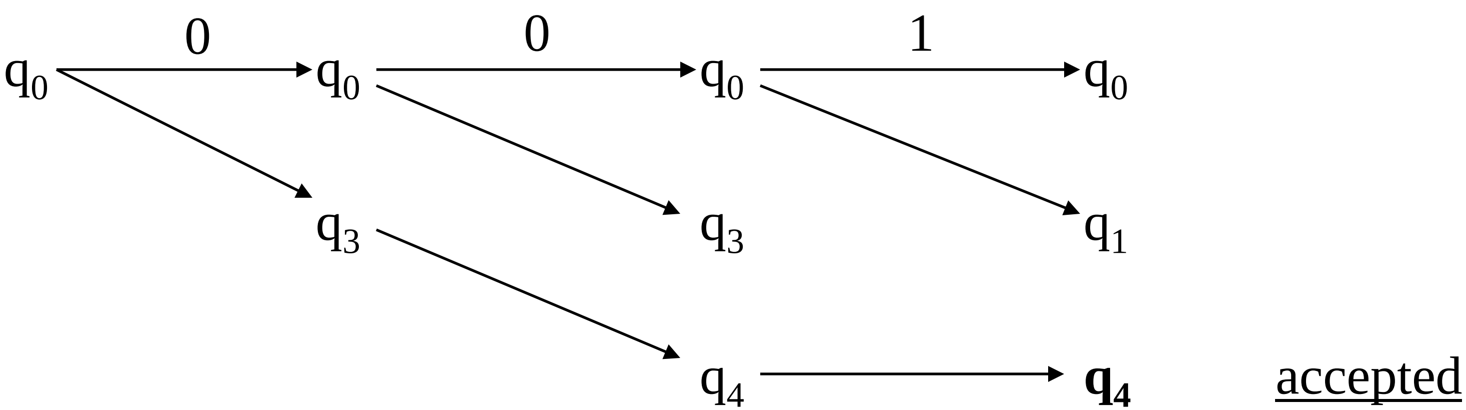
δ:

	0	1
q_0	$\{q_0, q_3\}$	$\{q_0, q_1\}$
q_1	$\{\}$	$\{q_2\}$
q_2	$\{q_2\}$	$\{q_2\}$
q_3	$\{q_4\}$	$\{\}$
q_4	$\{q_4\}$	$\{q_4\}$

- Notes:
  - δ(q,s) may not be defined for some *q* and *s* (*what does that mean*?)
  - δ(q,s) may map to multiple *q*'s
  - A string is said to be accepted *if there exists* a path from q_0 to some state in *F*
  - A string is rejected *if there exist NO path* to any state in *F*
  - The language accepted by an NFA is the set of all accepted strings

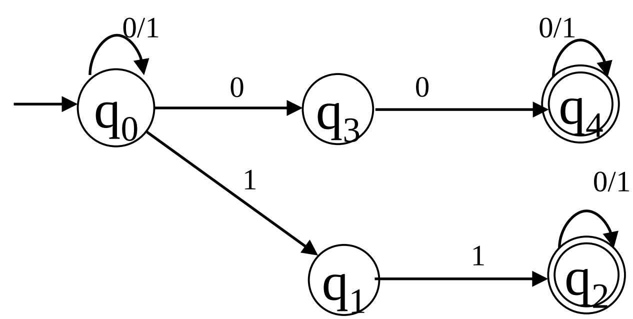
- Question: How does an NFA find the correct/accepting path for a given string?
  - NFAs are a non-intuitive computing model
  - You may use *backtracking* to find if there exists a path to a final state (following slide)
- Why NFA?
  - We are *primarily* interested in NFAs as language defining capability, i.e., do NFAs accept languages that DFAs do not?
  - Other secondary questions include practical ones such as whether or not NFA is easier to develop, or how does one implement NFA

- Determining if a given NFA (example #2) accepts a given string (001) can be done algorithmically:

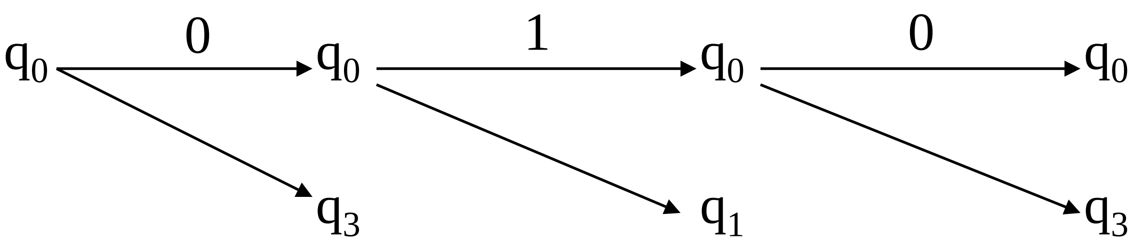


- Each level will have at most *n* states:

  Complexity: O(|x|*n), for running over a string *x*

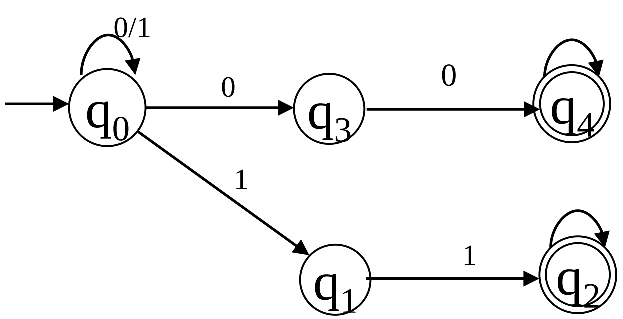


- Another example (010):



<u>not accepted</u>

- All paths have been explored, and none lead to an accepting state.

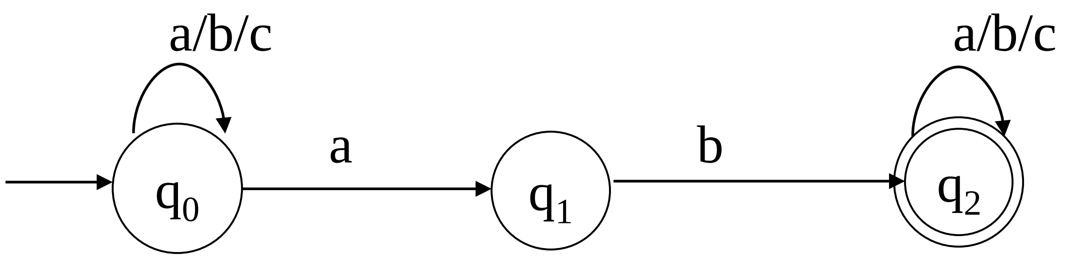


- Question: Why non-determinism is useful?
  - Non-determinism = Backtracking
  - Compressed information
  - Non-determinism hides backtracking
  - Programming languages, e.g., Prolog, hides backtracking => Easy to program at a higher level: *what we want to do, rather than how to do it*
  - Useful in algorithm complexity study

  - Is NDA more "powerful" than DFA, i.e., accepts type of languages that any DFA cannot?

- Let $\Sigma = \{a, b, c\}$. Give an NFA M that accepts:

$$L = \{x \mid x \text{ is in } \Sigma^* \text{ and } x \text{ contains } ab\}$$



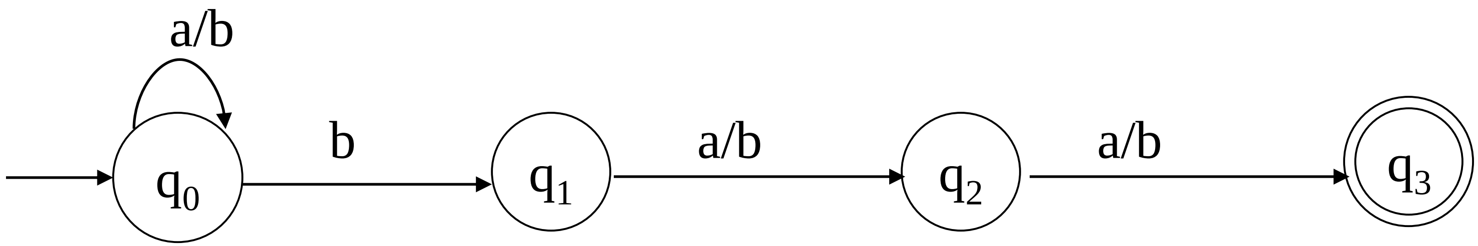
Is L a subset of L(M)? Or, does M accepts all string in L?

Is L(M) a subset of L? Or, does M rejects all strings not in L?

- Is an NFA necessary? Can you draw a DFA for this L?
- Designing NFAs is not as trivial as it seems: easy to create bug accepting string outside language

- Let $\Sigma = \{a, b\}$. Give an NFA M that accepts:

$$L = \{x \mid x \text{ is in } \Sigma^* \text{ and the third to the last symbol in } x \text{ is } b\}$$



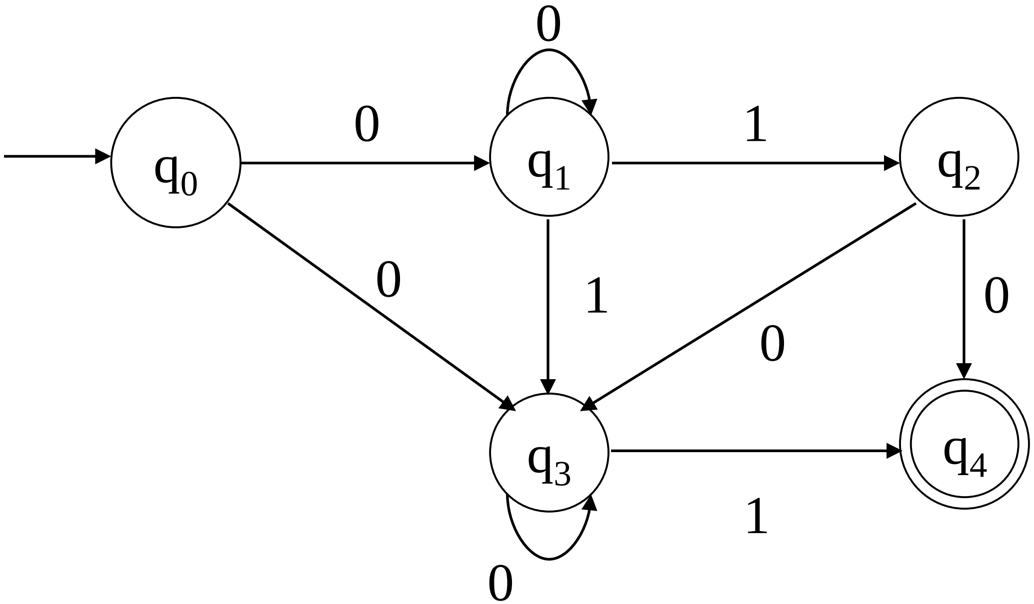
Is L a subset of L(M)?

Is L(M) a subset of L?

- Give an equivalent DFA as an exercise.

Extension of δ to Strings and Sets of States

- What we currently have: $\delta : (Q \times \Sigma) \rightarrow 2^Q$
- What we want (why?): $\delta : (2^Q \times \Sigma^*) \rightarrow 2^Q$
- We will do this in two steps, which will be slightly different from the book, and we will make use of the following NFA.



Extension of δ to Strings and Sets of States

- Step #1:

  Given δ: $(Q \times \Sigma) \rightarrow 2^Q$ define $\delta^\#$: $(2^Q \times \Sigma) \rightarrow 2^Q$ as follows:

  1) $\delta^\#(R, a) = \bigcup_{q \in R} \delta(q, a)$ for all subsets R of Q, and symbols a in Σ

- Note that:

  $$\begin{aligned} \delta^\#(\{p\}, a) &= \bigcup_{q \in \{p\}} \delta(q, a) && \text{by definition of } \delta^\#\text{, rule \#1 above} \\ &= \delta(p, a) \end{aligned}$$

- Hence, we can use δ for $\delta^\#$

  $\delta(\{q_0, q_2\}, 0)$
  $\delta(\{q_0, q_1, q_2\}, 0)$

  These now make sense, but previously they did not.

- Example:

$$\delta(\{q_0, q_2\}, 0) = \delta(q_0, 0) \cup \delta(q_2, 0)$$
$$= \{q_1, q_3\} \cup \{q_3, q_4\}$$
$$= \{q_1, q_3, q_4\}$$

$$\delta(\{q_0, q_1, q_2\}, 1) = \delta(q_0, 1) \cup \delta(q_1, 1) \cup \delta(q_2, 1)$$
$$= \{\} \cup \{q_2, q_3\} \cup \{\}$$
$$= \{q_2, q_3\}$$

- Step #2:

  Given δ: $(2^Q \times \Sigma) \rightarrow 2^Q$ define $\delta^\wedge$: $(2^Q \times \Sigma^*) \rightarrow 2^Q$ as follows:

  $\delta^\wedge(R,w)$ – The set of states M could be in after processing string w, having started from any state in R.

  Formally:

  2) $\delta^\wedge(R, \varepsilon) = R$ for any subset R of Q

  3) $\delta^\wedge(R,wa) = \delta(\delta^\wedge(R,w), a)$ for any w in Σ^*, a in Σ, and subset R of Q

- Note that:

  $\delta^\wedge(R, a) = \delta(\delta^\wedge(R, \varepsilon), a)$ by definition of $\delta^\wedge$, rule #3 above

  $\qquad\qquad = \delta(R, a)$ by definition of $\delta^\wedge$, rule #2 above

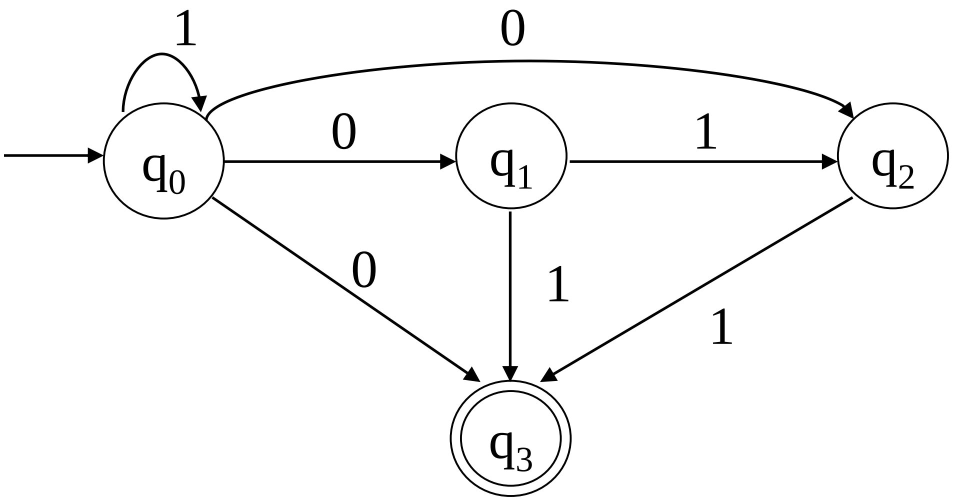
- Hence, we can use δ for $\delta^\wedge$

  $\delta(\{q_0, q_2\}, 0110)$

  $\delta(\{q_0, q_1, q_2\}, 101101)$

  These now make sense, but previously they did not.

- Example:



What is $\delta(\{q_0\}, 10)$?

Informally: The set of states the NFA could be in after processing 10, having started in state q_0, i.e., $\{q_1, q_2, q_3\}$.

Formally:

$$\begin{aligned}\delta(\{q_0\}, 10) &= \delta(\delta(\{q_0\}, 1), 0)\\ &= \delta(\{q_0\}, 0)\\ &= \{q_1, q_2, q_3\}\end{aligned}$$

Is 10 accepted? Yes!

- Example:

What is $\delta(\{q_0, q_1\}, 1)$?

$$
\begin{aligned}
\delta(\{q_0, q_1\}, 1) &= \delta(\{q_0\}, 1) \cup \delta(\{q_1\}, 1) \\
&= \{q_0\} \cup \{q_2, q_3\} \\
&= \{q_0, q_2, q_3\}
\end{aligned}
$$

What is $\delta(\{q_0, q_2\}, 10)$?

$$
\begin{aligned}
\delta(\{q_0, q_2\}, 10) &= \delta(\delta(\{q_0, q_2\}, 1), 0) \\
&= \delta(\delta(\{q_0\}, 1) \text{ U } \delta(\{q_2\}, 1), 0) \\
&= \delta(\{q_0\} \cup \{q_3\}, 0) \\
&= \delta(\{q_0,q_3\}, 0) \\
&= \delta(\{q_0\}, 0) \cup \delta(\{q_3\}, 0) \\
&= \{q_1, q_2, q_3\} \cup \{\} \\
&= \{q_1, q_2, q_3\}
\end{aligned}
$$

- Example:

$$
\begin{aligned}
\delta(\{q_0\}, 101) \quad &= \delta(\delta(\{q_0\}, 10), 1) \\
&= \delta(\delta(\delta(\{q_0\}, 1), 0), 1) \\
&= \delta(\delta(\{q_0\}, 0), 1) \\
&= \delta(\{q_1 , q_2, q_3\}, 1) \\
&= \delta(\{q_1\}, 1) \cup \delta(\{q_2\}, 1) \cup \delta(\{q_3\}, 1) \\
&= \{q_2, q_3\} \cup \{q_3\} \cup \{\} \\
&= \{q_2, q_3\}
\end{aligned}
$$

Is 101 accepted? Yes! q_3 is a final state.

Definitions for NFAs

- Let $M = (Q, \Sigma, \delta, q_0, F)$ be an NFA and let w be in Σ^*. Then w is *accepted* by M iff $\delta(\{q_0\}, w)$ contains at least one state in F.

- Let $M = (Q, \Sigma, \delta, q_0, F)$ be an NFA. Then the *language accepted* by M is the set:

  $L(M) = \{w \mid w$ is in Σ^* and $\delta(\{q_0\},w)$ contains at least one state in $F\}$

- Another equivalent definition:

  $L(M) = \{w \mid w$ is in Σ^* and w is accepted by $M\}$

Equivalence of DFAs and NFAs

- Do DFAs and NFAs accept the same *class* of languages?
  - Is there a language L that is accepted by a DFA, but not by any NFA?
  - Is there a language L that is accepted by an NFA, but not by any DFA?

- Observation: Every DFA is an NFA, DFA is only restricted NFA.

- Therefore, if L is a regular language then there exists an NFA M such that L = L(M).

- It follows that NFAs accept all regular languages.

- But do NFAs accept more?

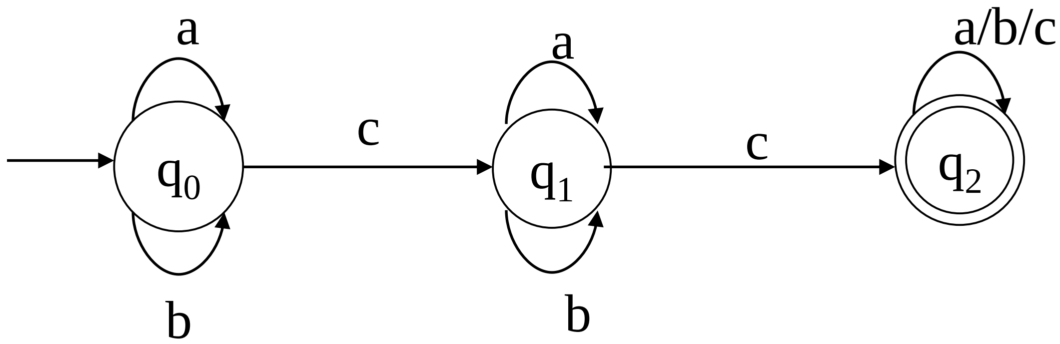
- Consider the following DFA: 2 or more c's

$Q = \{q_0, q_1, q_2\}$

$\Sigma = \{a, b, c\}$

Start state is q_0

$F = \{q_2\}$

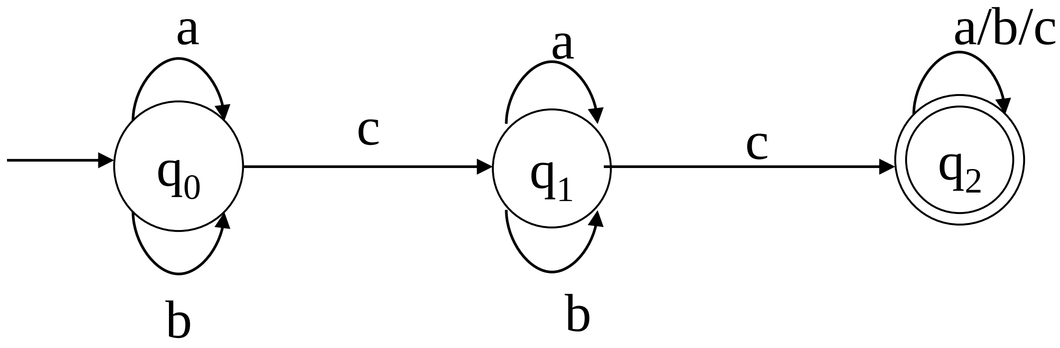


δ:

	a	b	c
q_0	q_0	q_0	q_1
q_1	q_1	q_1	q_2
q_2	q_2	q_2	q_2

- An Equivalent NFA:

$Q = \{q_0, q_1, q_2\}$
$\Sigma = \{a, b, c\}$
Start state is q_0
$F = \{q_2\}$



δ:

	a	b	c
q_0	$\{q_0\}$	$\{q_0\}$	$\{q_1\}$
q_1	$\{q_1\}$	$\{q_1\}$	$\{q_2\}$
q_2	$\{q_2\}$	$\{q_2\}$	$\{q_2\}$

- **Lemma 1:** Let M be an DFA. Then there exists a NFA M' such that L(M) = L(M').

- **Proof:** Every DFA is an NFA. Hence, if we let M' = M, then it follows that L(M') = L(M).

  The above is just a formal statement of the observation from the previous slide.

- **Lemma 2:** Let M be an NFA. Then there exists a DFA M' such that L(M) = L(M').

- **Proof:** (sketch)

Let $M = (Q, \Sigma, \delta, q_0, F)$.

Define a DFA $M' = (Q', \Sigma, \delta', q'_0, F')$ as:

$Q' = 2^Q$
$= \{Q_0, Q_1, \ldots,\}$

Each state in M' corresponds to a subset of states from M

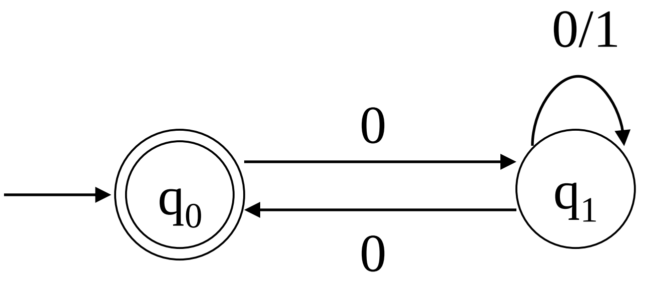
where $Q_u = [q_{i0}, q_{i1}, \ldots q_{ij}]$

$F' = \{Q_u \mid Q_u$ contains at least one state in $F\}$

$q'_0 = [q_0]$

$\delta'(Q_u, a) = Q_v$ iff $\delta(Q_u, a) = Q_v$

- Example: empty string or start and end with 0



$Q = \{q_0, q_1\}$

$\Sigma = \{0, 1\}$

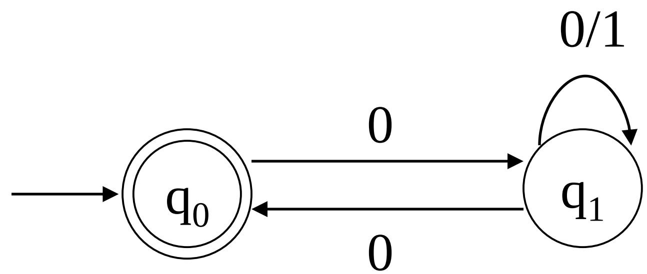
Start state is q_0

$F = \{q_0\}$

δ:

	0	1
q_0	$\{q_1\}$	$\{\}$
q_1	$\{q_0, q_1\}$	$\{q_1\}$

- Example of creating a DFA out of an NFA (as per the constructive proof):

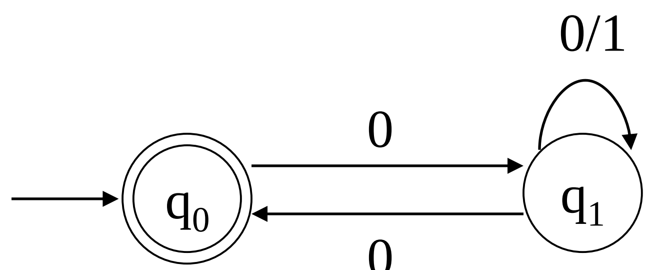


-->q_0	$\{q_1\}$	$\{\}$
q_1	$\{q_0, q_1\}$	$\{q_1\}$

δ for DFA:

	0	1
->q_0	$\{q_1\}$ write as $[q_1]$	$\{\}$ write as $[\ ]$
$[q_1]$		
$[\ ]$		

- Example of creating a DFA out of an NFA (as per the constructive proof):

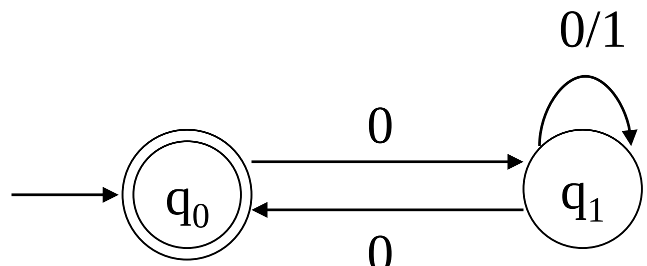


$\{q_1\}$	$\{\}$
$\{q_0, q_1\}$	$\{q_1\}$

δ:

	0	1
$\to q_0$	$\{q_1\}$ write as $[q_1]$	$\{\}$
$[q_1]$	$\{q_0,q_1\}$ write as $[q_{01}]$	$\{q_1\}$
$[\ ]$		
$[q_{01}]$		

- Example of creating a DFA out of an NFA (as per the constructive proof):

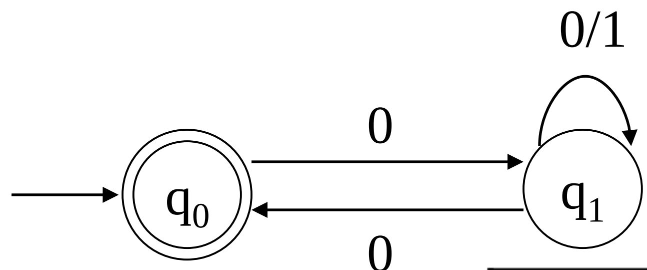


$\{q_1\}$	$\{\}$
$\{q_0, q_1\}$	$\{q_1\}$

δ:

	0	1
->q_0	$\{q_1\}$ write as $[q_1]$	$\{\}$
$[q_1]$	$\{q_0,q_1\}$ write as $[q_{01}]$	$\{q_1\}$
[ ]	[ ]	[ ]
$[q_{01}]$		

- Example of creating a DFA out of an NFA (as per the constructive proof):

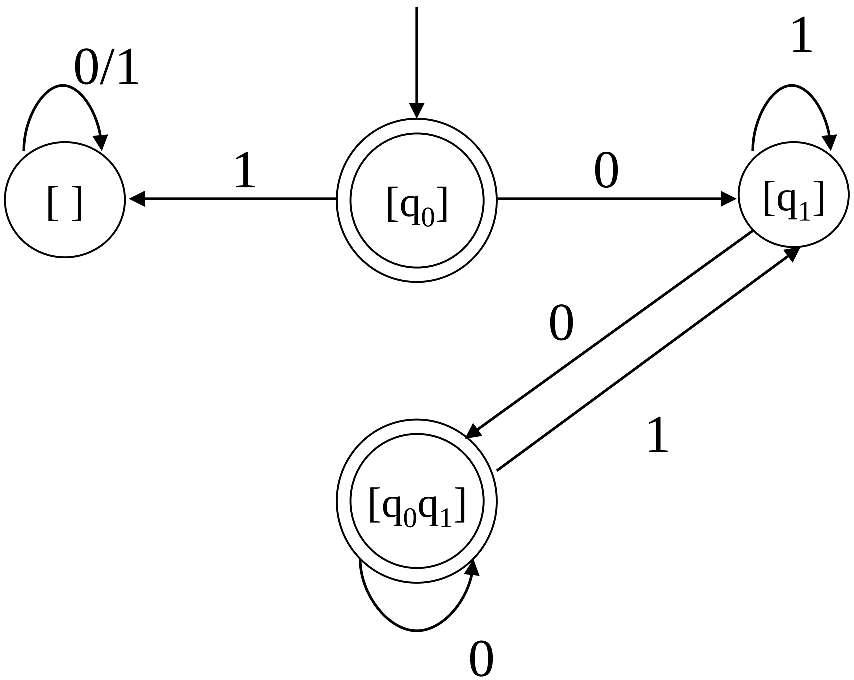


$\{q_1\}$	$\{\}$
$\{q_0, q_1\}$	$\{q_1\}$

δ:

	0	1
$\to q_0$	$\{q_1\}$ write as $[q_1]$	$\{\}$
$[q_1]$	$\{q_0,q_1\}$ write as $[q_{01}]$	$\{q_1\}$
$[\ ]$	$[\ ]$	$[\ ]$
$[q_{01}]$	$[q_{01}]$	$[q_1]$

- Construct DFA M' as follows:



$\delta(\{q_0\}, 0) = \{q_1\}$	=>	$\delta'([q_0], 0) = [q_1]$
$\delta(\{q_0\}, 1) = \{\}$	=>	$\delta'([q_0], 1) = [\ ]$
$\delta(\{q_1\}, 0) = \{q_0, q_1\}$	=>	$\delta'([q_1], 0) = [q_0q_1]$
$\delta(\{q_1\}, 1) = \{q_1\}$	=>	$\delta'([q_1], 1) = [q_1]$
$\delta(\{q_0, q_1\}, 0) = \{q_0, q_1\}$	=>	$\delta'([q_0q_1], 0) = [q_0q_1]$
$\delta(\{q_0, q_1\}, 1) = \{q_1\}$	=>	$\delta'([q_0q_1], 1) = [q_1]$
$\delta(\{\}, 0) = \{\}$	=>	$\delta'([\ ], 0) = [\ ]$
$\delta(\{\}, 1) = \{\}$	=>	$\delta'([\ ], 1) = [\ ]$

- **Theorem:** Let L be a language. Then there exists an DFA M such that L = L(M) iff there exists an NFA M' such that L = L(M').

- **Proof:**

  (if) Suppose there exists an NFA M' such that L = L(M'). Then by Lemma 2 there exists an DFA M such that L = L(M).

  (only if) Suppose there exists an DFA M such that L = L(M). Then by Lemma 1 there exists an NFA M' such that L = L(M').

- **Corollary:** The NFAs define the regular languages.

- Note: Suppose R = {}

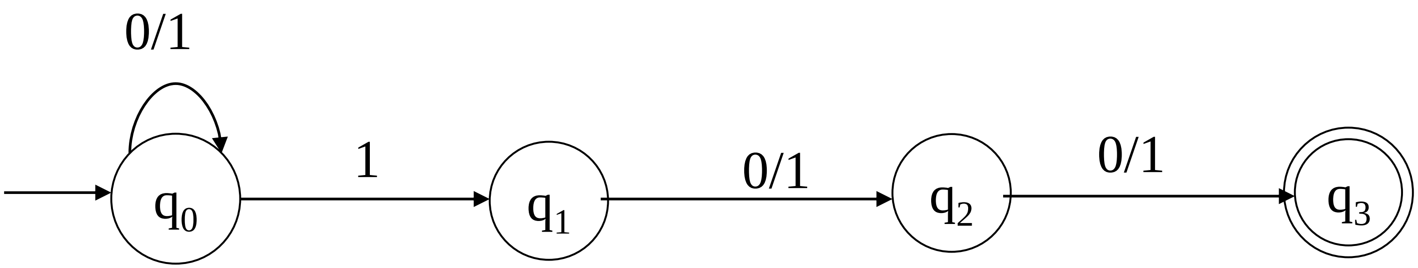
$$
\begin{aligned}
\delta(R, 0) &= \delta(\delta(R, \varepsilon), 0) \\
&= \delta(R, 0) \\
&= \bigcup_{q \in R} \delta(q, 0) \\
&= \{\} \qquad\qquad \text{Since } R = \{\}
\end{aligned}
$$

- Exercise - Convert the following NFA to a DFA:

$Q = \{q_0, q_1, q_2\}$
$\Sigma = \{0, 1\}$
Start state is q_0
$F = \{q_0\}$

δ:	0	1
q_0	$\{q_0, q_1\}$	{ }
q_1	$\{q_1\}$	$\{q_2\}$
q_2	$\{q_2\}$	$\{q_2\}$

- Problem: Third symbol from last is 1



Now, can you convert this NFA to a DFA?

NFAs with ε Moves

- An NFA-ε is a five-tuple:

  $M = (Q, \Sigma, \delta, q_0, F)$

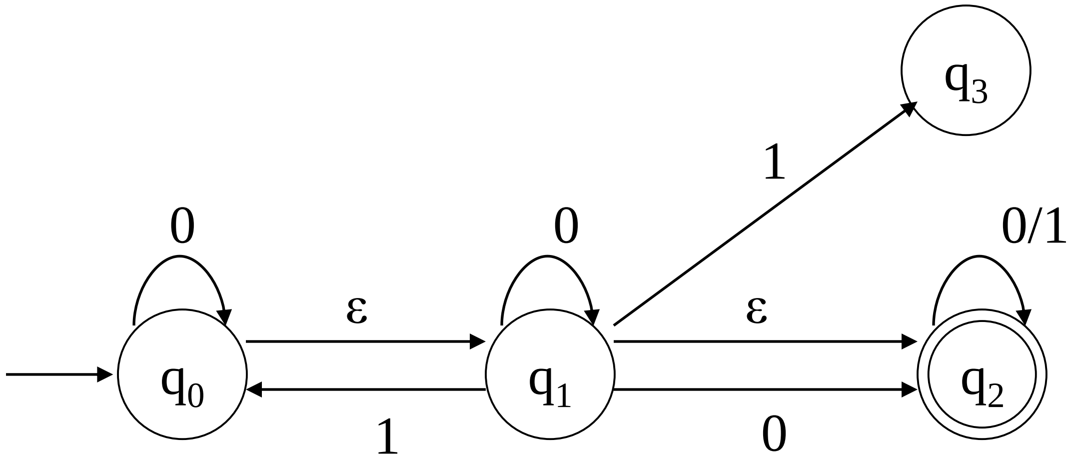
Q	A <u>finite</u> set of states
Σ	A <u>finite</u> input alphabet
q_0	The initial/starting state, q_0 is in Q
F	A set of final/accepting states, which is a subset of Q
δ	A transition function, which is a total function from Q x Σ U {ε} to 2^Q

  $\delta: (Q \times (\mathbf{\Sigma \cup \{\varepsilon\}})) \rightarrow 2^Q$

  $\delta(q,s)$ -The set of all states p such that there is a transition labeled a from q to p, where a is in Σ U {ε}

- Sometimes referred to as an NFA-ε other times, simply as an NFA.

- Example:



δ:	0	1	ε
q_0	$\{q_0\}$	$\{\ \}$	$\{q_1\}$
q_1	$\{q_1, q_2\}$	$\{q_0, q_3\}$	$\{q_2\}$
q_2	$\{q_2\}$	$\{q_2\}$	$\{\ \}$
q_3	$\{\ \}$	$\{\ \}$	$\{\ \}$

- A string $w = w_1w_2\ldots w_n$ is processed as $w = \varepsilon^*w_1\varepsilon^*w_2\varepsilon^* \ldots \varepsilon^*w_n\varepsilon^*$

- Example: all computations on 00:

$$\begin{array}{cccc} & 0 & \varepsilon & 0 \\ q_0 & q_0 & q_1 & q_2 \end{array}$$

:

Informal Definitions

- Let $M = (Q, \Sigma, \delta, q_0, F)$ be an NFA-ε.

- A String w in Σ^* is *accepted* by M iff there exists a path in M from q_0 to a state in F labeled by w and zero or more ε transitions.

- The language accepted by M is the set of all strings from Σ^* that are accepted by M.

ε-closure

- Define ε-closure(q) to denote the set of all states reachable from q by zero or more ε transitions.

- Examples: (for the previous NFA)

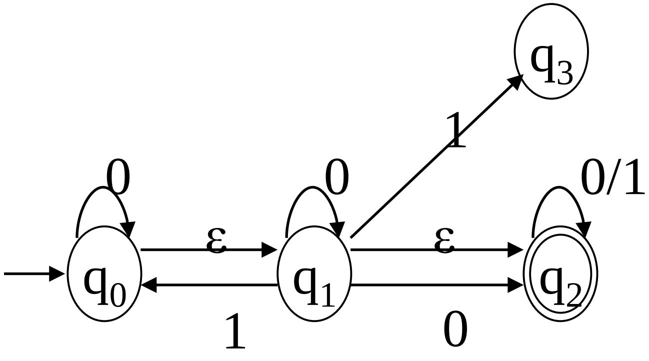
  ε-closure(q_0) = {q_0, q_1, q_2}
  ε-closure(q_1) = {q_1, q_2}
  ε-closure(q_2) = {q_2}
  ε-closure(q_3) = {q_3}

- ε-closure(q) can be extended to sets of states by defining:

$$\varepsilon\text{-closure}(P) = \bigcup_{q \in P} \varepsilon\text{-closure}(q)$$

- Examples:

  ε-closure({q_1, q_2}) = {q_1, q_2}
  ε-closure({q_0, q_3}) = {q_0, q_1, q_2, q_3}

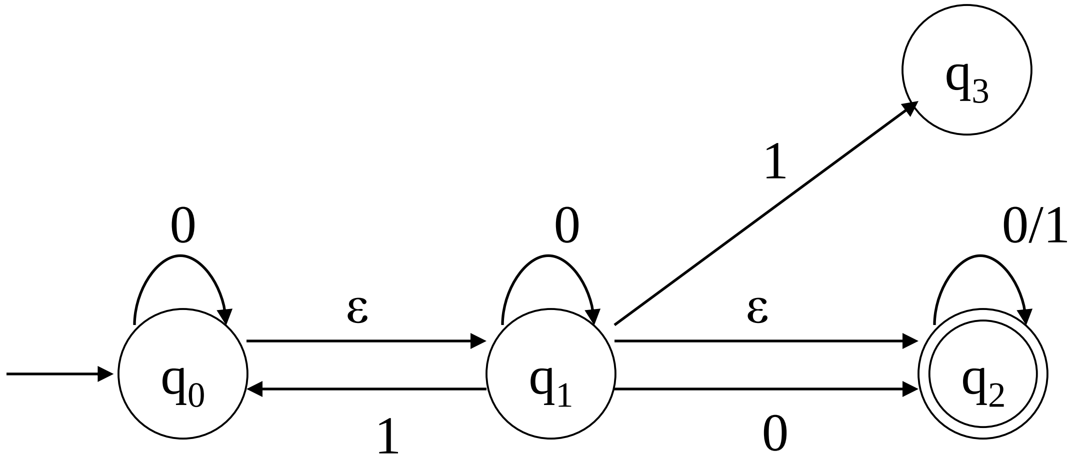


Extension of δ to Strings and Sets of States

- What we currently have: $\delta : (Q \times (\Sigma \cup \{\varepsilon\})) \rightarrow 2^Q$

- What we want (why?): $\delta : (2^Q \times \Sigma^*) \rightarrow 2^Q$

- As before, we will do this in two steps, which will be slightly different from the book, and we will make use of the following NFA.



- Step #1:

  Given δ: $(Q \times (\Sigma \cup \{\varepsilon\})) \rightarrow 2^Q$ define $\delta^{\#}$: $(2^Q \times (\Sigma \cup \{\varepsilon\})) \rightarrow 2^Q$ as follows:

  1) $\delta^{\#}(R, a) = \bigcup_{q \in R} \delta(q, a)$ for all subsets R of Q, and symbols *a* in $\Sigma \cup \{\varepsilon\}$

- Note that:

$$\delta^{\#}(\{p\}, a) = \bigcup_{q \in \{p\}} \delta(q, a) \text{ by definition of } \delta^{\#}\text{, rule \#1 above}$$
$$= \delta(p, a)$$

- Hence, we can use δ for $\delta^{\#}$

$\delta(\{q_0, q_2\}, 0)$	These now make sense, but previously
$\delta(\{q_0, q_1, q_2\}, 0)$	they did not.

- Examples:

What is $\delta(\{q_0, q_1, q_2\}, 1)$?

$$
\begin{aligned}
\delta(\{q_0, q_1, q_2\}, 1) &= \delta(q_0, 1) \cup \delta(q_1, 1) \cup \delta(q_2, 1) \\
&= \{\ \} \cup \{q_0, q_3\} \cup \{q_2\} \\
&= \{q_0, q_2, q_3\}
\end{aligned}
$$

What is $\delta(\{q_0, q_1\}, 0)$?

$$
\begin{aligned}
\delta(\{q_0, q_1\}, 0) &= \delta(q_0, 0) \cup \delta(q_1, 0) \\
&= \{q_0\} \cup \{q_1, q_2\} \\
&= \{q_0, q_1, q_2\}
\end{aligned}
$$

- Step #2:

  Given δ: $(2^Q \times (\Sigma \cup \{\varepsilon\})) \rightarrow 2^Q$ define $\delta^\wedge$: $(2^Q \times \Sigma^*) \rightarrow 2^Q$ as follows:

  $\delta^\wedge(R,w)$ – The set of states M could be in after processing string w, having starting from any state in R.

  Formally:

  2) $\delta^\wedge(R, \varepsilon) = \varepsilon\text{-closure}(R)$ - for any subset R of Q

  3) $\delta^\wedge(R,wa) = \varepsilon\text{-closure}(\delta(\delta^\wedge(R,w), a))$ - for any w in Σ^*, a in Σ, and subset R of Q

- Can we use δ for $\delta^\wedge$?

- Consider the following example:

$\delta(\{q_0\}, 0) = \{q_0\}$

$$
\begin{aligned}
\delta^{\wedge}(\{q_0\}, 0) &= \varepsilon\text{-closure}(\delta(\delta^{\wedge}(\{q_0\}, \varepsilon), 0)) && \text{By rule \#3} \\
&= \varepsilon\text{-closure}(\delta(\varepsilon\text{-closure}(\{q_0\}), 0)) && \text{By rule \#2} \\
&= \varepsilon\text{-closure}(\delta(\{q_0, q_1, q_2\}, 0)) && \text{By } \varepsilon\text{-closure} \\
&= \varepsilon\text{-closure}(\delta(q_0, 0) \cup \delta(q_1, 0) \cup \delta(q_2, 0)) && \text{By rule \#1} \\
&= \varepsilon\text{-closure}(\{q_0\} \cup \{q_1, q_2\} \cup \{q_2\}) \\
&= \varepsilon\text{-closure}(\{q_0, q_1, q_2\}) \\
&= \varepsilon\text{-closure}(\{q_0\}) \cup \varepsilon\text{-closure}(\{q_1\}) \cup \varepsilon\text{-closure}(\{q_2\}) \\
&= \{q_0, q_1, q_2\} \cup \{q_1, q_2\} \cup \{q_2\} \\
&= \{q_0, q_1, q_2\}
\end{aligned}
$$

- So what is the difference?

$\delta(q_0, 0)$ - Processes 0 as a single symbol, without ε transitions.

$\delta^{\wedge}(q_0, 0)$ - Processes 0 using as many ε transitions as are possible.

- Example:

$\delta^{\wedge}(\{q_0\}, 01) = \varepsilon\text{-closure}(\delta(\delta^{\wedge}(\{q_0\}, 0), 1))$ By rule #3

$= \varepsilon\text{-closure}(\delta(\{q_0, q_1, q_2\}), 1)$ Previous slide

$= \varepsilon\text{-closure}(\delta(q_0, 1) \cup \delta(q_1, 1) \cup \delta(q_2, 1))$ By rule #1

$= \varepsilon\text{-closure}(\{\ \} \cup \{q_0, q_3\} \cup \{q_2\})$

$= \varepsilon\text{-closure}(\{q_0, q_2, q_3\})$

$= \varepsilon\text{-closure}(\{q_0\}) \cup \varepsilon\text{-closure}(\{q_2\}) \cup \varepsilon\text{-closure}(\{q_3\})$

$= \{q_0, q_1, q_2\} \cup \{q_2\} \cup \{q_3\}$

$= \{q_0, q_1, q_2, q_3\}$

Definitions for NFA-ε Machines

- Let $M = (Q, \Sigma, \delta, q_0, F)$ be an NFA-ε and let w be in Σ^*. Then w is *accepted* by M iff $\delta^\wedge(\{q_0\}, w)$ contains at least one state in F.

- Let $M = (Q, \Sigma, \delta, q_0, F)$ be an NFA-ε. Then the *language accepted* by M is the set:

  $L(M) = \{w \mid w \text{ is in } \Sigma^* \text{ and } \delta^\wedge(\{q_0\},w) \text{ contains at least one state in } F\}$

- Another equivalent definition:

  $L(M) = \{w \mid w \text{ is in } \Sigma^* \text{ and } w \text{ is accepted by } M\}$

Equivalence of NFAs and NFA-εs

- Do NFAs and NFA-ε machines accept the same *class* of languages?
  - Is there a language L that is accepted by a NFA, but not by any NFA-ε?
  - Is there a language L that is accepted by an NFA-ε, but not by any DFA?

- Observation: Every NFA is an NFA-ε.

- Therefore, if L is a regular language then there exists an NFA-ε M such that L = L(M).

- It follows that NFA-ε machines accept all regular languages.

- But do NFA-ε machines accept more?

- **Lemma 1:** Let M be an NFA. Then there exists a NFA-ε M' such that L(M) = L(M').

- **Proof:** Every NFA is an NFA-ε. Hence, if we let M' = M, then it follows that L(M') = L(M).

  The above is just a formal statement of the observation from the previous slide.

- **Lemma 2:** Let M be an NFA-ε. Then there exists a NFA M' such that L(M) = L(M').

- **Proof:** (sketch)

  Let $M = (Q, \Sigma, \delta, q_0, F)$ be an NFA-ε.

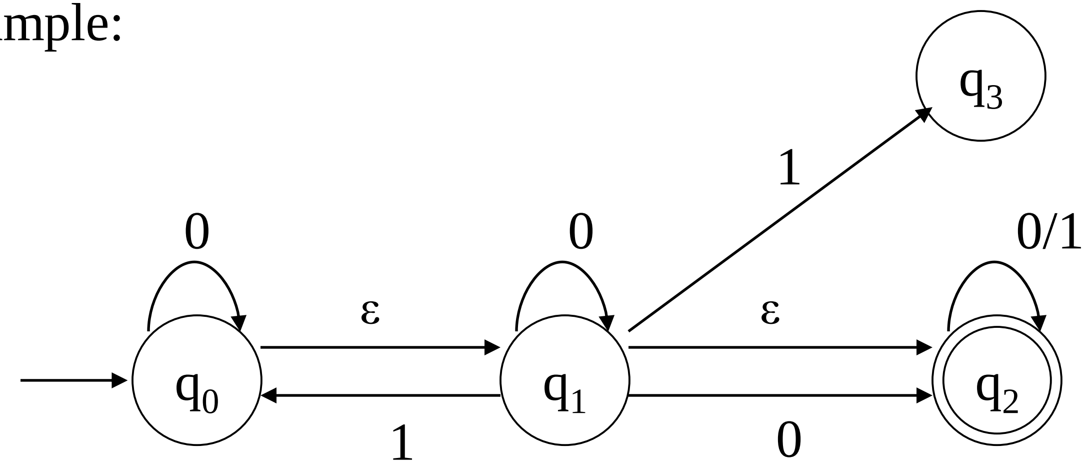
  Define an NFA $M' = (Q, \Sigma, \delta', q_0, F')$ as:

  $F' = F \cup \{q\}$ if ε-closure(q) contains at least one state from F
  $F' = F$ otherwise

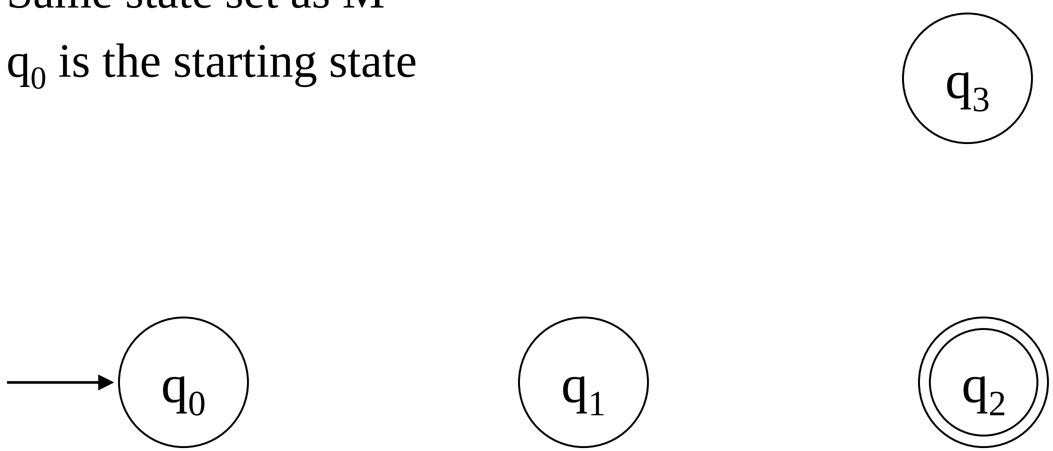
  $\delta'(q, a) = \delta^{\wedge}(q, a)$ - for all q in Q and a in Σ

- Notes:
  - $\delta' : (Q \times \Sigma) \rightarrow 2^Q$ is a function
  - M' has the same state set, the same alphabet, and the same start state as M
  - M' has no ε transitions

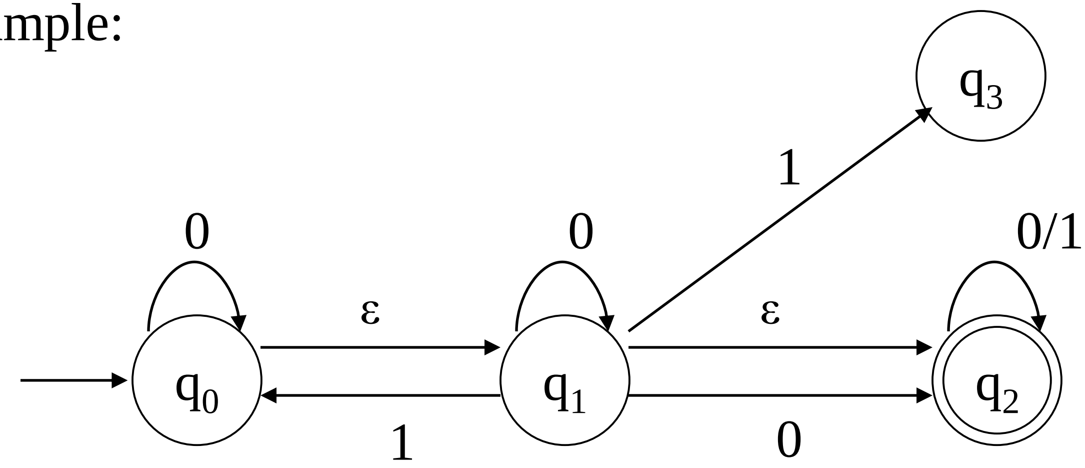
- Example:



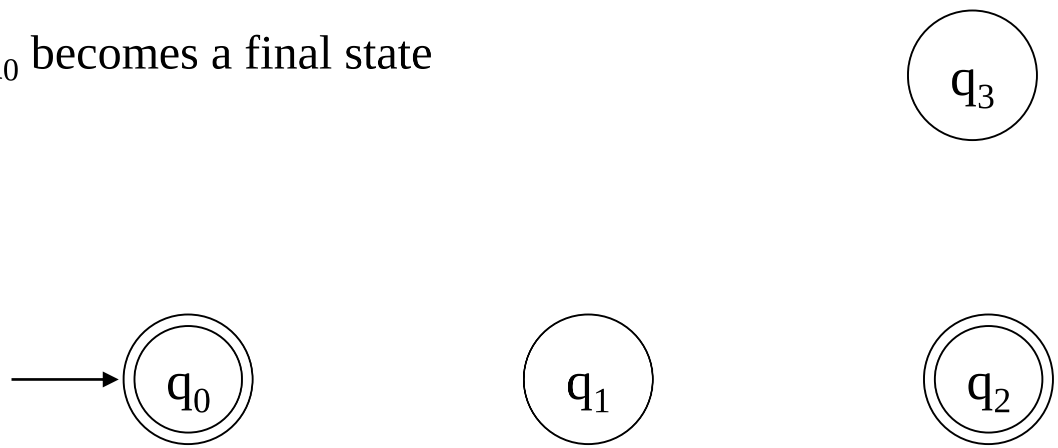
- Step #1:
  - Same state set as M
  - q_0 is the starting state



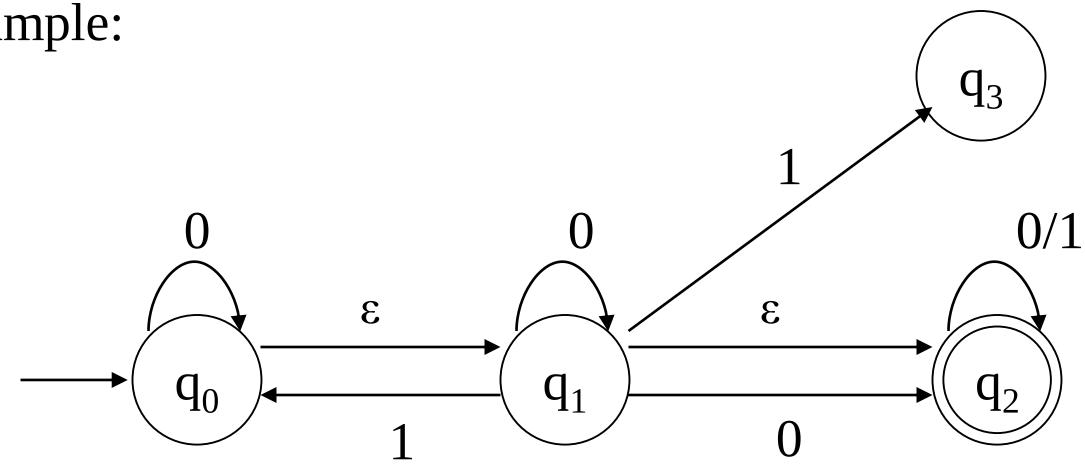
- Example:



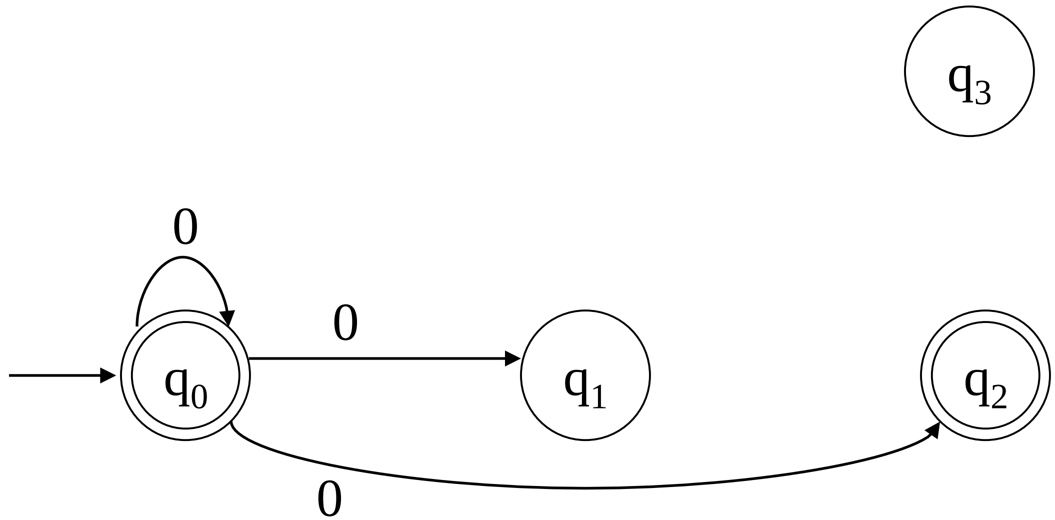
- Step #2:
  - q_0 becomes a final state



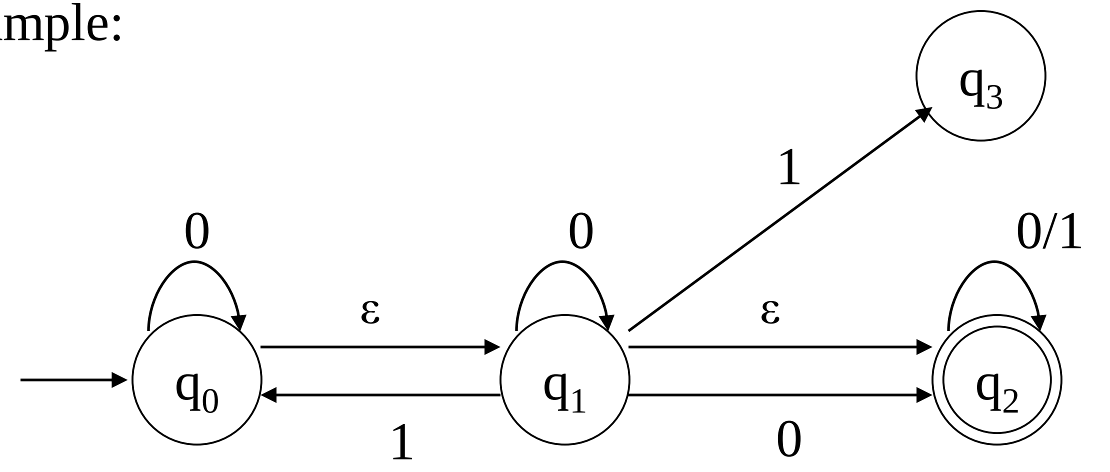
- Example:



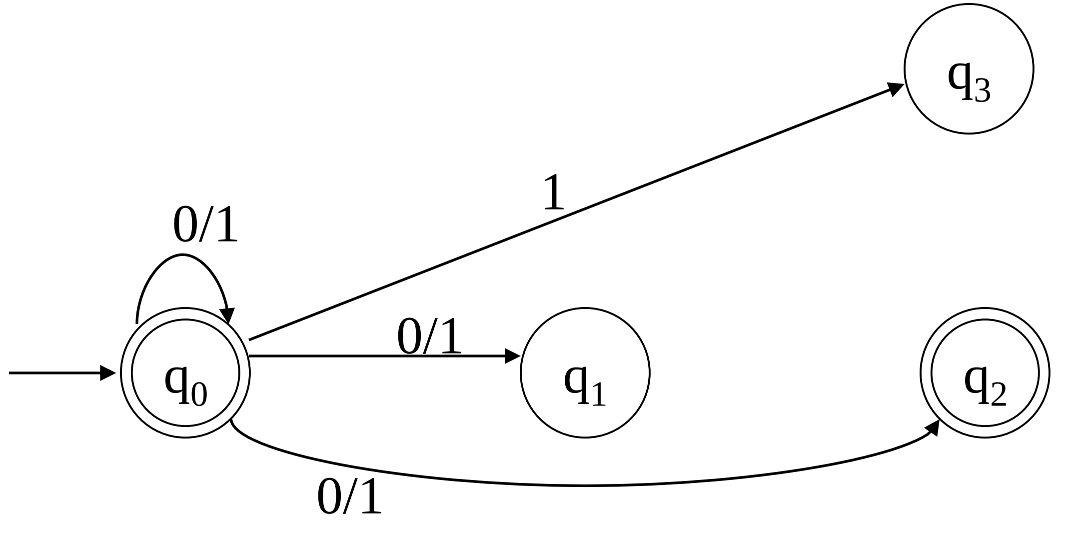
- Step #3:



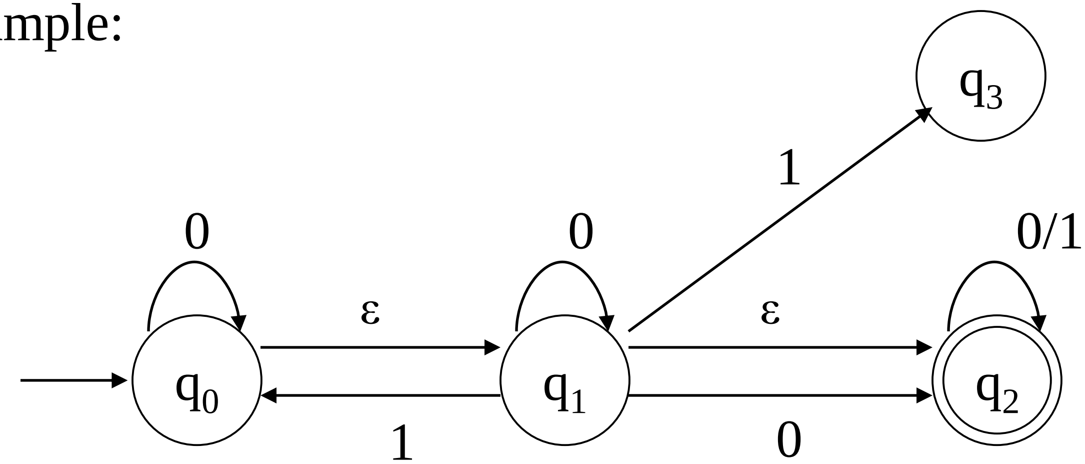
- Example:



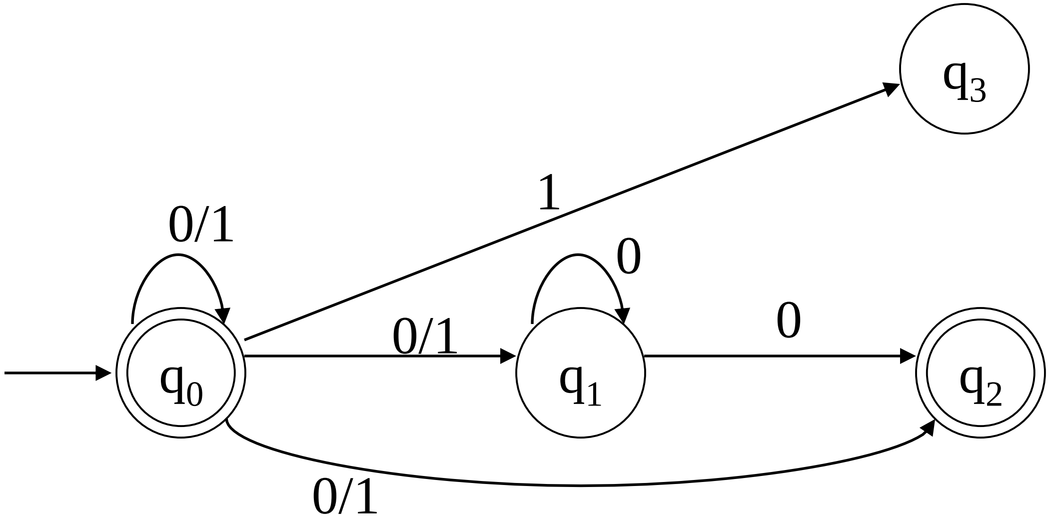
- Step #4:



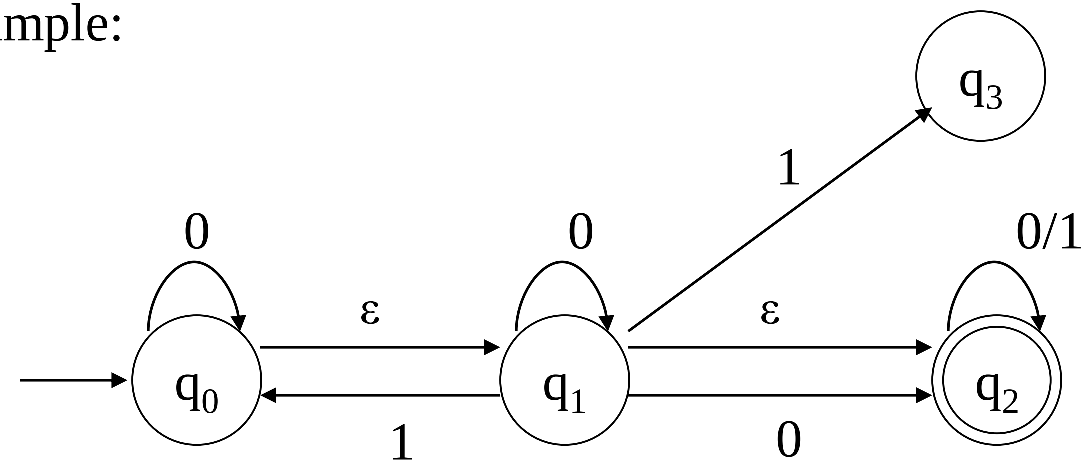
- Example:



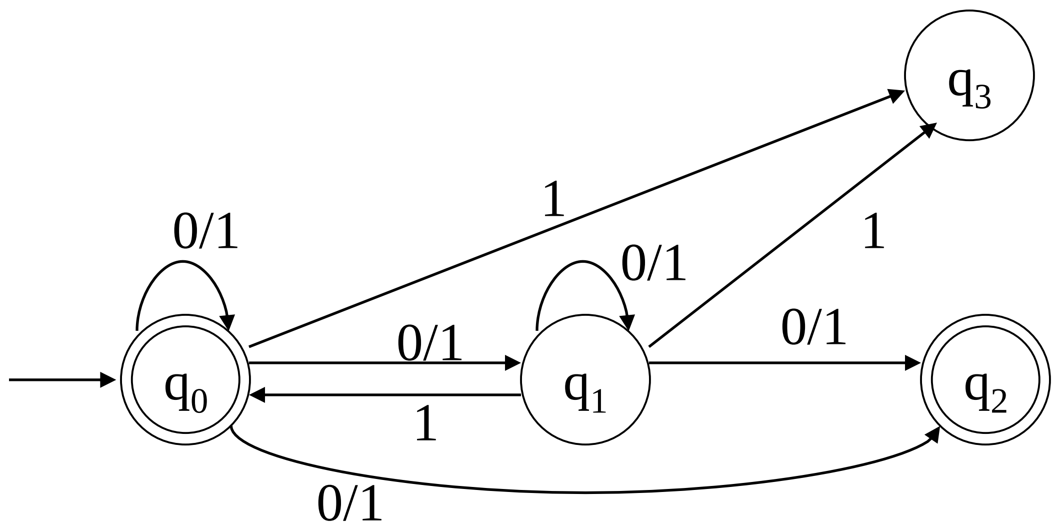
- Step #5:



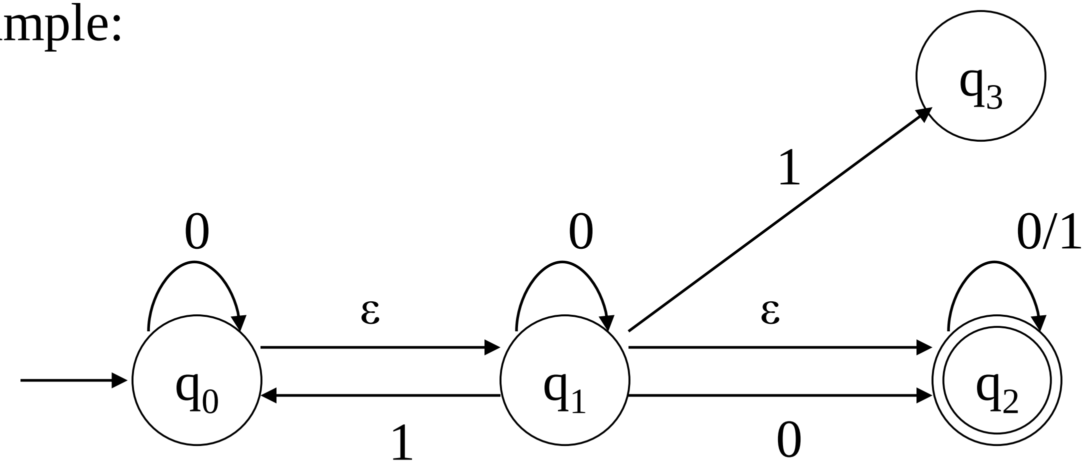
- Example:



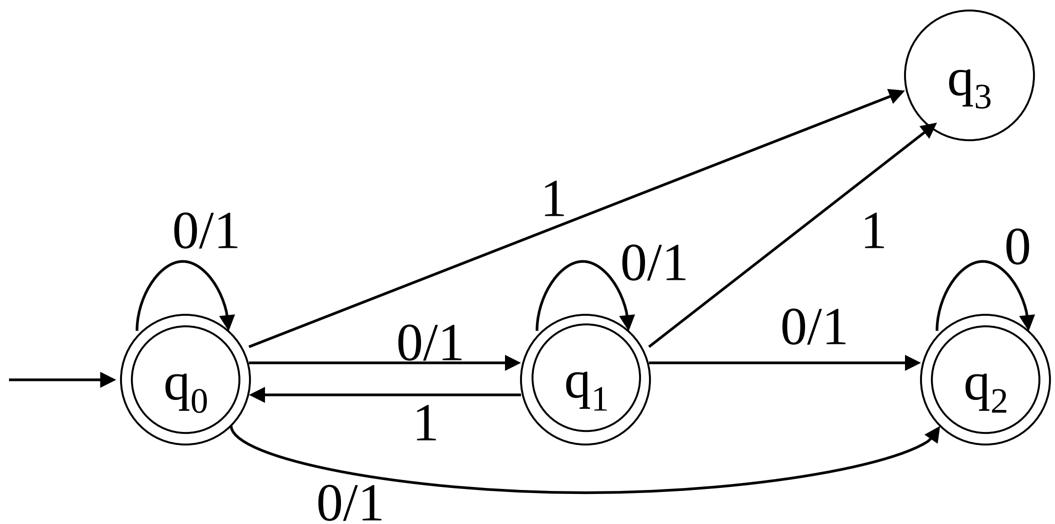
- Step #6:



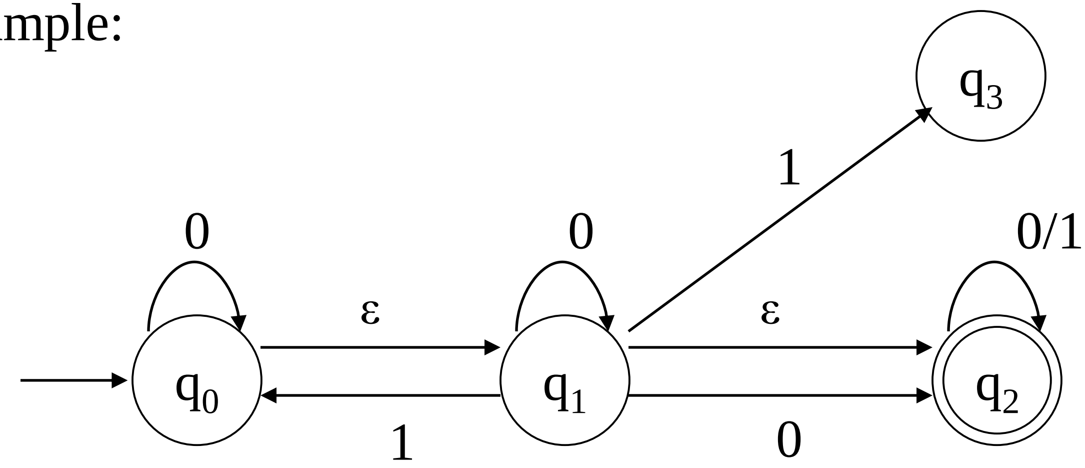
- Example:



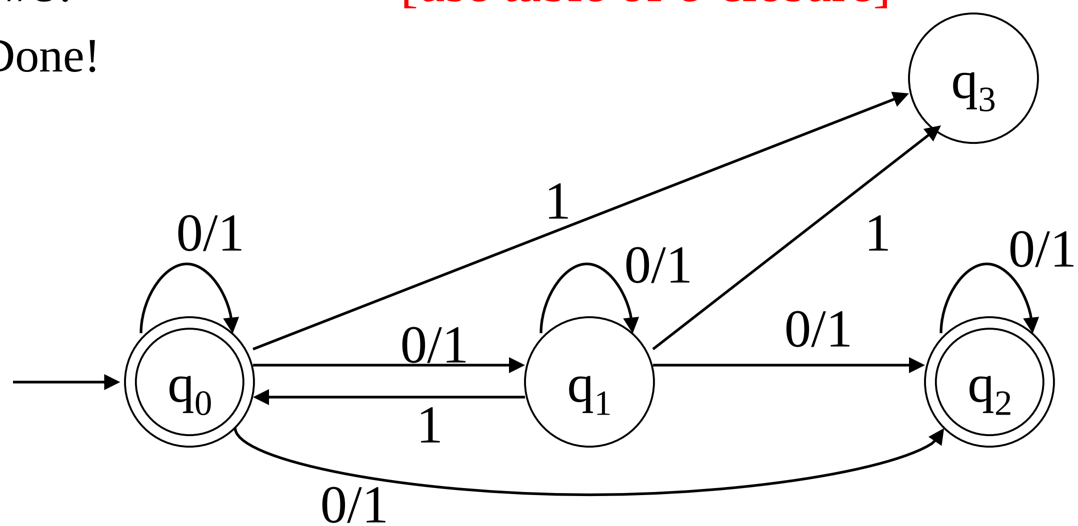
- Step #7:



- Example:



- Step #8: [use table of e-closure]
  - Done!



- **Theorem:** Let L be a language. Then there exists an NFA M such that L= L(M) iff there exists an NFA-ε M' such that L = L(M').

- **Proof:**

  (if) Suppose there exists an NFA-ε M' such that L = L(M'). Then by Lemma 2 there exists an NFA M such that L = L(M).

  (only if) Suppose there exists an NFA M such that L = L(M). Then by Lemma 1 there exists an NFA-ε M' such that L = L(M').

- **Corollary:** The NFA-ε machines define the regular languages.