

CS 240
Midterm Exam

Spring 2001
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No Additional Materials Allowed

2 hours

Answer each question in the space provided

Name: _____

I.D.: _____

Question	Out Of	Mark
1	19	
2	9	
3	12	
4	12	
5	9	
6	14	
7	9	
8	6	
Total	90	

1. [19 marks total] Asymptotic Analysis

- (a) [3 marks] Define Ω (“big-Omega”) notation and give an example of a $\Omega(n^3)$ algorithm. That is, write a few lines of pseudocode that will take $\Omega(n^3)$ time to execute.

- (b) [4 marks] Using Θ notation, give (and briefly justify) the order of magnitude of the following sum:

$$\sum_{i=0}^n \frac{i}{3^i}.$$

(c) [6 marks] Prove (by definition of O -notation) the following lemma:

If $f(x)$ is $O(h(x))$ and $g(x)$ is $O(h(x))$ then $af(x) + bg(x)$ is $O(h(x))$ for any constants a and b , ($a > 0, b > 0$). (That is, show that any linear combination of f and g is still $O(h)$).

(d) [6 marks] Prove (by definition of O -notation) the following lemma:

4^n is not $O(2^n)$ but 2^n is $O(4^n)$.

2. [9 marks total] Short Answer

- (a) [1 mark] Give the average case runtime (using O -notation) for interpolation search.
- (b) [1 mark] Give the worst case runtime (using O -notation) for interpolation search.
- (c) [1 mark] Give the average case runtime (using O -notation) for binary search.
- (d) [1 mark] Give the worst case runtime (using O -notation) for binary search.
- (e) [1 mark] Give the number of nodes that a perfect binary tree of height h contains.
- (f) [2 marks] Give the expected and worst case runtimes (using O -notation) for inserting into a binary search tree.
- (g) [2 marks] Give the expected and worst case runtimes (using O -notation) for inserting into an AVL tree.

3. [12 marks total] Runtime and Recurrence Relations

(a) [6 marks] Consider the following code fragment:

```
foo ( $A$  : array:  $i, j$  : integer)
  if  $i = j$  then
    output "Hello World"
  else
     $m \leftarrow \lfloor \frac{j-i+1}{2} \rfloor$ 

    foo( $A, i, i + m$ )
    foo( $A, i + m + 1, j$ )

     $\delta \leftarrow \frac{1}{j-i+1}$ 
     $x \leftarrow (j - i + 1)$ 
    while  $x > 0$ 
       $x \leftarrow x - \delta$ 
```

This routine takes an array together with start and end points of a subarray as input. Input size is to be measured by $n = j - i + 1$. Give a recurrence relation describing the runtime of foo as a function of n . You may assume that n is a power of 2. Explain. Be brief.

- (b) [6 marks] Prove that the recurrence relation $T(n) = 9T(n/3) + n$, $T(1) = 1$ is $O(n^2)$ by induction. You may assume n is a power of 3. Hint: It will not be good enough to show $T(n) \leq cn^2$, so make your inductive hypothesis stronger.

4. [12 marks total] Self-Organizing Lists

- (a) [4 marks] Consider the following list: 3 6 2 1 9 4 5. Give the list after looking up 1 2 3 4 5 (in that order) based on the

i. move-to-front heuristic

ii. exchange heuristic

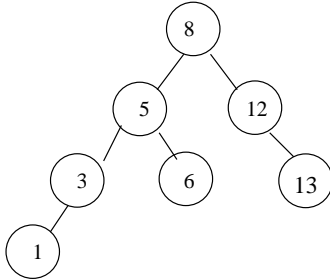
- (b) [8 marks] Suppose we have n elements to store in a dictionary and the probability of searching for element e is $p(e)$. Let $l(e)$ be the location (or index number) of where e is in the sequence. We know that the expected number of comparisons is then

$$\sum_e l(e) \cdot p(e).$$

Give the optimal ordering for elements in order to minimize the expected number of comparisons for a successful search. Prove that this ordering is minimal. (Hint: Consider swapping the position of any two elements i and j and compute the new cost).

5. [9 marks total] Binary Trees

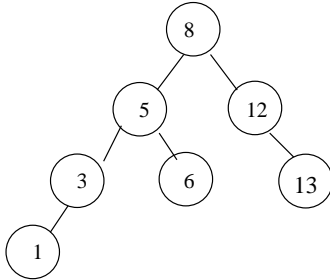
Here is a binary tree without the external nodes:



- (a) [4 marks] The tree is to be implemented using a “binary array” (that is, an array with a size close to a power of 2). Give this array with the elements in the correct positions. (Note: you don’t have to draw the actual array, just list the contents and the indices).
- (b) [3 marks] This tree could also be implemented with a linked structure in which each node stores an integer and 2 references (or pointers). Assume 4 bytes are required to store one integer. Also assume 4 bytes are required to store one reference (or pointer), regardless of whether or not the reference is null (or nil). Which representation uses more space to store the internal nodes, the array-based or linked-structure? Explain.
- (c) [2 marks] When we say that the runtime of a binary-tree based algorithm is $O(n)$, why does it not matter whether we explicitly state whether n is the number of internal, external or total number of nodes?

6. [14 marks total] AVL trees

Consider the following AVL tree (we call this the original tree):

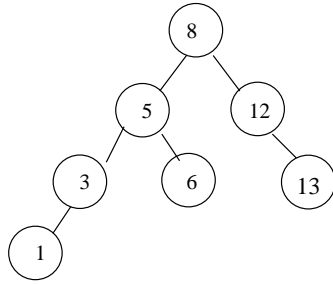


(a) [3 marks] Indicate the condition code for the above nodes in the following table:

1	3	5	6	8	12	13

(b) [2 marks] Draw the tree after an insertion of 4 into the original tree.

(c) [2 marks] Draw the tree after an insertion of 14 into the original tree.



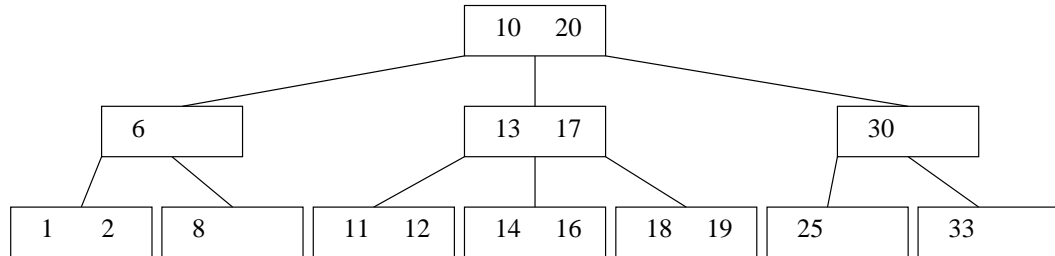
(d) [2 marks] Draw the tree after an insertion of 0 into the original tree.

(e) [2 marks] Draw the tree after a deletion of 5 into the original tree.

(f) [3 marks] Draw the tree after a deletion of 8 into the original tree.

7. [9 marks total] B-trees

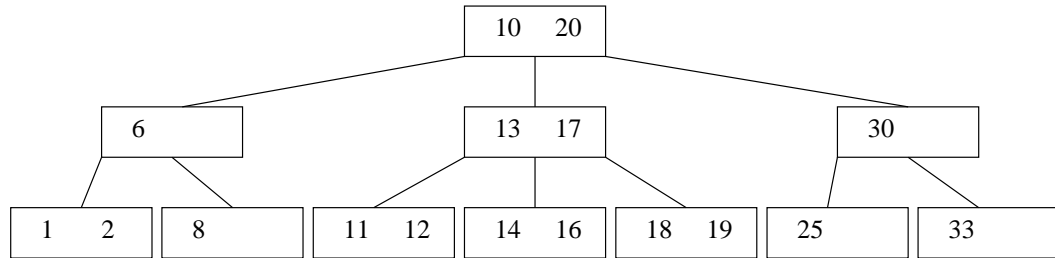
Consider the following B-tree:



(a) [1 mark] Suppose the root node is full. What is the order of this B-tree?

(b) [2 marks] Draw a new B-tree which is the original tree with 17 deleted.

(c) [2 marks] Draw a new B-tree which is the original tree with 8 deleted.



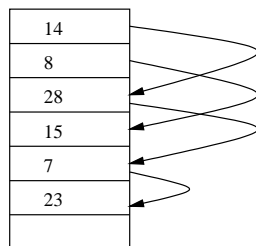
(d) [2 marks] Draw a new B-tree which is the original tree with 15 inserted.

(e) [2 marks] Draw a new B-tree which is the original tree with 25 deleted.

8. [6 marks total] Hashing

- (a) [4 marks] Give the average case runtime for separate chaining, assuming m buckets and n elements already inserted into the hash table. Be sure you give the successful and unsuccessful cases, and explain your answer (briefly).

- (b) [2 marks] Consider the following hash table, assuming coalesced chaining and assuming a hash function $h(k) = k \bmod 7$.



- i. Draw the hash table when 3 is inserted into the original hash table.

- ii. Draw the hash table when 28 is deleted from the original hash table.