Introduction to Feedback Control

Updated: November 1, 2023

Contents

Analysis of Feedback Control Systems

Routh-Hurwitz Criterion

October 27, 2023

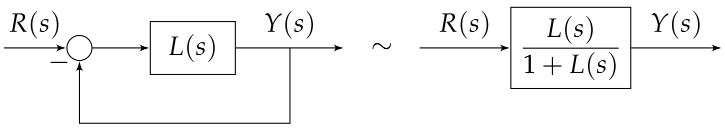


Figure 1: Transfer function of a closed-loop system

To assess the stability of a closed-loop system (Fig. 1) we need to determine its poles, i.e. the roots of the polynomial

$$1 + L(s) = 0. \tag{1}$$

Consider the polynomial

$$\pi(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0. \tag{2}$$

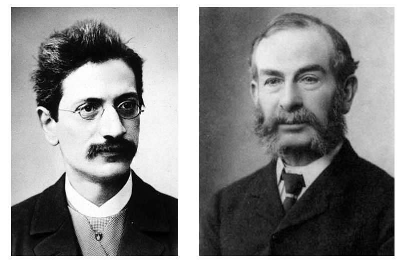


Figure 2: Adolf Hurwitz (1859–1919, left), German mathematician. His doctoral advisor was Felix Klein, who devised the *Klein bottle*. Edward Routh (1831–1907, right), English mathematician. He was born in Quebec.

Definition 1 *$\pi(s)$ is Hurwitz if all its roots have negative real part.*

Suppose $\pi(s)$ has r real roots, $\lambda_1, \ldots, \lambda_r$, and p complex conjugate pairs of roots, $\mu_1, \bar{\mu}_1, \ldots, \mu_p, \bar{\mu}_p$. Then, we can write

$$\pi(s) = \underbrace{(s-\lambda_1)\ldots(s-\lambda_r)}_{(\star)}\underbrace{(s-\mu_1)(s-\bar{\mu}_1)\ldots(s-\mu_p)(s-\bar{\mu}_p)}_{(\star\star)}. \tag{3}$$

$\pi(s)$ being Hurwitz means that $\lambda_i < 0,\ i = 1, \ldots, r$ and $\Re(\mu_i) < 0,\ i = 1, \ldots, p$. In such case, $\pi(s)$ has positive coefficients. This can be seen by expanding the polynomials $(\star)$ and $(\star\star)$ in (3) and realizing that both have positive coefficients, hence their product does too. Thus, if $\pi(s)$ is Hurwitz, necessarily its coefficients $a_i > 0,\ \forall i$.

Example 1

$s^4 + 3s^3 - 2s^2 + 5s + 6$	*Not Hurwitz as* $a_2 < 0$
$s^3 + 4s + 6$	*Not Hurwitz as* $a_2 = 0$
$s^3 + 5s^2 + 9s + 1$	*Don't know*

A NECESSARY AND SUFFICIENT CONDITION for a polynomial to be Hurwitz is provided by Routh's algorithm and the Routh-Hurwitz criterion. The first step of Routh's algorithm consists of building the following table—the *Routh table*—starting from the polynomial $\pi(s)$ in (2):

Step 1 of Routh's algorithm

$$
\begin{array}{c|ccc}
s^n & 1 & a_{n-2} & a_{n-4} \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} \\
s^{n-2} & r_{2,0} & r_{2,1} & r_{2,2} \\
s^{n-3} & r_{3,0} & r_{3,1} & r_{3,2} \\
\vdots & & & \\
s^2 & r_{n-2,0} & r_{n-2,1} & \\
s^1 & r_{n-1,0} & & \\
s^0 & r_{n,0} & &
\end{array}
$$

where

$$
\begin{aligned}
r_{2,0} &= -\frac{1}{a_{n-1}} \begin{vmatrix} 1 & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}, \qquad r_{2,1} = -\frac{1}{a_{n-1}} \begin{vmatrix} 1 & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}, \qquad \ldots \\
r_{3,0} &= -\frac{1}{r_{2,0}} \begin{vmatrix} a_{n-1} & a_{n-3} \\ r_{2,0} & r_{2,1} \end{vmatrix}, \qquad r_{3,1} = -\frac{1}{r_{2,0}} \begin{vmatrix} a_{n-1} & a_{n-5} \\ r_{2,0} & r_{2,2} \end{vmatrix}, \qquad \ldots,
\end{aligned} \tag{4}
$$

continuing along each row until a 0 appears, and terminating if a 0 appears in the first column.

Step 2 of Routh's algorithm

The second step of Routh's algorithm consists in applying the following criterion.

Theorem 1 (Routh-Hurwitz criterion)

(i) $\pi(s)$ *is Hurwitz* $\Leftrightarrow$ *All elements in the first column of the Routh table have the same sign*

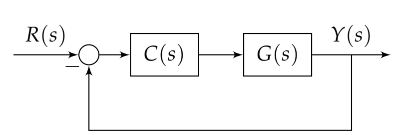
(ii) If the first column of the Routh table has no 0*'s, then*

- *(a) # sign changes = # roots with positive real part*
- *(b)* $\nexists$ *roots on the imaginary axis*

Using Routh's algorithm, we can predict whether a polynomial is Hurwitz without explicitly computing its roots. This is convenient to ensure the stability of a system, in particular of a closed loop system, whose poles are the roots of $1 + L(s)$.

Example 2 (P-control design using Routh's algorithm) *We are given a system with transfer function*

$$
G(s) = \frac{1}{s^4 + 6s^3 + 11s^2 + 6s}. \tag{5}
$$



The system is not stable and we would like to design a controller $C(s) = K$ *for some* $K \in \mathbb{R}$ *in order to stabilize it. The transfer function of the closed-loop system is*

$$
\frac{Y(s)}{R(s)} = \frac{K}{s^4 + 6s^3 + 11s^2 + 6s + K}. \tag{6}
$$

The Routh table is

$$
\begin{array}{c|ccc}
s^4 & 1 & 11 & K \\
s^3 & 6 & 6 & 0 \\
s^2 & 10 & K & 0 \\
s^1 & 6-\frac{3}{5}K & 0 & 0 \\
s^0 & K & &
\end{array}
$$

For the closed-loop system to be stable, the Routh-Hurwitz criterion says that we need $6-\frac{3}{5}K>0$ *and* $K>0$*, i.e.* $0<K<10$*.*

Nyquist Plot

October 30, 2023

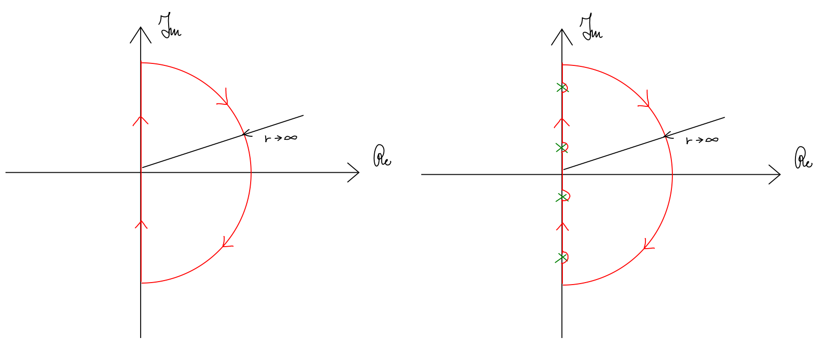


Figure 3: Nyquist contour (left) and its modified version to account for poles on the imaginary axis.

Figure 4: Harry Nyquist (1889–1976), Swedish-American physicist. Nyquist is also a programming language for sound synthesis.

Definition 2 *The Nyquist plot of a transfer function* $L(s)$ *is the image of the Nyquist contour (Fig. 3, on the left).*

If $L(s)$ has poles on the imaginary axis, then we modify the Nyquist contour to avoid these poles by going around them on infinitesimal semicircles on the right half plane (Fig. 3, on the right).

Note 1

(i) $L(-j\omega)=\overline{L(j\omega)}$

(ii) If $L(s)$ *is strictly proper, then* $L(\infty)=0$*, i.e. the whole semicircle of infinite radius is mapped to 0*

Example 3

$$L(s)=\frac{\mu}{1+\tau s},\quad \mu>0,\tau>0. \tag{7}$$

Theorem 2 (Nyquist criterion) *The closed-loop system in Fig. 1 is stable if and only if* $N=P$*, where*

- P *is the number of poles of* $L(s)$ *with positive real part*
- N *is the number of loops that the Nyquist plot of* $L(s)$ *makes around the point -1 in the complex plane (positive if counterclockwise, negative if clockwise);* N *is undefined if the Nyquist plot passes through -1*

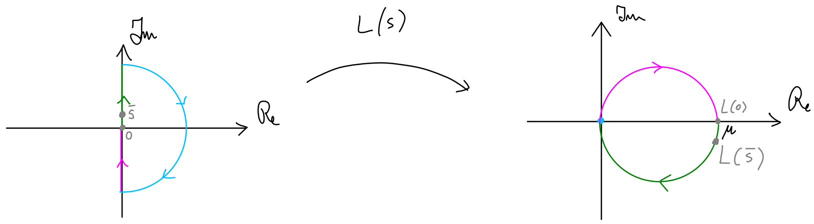


Figure 5: Nyquist plot of the transfer function (7).

Note 2

(i) If N is undefined, the closed-loop can be stable or unstable

(ii) If N is well-defined and $N \neq P$, then the closed-loop system is unstable and it has $P - N$ poles with positive real part

Example 3 (Continued) *$P = 0$ and $N = 0$. By the Nyquist criterion, the negative feedback interconnection of (7) is stable.*

Exercise 1 *Applying the Nyquist criterion, determine the stability property of the closed-loop system in Fig. 1, where $L(s) = \frac{\mu}{1+\tau s}$, where $\mu < 0$ and $\tau > 0$.*

Corollary 1 (to Theorem 2) *Given a stable $L(s)$, the following are sufficient conditions for the closed-loop system in Fig. 1 to be stable.*

- $|L(j\omega)| < 1 \ \forall \omega$
- $|\angle L(j\omega)| < 180° \ \forall \omega$

Stability Margins

November 1, 2023

Assume we designed a controller $C(s)$ for a system $G(s)$ so that the Nyquist plot of $L(s) = C(s)G(s)$ is the one in Fig. 6. The further the Nyquist plot from the point -1, the higher the safety margin with respect to perturbations of $L(s)$ (coming, for instance, from unmodeled dynamics). This margin can be conveniently decomposed into two stability margins which can be read directly from the Bode plots of $L(s)$.

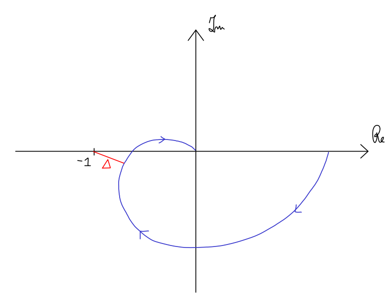


Figure 6: Δ is a stability margin.

ASSUME $L(s)$ HAS POSITIVE STEADY STATE GAIN, no unstable poles, and that its Nyquist plot intersects the negative real axis once, on the right of -1. Let ω_π be the frequency such that $\angle L(j\omega_\pi) = -180°$. The gain margin, k_m, is defined as follows (Fig. 7, on the left):

$$k_m = \frac{1}{|L(j\omega_\pi)|}, \tag{8}$$

and it represents the maximum multiplicative factor on the gain of $L(s)$ at ω_π that the system can tolerate before becoming unstable.

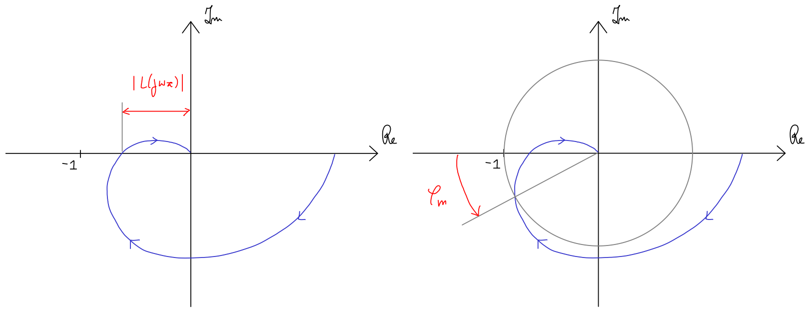


Figure 7: Gain margin, $k_m = \frac{1}{|L(j\omega_\pi)|}$, on the left, and phase margin $\varphi_m = 180° - |\angle L(j\omega_c)|$, on the right.

Assume the Nyquist plot of $L(s)$ crosses the unit circle only once, from outside to inside. Let ω_c be the frequency such that $|L(j\omega_c)| = 1$—ω_c is the *crossover* frequency. The phase margin, φ_m, is defined as follows (Fig. 7, on the right):

$$\varphi_m = 180° - |\angle L(j\omega_c)|, \tag{9}$$

and it represents the maximum (negative) phase shift of $L(s)$ at ω_c that the system can tolerate before becoming unstable.

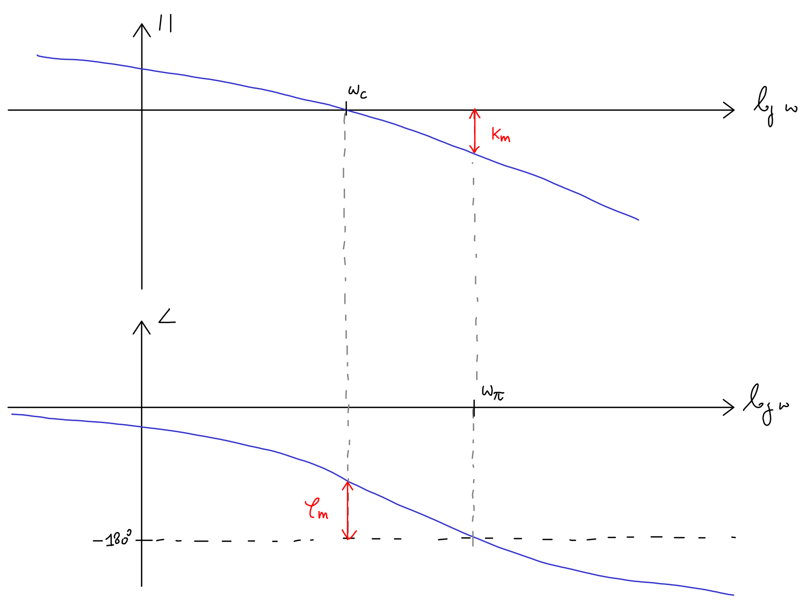


Figure 8: Reading k_m and φ_m on the Bode plots in correspondence of ω_π and ω_c, respectively.

As anticipated, the gain and phase margins can be read off from the Bode plots of $L(s)$, as shown in Fig. 8.

Positive gain (in decibels) and phase margins of $L(s)$ ensure that the closed-loop system is stable.

Note 3 *Just like for the Nyquist criterion, using the gain and phase margins we are able to predict the stability of the closed-loop system by evaluating metrics defined on the open-loop system.*