# Responses of linear, time-invariant (LTI) systems

Because the mathematics is so much simpler there, engineers often work in the Laplace domain. This requires an understanding of its relationship to the time domain. In this section, we'll examine how the form of the transfer function determines the form of the time-domain responses of LTI systems.

Consider finding the impulse response of a system with a given transfer function

$$h(t) = \mathcal{L}^{-1}\{H(s)\} \ .$$

Partial-fractions decomposition starts by factoring the denominator of $H(s)$ – called the system's *characteristic polynomial*. Its roots – the poles of the transfer function – determine the terms in the partial-fractions decomposition, and hence the terms of the response $h(t)$.

**Example** Consider a mass/spring/damper system with no damping:

$$m\ddot{y} + ky = f$$

Taking Laplace transforms with zero-valued initial conditions,

$$Y(s) = \frac{1}{ms^2 + k} F(s) \ .$$

The transfer function is therefore

$$H(s) = \frac{1}{ms^2 + k} \ .$$

The system's characteristic polynomial is $ms^2 + k$, and its roots (the poles of the transfer function) are $s = \pm j\sqrt{\frac{k}{m}}$. The impulse response is

$$
\begin{aligned}
h(t) &= \frac{1}{\sqrt{mk}} \sin \sqrt{\frac{k}{m}} t \\
&= \frac{1}{\sqrt{mk}} \frac{e^{j\sqrt{\frac{k}{m}}t} - e^{-j\sqrt{\frac{k}{m}}t}}{2j} \; .
\end{aligned}
$$

So, the poles at $s = \pm j\sqrt{\frac{k}{m}}$ give rise to sinusoidal oscillations at an angular frequency of $\sqrt{\frac{k}{m}}$ radians per second, which represents a natural vibrational mode, or resonance, of the system. □

As the example illustrates, the roots of the characteristic polynomial, or the poles of the transfer function, generalize the notion of *natural frequencies* of the system. They are associated with its "intrinsic dynamics."

Consider the response to a general input:

$$
Y(s) = H(s)F(s) \; .
$$

Here, a partial-fractions expansion will be based on a factorization of the denominator of $H(s)F(s)$. In general, its roots will include poles of both $H(s)$ and $F(s)$. Consequently, $y(t)$ will include terms of forms that appear in $f(t)$ and other terms, such as the sinusoidal oscillations of the example, that are "generated" by the dynamics of the system – namely, those associated with the poles of the transfer function.

To develop an understanding of the form of such responses, we'll look in particular at the impulse response and the step response (the response to a unit step), and study the relationship between the system's poles and the form of those responses. If we examine two particular examples – the so-called "standard" first- and second-order systems – we'll be able to extrapolate from there.

## The standard first-order system

Let

$$H(s) = \frac{K}{s\tau + 1}\ , K, \tau > 0\ .$$

Examples of such transfer functions include that of the RC circuit seen earlier.

The **impulse response** of a system with such a transfer function is, by definition, the inverse transform of $H(s)$:

$$h(t) = \frac{K}{\tau} e^{-\frac{t}{\tau}},\ t \geq 0\ .$$

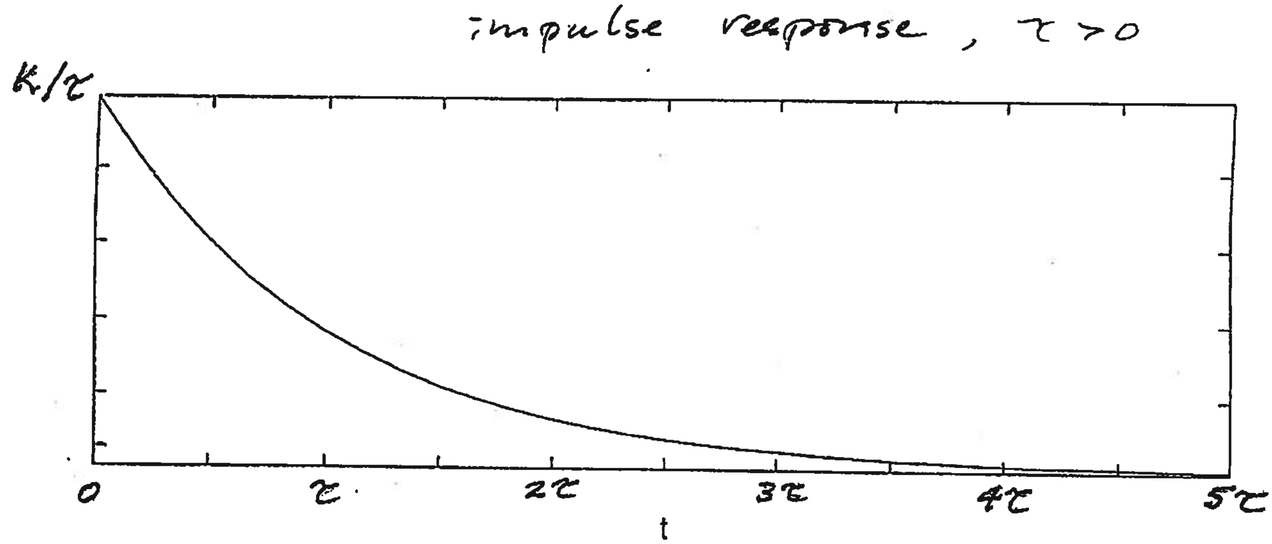
We see that the impulse response is a decaying exponential, whose rate of decay is determined by the parameter $\tau$, which for that reason is called the *time constant*.

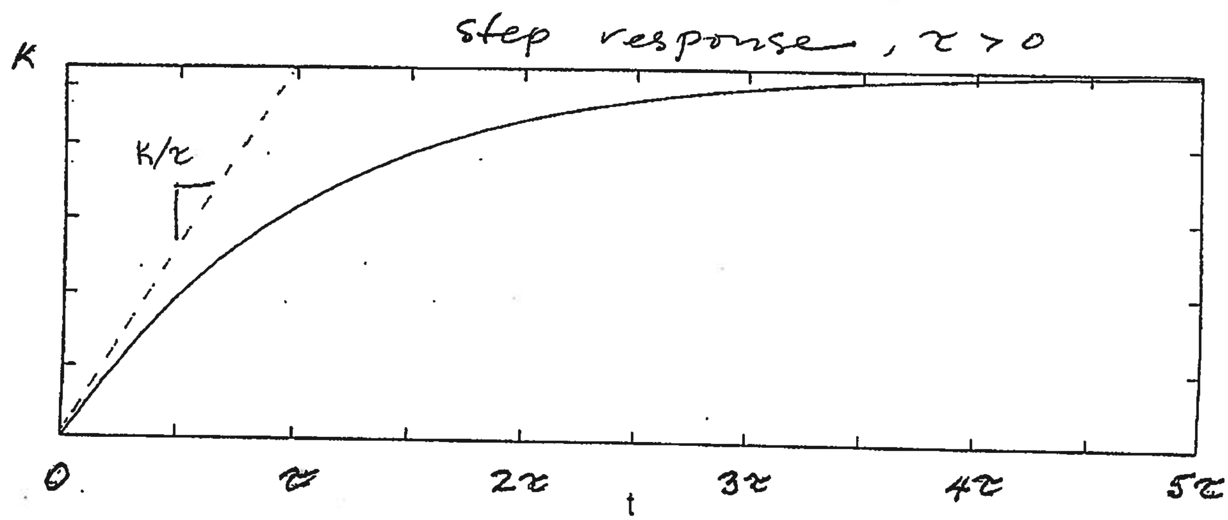
More quantitatively, if $t = 3\tau$, $e^{-t/\tau} = e^{-3} \approx 0.05$; if $t = 4\tau$, $e^{-t/\tau} \approx 0.02$. So the transient term decays to 5 percent of its initial value after 3 time constants, and to 2 percent after 4 time constants.

The **step response** of a system with the above transfer function is given by

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{H(s)\frac{1}{s}\}\ , \\ &= K[1 - e^{-t/\tau}],\ t \geq 0\ . \end{aligned}$$

Because the "final value" of the response to a unit step is $K$, that parameter is called the system's "d.c. gain."


impulse response , τ > 0
K/τ
0
τ
2τ
3τ
4τ
5τ
t


step response , τ > 0
K
K/τ
0
τ
2τ
3τ
4τ
5τ
t

This notion of d.c. gain is a more general one. For any transfer function whose poles all lie to the left of the imaginary axis, the final value of the step response is given by the final-value theorem:

$$\lim_{t\to\infty} y(t) = \lim_{s\to\infty} sY(s) = \lim_{s\to\infty} sH(s)\frac{1}{s} = H(0) \ .$$

The quantity $H(0)$ is consequently called the **d.c. gain**.

The above analysis of the standard second-order system also illustrates another more general fact. The step response, $\mathcal{L}^{-1}\{H(s)\frac{1}{s}\}$ is, by the differentiation/integration property of the Laplace transform, simply the integral of the impulse response $\mathcal{L}^{-1}\{H(s)\}$.

We have suggested that the point of this section is to develop an understanding of the relationship between the Laplace domain and the time domain, and that the transfer function poles play an important part in that relationship. Note that the pole of the standard first-order transfer function lies at $-1/\tau$. Hence, the further the pole is from the imaginary axis, the faster the transient part of the response, $e^{-t/\tau}$, decays. Control engineers use the distance of the poles from the imaginary axis as a proxy for the rate of decay of the real exponential factors in the time-domain response. If they wish the response of a system to converge faster toward its steady-state value, they will try to arrange for its poles to lie further from the imaginary axis.

As an illustration, let's look at an example of cruise-control design.

## Cruise-control example

Let the mass of a car be $m$, and the coefficient of a linear frictional force be $b$. If the net force exerted on the car in the direction of travel is $f$, and the resulting speed of the car is $v$, then the longitudinal motion of the car can be modelled as

$$m\dot{v} + bv = f \ .$$

The force $f$ will include a force $u$ exerted on the tires of the car by the road (in equal and opposite reaction to that exerted on the road by the tires) and another net force $d$ due to other factors such as gravity, if the car is on an incline:

$$m\dot{v} + bv = u + d\ .$$

Control engineers often use the symbol $u$ to denote a "control" input, that they can manipulate (in this case via the throttle), and $d$ for "disturbance" inputs, which are out of their control.

The relevant vehicle dynamics can then be represented by a transfer function:

$$V(s) = \frac{1}{ms+b}F(s) = \frac{1}{ms+b}[U(s) + D(s)]\ .$$

The purpose of a cruise control is to ensure that the speed $v(t)$ approximates some desired "reference" signal $r(t)$, specified in one way or another by the driver. For a variety of reasons that will be explored in SE380, it is practical to control the speed by means of feedback, used to generate an "error" signal $e(t) = r(t) - v(t)$, as in Figure 1.

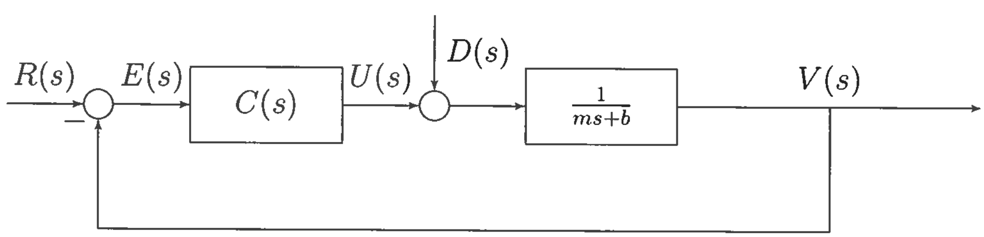



Figure 1: "Block diagram" of cruise control.

The block labelled $C(s)$ represents a controller – a system that takes the error signal as its input and (effectively) the throttle position as its output. The engineer is free to design and implement this part of the system. The problem is to design its transfer function $C(s)$ in such a way that the speed $v(t)$ "tracks" the reference input $r(t)$.

### Proportional control

The simplest possible choice for $C(s)$ is a constant, $K_p$. In this case, the applied throttle input is directly proportional to the error signal: the bigger the difference between the desired speed and the measured speed, the more throttle is applied.

In that case, it is easy to find the transfer function $H_{RV}(s)$ relating $R(s)$ and $V(s)$. For this, we set $D(s) \equiv 0$: we can always find, in similar fashion, a second transfer function $H_{DV}(s)$ relating $D(s)$ to $V(s)$, and then express the overall response of the system as

$$V(s) = H_{RV}(s)R(s) + H_{DV}(s)D(s) \; .$$

Confirm that this is just an application of the principle of superposition to the LTI system with output $v(t)$ and input $\begin{bmatrix} r(t) \\ d(t) \end{bmatrix}$.

To find $H_{RV}(s)$, let $P(s) = \frac{1}{ms+b}$. We have:

$$V(s) = K_p P(s) E(s) \; ,$$

$$E(s) = R(s) - V(s) \; .$$

Eliminating the transform of the error signal, we find

$$V(s) = \frac{KP(s)}{1 + K_p P(s)} R(s) = \frac{K_p}{ms + b + K_p} R(s) = \frac{K}{s\tau + 1} R(s) \; ,$$

where the time constant is $\tau = \frac{m}{b+K_p}$, and the d.c. gain $K = \frac{K_p}{b+K_p}$. The pole of the transfer function lies at $-\frac{1}{\tau} = -\frac{b+K_p}{m}$.

This looks promising. We wouldn't want the time constant to be too great – it should certainly be on the order of seconds, not minutes. But we can decrease it by increasing $K_p$, which the designer is free to choose. Increasing $K_p$ shifts the transfer function's pole at $-\frac{1}{\tau}$ to the left.

In the usual approach to control design, we would have a precise specification characterizing the speed at which the exponential transient term in the system's response should decay to zero. That specification would be translated into a constraint specifying how far to the left of the imaginary axis the pole should lie, and a value of $K_p$ would be chosen accordingly. The larger $K_p$, the further to the left lies the pole, and the shorter is the time constant.

Moreover, increasing $K_p$ has the effect of moving the d.c. gain closer to 1, which would be the ideal value, given that we wish $v(t)$ to follow $r(t)$.

Note however that, provided there is some "drag" or friction ($b \neq 0$), it is impossible to achieve a d.c. gain of unity with this controller. If the throttle input is directly proportional to the error signal, then an error of zero implies a throttle input of zero. In the presence of friction, that means that the car will slow.

### Integral control

On the other hand, if the controller does not simply multiply the error signal by a constant, but rather multiplies its integral by some constant, then it is in principle possible that the error signal could go to zero without the throttle input doing the same.

In this case we have $C(s) = \frac{K_i}{s}$, so (with $D(s) \equiv 0$),

$$V(s) = \frac{K_i}{s} P(s) E(s) \ ,$$

$$E(s) = R(s) - V(s) \ .$$

Eliminating the transform of the error signal, we find

$$V(s) = \frac{K_i P(s)}{s + K_i P(s)} R(s) = \frac{K_i}{ms^2 + bs + K_i} R(s) = \frac{K_i/m}{s^2 + (b/m)s + K_i/m} R(s) \ ,$$

Provided the poles of the transfer function lie to the left of the imaginary axis, this does give us a d.c. gain of unity. Let $R(s) = \frac{1}{s}$. Then, the conditions of the final value theorem are satisfied by $V(s)$, and

$$\lim_{t\to\infty} v(t) = \lim_{s\to 0} sV(s) = \lim_{s\to 0} s\left(\frac{K_i/m}{s^2+(b/m)s+K_i/m}\right)\frac{1}{s} = 1\ .$$

The tracking of a step reference is therefore asymptotically "perfect."

As far as other characteristics of the response are concerned, this transfer function is of the form of the "standard" second-order system.

## The standard second-order system

Let

$$H(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2},\quad \omega_n > 0\ .$$

Examples of systems with transfer functions of this form include some mass/spring/damper systems, RLC circuits, and the cruise-control example under integral control.

The poles of the system lie at

$$s = -\zeta\omega_n \pm \sqrt{\zeta^2\omega_n^2-\omega_n^2} = \omega_n(-\zeta \pm \sqrt{\zeta^2-1}\ .$$

If $\zeta > 1$, the poles are real and distinct. Partial-fractions expansion then decomposes the transfer function into a sum of two standard first-order systems. If $\zeta = 1$, there is a repeated real pole at $-\omega_n$, and the $t$-multiplication property shows that the impulse response, instead of being a simple decaying exponential, is a decaying exponential multiplied by $t$. (Confirm this.)

We shall concentrate on the so-called **underdamped** case, where $0 < \zeta < 1$. In this case the poles are complex, lying at $-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$.

The **impulse response** of this underdamped system is

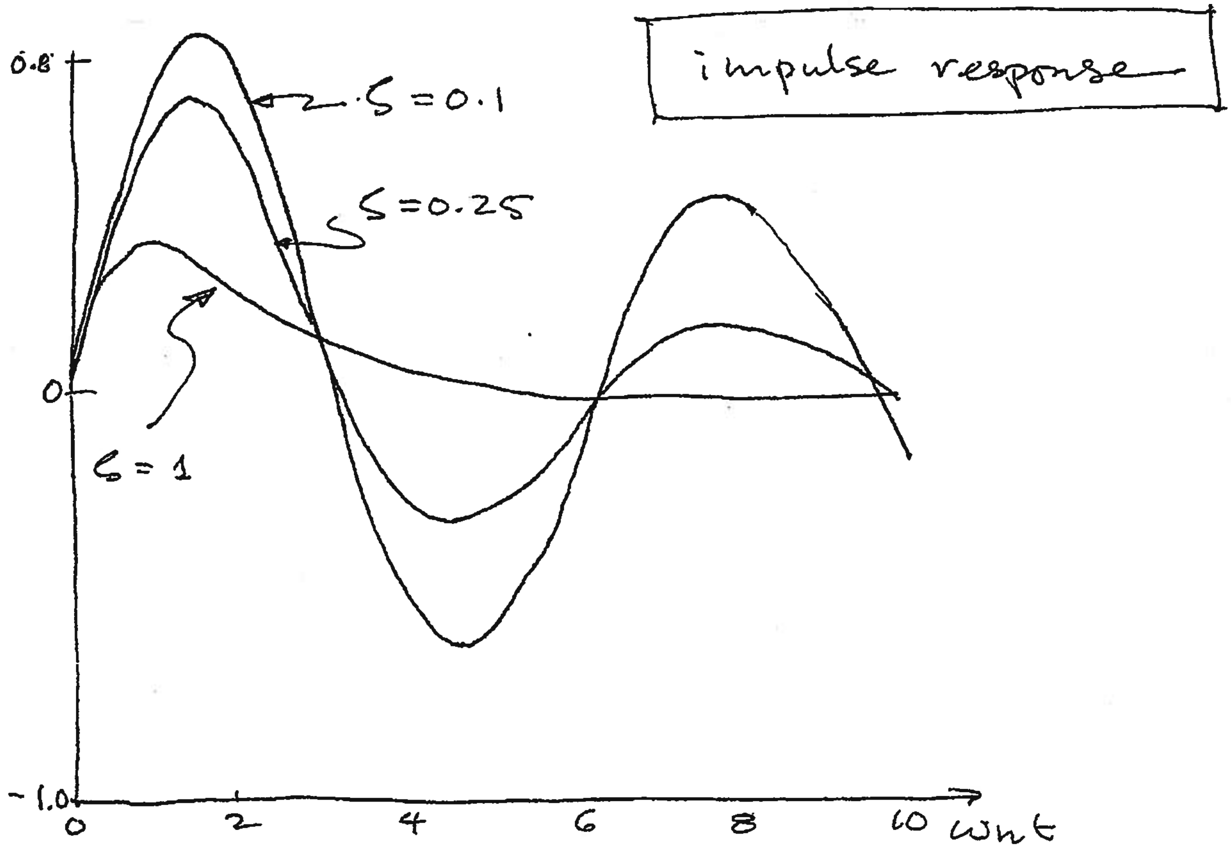
$$
\begin{aligned}
h(t) &= \mathcal{L}^{-1}\{H(s)\} \\
&= \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\omega_n t} \sin \omega_n \sqrt{1-\zeta^2} t \ .
\end{aligned}
$$

This has the form of a damped sinusoid.

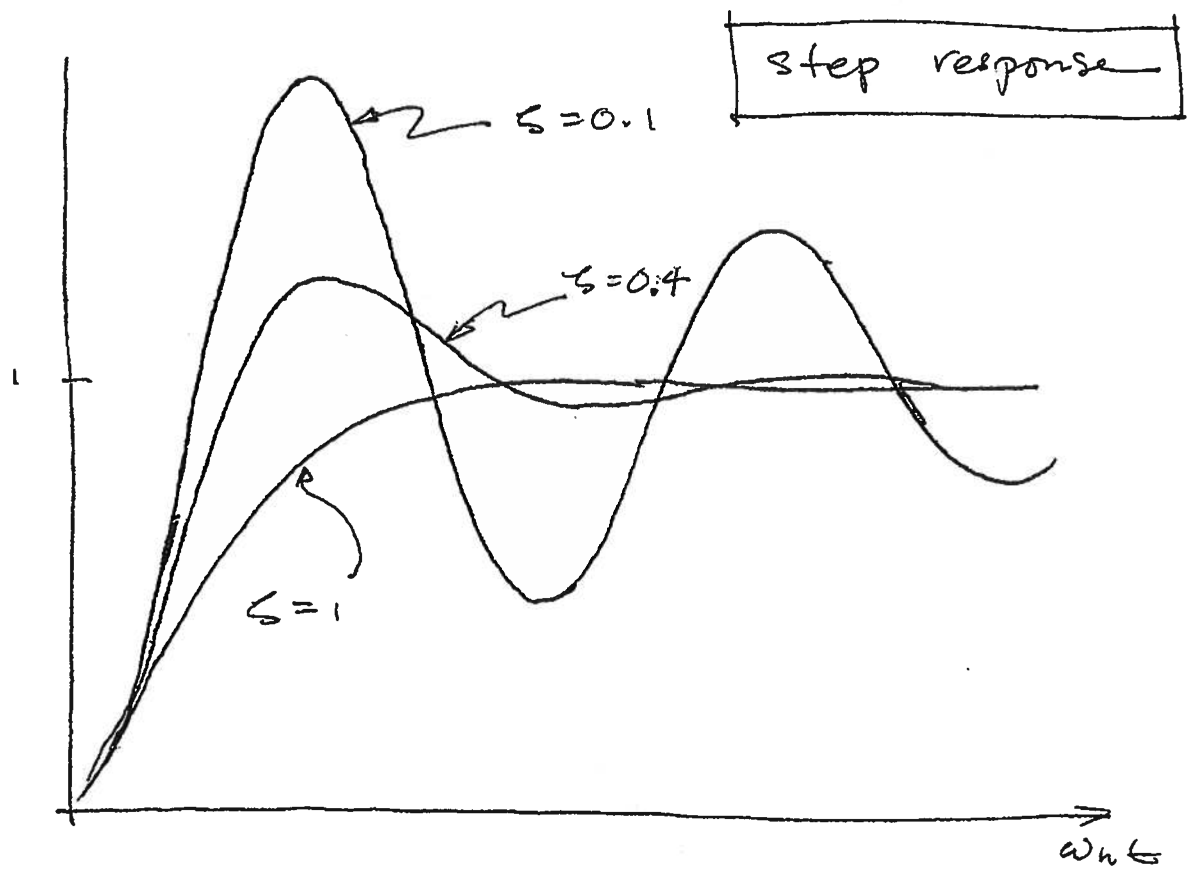
The **step response** is

$$
\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{H(s)\frac{1}{s}\} \\
&= 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2} t + \theta) \ ,
\end{aligned}
$$

where $\theta = \arccos \zeta$. As we've already observed in general, this is the integral of the impulse response.



impulse response
ζ = 0.1
ζ = 0.25
ζ = 1
0.8
0
-1.0
0
2
4
6
8
10
ωnt

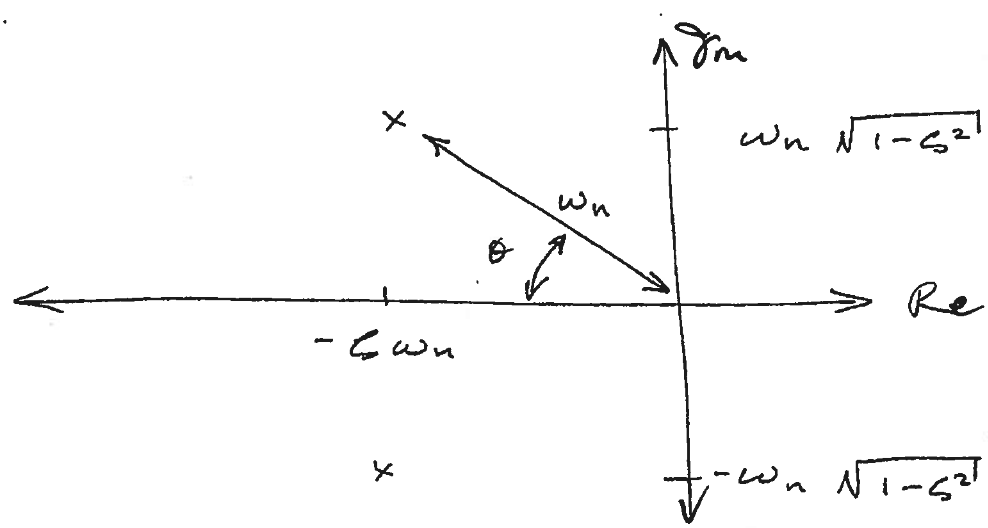



step response
ζ = 0.1
ζ = 0.4
ζ = 1
1
ωnt

As $\zeta$ tends to zero, the impulse response looks like an undamped sinusoid, with an angular frequency of oscillation of $\omega_n$, like our undamped spring/mass example. For this reason, $\omega_n$ ("omega-n") is called the *natural frequency*, and the dimensionless quantity $\zeta$ ("zeta") the *damping ratio*.

It's convenient to plot the responses as functions of $\omega_n t$, because wherever $t$ appears in the expressions, it's multiplied by $\omega_n$. This shows that $\omega_n$ acts as a time-scaling factor. For a fixed value of $\zeta$, the larger $\omega_n$, the faster the response. The plots show that, for a fixed $\omega_n$, the larger $\zeta$, the less oscillatory the response.

We can conveniently relate the responses to the pole locations in two different ways, essentially in rectangular and in polar coordinates. First, note that the real part of the poles, $-\zeta\omega_n$, appears as the coefficient of $t$ in the real exponential factors, and the (absolute value of the) imaginary part, $\omega_n\sqrt{1-\zeta^2}$, as the angular frequency of the sinusoidal oscillations. Thus, the further the poles are from the imaginary axis, the greater the decay rate of the real exponentials; and the further the poles from the real axis, the greater the frequency of oscillation.




But note that the modulus of the poles is $\sqrt{(\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)} = \omega_n$, and their angle relative to the negative real axis is $\arccos\zeta = \theta$. Thus, given the above observations, the further the poles are from the origin, the faster the response; and the smaller the angle $\theta$, the less oscillatory the response.

**Example** For the cruise-control system, under integral control, the controller gain $K_i$ affects the value of the natural frequency, but not its product with the damping ratio. So the tuning of the parameter can increase the speed of the response, but only at the price of increasing the amount of oscillation.

We can change this by combining a proportional and an integral term in the controller, so that $C(s) = K_p + K_i/s$. Doing so, we find that

$$V(s) = \frac{(K_i/m)((K_p/K_i)s + 1)}{s^2 + (b + K_p/m)s + K_i/m} \ .$$

(Confirm this.) The characteristic polynomial is of the form of that of the standard second-order system, and it shows that the integral gain $K_i$ determines the natural frequency, while $K_p$ determines its product with the damping ratio.

However, the numerator is no longer of the form of that of the standard system. In particular, it now has a finite zero. Before tuning the controller parameters, we should understand how this zero will affect the response. But as claimed earlier, having analyzed the standard first- and second-order systems, we can extrapolate to other transfer functions.

## Analysis of more complex transfer functions

### Effect of a finite zero

Consider a transfer function of the form of that of the cruise-control system with proportional-integral control:

$$H_z(s) = \frac{\omega_n^2(\frac{s}{\alpha\zeta\omega_n} + 1)}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \ \ \alpha > 0 \ .$$

Here, we have introduced a parameter $\alpha$ that is the ratio between the real zero and the real part of the poles. This turns out to be the key factor in determining the importance of the effect of the zero.

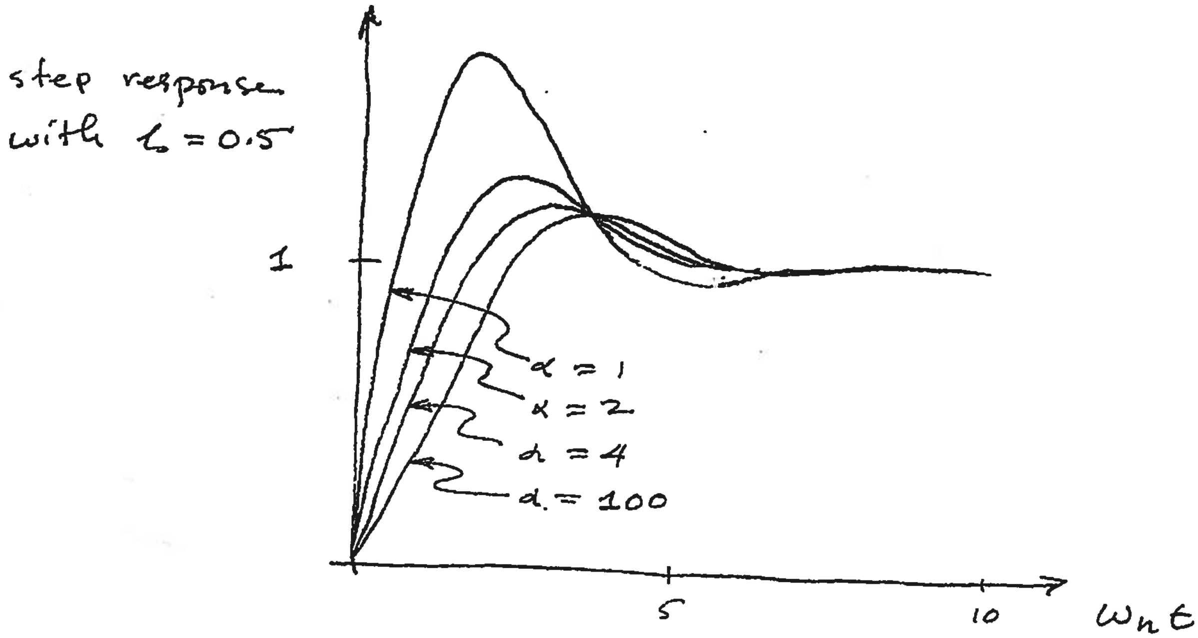
For simplicity, let's just look at the **step response**:

$$
\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{H_z(s)\frac{1}{s}\right\} \\
&= \mathcal{L}^{-1}\left\{\left(\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\right)\frac{1}{s}+\frac{s}{\alpha\zeta\omega_n}\left(\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\right)\frac{1}{s}\right\}
\end{aligned}
$$

Here, the first term will just give rise to the step response of the standard second-order system; and the second gives rise to $\frac{1}{\alpha\zeta\omega_n}$ times the impulse response of the standard second-order system. It follows that if the zero is sufficiently far from the imaginary axis (in comparison with the poles) – in other words, if $\alpha$ is sufficiently large – then the second term should be negligible, and the zero should not affect the response very appreciably. In practice, if the zero is, say, 10 times further from the imaginary axis than the poles, the system's response should resemble that of the standard second-order system.

On the other hand, the closer the zero to the imaginary axis, and the smaller $\alpha$, the more significant will be the factor multiplying the impulse response of the standard system, and the more important the effect of the zero on the response. In particular, the response will look more oscillatory than that of the standard system.

**Example** So, if we employ proportional-integral control in the cruise system, if we tune the controller gains in such a way that the value of $\alpha$ is "small," the step response may subject the car's occupants to a sudden jerk, and the speed may initially overshoot the desired value considerably. Care should be taken to avoid this effect. □




**Effect of a third pole**

To see the effect on system response of an added pole, consider the transfer function

$$H_z(s) = \frac{\omega_n^2}{(\frac{s}{\alpha\zeta\omega_n}+1)(s^2+2\zeta\omega_n s+\omega_n^2)}, \quad \alpha > 0$$
$$= \frac{a}{\frac{s}{\alpha\zeta\omega_n}+1} + \frac{bs+c}{s^2+2\zeta\omega_n s+\omega_n^2},$$

by an appropriate partial-fractions decomposition, where

$$a = \frac{1}{1-2\alpha\zeta^2+\alpha^2\zeta^2}$$
$$b = -\alpha\zeta\omega_n a, \quad \text{and}$$
$$c = \alpha(\alpha-2)\zeta^2\omega_n^2 a\ .$$

In this case, $\alpha$ is the ratio between the real pole and the real parts of the complex poles. As this becomes small, we have

$$\alpha \longrightarrow 0 \iff a \longrightarrow 1,\ b, c \longrightarrow 0\ .$$

So the response approaches that of a standard first-order system when the real pole is much closer to the imaginary axis than the complex poles.

On the other hand, is $\alpha$ becomes large,

$$\begin{aligned} \alpha \longrightarrow \infty \Longleftrightarrow a &\longrightarrow \frac{1}{\alpha^2\zeta^2} \longrightarrow 0\ , \\ b &\longrightarrow -\frac{\alpha\zeta\omega_n}{\alpha^2\zeta^2} \longrightarrow 0\ , \quad \text{and} \\ c &\longrightarrow \frac{\alpha^2\zeta^2\omega_n^2}{\alpha^2\zeta^2} = \omega_n^2\ . \end{aligned}$$

Thus, when the real pole is sufficiently far from the imaginary axis in comparison to the complex poles, the response approximates that of the standard second-order.system.

Owing to results of this sort, we say that a real pole or a pair of complex poles is *dominant* if it is much closer (say, 5 or 10 times closer, as a rule of thumb) to the imaginary axis than all other poles. In that case, the time-domain response of the system can be reasonably approximated by neglecting all poles other than the dominant one(s).