Frequency Responses of LTI systems

We've seen how we can transform problems in analysis of LTI systems (and initial-value problems with linear, constant-coefficient ODEs) to the Laplace domain, where they can be solved by purely algebraic means.

This was a consequence of the fact that the action of an LTI system on an exponential input is merely algebraic — the system simply multiplies the exponential input by a corresponding value of the system transfer function.

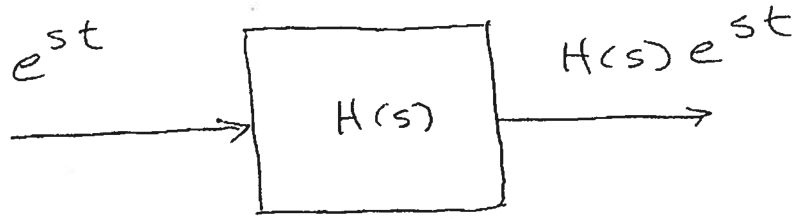
We can also transform problems to the "frequency domain," by focusing on the way an LTI system responds to purely sinusoidal inputs.

For example, in the design of feedback control systems, specifications (or "requirements," in CS parlance) are typically formulated in the time domain, but design is often carried out either in the Laplace domain or the frequency domain. Much communication engineering is carried out in the frequency domain (which is why we all commonly use terms such as "spectrum" and "bandwidth").

Engineers develop an intuition for working on these alternative domains: control engineers see how a system will behave simply by looking at the poles and zeros of its transfer function.

Frequency Response

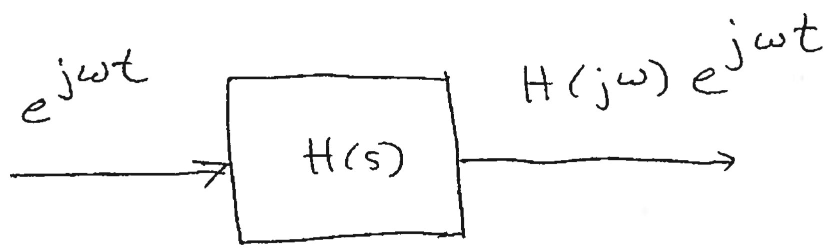
We've seen that the response of an LTI system to a (two-sided) exponential input e^{st} is simply the value of the transfer function at s multiplied by the input signal:



To focus on sinusoids, we'll simply consider the special case where s is purely imaginary:

$$e^{j\omega t} = \cos\omega t + j\sin\omega t\,.$$

Because the response to $e^{j\omega t}$ is completely determined by the value of the transfer function at $s = j\omega$, we call this quantity, $H(j\omega)$, the <u>frequency response</u> of the LTI system:



To see more clearly the system's effect on the sinusoidal input, think of the frequency response in terms of its modulus and angle:

$$H(j\omega) = |H(j\omega)|\, e^{j\angle H(j\omega)}$$

The output sinusoid is

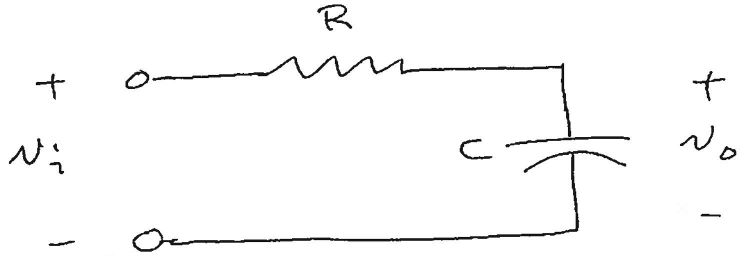
$$H(j\omega)\, e^{j\omega t} = |H(j\omega)|\, e^{j\angle H(j\omega)}\, e^{j\omega t}$$

$$= |H(j\omega)|\, e^{j(\omega t + \angle H(j\omega))}$$

So, the modulus $|H(j\omega)|$ of the frequency response is the factor by which the amplitude of the input sinusoid is multiplied ...

... and the angle $\angle H(j\omega)$ is the <u>phase shift</u>.

Example:



Our familiar RC circuit has the transfer function

$$\frac{V_o(s)}{V_i(s)} = H(s) = \frac{1}{RCs + 1}$$

Its frequency response is

$$H(j\omega) = \frac{1}{RCj\omega + 1} .$$

Suppose $RC = 0.01$ (seconds) and consider an input of

$$v_i(t) = 0.50 \sin 100 \; t \text{ volts.}$$

What is the output sinusoid?

With $\omega = 100$ radians / second,
the value of the frequency response
$H(j\omega)$ is

$$\frac{1}{0.01\, j100 + 1}$$

$$= \frac{1}{1 + j}$$

$$= \frac{1}{\sqrt{2}\, e^{j45^\circ}}$$

$$= \frac{1}{\sqrt{2}}\, e^{-j45^\circ}$$

So the input sinusoid's amplitude
will be multiplied by $\frac{1}{\sqrt{2}}$,
and its phase is shifted by -45°:

$$v_{out}(t) = \frac{0.50}{\sqrt{2}} \sin(100t - 45^\circ)$$
volts

Our result on responses of LTI systems to exponential inputs assumes "two-sided" exponentials. If instead the input is of the form

$$e^{j\omega t} u_{-1}(t)$$

it may provoke transients that are not of the form $e^{j\omega t}$. However, if $H(s)$ is stable, then in "steady state" — that is, neglecting transient terms, which decay to 0 as $t \to \infty$ — the output will be

$$H(j\omega) e^{j\omega t} u_{-1}(t)$$

In fact, if $H(s)$ is stable, its region of convergence includes the imaginary axis. We can then find $h(t)$ by integrating $H(s)$ along the imaginary axis $s = j\omega$.

It follows that the frequency response contains just as much information as the impulse response or the transfer function. So, for systems with stable transfer functions, the "frequency domain" is equivalent to the time domain and the Laplace domain.

Bode plots

These are a means of representing frequency response graphically — and of understanding the form of the frequency response, even for complex transfer functions.

A Bode plot (Hendrik Bode, 1905-1982) consists of two curves:

$$|H(j\omega)| \text{ in "decibels" vs. } \log_{10}\omega$$

$$\& \quad \angle H(j\omega) \text{ vs. } \log_{10}\omega$$

(respectively called the "magnitude" and "phase" curves).

Its particular form

a) allows the curves to be approximated in piecewise-linear form; and

b) allows plots for complex transfer functions to be obtained by summing plots of simpler factors.

Decibels

- One-tenth of a bel, of course.
- Named after Alexander Graham Bell, the bel is the base-10 logarithm of the power of two signals.

Power is typically proportional to the square of the amplitude.

For example, the power dissipated by a resistor is $v \cdot i$, the product of the voltage drop across the resistor with the current flowing through it. By Ohm's law,

$$v \cdot i = \frac{v^2}{R} = i^2 R,$$

where R is the resistance.

So, in bels, the ratio of the power of two signals f_1 and f_2 typically $\log_{10} \frac{f_1^2}{f_2^2} = 2 \log_{10} \frac{f_1}{f_2}$.

It's more common to use decibels (dB), in which case the ratio is

$$10 \log_{10} \frac{f_1^2}{f_2^2} \text{ dB} = 20 \log_{10} \frac{f_1}{f_2} \text{ dB}.$$

We know that the ratio of the amplitude of input & output sinusoids of an LTI system with transfer function $H(s)$ is $|H(j\omega)|$ (where ω is the angular frequency of the sinusoids); in decibels this is

$$20 \log |H(j\omega)|$$

Bode plots for simple transfer functions

Let's start with an example even simpler than our standard 1st-order transfer function.

Suppose $H(s) = \frac{1}{s}$.

The frequency response is then

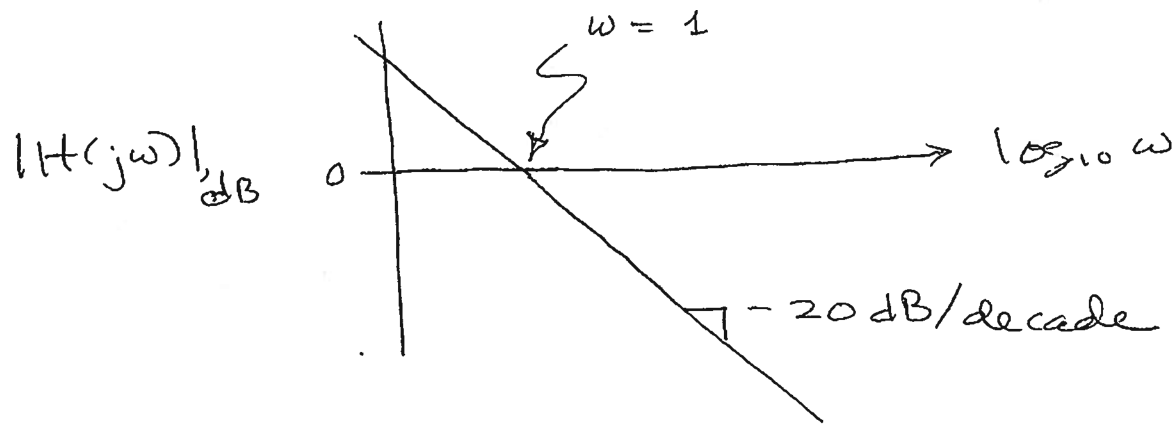
$$H(j\omega) = \frac{1}{j\omega}.$$

Magnitude:

$$\begin{aligned} &20 \log_{10} |H(j\omega)| \\ &= 20 \log_{10} \frac{1}{\omega} \qquad (\omega > 0) \\ &= -20 \log_{10} \omega \end{aligned}$$

- plotted vs. $\log_{10} \omega$, this gives a straight line, with a slope of -20 decibels per decade (i.e. per factor of 10 increase in ω).

This straight line intersects the horizontal (0 dB) axis when $\log_{10}\omega = 0$ – i.e., $\omega = 1$ (radian/second)



This is only one half of the Bode plot; the other half gives

$$\angle H(j\omega) \quad \text{vs.} \quad \log_{10}\omega$$

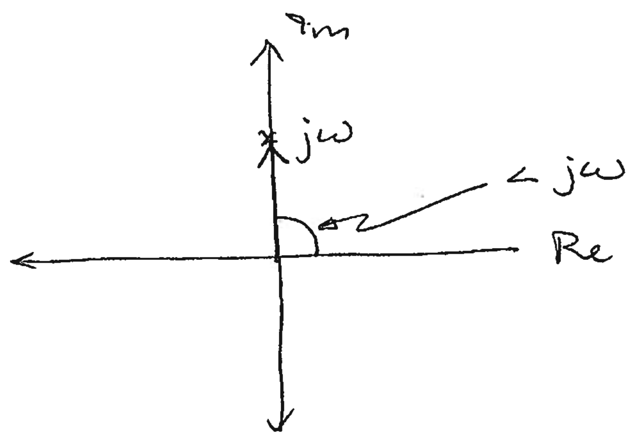
For our example,

$$\angle H(j\omega) = \angle \frac{1}{j\omega}$$

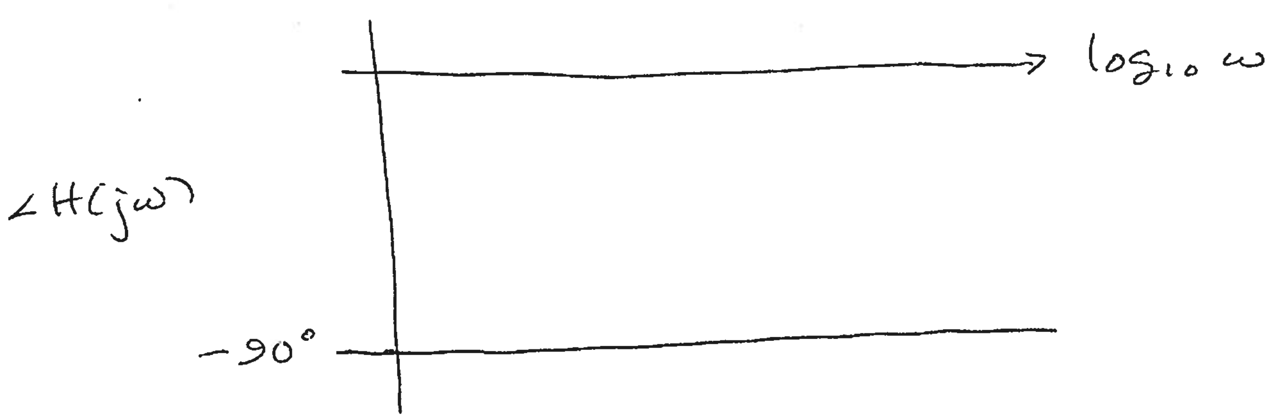
$$= \angle 1 - \angle (j\omega)$$

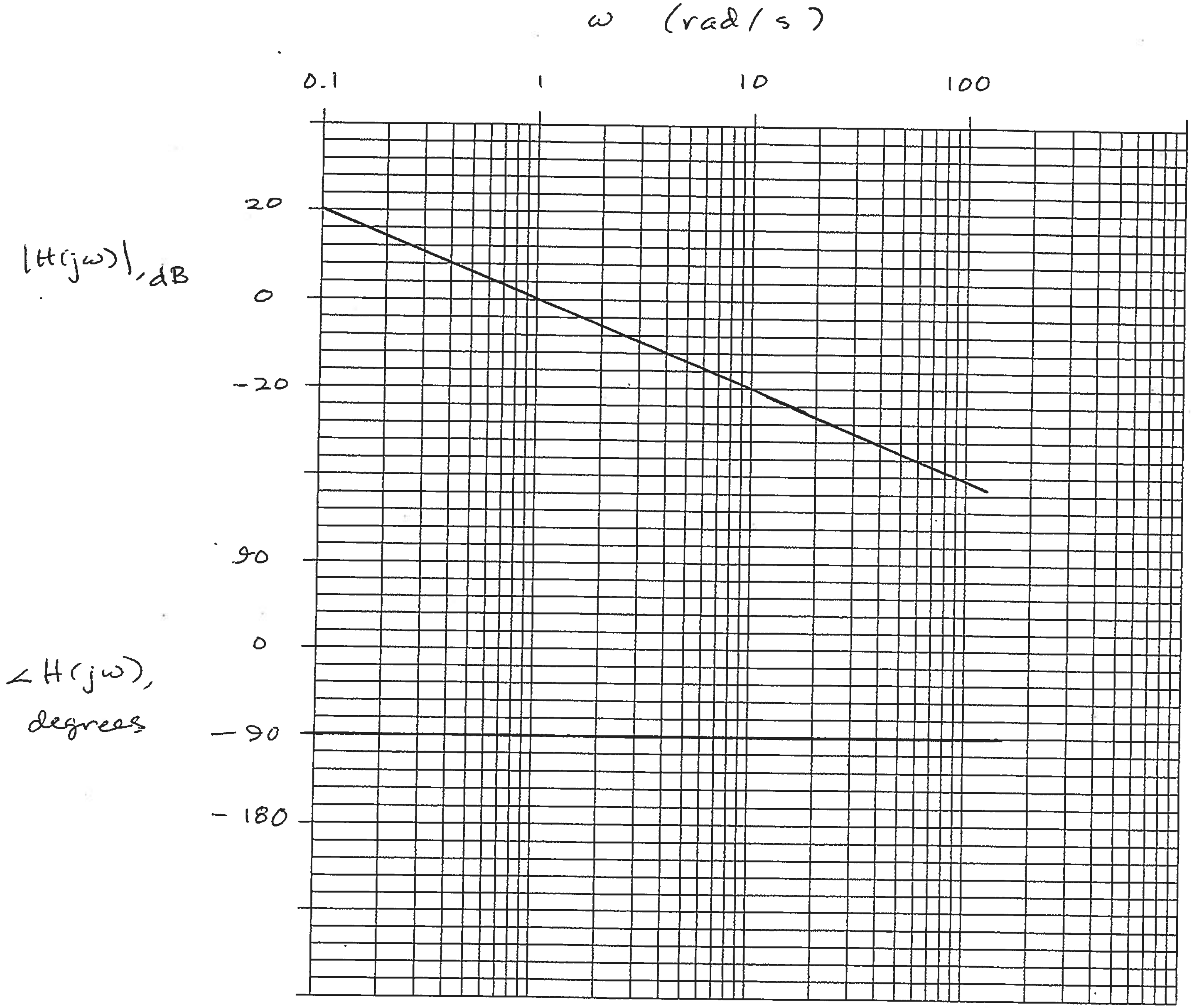
$$= 0 - \angle j\omega$$

Now, for $\omega > 0$, $j\omega$ lies on the upper half of the imaginary axis:



So $\angle j\omega = 90°$, and therefore $\angle H(j\omega) = -90°$:





Bode plot for $H(s) = \frac{1}{s}$

Frequency response of standard systems

1^{st}-order:

$$H(s) = \frac{K}{s\tau + 1}, \quad K, \tau > 0$$

$$\Rightarrow \quad H(j\omega) = \frac{K}{1 + j\omega\tau} = \frac{K/\tau}{j\omega - (-\frac{1}{\tau})}$$

magnitude:

$$|H(j\omega)|_{dB} = 20 \log_{10} K - 20 \log_{10} \sqrt{1 + (\omega\tau)^2}$$

phase:

$$\angle H(j\omega) = \angle K/\tau - \angle \left(j\omega - (-1/\tau)\right)$$

$$= -\angle \left(j\omega - (-\tfrac{1}{\tau})\right)$$

How to plot these?

- Draw low- and high-frequency asymptotes.

- <u>magnitude</u>:
  - for $\omega << \frac{1}{\tau}$,

$$|H(j\omega)|_{dB} \sim 20\log_{10}K - 20\log_{10}1$$

$$= 20\log_{10}K$$

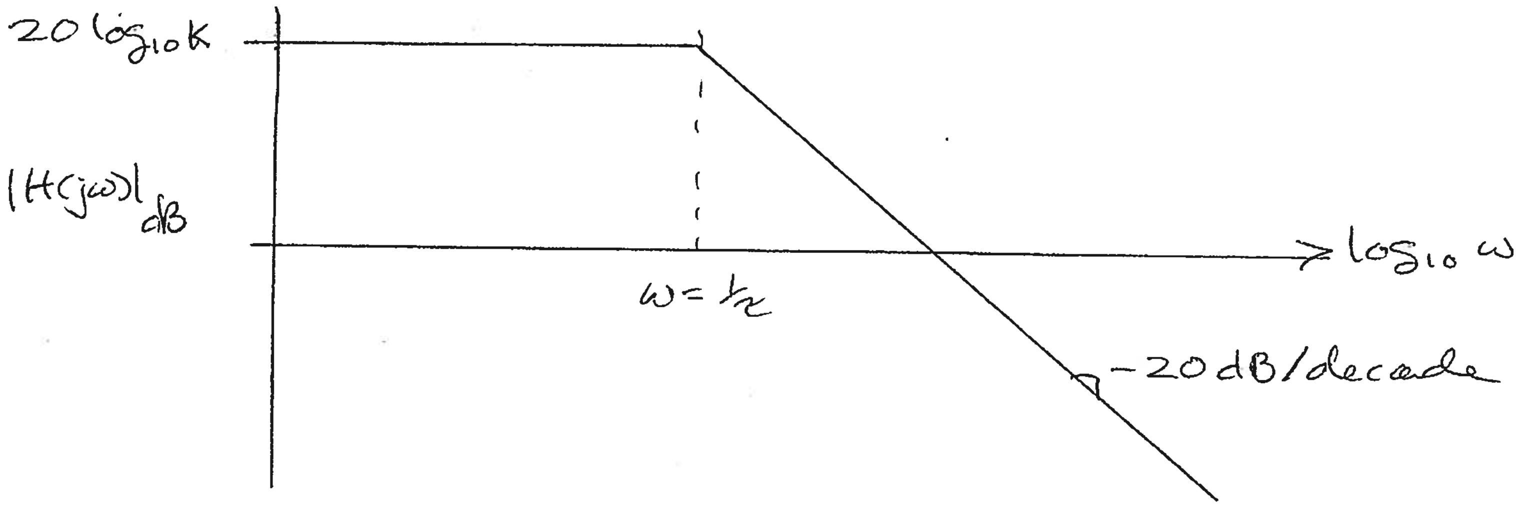
(constant)

  - for $\omega >> 1/\tau$,

$$|H(j\omega)|_{dB} \sim 20\log_{10}K - 20\log_{10}\omega\tau$$

$$= 20\log_{10}K + 20\log_{10}1/\tau - 20\log_{10}\omega$$

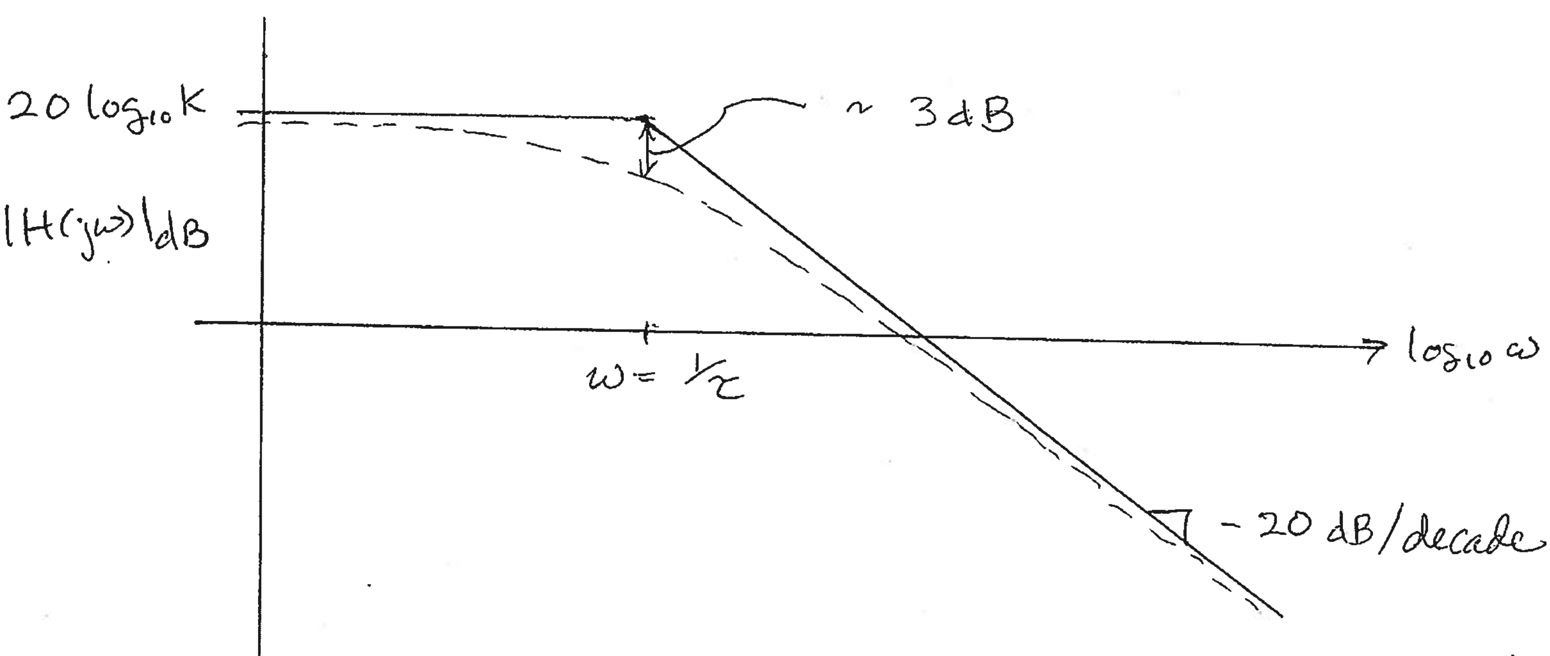
(straight line, with slope of -20 dB / decade; intersects low-frequency asymptote when $\omega = 1/\tau$).



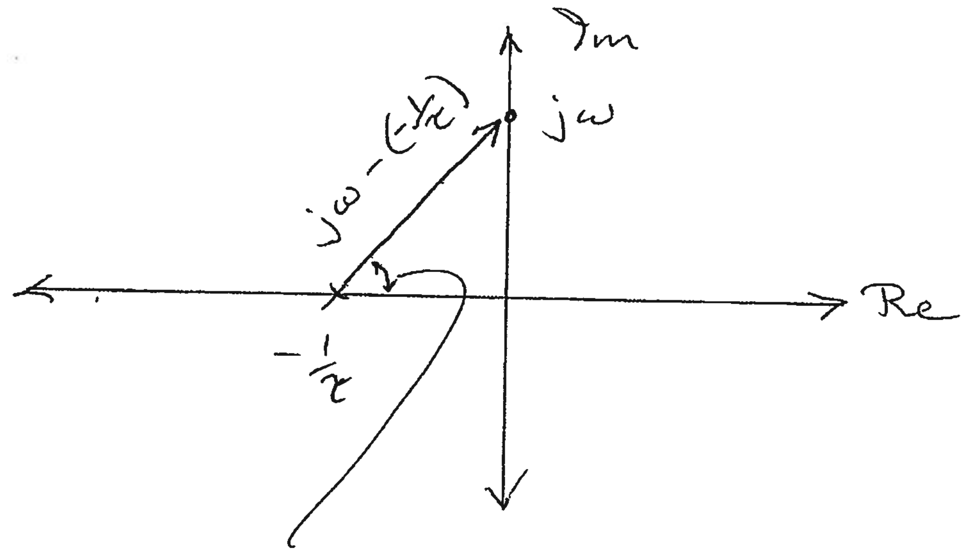
→ 'piecewise - linear' approximation of true curve.

- Where does true curve lie when $\omega = \frac{1}{\tau}$?

$$|H(j\omega)|_{dB} = 20\log_{10} K - 20\log_{10}\sqrt{2}$$
$$\approx 20\log_{10} K - 3\text{ dB}$$



- to see the shape of the <u>phase</u> curve, it may help to visualize $\angle(j\omega - (-1/\tau))$



$\angle(j\omega - (-1/\tau))$

- as ω varies from 0 to $+\infty$,

  $\angle(j\omega - (-1/\tau))$ varies from 0 to $90°$

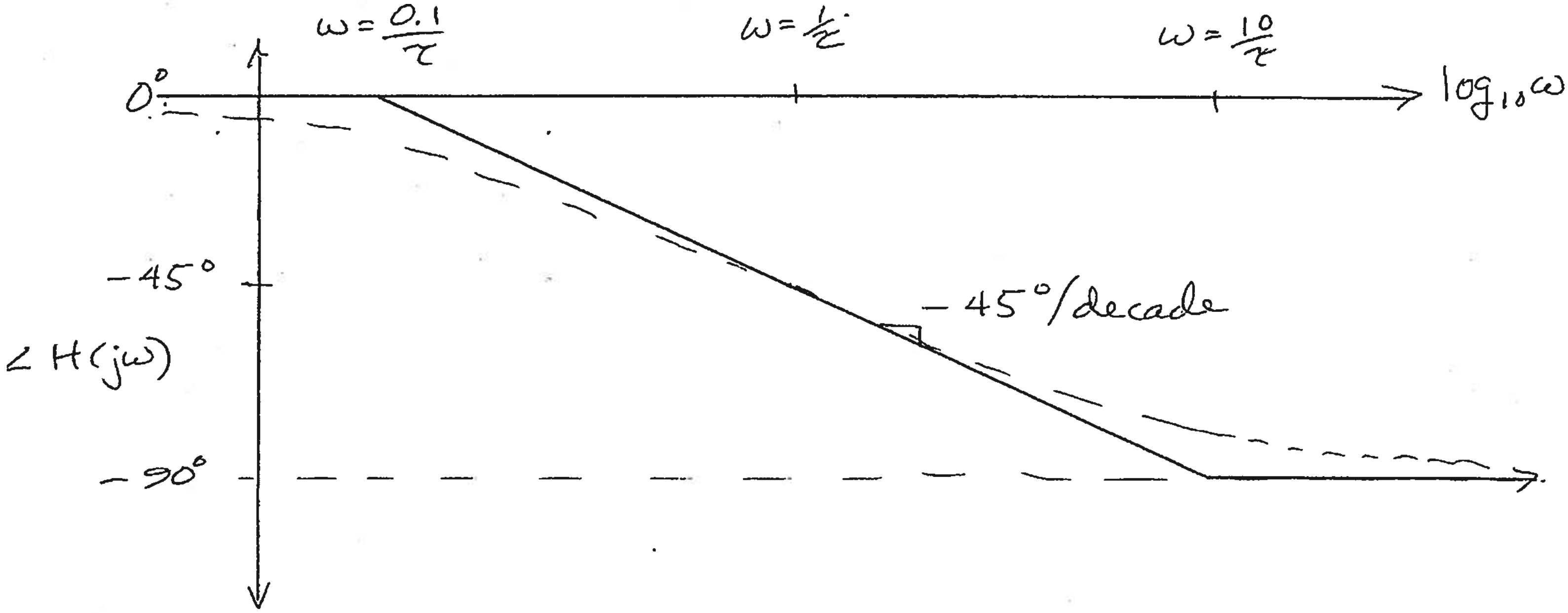
  $\angle H(j\omega)$ varies from 0 to $-90°$

- the exact value of $\angle(j\omega - (-1/\tau))$

  is

$$\operatorname{Tan}^{-1} \frac{\omega}{1/\tau}$$

... but, as with the magnitude curve, we'll use a "piecewise-linear" approximation:



$\omega = \frac{0.1}{\tau}$
$\omega = \frac{1}{\tau}$
$\omega = \frac{10}{\tau}$
0°
$\log_{10}\omega$
-45°
-45°/decade
$\angle H(j\omega)$
-90°

What is the Bode plot for

$$G(s) = \frac{s\tau + 1}{K} \quad ?$$

Ans:

$$|G(j\omega)|_{dB} = -|H(j\omega)|_{dB}$$

$$\angle G(j\omega) = -\angle H(j\omega),$$

so just "flip" the preceding curves.

2nd - order : $H(s) = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

freq. resp.:

$$H(j\omega) = \frac{\omega_n^2}{(\omega_n^2 - \omega^2) + 2j\zeta\omega_n\omega}$$

"gain"

$$|H(j\omega)|_{dB} = 20\log_{10}\omega_n^2 - 2\log_{10}\sqrt{(\omega_n^2-\omega^2) + 4\zeta^2\omega_n^2\omega^2}$$

"phase"

$$\angle H(j\omega)|_{dB} = \angle\omega_n^2 - \angle[(\omega_n^2-\omega^2) + 2j\zeta\omega_n\omega]$$

$$= 0^\circ - \mathrm{Tan}^{-1}\frac{2\zeta\omega_n\omega}{(\omega_n^2-\omega^2)}$$

- asymptotes:

$\omega \to 0$:

$$|H(j\omega)|_{dB} \xrightarrow{\omega \to 0} 0 \text{ dB}$$ (dc gain in dB)

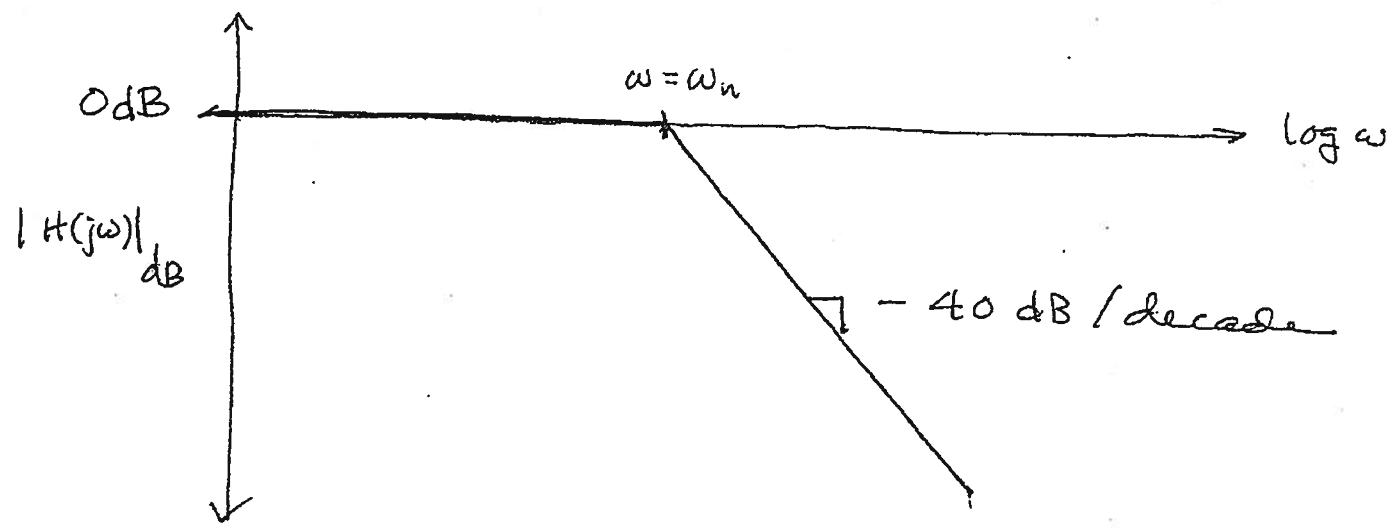
$$\angle H(j\omega) \xrightarrow{\omega \to 0} 0°$$

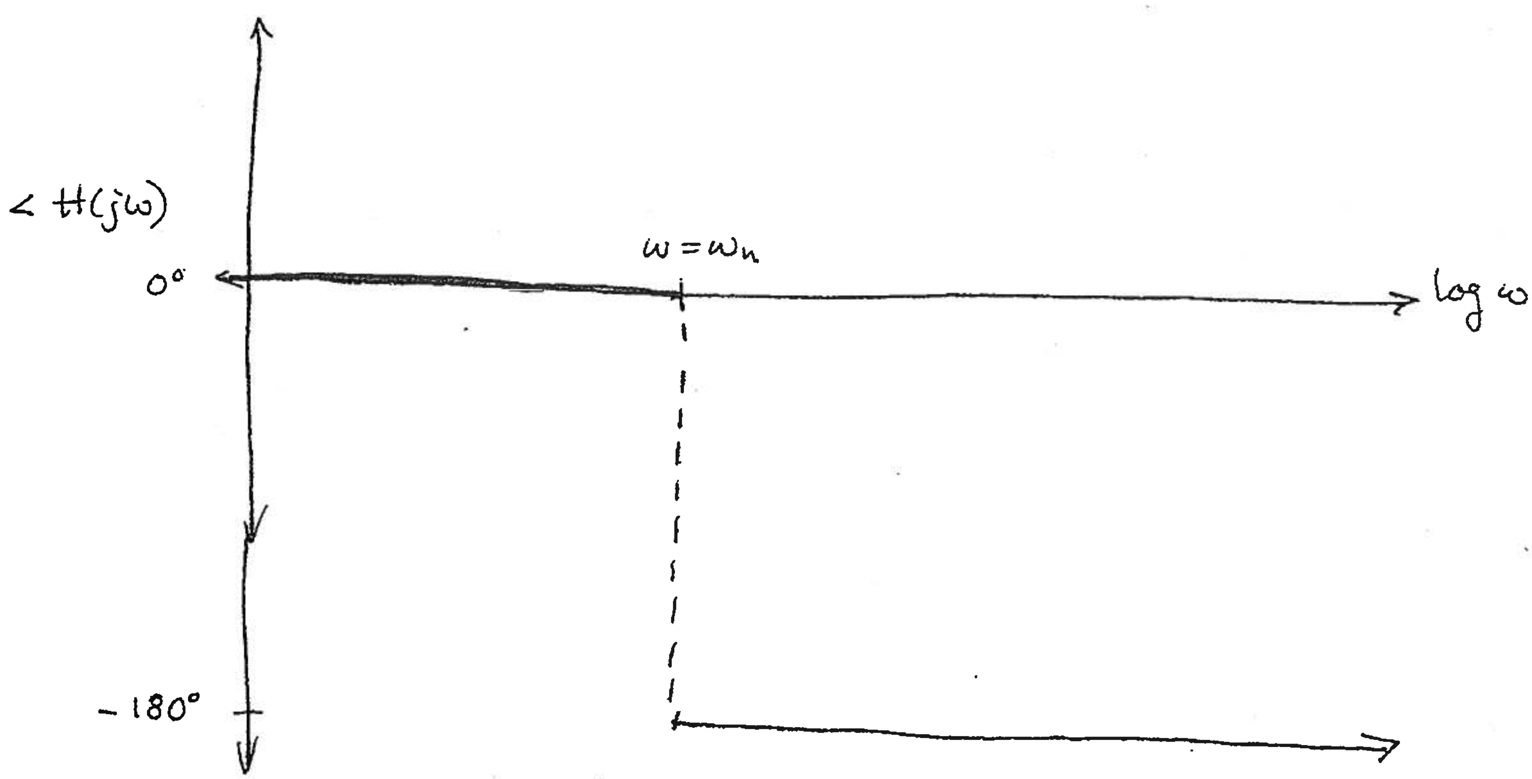
$\omega \to \infty$:

$$|H(j\omega)|_{dB} \longrightarrow \left|\frac{\omega_n^2}{\omega^2}\right|_{dB}$$

$$= 40 \log_{10} \omega_n - 40 \log_{10} \omega$$

$$\angle H(j\omega) \longrightarrow \angle -\frac{\omega_n^2}{\omega^2} = -180°$$


0dB
|H(jω)|dB
ω = ωn
log ω
−40 dB / decade


∠H(jω)
0°
ω = ωn
log ω
−180°

- The exact curves depend on ζ . . .

. . . if $\zeta < \frac{1}{\sqrt{2}}$, the gain curve exhibits a peak value of

$$|H(j\omega)| = M_p = \frac{1}{2\zeta\sqrt{1-\zeta^2}},$$

which occurs at a frequency

$$\omega_p = \omega_n \sqrt{1-2\zeta^2}$$

. . . as ζ decreases, the phase change becomes more abrupt.

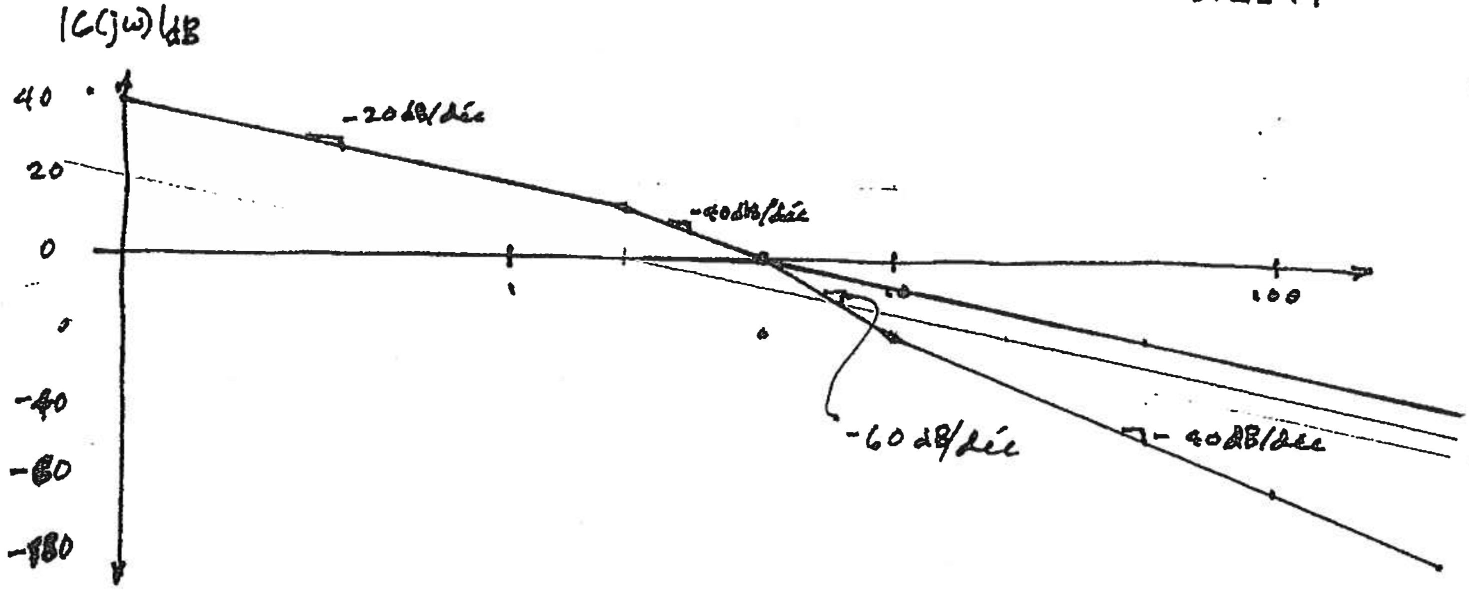
Example:

$$G(s) = \frac{10\ (s+10)}{s\ (s+2)(s+5)}$$

$$= \frac{1}{s} \cdot \frac{2}{s+2} \cdot \frac{5}{s+5} \cdot \frac{s+10}{1}$$

$$= G_1(s) \cdot G_2(s) \cdot G_3(s) \cdot G_4(s)$$

où $G_1(s) = \frac{1}{s}$, $G_2 = \frac{1}{0.5s+1}$, $G_3(s) = \frac{1}{0.2s+1}$, $G_4(s) = \frac{0.1}{0.}$



$$\left(|G(j\omega)|_{dB} = |G_1(j\omega)|_{dB} + |G_2(j\omega)|_{dB} + |G_3(j\omega)|_{dB} + |G_4(j\omega)|_{dB} \right)$$