University of Waterloo

# University of Waterloo
# Final Examination

Term: Fall Year: 2002

**Student Name**

**UW Student ID Number**

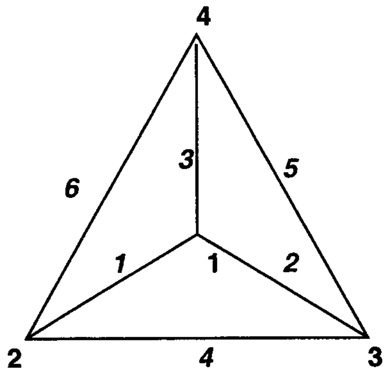
| | |
|---|---|
| Course Abbreviation and Number | CS370 |
| Course Title | Numerical Computation |
| Section(s) | 01 and 02 |
| Sections Combined Course(s) | |
| Section Numbers of Combined Course(s) | |
| Instructor | D Mohaplova and B Simpson |

| | |
|---|---|
| Date of Exam | Monday Dec 16 |
| Time Period | Start time: 9:00 am End time: noon |
| Duration of Exam | 3 hours |
| Number of Exam Pages (including this cover sheet) | |
| Exam Type | |
| Additional Materials Allowed | printed course notes, one text book, hand calculators |

**Marking Scheme:**

| Question | Score | Question | Score |
|---|---|---|---|
| 1 | /2 | 6 | /11 |
| 2 | /9 | 7 | /6 |
| 3 | /12 | 8 | /7 |
| 4 | /10 | 9 | /7 |
| 5 | /11 | 10 Exam Total | /75 |

1. **(2 marks)** Consider a circuit with the graph representation and general equations




$$Isum = \bar{A}\bar{V}$$

for

$$\bar{A} = \begin{pmatrix} 1.5 & -.5 & -.75 & -.25 \\ -.5 & 0.9 & -.1 & -.3 \\ -.75 & -.1 & 1.05 & -.2 \\ -.25 & -.3 & -.2 & .75 \end{pmatrix}$$

Consider the scenario in which

- node 1 has voltage 5 volts
- nodes 2,3, and 4 have $ISum_k = -30\ amps$ , $k = 2, 3, 4$

Write out the coefficient matrix for the equations for unknown voltages $V_2$ $V_3$ and $V_4$.

2. **(9 marks)** A floating point number system can be represented by four integers $(\beta, t, L, U)$ such that a floating point number $x$ has the form

$$x = \pm .b_1 b_2 \cdots b_t \times \beta^e.$$

(a) If $\beta = 10$, what are the smallest values of $t$ and $U$, and the largest value of $L$ such that $x_1 = 76.12$ and $x_2 = 0.000456$ can be represented exactly.

(b) For the floating point number system in part (a), what is the machine epsilon $E$.

(c) Let $x_3 = 76.11$. The following two expressions in real arithmetic:

$$exp1 = (x_1 + x_2) - x_3 \; ; \; exp2 = (x_1 - x_3) + x_2$$

are equivalent in the sense that they produce the same result for any $x_1, x_2, x_3$.
If they are implemented using the floating point number system in part (a), and evaluated with the assigned values of $x_k$, we get

$$\begin{aligned} ans1 &= (x_1 \oplus x_2) \ominus x_3 = 0.01000 \\ ans2 &= (x_1 \ominus x_3) \oplus x_2 = 0.01046 \end{aligned}$$

i. Compute the errors in using the FPNS arithmetic of part (a) in the *first* operation of $exp1$, and $exp2$.

ii. Which of $ans1$ or $ans2$ is a more accurate answer?

iii. Estimate the number of floating point numbers between $ans1$ and $ans2$.

3. **(12 marks) Parametric Curves** The parametric curve in Figure 1 has been interpolated from data points in Table 1.

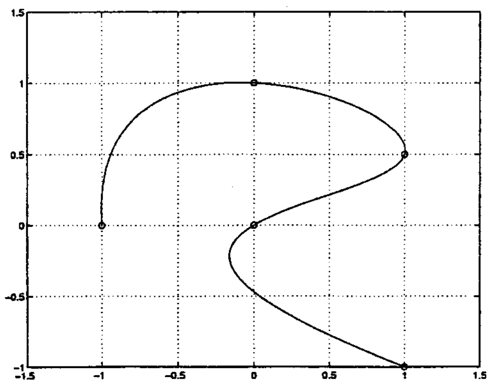



Figure 1

| i | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $t_i$ | 0 | 0.25 | 0.5 | 0.75 | 1 |
| $x_i$ | -1 | 0 | 1 | 0 | 1 |
| $y_i$ | 0 | 1 | 0.5 | 0 | -1 |

(1)

(a) Assume that the interpolation has been done using **interpolating polynomials** the form:

$$px(t) = ax_1 + ax_2t + ax_3t^2 + \cdots + ax_nt^{n-1}$$

$$py(t) = ay_1 + ay_2t + ay_3t^2 + \cdots + ay_nt^{n-1}$$

i.e. polynomials in monomial form with coefficients $ax_k$ for $px(t)$ and $ay_k$ for $py(t)$.

i) What is the value of $n$ required for $px(t)$ and $py(t)$ to interpolate the data in the Table (1)?

ii) The coefficients $a = (a_1, a_2 ... a_n)$ that are required for each $p(t)$ can be computed by solving a linear system of equations $Va = b$.
Write out the coefficient matrix of coefficients, $V$, for these equations.
You may express the matrix entries as powers of real numbers.

iii) Let the system of equations for the coefficients $ax = (ax_1, ax_2...ax_n)$ of $px(t)$ be $Vax = bx$, and similarly for $ay$. What are the right hand side vectors, $bx$, $by$?

iv) Write an efficient Matlab computation of the matrix of coefficients, $[ax, ay]$ from $V$ , $bx$, and $by$ of parts ii) and iii)

(b) You are to compute and evaluate a cubic spline parametric curve that interpolates the data in the table.

i) Write the MATLAB computation to compute the ppform of the interpolating cubic spline for the data $(t_i, x_i), i = 1, \ldots, 5$ of the Table (1). Use the default spline boundary conditions. Repeat for the data $(t_i, y_i), i = 1, \ldots, 5$.

ii) Write a simple MATLAB script that evaluates the cubic spline parametric curve constructed in part b i) at a given array of evaluation points, $tref$, and plots the curve.

4. **(10 marks)** Let $A$ be a $N \times N$ matrix and let $L$ and $U$ be the triangular factors of $PA$, for $P, L, U$ computed by Gaussian elimination with partial pivoting.

(a) Mark, by circling, the following statements as True of False **if $A$ is a singular matrix**

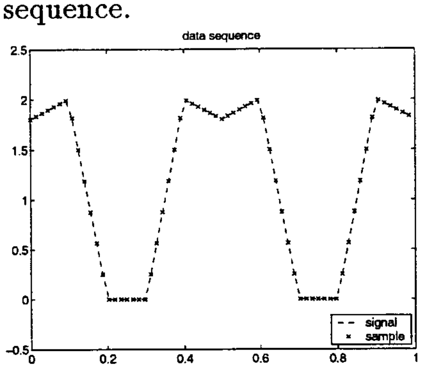
| | | |
|---|---|---|
| there is no matrix $P$ such that $PA = LU$ | True | False |
| $L_{k,k} = 0$ for at least one $k$ | True | False |
| $U$ is the zero matrix | True | False |

(b) Let $F$ be a floating point number system with about $t$ digits of accuracy, base $\beta = 10$. In other word, if $E$ is the machine epsilon for $F$, then $log_{10}(E) \approx -t$.
Let $T$ be the logarithm base 10 of a condition number of $A$.
You may assume that if we compute $P, L, U$ in the arithmetic of $F$, we are only guaranteed that $t - T$ digits of $U_{k,k}$ are correct, $k = 1 \ldots N$.

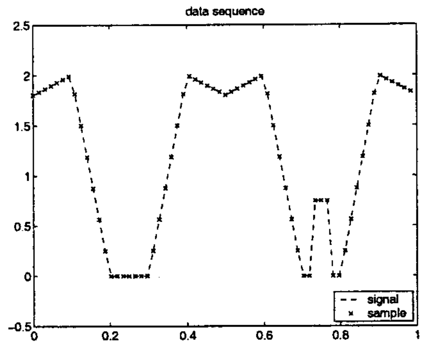
Write a Matlab function that takes as input a square array, `A`, and outputs `[U,m]` where

- `U` is the `N` vector of the computed $U_{k,k}$ values
- if $A$ is numerically singular, then `m = 0`, otherwise, `m` is the number of digits in the $U_{k,k}$ guaranteed to be correct.

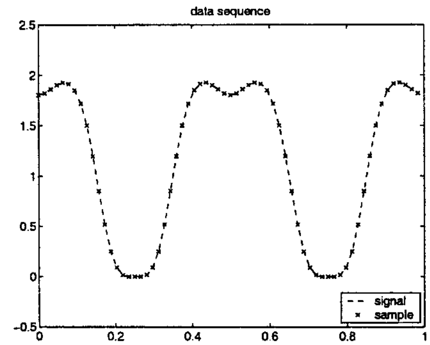
5. **(11 marks)** In this questions you are to match the following three data sequences (marked A), B), C)) with three of the four graphs of the sizes of coefficients in the Fourier transform of some data sequence.




A)



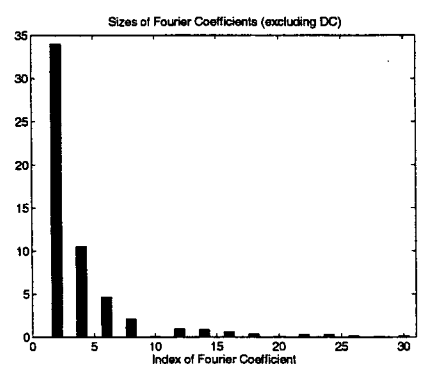

B)




C)

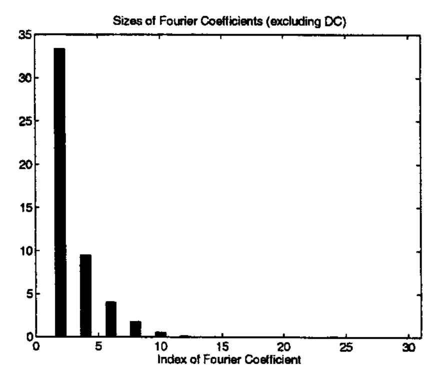
a) Each of the following graphs shows the size of the first 32 Fourier coefficients of some data sequence - excluding $F_0$, the D-C coefficient. In other words, they show the power spectrum of the Fourier transform of some data sequence For each one, if you think it is the power spectrum of one of the data sequences above, then circle the letter for that sample. Otherwise, circle 'none'.
On the following page, give a brief justification for your choice.

Spectrum 1:



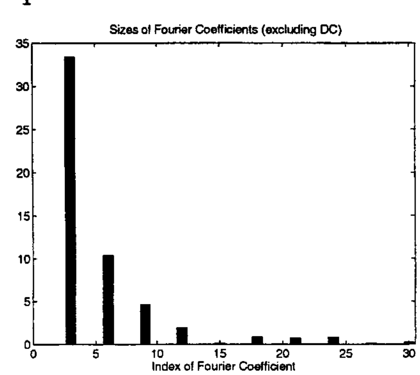

A) B) C) none

Spectrum 2:



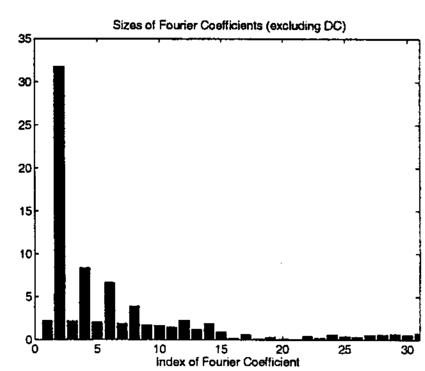

A) B) C) none

Spectrum 3:




A) B) C) none

Spectrum 4:




A) B) C) none

*Continued*

b) For each of your decisions in part a), give a brief justification.

**Spectrum 1**

**Spectrum 2**

**Spectrum 3**

**Spectrum 4**

6. **(11 marks)**

Let $N$ be an even integer and let $W = e^{i2\pi/N}$ be the basic $N$th root of unity. Let $\{f_n\}, n = 0 \dots N-1$ be a sequence of real numbers. Let $\{F_n\}, n = 0 \dots N-1$ be the Fourier transform of $\{f_n\}$, computed by

$$F_k = \sum_{n=0}^{N-1} f_n \bar{W}^{nk} \tag{2}$$

(a) Show that $W^{N/2} = -1$

(b) Show that $F_{N-k} = \bar{F}_k$ for $1 \leq k \leq N/2 - 1$

(c) Consider computing all the Fourier coefficients of even index, $\{F_{2k}\}$, by formula (2). Assume all the complex numbers $\bar{W}^{nk}$ have already been computed. The number of real arithmetic multiplications required has the form

$$C\,N^2 + O(N).$$

What is the value of $C$? Show your work.

(d) Write out another formula for computing $\{F_{2k}\}$ that is more efficient. What is the approximate number of real arithmetic multiplications required for this formula?

7. **(6 marks)** This question concerns termination events for IVPs. It is based on the novelty golf driving range model. We first review the notation for this model.

The modeling description uses a $x$ axis for the horizontal extent of the ball's trajectory on the driving range, and a $y$ axis for the vertical direction above the driving range. The ball trajectory, $(x(t), y(t))$, is assumed to stay in this plane.

The state variables of the standard first order form are:

$$z_1 \equiv x \quad z_2 \equiv y \quad z_3 \equiv dy/dt$$

Draw a straight line from the word description of a termination event for the ball trajectory on the left to an appropriate event function on the right.

**Evaluation of $E(t, z)$**

1) if $z_1 > 100$ and $z_2 < 5$ then 1 else -1

A) the ball hits the ground

2) $z_3$

B) the ball is 100 feet from the start

3) $100 - z_2$

4) $z_2$

C) the ball reaches it highest point

5) $\sqrt{z_1^2 + z_2^2} - 100$

D) the ball hits a fence 5 feet high and 100 feet from the starting point

6) if $|z_1 - 100| < .1$ and $z_2 < 5$ then 1 else -1

8. **(7 marks)** Consider a trajectory $(x(t), y(t))$ determined by modeling equations

$$\begin{aligned} dx(t)/dt &= 1 - (x^2(t) + y^2(t)) && (3) \\ dy^2(t)/dt^2 &= 2(y(t) - x(t)) && (4) \end{aligned}$$

$x$ and $y$ are measured in meters.

(a) Identify and write down a systems dynamics function, $f(t, z)$, for a system of ordinary differential equations in standard first order form for model equations (3).

(b) Write a terminal event function for terminating an IVP involving $f(t, z)$ when $y(t) > 2$.

(c) Write an odefile that informs a Matlab time stepping procedure about the systems dynamics function, $f(t, z)$ and the terminal event of part b).

9. **(7 marks)** Consider the initial value problem for standard first order form with $dy/dt = f(t, y)$ for $f(t, y) = 2t - y$ and initial conditions $t_0 = 1$ ; $y(t_0) = 1.5$.

(a) Compute two steps of of the time stepping solution by Euler's method , $y^{(1)}, y^{(2)}$, using constant step size $h = .2$

(b) The following simplified time stepping method computes an approximate solution, $(t_k, y^{(k)}), k = 0 \cdots N$ to an initial value problem.
It takes a fixed number $= N$ time steps. It uses an unspecified method to update $y^{(k)}$.
At each time step, it uses the local error at that time step to choose a suitable time step for the *next* step.

**initialize** $t_0, h_0, y^{(0)}$
**for** $k$ **= 1 to** $N$
  **compute** $y^{(k)}$ **from** $t_{k-1}, y^{(k-1)}, h_{k-1}$
  $t_k = t_{k-1} + h_{k-1}$
  **update** $h_k$
**end for**

To update $h_k$, the method computes an estimate, $ErrEst$, of the size of the local error at the $k$th step and compares it to an absolute error tolerance, $tol$. So this update has the form

**if** $ErrEst > tol$
  **then A**
  **else B**
**end if**

Indicate how you would choose $h_k$ in cases A and B.

*Continued*