My grades for Assignment 1

Q1

10 / 10

1 Taylor series approximation [10pts]

Show that $(1+x)^{\frac{p}{q}}$, where p and q are integers, can be approximated as $1+\frac{p}{q}x$ for small x.

MATH 213 - Assignment 1

Bilal Khan
bilal.khan1@uwaterloo.ca

January 18, 2023

1 Question 1

$$f(x) = (1+x)^{p/q}$$

For small x ($x << 1$), we can use the Taylor series expansion of $f(x)$ to approximate $f(x)$:

$$f(x_0 + a) = f(x_0) + a \cdot f'(x_0)$$
$$f(0 + x) = f(0) + x \cdot f'(0)$$
$$f(0) = (1+0)^{p/q} = 1$$
$$f'(x) = \frac{p}{q}(1+x)^{p/q-1}$$
$$f'(0) = \frac{p}{q}(1+0)^{p/q-1} = \frac{p}{q}$$
$$f(x) \approx 1 + x \cdot \frac{p}{q}$$

1

Q2

20 / 20

2 Oscillating electron [20pts]

An electron is placed in the middle of an imaginary line connecting two fixed negative charges, each with magnitude Q, positioned at a distance $2a$ from each other.
a) Write down the differential equation that would describe the displacement $x(t)$ of the electron from its initial position as a function of time. Assume the displacement is along the line connecting the two fixed charges Q.
b) Show that for small enough displacements (specify what you mean by small), the force on the electron will scale linearly with its displacement and thus the electron would behave like a harmonic oscillator (mass on a spring) if you displace it and then release.
c) What would be the angular frequency of the electron's oscillation in part b)?

---

2 Question 2

2.1 (a)

Let Q_e be the charge of the electron, and we know that the force on the electron from a charge is given by the Couloumb's law: $F = \frac{cQQ_e}{r^2}$, where c is the Coulomb constant. Assume that the electron is placed location $x(t)$, the distance from each charge is then $a + x(t)$ and $a - x(t)$, respectively. We know that the forces on the electron from each charge act in opposite directions, and the net force is given by $F = ma$, so we can write:

$$F = ma = \frac{cQQ_e}{(a + x(t))^2} - \frac{cQQ_e}{(a - x(t))^2}$$
$$a = \frac{cQQ_e}{(a + x(t))^2} - \frac{cQQ_e}{(a - x(t))^2}$$

We can see that $a = \frac{d^2x}{dt^2}$, so we can write our differential equation as:

$$\frac{d^2x}{dt^2} = \frac{cQQ_e}{(a + x(t))^2} - \frac{cQQ_e}{(a - x(t))^2}$$

2.2 (b)

For small displacements, i.e. where $t << 1$, and assuming that $y = x(t) << 1$ for small t, we can approximate $(a \pm y)^{-2}$ with its Taylor series expansion:

$$\begin{aligned}
f(y) &= (a + y)^{-2} \\
f(0) &= (a + 0)^{-2} = a^{-2} \\
f'(y) &= -2(a + y)^{-3} \\
f'(0) &= -2(a + 0)^{-3} = -2a^{-3} \\
f(0 + x(t)) &\approx f(0) + x(t) \cdot f'(0) = a^{-2} - 2a^{-3}x(t)
\end{aligned}$$

$$\begin{aligned}
g(y) &= (a - y)^{-2} \\
g(0) &= (a - 0)^{-2} = a^{-2} \\
g'(y) &= -2(a - y)^{-3}(-1) \\
g'(0) &= -2(a + 0)^{-3}(-1) = 2a^{-3} \\
g(0 + x(t)) &\approx g(0) + x(t) \cdot g'(0) = a^{-2} + 2a^{-3}x(t)
\end{aligned}$$

2

$$
\begin{aligned}
\frac{d^2x}{dt^2} &= \frac{cQQ_e}{(a+x(t))^2} - \frac{cQQ_e}{(a-x(t))^2} \\
&\approx cQQ_e(a^{-2} - 2a^{-3}x(t)) - cQQ_e(a^{-2} + 2a^{-3}x(t)) \\
&= cQQ_e(a^{-2} - 2a^{-3}x(t) - a^{-2} - 2a^{-3}x(t)) \\
&= cQQ_e(-2a^{-3}x(t) - 2a^{-3}x(t)) \\
&= cQQ_e(-4a^{-3}x(t)) \\
&= -(4cQQ_e a^{-3})x(t)
\end{aligned}
$$

$$
\begin{aligned}
F &= m\frac{d^2x}{dt^2} \\
&= -(4cQQ_e a^{-3}m)x(t)
\end{aligned}
$$

$k = 4cQQ_e a^{-3}m$, which is a constant, so we can see that the force will scale linearly with the displacement and the electron behaves like a harmonic oscillator.

2.3 (c)

The equation for the angular frequency of a harmonic oscillator is given by $\omega = \sqrt{\frac{k}{m}}$, so we can write:

$$
\omega = \sqrt{\frac{-(4cQQ_e a^{-3})m}{m}} = \sqrt{-4cQQ_e a^{-3}}
$$

Where Q is the magnitude of the charge, Q_e is the charge of the electron, a is one half of the distance between the charges, and c is the Coulomb constant.

3

Q3

20 / 20

3 Circumference of a circle [20 pts]

Use the formula for path length introduced in class to calculate the circumference of a circle with radius r=1m.
Hint: A circle can be described as a set of points that have a constant distance from the origin: $x^2 + y^2 = r^2$

3 Question 3

Since the start and ending points of y are the same, we calculate the path length of the circle in two parts; The part above the x-axis (from $x = 1$ to $x = -1$) and the part below the x-axis (from $x = -1$ to $x = 1$).

$$S = -\int_{1}^{-1} \sqrt{1 + (f'(x))^2} dx + \int_{-1}^{1} \sqrt{1 + (f'(x))^2} dx$$

$$\begin{aligned} r^2 &= x^2 + y^2 \\ 1 &= x^2 + y^2 \\ y &= \pm\sqrt{1 - x^2} \end{aligned}$$

$$y' = \frac{dy}{dx} \pm (1 - x^2)^{\frac{1}{2}} = \frac{\pm x}{\sqrt{1 - x^2}}$$

The domain of y is positive from $x = 1$ to $x = -1$ and negative from $x = -1$ to $x = 1$, so we use the appropriate y values in computing the antiderivative for the path length.

$$\begin{aligned} \int \sqrt{1 + (f'(x))^2} dx &= \int \sqrt{1 + \left(\frac{\pm x}{\sqrt{1 - x^2}}\right)^2} dx \\ &= \int \sqrt{1 + \frac{x^2}{1 - x^2}} dx \\ &= \int \sqrt{\frac{1 - x^2}{1 - x^2} + \frac{x^2}{1 - x^2}} dx \\ &= \int \sqrt{\frac{1}{1 - x^2}} dx \\ &= \int \frac{1}{\sqrt{1 - x^2}} dx \\ &= \sin^{-1}(x) + C \end{aligned}$$

$$\begin{aligned} S &= -\int_{1}^{-1} \frac{1}{\sqrt{1 - x^2}} dx + \int_{-1}^{1} \frac{1}{\sqrt{1 - x^2}} dx \\ &= -\sin^{-1}(x)\big|_{1}^{-1} + \sin^{-1}(x)\big|_{-1}^{1} \\ &= -\left(\sin^{-1}(-1) - \sin^{-1}(1)\right) + \left(\sin^{-1}(1) - \sin^{-1}(-1)\right) \\ &= -\left(-\frac{\pi}{2} - \frac{\pi}{2}\right) + \left(\frac{\pi}{2} - -\frac{\pi}{2}\right) \\ &= 2\pi \end{aligned}$$

4

Q4

25 / 30

4 Carbon dating [30 pts]

Radiocarbon dating (also referred to as carbon dating) is a method for determining the age of an object containing organic material by using the properties of a radioactive isotope of carbon, ^{14}C. It is based on the fact that ^{14}C is constantly being created in the Earth's atmosphere by the interaction of cosmic rays with atmospheric nitrogen. The resulting ^{14}C combines with atmospheric oxygen to form radioactive carbon dioxide, which is incorporated into plants by photosynthesis; animals then acquire ^{14}C by eating the plants. When the animal or plant dies, it stops exchanging carbon with its environment, and thereafter the amount of ^{14}C it contains begins to decrease as the ^{14}C undergoes radioactive decay. Measuring the amount of ^{14}C in a sample from a dead plant or animal, such as a piece of wood or a fragment of bone, provides information that can be used to calculate when the animal or plant died. The older a sample is, the less ^{14}C there is to be detected because the half-life of ^{14}C (the period of time after which half of the ^{14}C inside a given sample will have decayed) is about 5,730 years. (Source: wikipedia.org)
a) Assuming the number N of ^{14}C atoms in a sample decays exponentially, it can be described with differential equation of the form $\frac{dN}{dt} = -\lambda N(t)$. Find the value of λ (make sure to specify the units) based on the half-life of ^{14}C being 5,730 years.
b) The abundance of ^{14}C in the atmosphere is about 1 atom of ^{14}C per 10^{12} carbon atoms. Modify the differential equation from part a, so that it describes the number of ^{14}C atoms in the atmosphere (as opposed to in a sample). Describe the role of the new term(s) added to the original equation and estimate the value(s) of the constants in these terms (make sure to specify their units). You can assume that out of every 1,000,000 molecules of air, roughly 781,000 are N_2, 209,000 are O_2, and (at least historically) 300 are CO_2.
c) Estimate the abundance of ^{14}C (how many ^{14}C per 10^{12} carbon atoms) in a parchment from 3rd century BCE, such as the Dead Sea Scrolls.

---

4 Question 4

4.1 (a)

$$\begin{aligned}
\frac{dN}{dt} &= -\lambda N(t) \\
\frac{dN}{N} &= -\lambda dt \\
\int \frac{dN}{N} &= -\int \lambda dt \\
\ln N + C_1 &= -\lambda t + C_2 \\
\ln N &= -\lambda t + C_3 \\
N &= e^{-\lambda t + C_3} \\
N(t) &= Ce^{-\lambda t} \\
N(0) &= ce^{-\lambda \cdot 0} = c
\end{aligned}$$

Using our values for the half-life of Carbon-14, we have that if we originally have 1 unit of carbon-14,

$$\begin{aligned}
N(0) &= 1 = c \\
N(5730) &= 0.5 \\
\frac{1}{2} &= 1 \cdot e^{-\lambda \cdot 5730} \\
\ln \frac{1}{2} &= -\lambda \cdot 5730 \\
\lambda &= \frac{-\ln \frac{1}{2}}{5730}
\end{aligned}$$

The value of λ is $\frac{-\ln \frac{1}{2}}{5730}$ with units of years.

4.2 (b)

We know that the abundance of carbon atoms in the atmosphere has remained constant (until recent times wrt global warming and such) at about 300 atoms of carbon (in the form of CO2) for every million molecules of air and nitogen being at about 7.81×10^5 molecules per 1 million molecules of air. We also know that the abundance of carbon 14 in the atmosphere is 1 atom of carbon-14 per 10^{12} atoms of carbon. We are given that carbon 14 is formed in the atmosphere by cosmic rays breaking down atmospheric nitrogen. We can use the fact that the abundance of carbon 14 in the atmosphere is constant to assume that the differential equation describing the abundance of carbon 14 is at steady state with the decay of carbon 14 being balanced by the formation of carbon 14.

$$\frac{dN}{dt} = -\lambda N(t) + b = 0$$

5

4.3 (c)

We know from part (a) that $\lambda = \frac{-\ln\frac{1}{2}}{5730}$. It has been 2323 years since 300 BCE and we know that the abundance of carbon-14 is 1 atom of carbon-14 per 10^{12} atoms of carbon. Applying the equation that we got in part (a) we have that:

$$\begin{aligned} N(t) &= Ce^{-\lambda t} \\ N(0) &= 1 = c \\ N(2323) &= 1 \cdot e^{-\lambda \cdot 2323} \\ &= 1 \cdot e^{\ln 1/2 * 2323/5730} \\ &= 0.755 \end{aligned}$$

The abundance of carbon-14 in the parchment today is 0.755 atoms of carbon-14 per 10^{12} atoms of carbon.

Q5

20 / 20

The amazon problem
The two ends of a cable that is
80 meters long are anchored
on two poles sticking out from

a lake (each end on a separate pole). Each anchoring point is 50 meters above the surface of the water. What is the distance between the two poles (to one decimal point) if the centre of the cable is
(a) 20 meters off the ground and
(b) 10 meters off the ground?
Solve this problem using the formulas derived in lecture 3 (you do not have to re-derive and solve the whole differential equation but can just start with the result that the lecture arrived at).

---

5 Question 5

5.1 (a)

From the lecture we are given that the height of the rope can be modeled by a catenary with $y(x) = a\cosh(x/a) + b$. Taking $x = 0$ to be the location of the center of the cable above the ground and d to be the distance between the two poles, we have that:

$$y(0) = 20 = a\cosh(0) + b = a + b$$

$$\begin{aligned} y(d/2) = 50 &= a\cosh(d/2a) + b \\ 50 &= a\cosh(d/2a) + 20 - a \end{aligned}$$

The cable is 80m long and so the center of the cable is at the 40m mark. We can connect the path length defined by $y(x)$ to pillar spacing to get the distance between the two poles as two times the value of x such that the path length is 40m. We use the hyperbolic trignometric identity $\cosh^2(x) - \sinh^2(x) = 1$.

$$y'(x) = \sinh(x/a)$$

$$\begin{aligned} \cosh^2(x) - \sinh^2(x) &= 1 \\ 1 + \sinh^2(x) &= \cosh^2(x) \end{aligned}$$

$$\begin{aligned} S = 40 &= \int_0^{d/2} \sqrt{1 + (f'(x))^2}dx \\ &= \int_0^{d/2} \sqrt{1 + \sinh^2(x/a)}dx \\ &= \int_0^{d/2} \sqrt{\cosh^2(x/a)}dx \\ &= \int_0^{d/2} \cosh(x/a)dx \\ &= a\sinh(x/a)\Big|_0^{d/2} \\ &= a\sinh(d/2a) - a\sinh(0) \\ 40 &= a\sinh(d/2a) \\ \sinh(d/2a) &= \frac{40}{a} \end{aligned}$$

7

$$
\begin{aligned}
y(0) = 20 &= a + b \\
b &= 20 - a
\end{aligned}
$$

$$
\begin{aligned}
y(d/2) = 50 &= a\cosh(d/2a) + b \\
50 &= a\cosh(d/2a) + 20 - a \\
30 + a &= a\cosh(d/2a) \\
\cosh(d/2a) &= \frac{30 + a}{a}
\end{aligned}
$$

$$
\begin{aligned}
\cosh^2(x) - \sinh^2(x) &= 1 \\
\cosh^2(d/2a) - \sinh^2(d/2a) &= 1 \\
\left(\frac{30 + a}{a}\right)^2 - \left(\frac{40}{a}\right)^2 &= 1 \\
\left(\frac{a^2 + 60a + 900}{a^2}\right) - \left(\frac{1600}{a^2}\right) &= 1 \\
a^2 + 60a - 700 &= a^2 \\
60a &= 700 \\
a &= 35/3
\end{aligned}
$$

We get that $a = 35/3$ and $b = 20 - 35/3 = 25/3$. We can then use these values to find the distance between the two poles.

$$
\begin{aligned}
y(d/2) = 50 &= (35/3)\cosh(d\frac{3}{2 \cdot 35}) + \frac{25}{3} \\
\frac{50 - 25/3}{\frac{35}{3}} &= 25/7 = \cosh(d\frac{3}{70}) \\
\cosh^{-1}(\frac{25}{7}) &= d\frac{3}{70} \\
d &\approx 45.4\text{m}
\end{aligned}
$$

5.2 (b)

Reusing our work from the previous section and noting that the height of the center is now 10m off the ground, we have that:

$$\begin{aligned} 10 &= a + b \\ b &= 10 - a \end{aligned}$$

$$\sinh(d/2a) = \frac{40}{a}$$

$$\begin{aligned} y(d/2) = 50 &= a\cosh(d/2a) + b \\ 50 &= a\cosh(d/2a) + 10 - a \\ 40 + a &= a\cosh(d/2a) \\ \cosh(d/2a) &= \frac{40+a}{a} \end{aligned}$$

$$\begin{aligned} \cosh^2(x) - \sinh^2(x) &= 1 \\ \cosh^2(d/2a) - \sinh^2(d/2a) &= 1 \\ \left(\frac{40+a}{a}\right)^2 - \left(\frac{40}{a}\right)^2 &= 1 \\ a^2 + 80a + 1600 - 1600 &= a^2 \\ 80a &= 0 \end{aligned}$$

If we try to solve this equation we will see that there is no solution. This is because the length of the cable is $2 * 40 = 80$m and we expect each path length to be 40m. If the center of the cable is 10m off the ground, then the vertical drop of the cable is the same as the path length, requiring the cable to go straight up and down without any travel in the x direction which would require both poles to overlap exactly which is not possible.

Q6

17 / 20

6 Extra credit: Swiss-cheese car [20 pts]

Make a list of measures and technologies layered in a modern car to prevent injury of its human occupant from a frontal collision. Group these measures and technologies according to their locations in the "Hierarchy of Controls" pyramid described in Lecture 3.

6 Question 6

Elimination of hazard

1. Drive less often
2. Avoid crowded highways
3. Obey speed limits

Engineering controls

1. Install rumble strips
2. Install guardrails
3. Install median barriers
4. Build cars with fewer blind spots

Administratice controls

1. Avoid driving at night
2. Avoid driving in bad weather
3. Avoid driving when tired
4. Avoid texting and driving

Provide PPE

1. Seatbelts
2. Airbags
3. Automatic emergency braking