SE 380 — HW 2

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September 28, 2023

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1.1 a

Considering the state space model

$$
\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{aligned}
$$

where $x \in \mathbb{R}^2$, $u \in \mathbb{R}^2$, and $y \in \mathbb{R}$, find values for A, B, C, D such that the corresponding transfer function is

$$G(s) = \frac{\mu}{1 + \tau s}$$

Taking the Laplace transform of the above equations, we get

$$
\begin{aligned}
sX(s) - x(0) &= AX(s) + BU(s) \\
Y(s) &= CX(s) + DU(s)
\end{aligned}
$$

Solving this, assuming initial state is zero, we get

$$
\begin{aligned}
sX(s) - x(0) &= AX(s) + BU(s) \\
sX(s) - AX(s) &= BU(s) \\
X(s)(sI - A) &= BU(s) \\
X(s) &= \frac{BU(s)}{sI - A} \\
Y(s) &= \left(\frac{CB}{sI - A} + D\right) U(s) \\
H(S) &= \frac{Y(s)}{U(s)} = \frac{CB}{sI - A} + D \\
\frac{CB}{sI - A} &= \frac{\mu}{1 + \tau s}
\end{aligned}
$$

We want $CB = \mu$, so one possible solution is

$$
C = \begin{bmatrix} \mu & 0 \end{bmatrix} B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
$$

The $1 + \tau s$ term can be made simpler by rearranging it to $s + \frac{1}{\tau}$, and we want the same $x_1(t)$ state value to change as in B so we want A to be

$$
A = \begin{bmatrix} -\frac{1}{\tau} & 0 \\ 0 & 0 \end{bmatrix}
$$

D is zero because we do not use it in the transfer function.

1.2 b

No, there are multiple ways to set up the state space matrices to get the same transfer function. For example, we could have chosen $x_2(t)$ to be the state variable we use and then choose C to be $\begin{bmatrix} 0 & \mu \end{bmatrix}$ and B to be $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and A to be $\begin{bmatrix} 0 & 0 \\ -\frac{1}{\tau} & 0 \end{bmatrix}$, and D to be zero. This would have given us the same transfer function.

2 2

2.1 a

Consider a first-order system with transfer function given by $H(s) = \dfrac{\mu}{1 + \tau s}$. Compute the response $y_1(t)$ to a step input $u_1(t) = H(t)$ and the response of $y_2(t)$ to a ramp input $u_2(t) = tHTt)$.

The laplace transform of $u_1(t)$ is $U_1(s) = \dfrac{1}{s}$ and the Laplace transform of $u_2(t)$ is $U_2(s) = \dfrac{1}{s^2}$

$$
\begin{aligned}
Y_1(s) &= H(s)U_1(s) \\
&= \frac{\mu}{1+\tau s}\frac{1}{s} \\
&= \frac{\mu}{s(1+\tau s)} \\
Y_2(s) &= H(s)U_2(s) \\
&= \frac{\mu}{1+\tau s}\frac{1}{s^2} \\
&= \frac{\mu}{s^2(1+\tau s)}
\end{aligned}
$$

We can compute the inverse Laplace transforms on their partial fraction decompositons.

$$
\begin{aligned}
Y_1(s) &= \frac{\mu}{s(1+\tau s)} \\
\frac{\mu}{s(1+\tau s)} &= \frac{A}{s} + \frac{B}{1+\tau s} \\
\mu &= A(1+\tau s) + Bs \\
\mu &= A && \text{(s = 0)} \\
\mu &= \mu(1+\tau s) + Bs \\
\mu &= \mu + \mu\tau s + Bs \\
0 &= s(\mu\tau + B) && (s \neq 0) \\
B &= -\mu\tau \\
Y_1(s) &= \frac{\mu}{s} - \frac{\mu\tau}{1+\tau s} \\
y_1(t) &= (\mu - \mu e^{-t/\tau})u_1(t)
\end{aligned}
$$

$$
\begin{aligned}
Y_2(s) &= \frac{\mu}{s^2(1+\tau s)} \\
\frac{\mu}{s^2(1+\tau s)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{1+\tau s} \\
\mu &= As(1+\tau s) + B(1+\tau s) + Cs^2 \\
\mu &= B && (s=0) \\
\mu &= As(1+\tau s) + \mu(1+\tau s) + Cs^2 \\
\mu &= C(-1/\tau)^2 && (s=-1/\tau) \\
\mu &= C/\tau^2 \\
C &= \mu\tau^2 \\
\mu &= As(1+\tau s) + \mu(1+\tau s) + \mu\tau^2 s^2 \\
\mu &= As(1+\tau s) + \mu + \mu\tau s + \mu\tau^2 s^2 \\
0 &= As(1+\tau s) + (1+\tau s)\mu\tau s \\
-(1+\tau s)\mu\tau s &= As(1+\tau s) \\
-\mu\tau &= A \\
Y_2(s) &= \frac{-\mu\tau}{s} + \frac{\mu}{s^2} + \frac{\mu\tau^2}{1+\tau s} \\
y_2(t) &= (-\mu\tau + \mu t + \mu\tau e^{-t/\tau})u_2(t)
\end{aligned}
$$

2.2 b

2.2.1 i

$u_1(t)$ is positive and 1 for all values ≥ 1 so its value is fixed. As $t \to \infty$, $y_1(t) \to \mu$ for all values of t and so the limit goes to 1. For the abs of the difference to go to zero, we need $\mu = 1$.

2.2.2 ii

$u_2(t)$ is positive and t for all values $t \geq 1$. As $t \to \infty$, the terms in

$$|-\mu\tau + \mu t + \mu\tau e^{-t/\tau} - t|$$

go to

$$|-\mu\tau + t(\mu - 1)|$$

This goes to zero as $t \to \infty$ if $\mu = 1$ and $\tau = 0$.

The first example is about tracking the error between a constant reference given by the input (a constant value of 1 from the heaviside function) and so our system's error will go to zero if the gain of our transfer function μ is also one.

The second example is about tracking the error between a ramp reference with a slope of t given by the input (the ramp) and so our system's error will go to zero if the gain of our transfer function μ is one and the time constant τ is zero. Setting μ to one ensures that the middle term in our response tracks the input and leads to zero error. However, this is *probably* a case that is not physically feasible since that would mean that the damped exponential in our response would immediately go to zero. In more ,realistic cases, we would still set μ to one but we would set τ to a small value so that the damped exponential would go to zero quickly and the first term $\mu\tau$ would also reduce the error to a constant τ value.

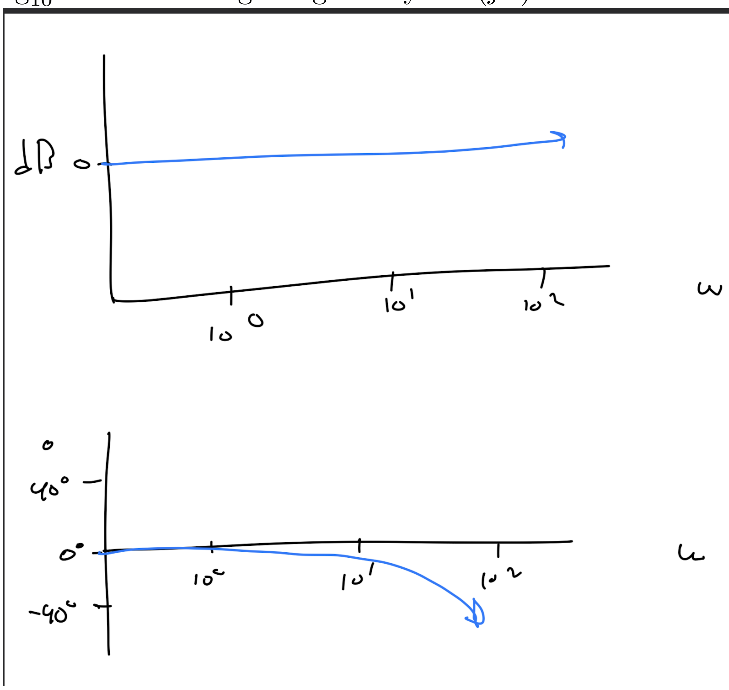
3 3

3.1 a

Given the equation $y(t) = u(t - \tau)$ where $\tau > 0$, its Laplace transform is $Y(s) = e^{-\tau s}U(s)$. Its transfer function is given by $H(s) = e^{-s\tau}$

3.2 b

$H(j\omega) = e^{-j\omega\tau}$. The magnitude in decibels is given by $20\log_{10}|H(j\omega)| = 20\log_{10}|e^{-j\omega\tau}| = 20\log_{10} 1 = 0$. The angle is given by $\angle H(j\omega) = \angle e^{-j\omega\tau} = -\omega\tau$.



3.3 c

$H(s) = e^{-\tau/s}$. Taking the Laplace transform of the input, $U(s) = \dfrac{0.15 \times 2\pi}{s^2 + (2\pi)^2}$. The output is given by $Y(s) = H(s)U(s) = \dfrac{0.15 \times 2\pi e^{-\tau/s}}{s^2 + (2\pi)^2}$. Taking the inverse Laplace transform, we get apply the rules for sin and time shifting to get $y(t) = 0.15 \times \sin(2\pi(t - \tau))$.

3.4 d

Yes it holds because the output is a sinusoid with a modified amplitude that has been phase shifted, but at the same frequency.