UNIVERSITY OF WATERLOO
FINAL EXAMINATION
FALL TERM 2008

Surname: ____________________

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Course Number	MATH 239
Course Title	Introduction to Combinatorics
Instructor	Professors Christian, Goulden, Jao, Purbhoo
Date of Exam	December 15, 2008
Time Period	9:00 — 11:30 am
Number of Exam Pages (including this cover sheet)	16
Exam Type	Closed Book
Additional Materials Allowed	Calculator with pink tie sticker.
Additional Instructions	Please check the box beside your lecture section.

Problem	Value	Mark Awarded	Problem	Value	Mark Awarded
1	8		6	13	
2	14		7	13	
3	6		8	16	
4	8		9	6	
5	16		Total	100	

1. [8 marks] Let c_n be the number of compositions of n which have an even number of parts, and in which no part is equal to 1 (for example, $(7, 5)$ and $(2, 5, 2, 3)$ are compositions of 12 of this type). Determine the generating function $\sum_{n\geq 0} c_n x^n$, expressed as a rational function of x.

2. (a) [4 marks] Give decompositions for the following sets of $\{0,1\}$-strings. All strings should be uniquely created.

   i. The set A in which all blocks have length 1 or 2.

   ii. The set B in which no block has length 1 or 2.

   (b) [2 marks] Find a string that is not in A and not in B.

2. (continued)

   (c) [2 marks] Let $S = \{0, 00, 001\}^\star$. Determine whether the elements of S are uniquely created.

   (d) [6 marks] Determine the generating function for S (given in part (c) above) with respect to length.

3. [6 marks] Suppose that

$$\sum_{n\geq 0} a_n x^n = \frac{1 - x + x^3}{1 - 2x + x^2 - x^3}.$$

Find a linear recurrence for a_n, and enough initial conditions to uniquely specify the a_n.

4. [8 marks] Prove that an edge e of a connected graph G is a bridge if and only if e is contained in every spanning tree of G.

5. (a) [7 marks] Prove that if G is a bipartite graph, and each vertex of G has degree at least 3, then G contains a path of length at least 5 (recall that the length of a path is the number of edges in the path).

   (b) [3 marks] Give an example of a bipartite graph G, such that each vertex of G has degree at least 3 and there are no paths of length 6 in G.

5. (continued)

   (c) [6 marks] Prove that if G is a planar bipartite graph, and each vertex of G has degree at least 3, then G contains a path of length at least 6.

6. A planar embedding P has 102 vertices and 300 edges.

   (a) [3 marks] Using Euler's formula, prove that P has at least 200 faces.

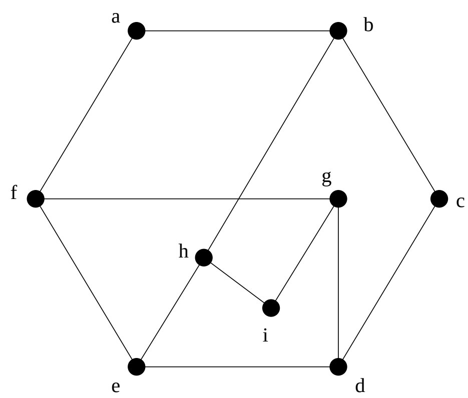
   (b) [3 marks] Prove that P has at least one cycle.

6. (continued)

   (c) [4 marks] Prove that every face of P has degree 3.

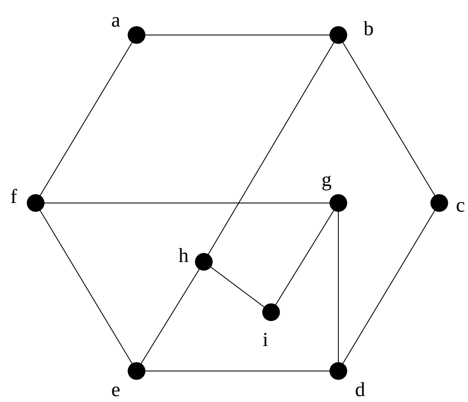
   (d) [3 marks] Prove that P is connected.

7. (a) [8 marks] Determine whether the graph H drawn below is planar.

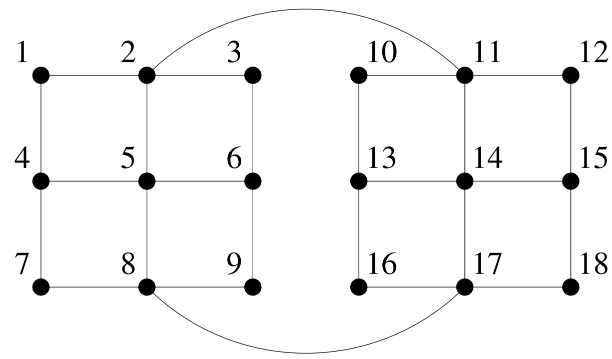


7. (continued)

(b) [5 marks] Show that H is 3-colourable. (A new drawing of H is given for your convenience.)



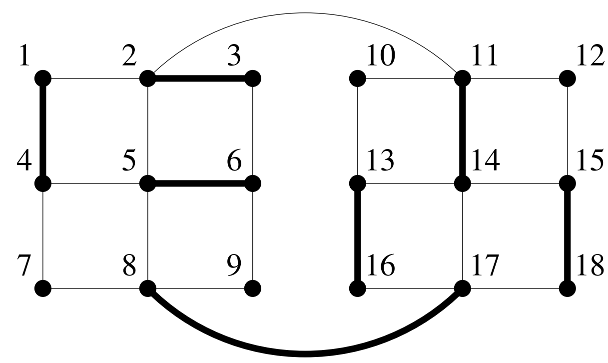
8. (a) [7 marks] Let G be the graph drawn below. Use the breadth first search algorithm to prove that G is a bipartite graph. Use vertex 1 as the root of your breadth first search tree; when considering the vertices adjacent to the vertex being examined, add them to the tree in increasing order of label. Give a list of the vertices of G in the order that they join the tree.



8. (continued)

(b) [9 marks] Starting from the matching M denoted below with thick edges, use the bipartite matching algorithm to find both a maximum matching and a minimum cover of G (the graph that you proved was bipartite in part (a) above). Use the bipartition A, B for which vertex 1 is contained in A. At each stage of the algorithm, give the sets X_0, X and Y. You may find the extra drawings of G helpful in any iterations of the algorithm that are required.

First iteration:

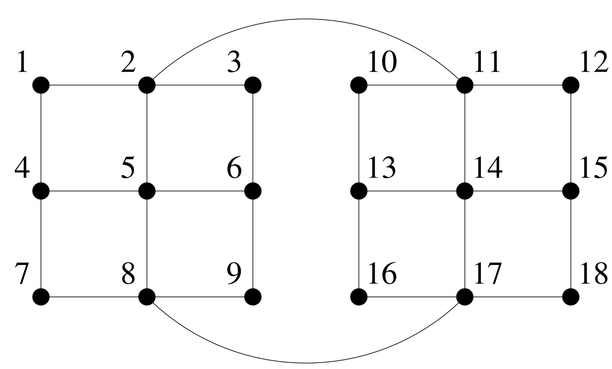


X_0 :

X :

Y :

Second iteration (if needed):



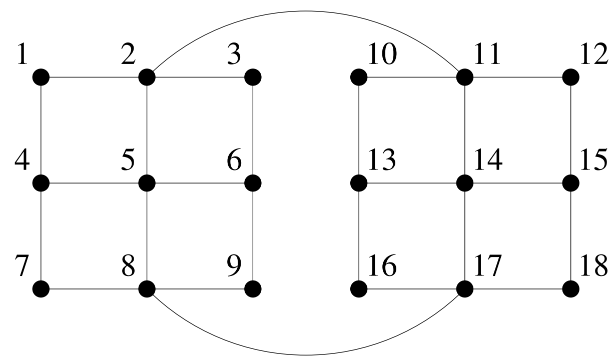
X_0 :

X :

Y :

8. (b) (continued)

Third iteration (if needed):

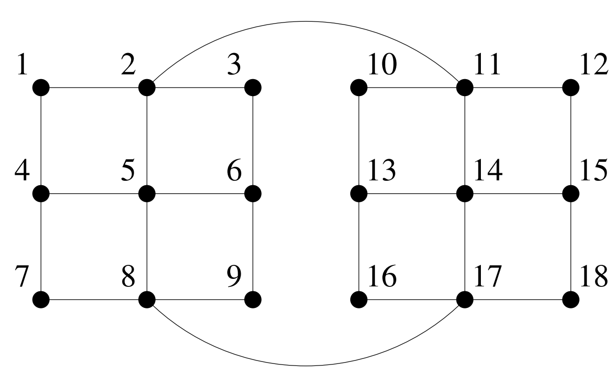


X_0 :

X :

Y :

Fourth iteration (if needed):



X_0 :

X :

Y :

9. [6 marks] For the bipartite graph G with bipartition A, B drawn below, the set $C = \{2, 5, 6, a, c, f\}$ is a cover. Use this cover to find a subset D of A such that

$$|N(D)| < |D|.$$