Computer Science 370
Midterm Examination
Fall Term 2000

Date: October 25, 7:00-9:00 PM

This is an open book exam where all textbooks and notes are permitted. Do all questions and show *all* work. The total marks available are 50.

1. (10 marks) (Floating Point Arithmetic)

Let

$$I_n = \int_0^1 \frac{x^{2n+1}}{x^2 - 2} dx.$$

Then $I_0 = .20273...$ and we can use the identity $I_n = \frac{1}{2n} + 2I_{n-1}$ to compute the other values of this integral. Analyze the stability of this recursion for computing I_n for large n.

2. (10 marks) (Piecewise Polynomial Interpolation)

a) For a given set of knots x_i, $i = 0, \dots, N$, $x_0 < \dots < x_N$, let $L_i(x)$ be linear piecewise polynomials satisfying

$$\begin{aligned} L_i(x_k) &= 1 \text{ if } k = i \\ &= 0 \text{ if } k \neq i. \end{aligned} \tag{1}$$

The $L_i(x)$ are called the linear *Lagrange basis functions.* Give a formula for $L_i(x)$.

b) Suppose now that a new knot is inserted at

$$x_{i+3/4} = x_i + \frac{3}{4}(x_{i+1} - x_i)$$

with a known function value $f_{i+3/4} = f(x_{i+3/4})$. Verify that the three piecewise polynomials given by

$$\begin{aligned} \hat{L}_i(x) &= \tfrac{4}{3}L_i(x) \cdot (L_i(x) - \tfrac{1}{4}), \\ \hat{L}_{i+3/4}(x) &= \tfrac{16}{3}L_i(x) \cdot L_{i+1}(x), \\ \hat{L}_{i+1}(x) &= 4L_{i+1}(x) \cdot (L_{i+1}(x) - \tfrac{3}{4}) \end{aligned}$$

$(X_{k+\frac{3}{4}} - X_k)\,(X_k - X_{k+1})$

are quadratic Lagrange basis functions for the new set of points $x_0, x_{3/4}, x_1, \dots x_i, x_{i+3/4}, x_{i+1}, \dots, x_N$. Show *all* work.

$2X_{k+1} \cdot X_k - X_{k+1}^2 - X_k^2$

$4(L_{i+1}(x))^2 - 3L_i$

3. (10 marks) (Spline Interpolants)

Let $x_1 < \dots < x_n$ be n equally spaced points and $\{y_i\}_{i=1,\dots,n}$ be n additional values. A quadratic spline $S(x)$ is a piecewise quadratic polynomial which interpolates the points (x_i, y_i), and where both $S(x)$ and $S'(x)$ are continuous. Suppose that in the i-th interval we write

$a_i x - a_i x_i + (b_i x - b_i x_i)(x - x_{i+1}) + c_i x - c_i x_{i+1}$

$b_i x x - b_i x_{i+1} x - b_i x_i x + b_i x_i x_{i+1}$

$$\begin{aligned} S_i(x) &= a_i(x - x_i) + b_i(x - x_i)(x - x_{i+1}) + c_i(x - x_{i+1}) \\ &\quad \text{for } x \in [x_i, x_{i+1}] \end{aligned} \tag{2}$$

and that we impose the boundary condition $S'(x_1) = S'(x_n)$. Determine the equations for the spline parameters a_i, b_i, c_i.

$a_{i+1} X - a_{i+1} X_{i+1} + (b_{i+1} X - b_{i+1} X_{i+1})(X - X_{i+2}) + c_{i+1} X -$

$b_i X \cdot$

$b_{i+1} X^2 - b_{i+1} X X_{i+2} - b_{i+1} X_{i+1} X +$

$a_i X \cdot$

a_{i+1}

4. (10 marks) (Finite Difference Formulas)

Suppose a function $U(x)$ is known at three points x_i, x_{i+1}, x_{i+2} having spacing

$$\Delta x_{i+1} = x_{i+2} - x_{i+1} = 4\Delta x_i = 4(x_{i+1} - x_i).$$

Determine a *first* order difference approximation to $U''(x_i)$, using three values $U(x_i)$, $U(x_{i+1})$ and $U(x_{i+2})$. Be sure to include the truncation error term in your final expression.

5. (10 marks) (PDEs : discretization)

a) Let V be a function of three variables S_1, S_2, τ and consider the PDE given by

$$\frac{\partial V}{\partial \tau} = r_1 S_1 \frac{\partial V}{\partial S_1} + r_2 S_2 \frac{\partial V}{\partial S_2} - r_3 V$$

where r_1, r_2, r_3 are positive constants. Subdivide the S_1 and S_2 axes with increments ΔS and the τ axis with increment $\Delta\tau$ and set $V(i\Delta S, j\Delta S, k\Delta\tau) = V^k_{i,j}$. Discretize the PDE using forward differencing and keeping track of your error terms.

b) Give an upper bound for $\Delta\tau$ which ensures that the resulting iterative scheme is numerically stable.

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1. Analyze the stability.

$I_n = \frac{1}{2n} + 2I_{n-1}$ $\qquad I_0 = .20273\ldots$ ✓ known

$(I_0)_{Exact} = I_0$, $\quad (I_0)_A$ = approximate I_0

$e_0 = I_0 - (I_0)_A$

so. $(I_1)_A = \frac{1}{2} + 2(I_0)_A$

$= \frac{1}{2} + 2\cdot(I_0 - e_0)$

$= \frac{1}{2} + 2I_0 - 2e_0$

$= I_1 - 2e_0$

$\therefore |e_1| = |I_1 - (I_1)_A|$

$= |I_1 - I_1 + 2e_0|$

$= |2|\cdot|e_0| = 2\cdot|e_0|$

More generally.

$(I_n)_A = \frac{1}{2n} + 2\cdot(I_{n-1})_A$

$= \frac{1}{2n} + 2(I_{n-1} - e_{n-1})$

$= \frac{1}{2n} + 2I_{n-1} - 2e_{n-1}$

$= I_n - 2e_{n-1}$ ✓

$|e_n| = 2\cdot|e_{n-1}|$ ✓

$= 2^2|e_{n-2}|$

$= 2^3|e_{n-3}|$

$\vdots$

$= 2^n|e_0|$ ✓

So when n is very large, the error $|e_n|$ tends to very large.

So using $I_n = \frac{1}{2n} + 2I_{n-1}$ to compute the values of this integral tends to be UNSTABLE!!

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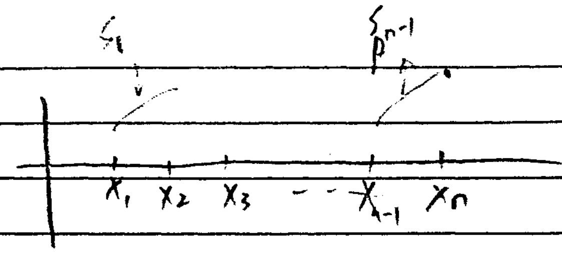
3.



The spline conditions are

① $S_i(x_i) = y_i$ for $i = 1, \dots, n-1$ $N-1$ equations
② $S_i(x_{i+1}) = y_{i+1}$ for $i = 1, \dots, n-1$ $N-1$ equations
③ $S_i'(x_{i+1}) = S_{i+1}'(x_{i+1})$ for $i = 1, \dots, n-2$, $N-2$ equations
④ Boundary condition

$S_1'(x_1) = S_{n-1}'(x_n)$ 1 equation

There are totally $N-3$ equations,
And there are $N-3$ unknowns.
Now we solve for parameters a_i, b_i, C_i.
From ①, ②.

$$S_i(x_i) = c_i(x_i - x_{i+1})$$
$$S_i(x_{i+1}) = a_i(x_{i+1} - x_i)$$

Denote $\Delta X_i = x_{i+1} - x_i$

$$\therefore C_i = \frac{S_i(x_i)}{-\Delta X_i} = \frac{y_i}{-\Delta X_i} \quad (*) \qquad a_i = \frac{y_i}{\Delta X_i} \quad (**)$$

$= \Delta x$ $= \Delta x$

Now.

$$S_i'(x) = a_i + 2b_i x - b_i x_i - b_i x_{i+1} + C_i$$
$$= a_i + C_i + b_i(2x - x_i - x_{i+1})$$

$$S_{i+1}'(x) = a_{i+1} + C_{i+1} + b_{i+1}(2x - x_{i+1} - x_{i+2}).$$

So. $S_i'(x_{i+1}) = a_i + C_i + b_i(2x_{i+1} - x_i - x_{i+1})$
$= a_i + C_i + b_i \Delta X_i$

$S_{i+1}'(x_{i+1}) = a_{i+1} + C_{i+1} + b_{i+1}(2x_{i+1} - x_{i+1} - x_{i+2})$
$= a_{i+1} + C_{i+1} - b_{i+1}\Delta X_{i+1}$

We have. $S_i'(x_{i+1}) = S_{i+1}'(x_{i+1})$ ✓

So. $a_i + C_i + b_i \Delta X_i = a_{i+1} + C_{i+1} - b_{i+1}\Delta X_{i+1}$

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$$b_i \Delta X_i + b_{i+1} \Delta X_{i+1} = a_{i+1} + c_{i+1} - a_i - c_i$$

Since a_i, c_i are easy to compute, as we did.

Also for $S_1'(X_1) = S_{n-1}'(X_n)$

$$S_1'(X_1) = a_1 + c_1 - b_1 \Delta X_1$$

$$S_{n-1}'(X_n) = a_{n-1} + c_{n-1} + b_{n-1} \Delta X_{n-1}$$

$$\Rightarrow \quad b_{n-1} \Delta X_{n-1} + b_1 \Delta X_1 = a_{n-1} + c_{n-1} - a_1 - c_1 .$$

So we have

$\Delta X_1 = \Delta X_2 = \cdots$

$$\begin{bmatrix} b_1 & b_2 & 0 & 0 & 0 & \cdots \\ 0 & b_2 & b_3 & 0 & 0 & \cdots \\ & & \ddots & & & \\ & & & & b_{n-2} & b_{n-1} \\ b_1 & & & & & b_{n-1} \end{bmatrix} \cdot \begin{bmatrix} \Delta X_1 \\ \Delta X_2 \\ \vdots \\ \Delta X_{n-1} \end{bmatrix} = \begin{bmatrix} a_2 + c_2 - a_1 - c \\ a_3 + c_3 - a_2 - c \\ \vdots \\ a_{n-1} + c_{n-1} - a_1 - \end{bmatrix}$$

got from boundary case

Substitute in a_i & c_i, we got

$$\begin{bmatrix} b_1 & b_2 & 0 & 0 & 0 & \cdots \\ 0 & b_2 & b_3 & 0 & \cdots & \\ & & & & & \\ & & & & b_{n-2} & b_{n-1} \\ b_1 & & & & & b_{n-1} \end{bmatrix} \cdot \begin{bmatrix} \Delta X_1 \\ \Delta X_2 \\ \vdots \\ \Delta X_{n-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{~~why~~ "0"?} \quad (***)$$

Equations (*) (**) (***) give the answer.

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