University of Waterloo
CS 341 Fall 2022
Written Assignment 5

Due Date: Monday, Dec 5 at 11:59pm to Crowdmark
All work submitted must be the student's own.

Question 1 [15 marks] 0-1 Double Knapsack

Consider the following problem:

> *Input*: Positive integers $v_1, \ldots, v_n, w_1, \ldots, w_n, W_1, W_2$.
> *Output*: Disjoint subsets $A, B \in \{1, \ldots, n\}$ such that $\sum_{k \in A} w_k \leq W_1$ and $\sum_{k \in B} w_k \leq W_2$, while maximizing the total value $\sum_{k \in A} v_k + \sum_{k \in B} v_k$.

[*Note*: Recall that the bit complexity of the input is polynomial in n and $\log U$, where U is the largest integer in the input.]

a) [6 marks] Convert the optimization problem above into a decision problem and show that the corresponding decision problem is in NP.

b) [3 marks] Show that if we could solve your decision problem in (a) in time polynomial in the number of input bits, then we could also compute the maximum total value in time polynomial in the number of input bits.

c) [6 marks] Show that if we could solve your decision problem in (a) in time polynomial in the number of input bits, then we could also compute the optimal subsets A and B in time polynomial in the number of input bits. You may use part (b) as a subroutine.

Question 2 [15 marks] SUBSET-GCD-DEC

One of the most ubiquitous operations in computer algebra is to compute the greatest common divisor (GCD) of a collection of positive integers. A question that arises is whether there exists a small cardinality subset of the numbers that has the same GCD as the entire set. Formally:

> SUBSET-GCD-DEC
> *Instance*: A set S of positive integers and an integer k.
> *Question*: Does there exist $S' \subset S$ with $\mid S' \mid \leq k$ such that GCD(S') = GCD(S)?

For example, consider $S_1 = \{42, 30, 70\}$ with $k = 2$. For S_1, the GCD(42, 30, 70) = 2 but no proper subset of S_1 of size $k = 2$ will have a GCD = 2.

As a second example, consider $S_2 = \{91, 104, 95\}$ with $k = 2$ and let $S' = \{91, 95\}$. For S_2, GCD(91, 104, 95) = 1 and GCD(91, 95) = 1 so S_2 has proper subset S' of size 2 where $\text{GCD}(S') = \text{GCD}(S_2)$.

Prove that SUBSET-GCD-DEC is NP-complete.

a) Define a suitable certificate and give a polynomial time certificate verification algorithm.

b) Reduce VERTEX-COVER-DEC to SUBSET-GCD-DEC.

Question 3 [15 marks] Art Gallery Problem

Suppose you are in charge of allocating paintings to art galleries. Specifically, there are t paintings, denoted $P_1, \ldots, P_t$, that a collector wishes to allocate to K art galleries, denoted $G_1, \ldots, G_k$. There are s art critics, denoted $A_1, \ldots, A_s$ and each critic A_i has selected a subset of paintings $Q_i \subseteq \{P_1, \ldots, P_t\}$ that they wish to view.

The collector is rather eccentric and wishes to ensure that each critic A_i is forced to visit at least two galleries in order to view all of the paintings in Q_i.

One way to formulate the GALLERY-ALLOCATION problem is as follows:

> *Input*: A set $P = \{P_1, \ldots, P_t\}$, s subsets $Q_1, \ldots, Q_s \subseteq P$ and a positive integer $K \geq 2$.
> *Output*: "Yes" if and only if there exists a function $f : P \longrightarrow \{1, \ldots, K\}$ such that $|\ \{f(P_j) : P_j \in Q_i\}\ | > 1$ for all i, $1 \leq i \leq s$.

Prove that the GALLERY-ALLOCATION problem is in NP.

a) Define a suitable certificate and give a polynomial time certificate verification algorithm.

b) Reduce 3-CNF-SAT to GALLERY-ALLOCATION.

Here are some hints to get you started:

The Reduction. We are given an instance of 3-CNF-SAT, i.e., a 3-CNF Boolean formula F using n variables $x_1, \ldots, x_n$, having m clauses $C_1, \ldots, C_m$. We want to construct an instance of GALLERY-ALLOCATION, which is defined by a set P, a collection of subsets of P, and an integer $K \geq 2$. Define $P = \{x_1, \overline{x_1}, \ldots, x_n, \overline{x_n}, z\}$ where z is a new element. For each clause C_i, create a subset Q_i consisting of the three literals in C_i, along with z. In addition, for each $i = 1, \ldots, n$, we create a subset $Q_{m+i} = \{x_i, \overline{x_i}\}$. Finally, we choose $K = 2$.

Question 4 [BONUS 5 marks] Approximation Algorithm

Consider the following problem to approximate the minimum weight connected subgraph:

> *Input*: An undirected graph $G = (V, E)$ with positive edge weights and a subset of vertices $S \subseteq V$.
> *Output*: A subset of edges $F \subseteq E$ such that for any two vertices $u, v \in S$, there is a path in the subgraph (V, F) between u and v. In addition, the sum of the weights of the edges in F should be minimal among all such subgraphs.

Give a polynomial time 2-approximation algorithm for this problem.

Hint: Construct a complete graph H on S such that a minimum spanning tree in H corresponds to a subgraph of G with the desired property.