CS442
Module 5 addendum: $[\![\hat{\ }]\!]$ inference

University of Waterloo

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$[\![\hat{\ }]\!]$

$[\![\hat{\ }]\!] = \lambda m.\, \lambda n.\, \lambda f.\, \lambda x.\, nmfx$

$W(\{\}, \lambda m.\, \lambda n.\, \lambda f.\, \lambda x.\, nmfx) = ...$

$W(\{\langle m, \alpha_m\rangle, \langle n, \alpha_n\rangle, \langle f, \alpha_f\rangle, \langle x, \alpha_x\rangle\}, n) = [], \alpha_n$

$W(\{\langle m, \alpha_m\rangle, \langle n, \alpha_n\rangle, \langle f, \alpha_f\rangle, \langle x, \alpha_x\rangle\}, m) = [], \alpha_m$

$W(\{\langle m, \alpha_m\rangle, \langle n, \alpha_n\rangle, \langle f, \alpha_f\rangle, \langle x, \alpha_x\rangle\}, nm) = ...$

$U(\alpha_n, \alpha_m \to \alpha_1) = [\alpha_m \to \alpha_1/\alpha_n]$

$W(\{\langle m, \alpha_m\rangle, \langle n, \alpha_n\rangle, \langle f, \alpha_f\rangle, \langle x, \alpha_x\rangle\}, nm) = [\alpha_m \to \alpha_1/\alpha_n], \alpha_1$

$W(\{\langle m, \alpha_m\rangle, \langle n, \alpha_m \to \alpha_1\rangle, \langle f, \alpha_f\rangle, \langle x, \alpha_x\rangle\}, (nm)f) = ...$

$U(\alpha_1, \alpha_f \to \alpha_2) = [\alpha_f \to \alpha_2/\alpha_1]$

$W(\{\langle m, \alpha_m\rangle, \langle n, \alpha_m \to \alpha_1\rangle, \langle f, \alpha_f\rangle, \langle x, \alpha_x\rangle\}, (nm)f) = [\alpha_f \to \alpha_2/\alpha_1], \alpha_2$

$W(\{\langle m, \alpha_m\rangle, \langle n, \alpha_m \to \alpha_f \to \alpha_2\rangle, \langle f, \alpha_f\rangle, \langle x, \alpha_x\rangle\}, ((nm)f)x) = ...$

$U(\alpha_2, \alpha_x \to \alpha_3) = [\alpha_x \to \alpha_3/\alpha_2]$

$W(\{\langle m, \alpha_m\rangle, \langle n, \alpha_m \to \alpha_f \to \alpha_2\rangle, \langle f, \alpha_f\rangle, \langle x, \alpha_x\rangle\}, ((nm)f)x) = [\alpha_x \to \alpha_3/\alpha_2], \alpha_3$

$\Gamma = \{\langle m, \alpha_m\rangle, \langle n, \alpha_m \to \alpha_f \to \alpha_x \to \alpha_3\rangle, \langle f, \alpha_f\rangle, \langle x, \alpha_x\rangle\}$ (in the body of $[\![\hat{\ }]\!]$)

$[\![\hat{\ }]\!] : \alpha_m \to (\alpha_m \to \alpha_f \to \alpha_x \to \alpha_3) \to \alpha_f \to \alpha_x \to \alpha_3$

$[\![2]\!]$

$[\![2]\!] = \lambda f.\, \lambda x.\, f(fx)$

$W(\{\}, \lambda f.\, \lambda x.\, f(fx)) = ...$

$W(\{\langle f, \alpha_f\rangle, \langle x, \alpha_x\rangle\}, f) = [], \alpha_f$

$W(\{\langle f, \alpha_f\rangle, \langle x, \alpha_x\rangle\}, x) = [], \alpha_x$

$W(\{\langle f, \alpha_f\rangle, \langle x, \alpha_x\rangle\}, fx) = ...$

$U(\alpha_f, \alpha_x \to \alpha_4) = [\alpha_x \to \alpha_4/\alpha_f]$

$W(\{\langle f, \alpha_f\rangle, \langle x, \alpha_x\rangle\}, fx) = [\alpha_x \to \alpha_4/\alpha_f], \alpha_4$

$W(\{\langle f, \alpha_x \to \alpha_4 \rangle, \langle x, \alpha_x \rangle\}, f(fx)) = ...$

$U(\alpha_x \to \alpha_4, \alpha_4 \to \alpha_5) = ...$

$U(\alpha_x, \alpha_4) = [\alpha_4/\alpha_x]$ (note: the order of this substitution is arbitrary, but it's written in the notes to "prefer" the left in this way)

$U(\alpha_4[\alpha_4/\alpha_x], \alpha_5[\alpha_4/\alpha_x]) = ...$

$U(\alpha_4, \alpha_5) = [\alpha_5/\alpha_4]$ (note: again, the order is arbtirary)

$U(\alpha_x \to \alpha_4, \alpha_4 \to \alpha_5) = [\alpha_4/\alpha_x][\alpha_5/\alpha_4]$

$W(\{\langle f, \alpha_x \to \alpha_4 \rangle, \langle x, \alpha_x \rangle\}, f(fx)) = [\alpha_4/\alpha_x][\alpha_5/\alpha_4], \alpha_5$

$\Gamma = \{\langle f, \alpha_5 \to \alpha_5 \rangle, \langle x, \alpha_5 \rangle\}$ (in the body of $[\![2]\!]$)

$[\![2]\!]$: $(\alpha_5 \to \alpha_5) \to \alpha_5 \to \alpha_5$

$[\![\hat{}]\!]$ $[\![2]\!]$ $[\![2]\!]$

Note: These components don't share any variables, so their substitutions can't interfere with each other. Formally, we should be passing their resulting substitutions into each other, but they're being left out because we know that will have no effect.

Note 2: As $\alpha_{anything}$ is a type variable, it can be substituted; there is no reason for the first and second $[\![2]\!]$ to share the same type variable, so a new name will be introduced when we get to the second one.

$W(\{\}, [\![\hat{}]\!]\ [\![2]\!]\ [\![2]\!]) = ...$

$W(\{\}, [\![\hat{}]\!]\ [\![2]\!]) = ...$

$U(\alpha_m \to (\alpha_m \to \alpha_f \to \alpha_x \to \alpha_3) \to \alpha_f \to \alpha_x \to \alpha_3, ((\alpha_5 \to \alpha_5) \to \alpha_5 \to \alpha_5) \to \alpha_6) = ...$

Adding parentheses for clarity:

$U(\alpha_m \to ((\alpha_m \to \alpha_f \to \alpha_x \to \alpha_3) \to (\alpha_f \to (\alpha_x \to \alpha_3))), ((\alpha_5 \to \alpha_5) \to (\alpha_5 \to \alpha_5)) \to \alpha_6) = ...$

$U(\alpha_m, (\alpha_5 \to \alpha_5) \to (\alpha_5 \to \alpha_5)) = [(\alpha_5 \to \alpha_5) \to (\alpha_5 \to \alpha_5)/\alpha_m]$

$U(((\alpha_m \to \alpha_f \to \alpha_x \to \alpha_3) \to (\alpha_f \to (\alpha_x \to \alpha_3)))[(\alpha_5 \to \alpha_5) \to (\alpha_5 \to \alpha_5)/\alpha_m],$
$\alpha_6[(\alpha_5 \to \alpha_5) \to (\alpha_5 \to \alpha_5)/\alpha_m]) = ...$

$U(((((\alpha_5 \to \alpha_5) \to (\alpha_5 \to \alpha_5)) \to \alpha_f \to \alpha_x \to \alpha_3) \to (\alpha_f \to (\alpha_x \to \alpha_3))), \alpha_6) =$
$[(((\alpha_5 \to \alpha_5) \to (\alpha_5 \to \alpha_5)) \to \alpha_f \to \alpha_x \to \alpha_3) \to (\alpha_f \to (\alpha_x \to \alpha_3))/\alpha_6]$

$U(\alpha_m \to (\alpha_m \to \alpha_f \to \alpha_x \to \alpha_3) \to \alpha_f \to \alpha_x \to \alpha_3, ((\alpha_5 \to \alpha_5) \to \alpha_5 \to \alpha_5) \to \alpha_6) =$
$[(\alpha_5 \to \alpha_5) \to (\alpha_5 \to \alpha_5)/\alpha_m][(((\alpha_5 \to \alpha_5) \to (\alpha_5 \to \alpha_5)) \to \alpha_f \to \alpha_x \to \alpha_3) \to (\alpha_f \to (\alpha_x \to \alpha_3))/\alpha_6]$

$W(\{\}, [\![\hat{}]\!]\ [\![2]\!]) =$
$[(\alpha_5 \to \alpha_5) \to (\alpha_5 \to \alpha_5)/\alpha_m][(((\alpha_5 \to \alpha_5) \to (\alpha_5 \to \alpha_5)) \to \alpha_f \to \alpha_x \to \alpha_3) \to (\alpha_f \to (\alpha_x \to \alpha_3))/\alpha_6],$
$(((\alpha_5 \to \alpha_5) \to (\alpha_5 \to \alpha_5)) \to \alpha_f \to \alpha_x \to \alpha_3) \to (\alpha_f \to (\alpha_x \to \alpha_3))$

A note so far: What we've done is established the type of α_m (it's just the type of $[\![2]\!]$), and established the result type of $[\![\hat{}]\!]$ $[\![2]\!]$. The intiminading type above is the result type of $[\![\hat{}]\!]$ $[\![2]\!]$: it's just the type of $[\![\hat{}]\!]$, where we've been more particular about the type of m (it's a Church numeral with α_5 as its base type).

Neither α_m nor α_6 will be used again, so I'll stop dragging those substitutions around at this point; formally, we would need those both at each step.

We'll give the second $[\![2]\!]$ the type $(\alpha_7 \to \alpha_7) \to (\alpha_7 \to \alpha_7)$, to distinguish its type variable from the first $[\![2]\!]$. (Ultimately, we would have done algorithm W separately for it, and so found a different type variable anyway).

$W(\{\}, (\ [\![\hat{\ }]\!]\ [\![2]\!]\ )\ [\![2]\!]\ ) = ...$

$U((((\alpha_5 \to \alpha_5) \to (\alpha_5 \to \alpha_5)) \to \alpha_f \to \alpha_x \to \alpha_3) \to (\alpha_f \to (\alpha_x \to \alpha_3)), ((\alpha_7 \to \alpha_7) \to (\alpha_7 \to \alpha_7)) \to \alpha_8) = ...$

$U(((\alpha_5 \to \alpha_5) \to (\alpha_5 \to \alpha_5)) \to \alpha_f \to \alpha_x \to \alpha_3, (\alpha_7 \to \alpha_7) \to (\alpha_7 \to \alpha_7)) = ...$

$U((\alpha_5 \to \alpha_5) \to (\alpha_5 \to \alpha_5), \alpha_7 \to \alpha_7) = ...$

$U(\alpha_5 \to \alpha_5, \alpha_7) = [\alpha_5 \to \alpha_5/\alpha_7]$ (as U is commutative)

$U(\alpha_f \to (\alpha_x \to \alpha_3)[\alpha_5 \to \alpha_5/\alpha_7], \alpha_7 \to \alpha_7[\alpha_5 \to \alpha_5/\alpha_7]) = ...$

$U(\alpha_f \to (\alpha_x \to \alpha_3), (\alpha_5 \to \alpha_5) \to (\alpha_5 \to \alpha_5)) = ...$

$U(\alpha_f, \alpha_5 \to \alpha_5) = [\alpha_5 \to \alpha_5/\alpha_f]$

$U(\alpha_x \to \alpha_3, \alpha_5 \to \alpha_5) = ...$ (neither side has α_f, so I skipped that substitution)

$U(\alpha_x, \alpha_5) = [\alpha_5/\alpha_x]$ (order is again arbitrary)

$U(\alpha_3, \alpha_5) = [\alpha_5/\alpha_3]$ (order is again arbitrary)

$U(\alpha_x \to \alpha_3, \alpha_5 \to \alpha_5) = [\alpha_5/\alpha_x][\alpha_5/\alpha_3]$

$U(\alpha_f \to (\alpha_x \to \alpha_3), (\alpha_5 \to \alpha_5) \to (\alpha_5 \to \alpha_5)) = [\alpha_5 \to \alpha_5/\alpha_f][\alpha_5/\alpha_x][\alpha_5/\alpha_3]$

...

$U((((\alpha_5 \to \alpha_5) \to (\alpha_5 \to \alpha_5)) \to \alpha_f \to \alpha_x \to \alpha_3) \to (\alpha_f \to (\alpha_x \to \alpha_3)), ((\alpha_7 \to \alpha_7) \to (\alpha_7 \to \alpha_7)) \to \alpha_8) =$
$[\alpha_5 \to \alpha_5/\alpha_f][\alpha_5/\alpha_x][\alpha_5/\alpha_3]$

(Note: At this point, we've found the relationship between all αs)

$W(\{\}, (\ [\![\hat{\ }]\!]\ [\![2]\!]\ )\ [\![2]\!]\ ) = [\alpha_5 \to \alpha_5/\alpha_f][\alpha_5/\alpha_x][\alpha_5/\alpha_3], \alpha_5$

α_m (the type of m) $= \alpha_m[(\alpha_5 \to \alpha_5) \to (\alpha_5 \to \alpha_5)/\alpha_m] = (\alpha_5 \to \alpha_5) \to (\alpha_5 \to \alpha_5)$
(i.e., a Church numeral with α_5 as its x type)

α_n (the type of n)
$= \alpha_n[\alpha_m \to \alpha_1/\alpha_n][(\alpha_5 \to \alpha_5) \to (\alpha_5 \to \alpha_5)/\alpha_m][\alpha_f \to \alpha_2/\alpha_1][\alpha_x \to \alpha_4/\alpha_f][\alpha_x \to \alpha_3/\alpha_2][\alpha_5/\alpha_x][\alpha_5/\alpha_4][\alpha_5/\alpha_3]$
$= ((\alpha_5 \to \alpha_5) \to (\alpha_5 \to \alpha_5)) \to (\alpha_5 \to \alpha_5) \to (\alpha_5 \to \alpha_5)$
(i.e., a Church numeral with $(\alpha_5 \to \alpha_5)$ as its x type)