Note: by the time-shift result ~~of assignment 2~~, and superposition, the coefficients of the Fourier series of the output of the full-wave rectifier should be

$$c_n = \left[1 + e^{j\frac{2\pi}{T}n\Delta}\right] \times \text{coeff}^{\underline{t}} \text{ of } \tfrac{1}{2}\text{-wave series},$$

where $\Delta = \frac{T}{2}$.

This gives,

for $n = 0$ — $c_n = 2$

for $n = \pm 1$ - $c_n = \left(1 + e^{\pm j\pi}\right)\dfrac{An}{4j} = 0$

for $|n| \geq 2$, $|n|$ even $\quad c_n = \left(1 + e^{j\pi n}\right)\dfrac{-A}{\pi(n^2-1)}$

$$= \frac{-2A}{\pi(n^2-1)}$$

for $|n| \geq 2$, $|n|$ odd $\rightarrow c_n = 0$

- This agrees with our direct calculation.

We motivated the use of Fourier series by pointing out that it would let us extend our frequency-response analysis to periodic signals other than pure sinusoids.

To illustrate, suppose that the Fourier series

$$\sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi}{T}nt}$$

represents the output of a rectifier, and that we now wish to filter that signal — perhaps to produce a dc power supply.

Suppose that the filter has a transfer function $H(s)$. Then its response to each of the sinusoids in the Fourier series is given by its frequency response; and by superposition, its response to the entire Fourier series is

$$\sum_{n=-\infty}^{\infty} c_n \, H\left(j\tfrac{2\pi}{T}n\right) e^{j\frac{2\pi}{T}nt}$$

$$= \sum_{n=-\infty}^{\infty} c_n \left| H\left(j\tfrac{2\pi}{T}n\right)\right| e^{j\left(\frac{2\pi}{T}nt + \angle H(j\frac{2\pi}{T}\right.}$$

— so we can easily find the output of the filter, in the form of a Fourier series.

We can then quantify the effect of the filtering in terms of amplitudes, or, using Parseval's Theorem, in terms of average power.

Example: Let's try filtering the output of the full-wave rectifier.

Call the input frequency ω_0

We wish to pass the dc component, but block the first harmonic of the input frequency at $2\omega_0$.

(Recall that the full-wave rectifier output has no frequency component at ω_0 itself.)

The dc component is $\frac{2}{\pi} A$,

so its power is $\frac{4}{\pi^2} A^2$.

The component at the frequency

$2\omega_0$ has an amplitude of

$$\frac{4}{\pi} A \cdot \frac{1}{3},$$

so its power is $\frac{8}{9} \frac{A^2}{\pi^2}$.

We need to reduce the power

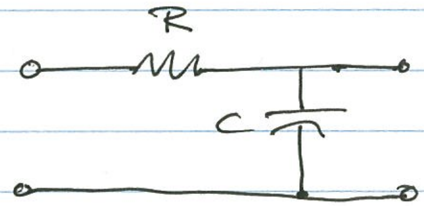
of the signal at $2\omega_0$ by a factor

of about 25, or $10 \log 25$ dB = 14 dB.

Let's be conservative and go for

20 dB.

If we use the following simple first-order filter:



Then the frequency $2\omega_o$ will have to be one decade above the "corner frequency" of the filter.

→ Place corner frequency at

$$\frac{2\omega_o}{10} \Longleftrightarrow \tau = RC = \frac{5}{\omega_o}$$

So the dc component of the filter output is

$$\frac{2A}{\pi} H(0) = \frac{2A}{\pi}$$

... and the amplitude of the component at the frequency $2\omega_0$ is

$$\frac{4A}{\pi} \cdot \frac{1}{3} \left| \frac{1}{1 + j5.2} \right|$$

$$\sim \frac{4}{3\pi} \cdot \frac{1}{10.5} \quad A$$

$$\sim 0.04 \quad A$$

The power in the dc

component is therefore

$$\frac{4A^2}{\pi^2}$$

while the power in the

$2\omega_0$ component is

$$\frac{1}{2} \cdot 16 \times 10^{-4} A^2$$

$$\sim 8 \times 10^{-4} A^2$$

Real sinusoidal form of Fourier series:

Suppose $f(t)$ is real-valued.

Then, for any n,

$$c_{-n} = c_n^*$$

It follows that the Fourier series is

$$\sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi}{T}nt}$$

$$= c_0 + \sum_{n=1}^{\infty} \left(c_n e^{j\frac{2\pi}{T}nt} + c_n^* e^{-j\frac{2\pi}{T}nt} \right)$$

$$= c_0 + \sum_{n=1}^{\infty} 2 \operatorname{Re} \left(c_n e^{j\frac{2\pi}{T}nt} \right)$$

$$= c_0 + \sum_{n=1}^{\infty} 2 \operatorname{Re} \left(|c_n| e^{j\left(\frac{2\pi}{T}nt + \angle c_n\right)} \right)$$

$$= c_0 + \sum_{n=1}^{\infty} 2 |c_n| \cos \left(\frac{2\pi}{T}nt + \angle c_n \right)$$

But this is

$$c_0 + \sum_{n=1}^{\infty} 2|c_n| \left(\cos \angle c_n \cos \frac{2\pi}{T} nt - \sin \angle c_n \sin \frac{2\pi}{T} nt \right)$$

$$= a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2\pi}{T} nt + b_n \sin \frac{2\pi}{T} nt \right),$$

where

$$a_0 = c_0 ,$$

and for $n \geq 1$,

$$a_n = 2 \operatorname{Re} c_n$$

$$b_n = -2 \operatorname{Im} c_n$$

It follows that

$$a_n = 2\,\mathrm{Re}\left[\frac{1}{T}\int_{-\frac{T}{2}}^{T/2} f(t)\,e^{-j\frac{2\pi}{T}nt}\,dt\right]$$

$$= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\frac{2\pi}{T}nt\;dt$$

& $$b_n = -2\,\mathrm{Im}\left[\frac{1}{T}\int_{-\frac{T}{2}}^{T/2} f(t)\,e^{-j\frac{2\pi}{T}nt}\,dt\right]$$

$$= \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin\frac{2\pi n}{T}t\,dt$$

$$\left(\&\ a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\,dt\right).$$

Gibbs' phenomenon

The partial sums of the square-wave Fourier series "overshoot" the square wave itself in the region of the discontinuities.

This overshoot doesn't tend to zero as the number of terms goes to infinity — it tends to something like 9% of the "jump."

This may seem surprising in light of the Dirichlet convergence theorem, but it is, of course, consistent.

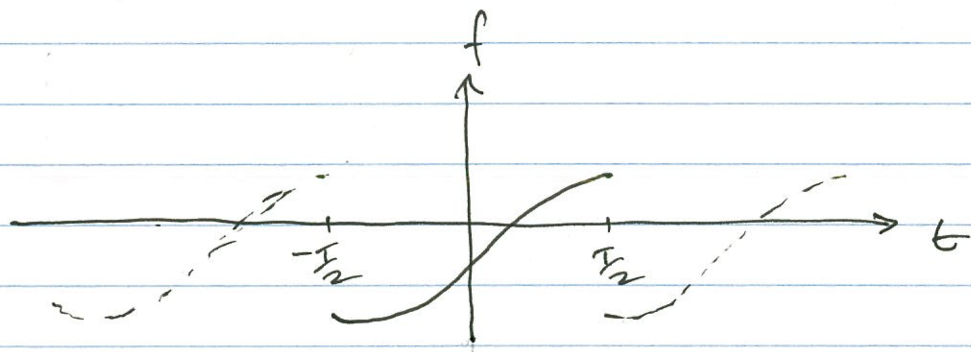
In fact, the same thing happens at discontinuities of any $f(t)$.

Though the Fourier series does not converge pointwise to $f(t)$ at every point (namely, not at discontinuities), the average power of the approximation error of the partial sums tends to zero — we say that the Fourier series "converges in mean square" to $f(t)$.

The Fourier transform

Suppose we wished to extend the Fourier series to functions that aren't periodic.

We can always approximate $f(t)$ over an integral $[-T/2, T/2]$ by pretending that it's periodic with period T.



This gives a Fourier series with frequency components at integer multiples of $\frac{2\pi}{T}$.

Call the frequency increment

$$\Delta\omega = \frac{2\pi}{T}$$

Then

$$f(t) \sim \sum_{n=-\infty}^{\infty} c_n e^{j\Delta\omega n t}$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (T c_n) e^{j\Delta\omega n t} \Delta\omega$$

Now, as $T \to \infty$ this sum should approach an integral

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{j\omega t} dt ,$$

where $\hat{f}(\omega)$ is what Tc_n approaches as $T \to \infty$; namely,

$$\hat{f}(\omega) = \frac{T}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\, e^{-j\omega t}\, dt$$

We call

$$\mathcal{F}\{f\} = \hat{f}(\omega)$$

the <u>Fourier transform</u> of $f(t)$, and we call the integral

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{j\omega t}\, d\omega$$

the Fourier transform <u>inversion integral</u>.

Note that the Fourier transform resembles the Laplace transform with $s = j\omega$.

Indeed, it has similar properties with respect to time-shifts, sinusoidal modulation, convolution and the like, as long as one restricts attention to functions satisfying

$$\int_{-\infty}^{\infty} |f(t)|\, dt < \infty.$$

It also satisfies a version of Parseval's Theorem:

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 \, d\omega$$