UNIVERSITY OF WATERLOO
FINAL EXAMINATION
WINTER TERM 2002

Surname: ______________________

First Name: ______________________

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Course Number	MATH 239
Course Title	Introduction to Combinatorics
Instructor	☐ Professor Chan 8:30 MWF ☐ Professor Goulden 10:30 MWF ☐ Professor Schellenberg 1:30 MWF
Date of Exam	April 16, 2002
Time Period	2:00 pm - 5:00 pm
Number of Exam Pages (including this cover sheet)	15 pages
Exam Type	Closed Book
Additional Materials Allowed:	Calculator

INSTRUCTIONS:

1. Please check that you have all 15 (including this cover page) pages of this examination.
2. Be sure to explain your solutions fully.
3. Define all symbols that you introduce in your solutions.
4. If you require more space, use the back of the *previous* page.

1. Let a_n be the number of compositions of n in which no part is equal to 3, for $n \geq 0$ (e.g. two of these compositions of $n = 15$ are (1, 5, 2, 1, 6) and (9, 2, 2, 2)).

[7] (a) Prove that

$$\sum_{n=0}^{\infty} a_n x^n = \frac{1-x}{1-2x+x^3-x^4}$$

[3] (b) From part (a), give a linear recurrence equation for a_n together with initial conditions to uniquely determine the sequence $\{a_n\}$.

[5] 2 (a) For each of the following sets, write down a decomposition that uniquely creates the elements of the sets.

(i) The set of binary strings in which the substring 1110 does not occur.

(ii) The set of binary strings in which the substring 0110 does not occur.

[4] 2 (b) Let b_n be the number of $\{0,1\}$-strings of length n, $n \geq 0$, with no consecutive 1's. (e.g., two of these strings with $n = 12$ are 001000100100, 100010100001). Prove that

$$\sum_{n=0}^{\infty} b_n x^n = \frac{1+x}{1-x-x^2}$$

[6] (c) Let c_n be the number of $\{0,1\}$-strings in which every block of 0's is followed immediately by a longer block of 1's (e.g., two of these strings with $n = 18$ are 001110001111001111, 111100011111101111). Prove that

$$\sum_{n=0}^{\infty} c_n x^n = \frac{1-x^2}{1-x-x^2}$$

[5] (d) From parts (a) and (b), prove that

$$b_n = c_n + c_{n+1}, n \geq 1.$$

3. For each $n \geq 1$, let G_n be the graph whose vertices are the n-subsets of $\{1, 2, \ldots, 2n-1\}$, and two such subsets are adjacent when they intersect in a single element of $\{1, 2, \ldots, 2n-1\}$.(e.g., in G_4, subsets $\{1, 3, 4, 6\}$ and $\{2, 4, 5, 7\}$ are adjacent.)

(a) Draw G_2 and G_3.

(b) How many vertices and edges does G_n have, $n \geq 1$?

(c) Find an odd cycle in G_5, thus proving that G_5 is not bipartite.

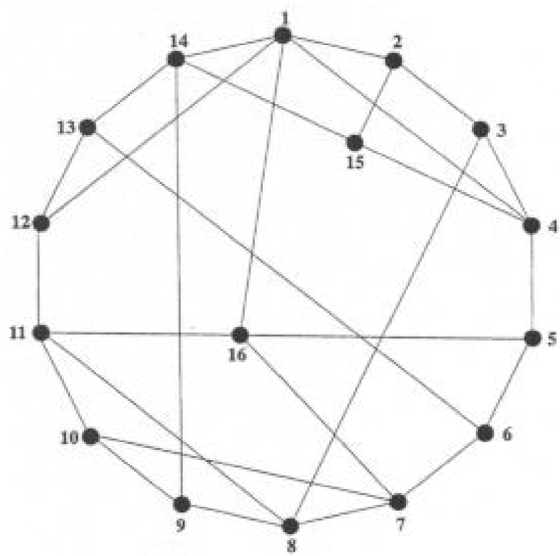
4 (a) Give an example of a path in the 8-cube from vertex 00000000 to vertex 11110000, of length 4.

(b) Give an example of a path in the 8-cube from 00000000 to 11110000, of length 8.

(c) Prove that there is no path in the 8-cube from 00000000 to 11110000 of odd length.

7] 5 (a) (i) Construct a breadth-first search tree for the graph G below, using vertex 1 as the root. At every stage, if there is a choice of vertices to enter the tree, choose the vertex of smallest label. With your tree, list the vertices in the order of entering the tree.

(ii) Use the breadth-first search tree in part (a) (i) to determine whether G is bipartite or not. If G is bipartite, give a bipartition of the vertices, and if G is not bipartite, give an odd cycle.



[5] (b) Prove that every graph with at most two odd cycles is 3-colourable.

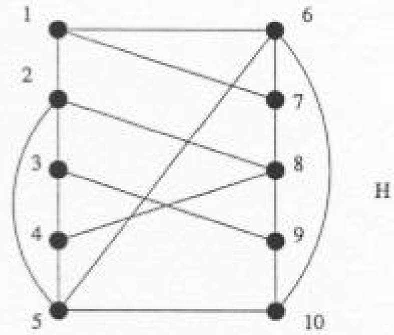
[2] 6 (a) State Euler's formula for a connected planar embedding.

(b) Suppose that a planar graph G with p vertices and $2p$ edges has a planar embedding in which every face has degree 3 or 4.

[7] (i) Prove that G has exactly 8 faces of degree 3.

(ii) Find the minimum value of p for which G exists, and give an example of a planar embedding of G with this minimum value of p.

[5] (c) Determine whether the graph H below is planar or not. Justify your answer.

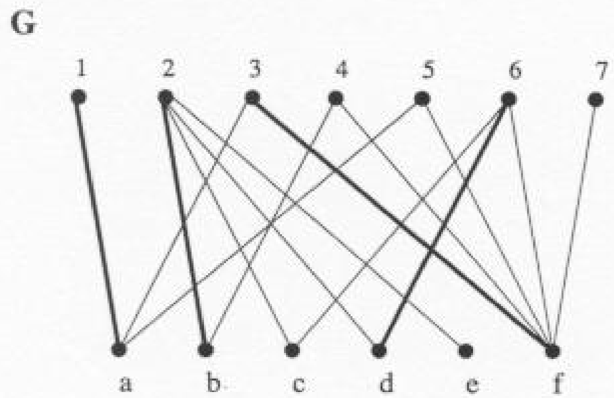


[2] 7. (a) State König's Theorem.

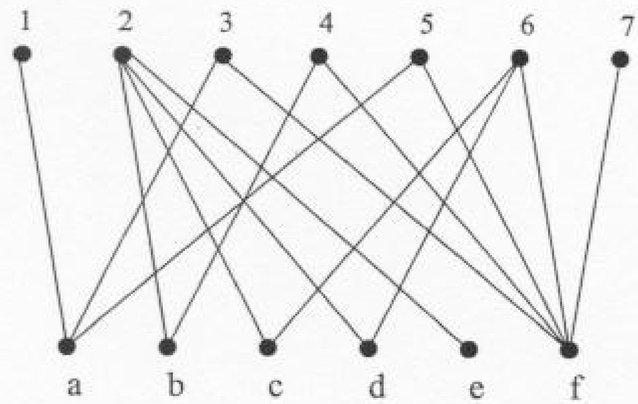
[6] (b) (i) Let J be a bipartite graph with $4k$ edges and maximum degree 4. Prove that J has a matching of size at least k, for all $k \geq 1$.

(ii) Give an example of a bipartite graph with 16 edges and maximum degree 4 that has no matching of size 5.

[6] (c) Beginning with the matching indicated by the thickened edges in the graph G below, apply the bipartite matching algorithm to find a maximum matching M and a minimum cover C. Show the sets X and Y in the table below. Notice that G is bipartite with bipartition (A, B), where $A = \{a, b, c, d, e, f\}$ and $B = \{1, 2, 3, 4, 5, 6, 7\}$. You may use the two copies of G printed on the next page as working copies in solving this problem.



X	Y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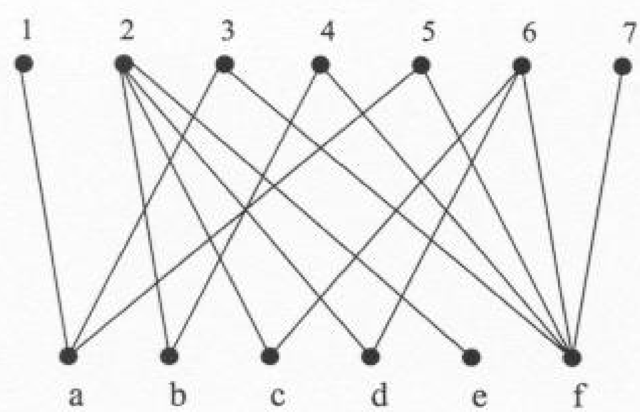
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8. Determine whether each of the following statements is true or false. If true, give a short proof; if false, provide a counterexample.

[4] (a) There is no tree having exactly 7 vertices of degree 1, 3 of degree 3 and 2 of degree 4.

[4] (b) If every vertex of graph G has even degree, then G has no bridges.

[4] (c) If every vertex of G has odd degree greater than one, then G has no bridges.