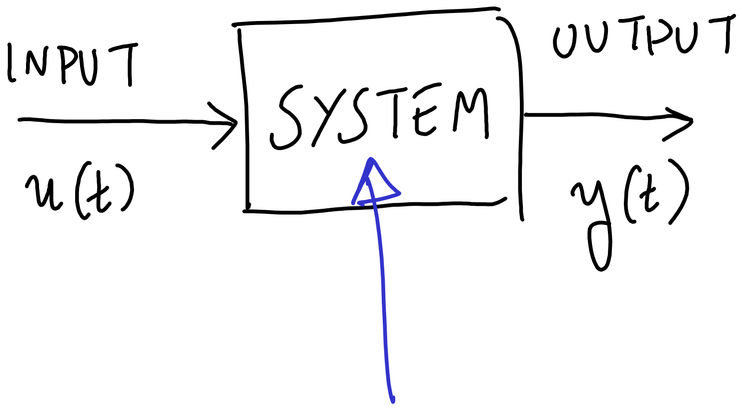
STATE SPACE MODELS



$u(t) \in \mathbb{R}^m$ INPUT at time t

$y(t) \in \mathbb{R}^p$ OUTPUT at time t

$$\underbrace{\begin{pmatrix} u \in \mathbb{R}^m \\ y \in \mathbb{R}^p \end{pmatrix}}$$

STATE $x(t) \in \mathbb{R}^n$

$x(t_0)$

$u(t),\ t \in [t_0, t_1]$ $\longrightarrow$ $x(t_1)$

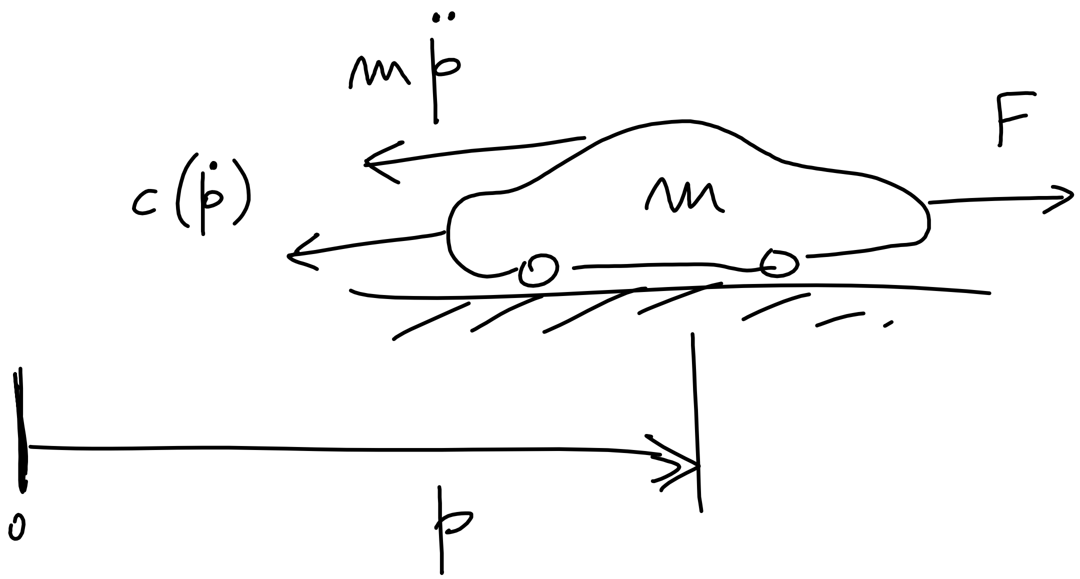
$$\dot{x}(t) = f(\underbrace{x(t)}_{\in \mathbb{R}^n}, \underbrace{u(t)}_{\in \mathbb{R}^m})$$

$$\begin{cases} \dot{x} = f(x, u) \\ y = h(x) \end{cases}$$

$$f : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$$
$$: (x, u) \longmapsto \dot{x}$$

$$h : \mathbb{R}^m \longrightarrow \mathbb{R}^p$$
$$: x \longmapsto y$$

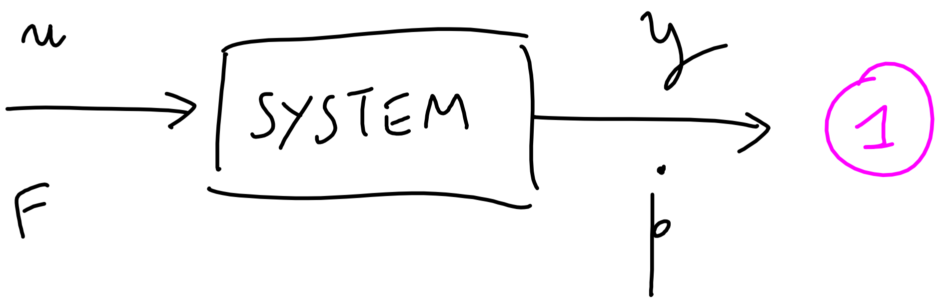
EXAMPLE Autonomous vehicle

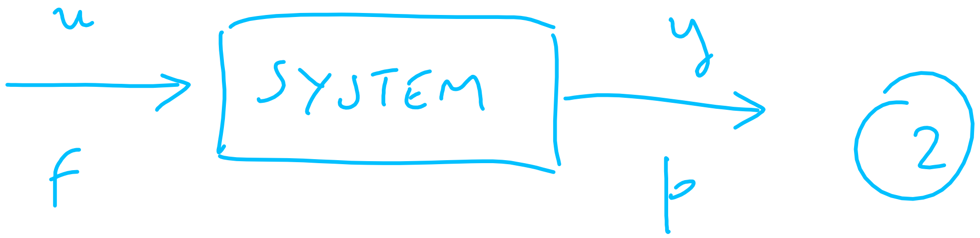


$$m\ddot{p} + c(\dot{p}) = F$$

$$x = \begin{bmatrix} p \\ \dot{p} \end{bmatrix} \in \mathbb{R}^2 \qquad (n = 2)$$

$$u = F$$





$$\dot{x} = f(x,u) = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dot{p} \\ \ddot{p} \end{bmatrix}$$

$$= \begin{bmatrix} x_2 \\ -\dfrac{c(x_2)}{m} + \dfrac{u}{m} \end{bmatrix}$$

1 $y = x_2 = h(x)$

2 $y = x_1 = h(x)$

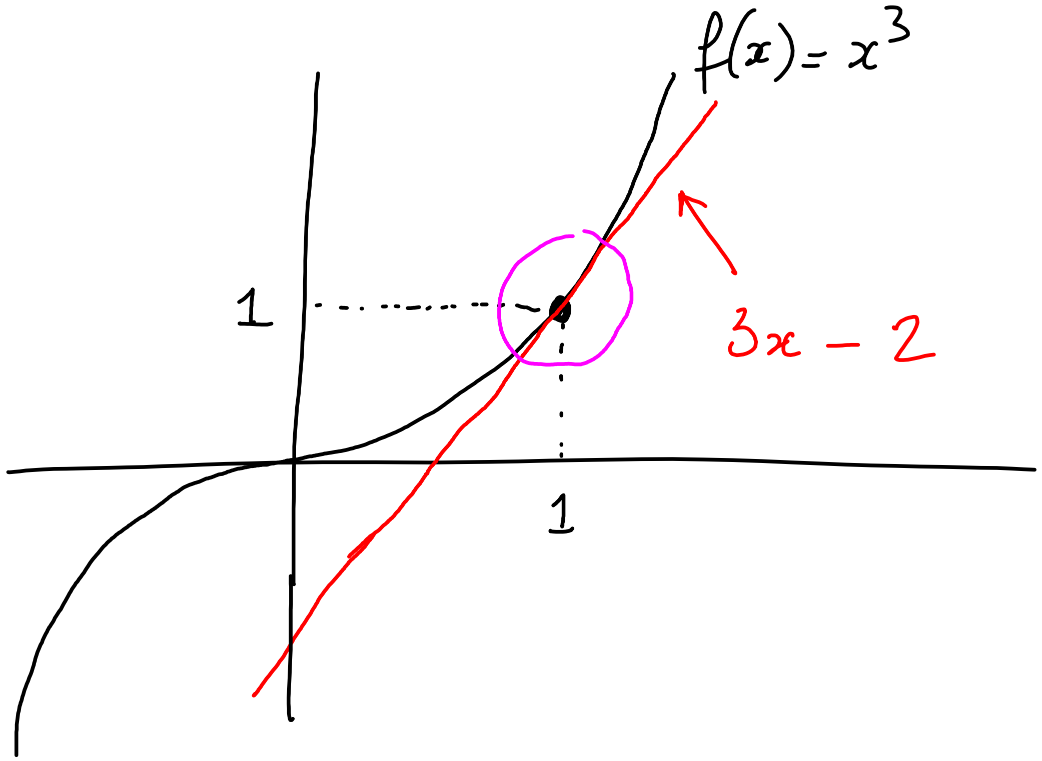
LINEAR SYSTEMS

$$\begin{cases} \dot{x} = f(x, u) \\ y = h(x) \end{cases}$$

FIND A LINEAR APPROXIMATION OF THE SYSTEM

EXAMPLE

LINEARIZE A FUNCTION



$$f(x) = x^3$$

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{d^n f}{dx^n} \right|_{x=\bar{x}} (x - \bar{x})^n$$

$$= f(\bar{x}) + f'(\bar{x})(x - \bar{x}) + \frac{1}{2} f''(\bar{x})(x-\bar{x})^2$$

$+ \dots$ higher order terms (h.o.t.)

$$
\begin{aligned}
f(x) &= f(\bar{x}) + f'(\bar{x})(x-\bar{x}) + \text{h.o.t.} \\
&= \bar{x}^3 + 3\bar{x}^2(x-\bar{x}) + \text{h.o.t.} \\
&= \underline{1} + 3(x-1) \\
&= 3x - 2
\end{aligned}
$$

$$\begin{cases} \dot{x} = f(x,u) \\ y = h(x) \end{cases} \qquad \text{around} \qquad \begin{aligned} x &= \bar{x} \\ u &= \bar{u} \end{aligned}$$

$x \in \mathbb{R}^n$

$u \in \mathbb{R}^m$

$y \in \mathbb{R}^p$

$$\Big\downarrow \text{ LINEARIZE } \quad (\text{1st order Taylor expansion})$$

$$\begin{cases} \underbrace{\dot{x}}_{m\times 1} = \underbrace{f(\bar{x},\bar{u})}_{m\times 1} + \underbrace{\overset{A}{\boxed{\left.\dfrac{\partial f}{\partial x}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}}}}}_{n\times n} \underbrace{(x-\bar{x})}_{n\times 1} + \underbrace{\overset{B}{\boxed{\left.\dfrac{\partial f}{\partial u}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}}}}}_{n\times m} \underbrace{(u-\bar{u})}_{m\times 1} + \text{h.o.t.} \\[2em] \underbrace{y}_{p\times 1} = \underbrace{h(\bar{x})}_{p\times 1} + \underbrace{\boxed{\left.\dfrac{\partial h}{\partial x}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}}}}_{p\times n \;\; C} \underbrace{(x-\bar{x})}_{n\times 1} + \underbrace{\boxed{\left.\dfrac{\partial h}{\partial u}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}}}}_{p\times m \;\; D} \underbrace{(u-\bar{u})}_{m\times 1} + \text{h.o.t.} \end{cases}$$

$$\begin{cases} \dot{x} \approx f(\bar{x},\bar{u}) + A(x-\bar{x}) + B(u-\bar{u}) \\ y \approx h(\bar{x}) + C(x-\bar{x}) + D(u-\bar{u}) \end{cases}$$

$$\delta x = x - \bar{x}$$

$$\delta u = u - \bar{u}$$

$$\begin{cases} \dot{x} - f(\bar{x},\bar{u}) \approx A\,\delta x + B\,\delta u \\ y - h(\bar{x}) \approx C\,\delta x + D\,\delta u \end{cases}$$

EQUILIBRIUM POINTS

Def $(\bar{x},\bar{u}) \in \mathbb{R}^n \times \mathbb{R}^m$ is an equilibrium configuration of $\dot{x} = f(x,u)$ if $f(\bar{x},\bar{u}) = 0$. $\bar{x} \in \mathbb{R}^n$ is called an equilibrium point.

1

$x(t) \equiv \bar{x}$ solves $\underbrace{\dot{x} = f(x, u)}$

$u(t) \equiv \bar{u}$

2

Systems linearized about eql. config. tells us some important properties of the nonlinear system.

3

$$\begin{cases} \dot{x} \approx f(\bar{x},\bar{u}) + A\,\delta x + B\,\delta u \\ y \approx h(\bar{x}) + C\,\delta x + D\,\delta u \end{cases}$$

$(\bar{x},\bar{u})$ eql. config

$$f(\bar{x},\bar{u}) = 0$$

$$\dot{\delta x} = \dot{x} - \dot{\bar{x}}$$

$$= \dot{x} - \cancel{f(\bar{x},\bar{u})}$$

$$= \dot{x}$$

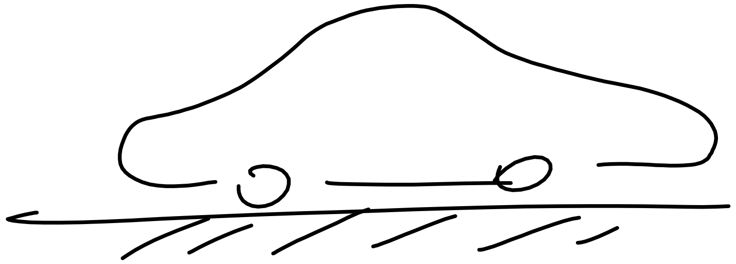
$$\delta y := y - h(\bar{x})$$

$$\boxed{\begin{cases} \dot{\delta x} \approx A\,\delta x + B\,\delta u \\ \delta y \approx C\,\delta x + D\,\delta u \end{cases}}$$

$$\begin{cases} \dot{x} = A x + B u \\ y = C x + D u \end{cases}$$

LINEAR TIME-INVARIANT SYSTEM (LTI)

EXAMPLE



$$\begin{cases} \dot{x} = \begin{bmatrix} x_2 \\ -\dfrac{c(x_2)}{m} + \dfrac{u}{m} \end{bmatrix} \\ \boxed{y = x_1} \end{cases}$$

ASSUME

$c(0) = 0$

$\dfrac{\partial c}{\partial x_2}(0) = 1$

$$f(x,u) = \begin{bmatrix} x_2 \\ -\dfrac{c(x_2)}{m} + \dfrac{u}{m} \end{bmatrix} \qquad x \in \mathbb{R}^2, \; u \in \mathbb{R}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\dfrac{1}{m}\dfrac{\partial c}{\partial x_2}(x_2) \end{bmatrix} =: A$$

$$\frac{\partial f}{\partial u} = \begin{bmatrix} \dfrac{\partial f_1}{\partial u} \\ \dfrac{\partial f_2}{\partial u} \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} =: B$$

EQL. CONFIG.

$$f(\bar{x}, \bar{u}) = 0$$

$$\begin{bmatrix} \bar{x}_2 \\ -\dfrac{c(\bar{x}_2)}{m} + \dfrac{\bar{u}}{m} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} \bar{x}_2 = 0 \\ \bar{u} = 0 \end{matrix}$$

$$c(\bar{x}_2)\Big|_{\bar{x}_2=0} = 0$$

$\bar{x}_1$??

ARBITRARY

Let's pick $\bar{x}_1 = 0$: $\quad \bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \bar{u} = 0$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -\dfrac{1}{m}\cancel{\dfrac{\partial c(0)}{\partial x_2}}^{1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\dfrac{1}{m} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix}$$

The vehicle, around $\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\bar{u} = 0$, can be modeled by:

$$\boxed{\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{m} \end{bmatrix} (x - \cancel{\bar{x}}) + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} (u - \cancel{\bar{u}}) \\ \\ y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}} (x - \cancel{\bar{x}}) + \underbrace{0} (u - \bar{u}) \end{cases}}$$

LTI model of the vehicle

$$h(x) = x_1$$

$$\overbrace{\frac{\partial h}{\partial x} = \begin{bmatrix} 1 & 0 \end{bmatrix}} =: C$$

$$\frac{\partial h}{\partial u} = 0 \;\overset{!!}{=:}\; D$$