UNIVERSITY OF WATERLOO
FINAL EXAMINATION
WINTER TERM 2003

Surname: ________________________

First Name: ________________________

Id.#: ________________________

Course Number	MATH 239
Course Title	Introduction to Combinatorics
Instructor	01 Professor Menezes 8:30 ☐ 02 Professor Irving 10:30 ☐ 03 Professor Goulden 1:30 ☐
Date of Exam	April 7, 2003
Time Period	2-5 p.m.
Number of Exam Pages (including this cover sheet)	14 pages
Exam Type	Closed Book

ADDITIONAL INSTRUCTIONS:

1. Write your name and Id.# in the blanks above. Put a check mark in the box next to your instructor's name and lecture time.
2. There are 14 pages to this exam including the cover page. Please be sure you have all 14 pages.
3. Answer each of the problems in the space provided; use the back of the previous page for additional space.
4. You may only use a non-programmable calculator. Show the reasoning used in any calculation.

Problem	Value	Mark Awarded	Problem	Value	Mark Awarded
1	11		6	14	
2	10		7	10	
3	9		8	13	
4	11		9	12	
5	10		TOTAL	100	

[2] 1(a) Give an example of a composition of 31 into 7 parts, where each part is congruent to 1 modulo 3.

[6] (b) Determine the number of compositions of 31 into 7 parts, where each part is congruent to 1 modulo 3.

[3] (c) Determine the number of compositions of 29 into 7 parts, where each part is congruent to 1 modulo 3.

[7] 2(a) Let a_n, $n \geq 0$, be the number of $\{0,1\}$–strings of length n in which neither 00011 nor 00111 occur as substrings. Prove that

$$\sum_{i\geq 0} a_i x^i = \frac{1}{1-2x+2x^5-x^6}.$$

[3] (b) From part (a), deduce a linear recurrence equation for a_n, with initial conditions to uniquely determine $\{a_n\}_{n\geq 0}$.

[9] 3. Solve the recurrence equation $c_n = c_{n-1} + 5\,c_{n-2} + 3\,c_{n-3}$, with initial conditions $c_0 = 1,\ c_1 = -6,\ c_2 = -5$.

[5] 4(a) Let n_i be the number of vertices of degree i in a tree with at least 2 vertices. Prove that

$$n_1 = 2 + n_3 + 2\,n_4 + 3\,n_5 + \ldots.$$

[6] (b) What is the smallest number M of vertices of degree 1 in a tree with 3 vertices of degree 3 and 2 vertices of degree 5 (and any number of vertices of degree i, for $i \neq 3, 5$) ? Justify your answer by proving that every such tree has at least M vertices of degree 1, and by drawing such a tree with exactly M vertices of degree 1.

[7] 5(a) Construct a breadth-first search tree for the graph H below, using the vertex labelled 1 as the root vertex. When considering the vertices adjacent to the vertex being examined, add them to the tree in increasing order of label. Give a list of the vertices in the order that they join the tree.

[3] (b) Use the breadth-first search tree from (a) to determine whether H is bipartite or not. If H is bipartite, find a bipartition; if H is not bipartite, find an odd cycle.

[6] 6(a) For each of the graphs below, determine if it is planar or not.

[5] (b) Prove that a connected planar embedding with all faces of degree 3, and all vertices of degree either 4 or 5, must have at least $s = 8$ faces.

[3] (c) Draw a connected planar embedding with all faces of degree 3, and all vertices of degree either 4 or 5, with exactly $s = 8$ faces.

[2] 7(a) State König's Theorem for bipartite graphs.

[8] (b) Determine a maximum matching and a minimum cover in the graph G below, by applying the maximum matching algorithm, beginning with the matching indicated by the thick edges in G. You may find the extra drawings of G helpful in any iterations of the matching algorithm that are required.

7(b) (Continued)

8(a) For each of the following statements, if the statement is true in general, give a proof, or, if the statement is not always true, give a counterexample.

[4] (i) Every connected graph with p vertices and p edges, for $p \geq 2$, contains exactly one cycle.

[4] (ii) For a graph of girth 5, the size of the minimum cover is never more than 5 larger than the size of the maximum matching.

[5] (b) Find a 3-colouring of the graph G below. Can it be coloured using fewer colours? Why or why not?

9. Let A_n, $n \geq 1$, be the graph whose vertices are the $\{0,1\}$–strings of length n, and two strings a and b are adjacent if the number of 1's in a plus the number of 1's in b is equal to n. (HINT: It may be useful to consider the two cases n even and n odd separately in parts (c) and (d) below.)

[3] (a) Draw the graphs $A_1,\ A_2,\ A_3$.

[3] (b) Determine all values of $n \geq 1$ for which A_n is connected.

[3] (c) Determine all values of $n \geq 1$ for which A_n is bipartite.

[3] (d) Determine all values of $n \geq 1$ for which A_n is planar.