Much of the following material is based on this reference:

Murray R. Spiegel, *Applied Differential Equations*, 2nd ed., Prentice-Hall, 1967.

Introduction

A differential equation is an equation containing derivatives of some dependent variables with respect to an independent variable.

Example: Constant percentage growth rates.

Consider

$$\frac{dx}{dt} = a x \, ,$$

where a is a positive real number.

The independent variable t might represent time...

... in which case, the equation says that the value of the dependent variable x grows at a rate proportional to the value of x.

- So x might represent, say, a population of animals in an environment of abundant resources: if there's plenty of food, and plenty of shelter from hazards, then the rate of population growth might be expected to be proportional to the population.

- Alternatively, x might represent the gross domestic product (GDP) of an economy with a constant percentage rate of annual growth. If GDP grows each year by a constant percentage, then, ignoring annual cycles and so forth, one might suppose that at every instant, growth is proportional to GDP.

- When is this equation satisfied?

- We can see what functions $x(t)$ satisfy this particular equation just by rearranging and integrating:

$$\frac{dx}{dt} = a x$$

$$\iff \frac{dx}{x} = a\,dt$$

$$\iff \int \frac{dx}{x} = \int a\,dt$$

$$\iff \ln x + c_1 = at + c_2$$

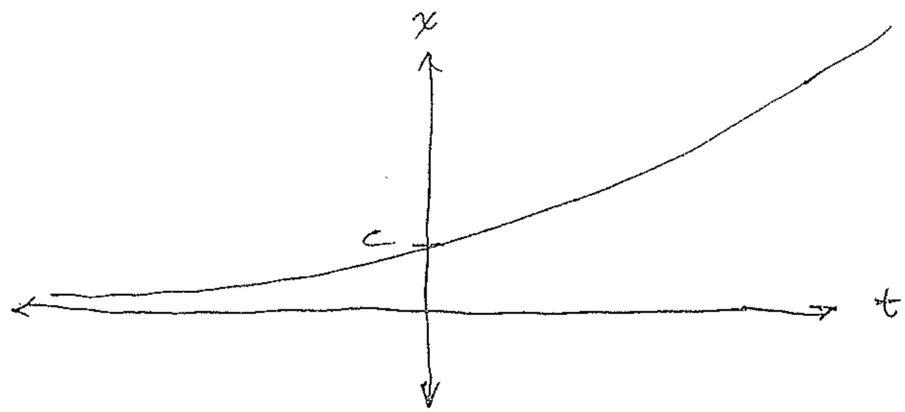
$$\iff \ln x = at + c_3$$

$$\iff x = e^{(at + c_3)}$$

$$\iff x = c\,e^{at}$$

(where c_1 and c_2 are constants of integration, $c_3 = c_2 - c_1$, and $c = e^{c_3}$).

- So our differential equation implies exponential growth of the value of x:



- In the economic example, suppose the annual growth rate is a constant 3% (of GDP) and t is in years.

Then

$$x = c\,(1.03)^t$$

$$= c\,e^{\ln(1.03)\,t}$$

so

$$a = \ln(1.03)$$

What about c?

- Setting $t = 0$,

$$x(0) = c\, e^{\ln(1.03) \cdot 0}$$
$$= c$$

So, in general

$$x = x(0)\, e^{\ln(1.03)\, t}$$

the value $x(0)$ is called an <u>initial condition</u>.

- Of course, we could find the value of c given the value of $x(t)$ at any time t:

Setting $t = 5$,

$$x(5) = c\, e^{\ln(1.03)\, 5}$$
$$= c\,(1.03)^5$$

$$\Longleftrightarrow \quad c = (1.03)^{-5}\, x(5)$$

- Specifications of values of $x(t^*)$ (and possibly of the values of derivatives of x) at a specific value t^* of the independent variable are called [4]<u>initial conditions</u>.

- Whatever the values of the constants a and c, the form of $x(t)$ shows that x grows without bound with increasing t.

- What if there are limits to growth?

Let's add another term to our equation:

$$\frac{dx}{dt} = ax - bx^2 \qquad (a, b > 0)$$

- When x is small, this equation approximates the previous one ...

  ... but when x is sufficiently large, the new term dominates the right-hand side (RHS).

- If x "starts out" small, it should grow exponentially until the bx^2 term becomes appreciable ...

  ... and eventually, the two RHS terms should cancel out.

At what level does this happen?

$$ax - bx^2 = 0$$

$$\Longleftrightarrow \quad x = 0 \quad \text{or} \quad x = \frac{a}{b}$$

So intuitively, the growth of x should "level off" at $\frac{a}{b}$.

- In the population example, $\frac{a}{b}$ might represent the largest population that the environment can sustain.

What's the form of $x(t)$ in this case?

$$\frac{dx}{dt} = ax - bx^2$$

$$\frac{dx}{ax - bx^2} = dt$$

$$\int \frac{dx}{ax - bx^2} = \int dt$$

How to integrate the left-hand side (LHS)?

- One way is to rewrite $\dfrac{1}{ax - bx^2}$

5 by the <u>method of partial fractions</u>.

$$\frac{1}{ax - bx^2} = \frac{1}{x(a - bx)}$$

$$= \frac{A}{x} + \frac{B}{a - bx} ,$$

for suitable values of A and B.

- An easy way to do the algebra is due to Heaviside:

  - to find A, multiply through by x, and then set $x = 0$

$$\frac{\cancel{x}}{\cancel{x}(a - b\cancel{x}_{\to 0})} = A + \cancel{\frac{Bx}{a - bx}}_{\to 0}$$

So $A = \dfrac{1}{a}$

  - for B, multiply by $a - bx$, and set $a - bx = 0$, or $x = \dfrac{a}{b}$:

$$\frac{\cancel{a-bx}}{x\,(\cancel{a-bx})} = A\,\cancel{\frac{a-bx}{x}}\to 0 + B$$

$$\text{So} \quad B = \left.\frac{1}{x}\right|_{x=\frac{a}{b}}$$

$$\Longleftrightarrow \quad B = \frac{b}{a}$$

This gives

$$\frac{1}{ax - bx^2} = \frac{1/a}{x} + \frac{b/a}{a-bx}$$

Plugging this into our equation, we have

$$\int \frac{1/a}{x}\,dx + \int \frac{b/a}{a-bx}\,dx = \int dt$$

$$\frac{1}{a}\ln|x| + \frac{b}{a}\left(\frac{-1}{b}\right)\ln|a-bx| = t + c_1$$

$$\frac{1}{a}\left(\ln|x| - \ln|a-bx|\right) = t + c_1$$

$$\ln\frac{|x|}{|a-bx|} = at + c_2$$

$$\frac{|x|}{|a-bx|} = c_3\, e^{at}$$

Suppose $0 < x < \frac{a}{b}$. Then

$$x = c_3\, e^{at}(a-bx)$$

$$x = \frac{a\, c_3\, e^{at}}{1 + b c_3\, e^{at}}$$

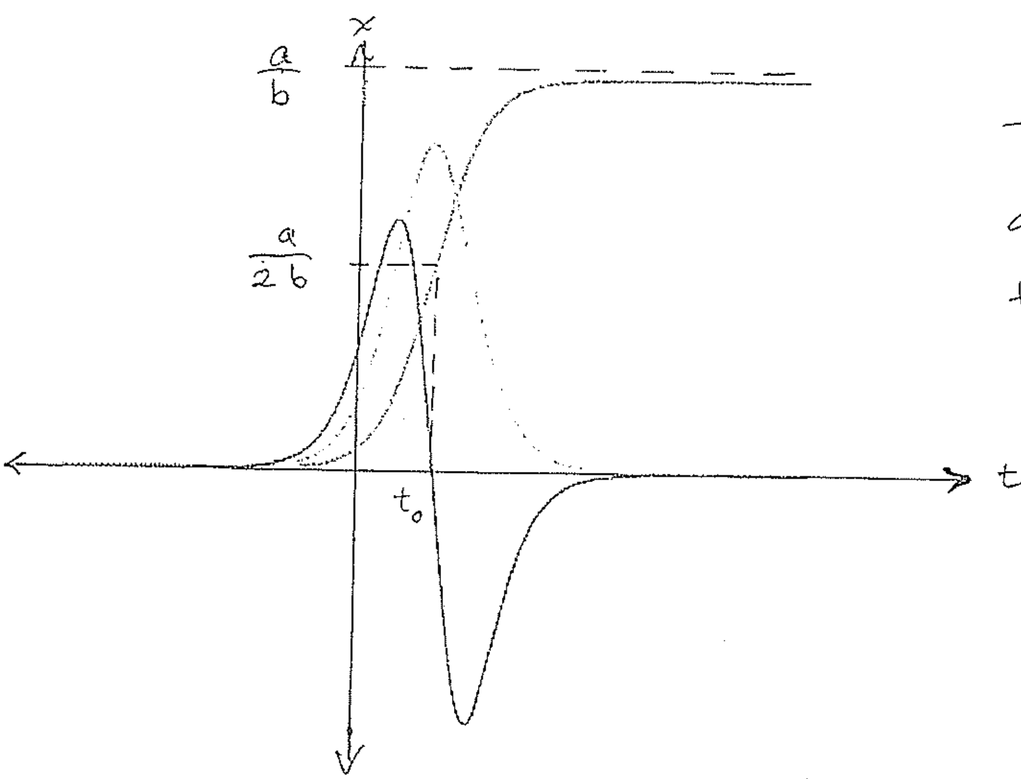
$$= \frac{a}{b}\,\frac{1}{1 + c\, e^{-at}}$$

Alternatively,

$$x = \frac{a}{b} \frac{1}{1 + e^{-a(t-t_0)}} .$$

where $e^{a t_0} = c \iff t_0 = \dfrac{\ln c}{a}$.

Our differential equation is known as the <u>Logistic Equation</u>, and $x(t)$ is the <u>logistic curve</u>:



The logistic curve and its 1st two derivatives

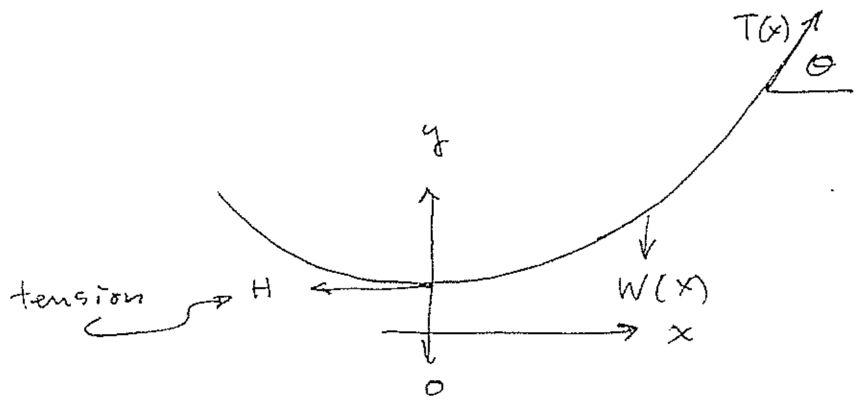
"Peak oil"

- the logistic equation is sometimes used as a simple model of resource depletion.
- Consumption initially increases exponentially (with population growth and economic growth) but must eventually level off.
- the original peak oil theory assumed that cumulative production satisfies the logistic equation.
- The rate of production must then be the 1st derivative of the logistic curve.
- Fitting the US production data predicted a peak at a time t_0 in the mid-seventies.
- Turned out to be accurate — at least for 'conventional' oil.

Lecture 2

The independent variable doesn't have to represent time:

Example: geometry of a hanging cable



Let the weight supported by the length of cable hanging between 0 and x be W(x). Then

$$T(x)\sin\theta = W(x)$$

$$T(x)\cos\theta = H$$

Dividing,

$$\tan\theta = \frac{W(x)}{H}$$

$$\Longleftrightarrow \quad \frac{dy}{dx} = \frac{W(x)}{H}$$

Now suppose that the cable supports a suspension bridge, so that $W(x)$ is directly proportional to x: $W(x) = wx$

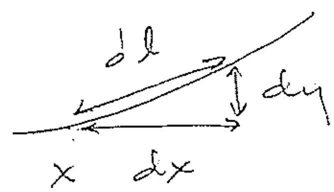
$$\frac{dy}{dx} = \frac{w}{H} x$$

$$dy = \frac{w}{H} x \, dx$$

$$y = \frac{w}{2H} x^2 + c$$

$\rightarrow$ the cable must have the shape of a parabola.

On the other hand, if the cable supports only its own weight, then $W(x)$ is proportional to the length l of cable from 0 to x:



$$dl = \sqrt{(dx)^2 + (dy)^2}$$

$$\frac{dl}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Now

$$\frac{dy}{dx} = \frac{W(x)}{H} = \frac{wl}{H}$$

$$\Longrightarrow \quad \frac{d^2y}{dx^2} = \frac{w}{H}\frac{dl}{dx} = \frac{w}{H}\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Let $\quad s = \dfrac{dy}{dx} \quad \Longrightarrow \quad \dfrac{ds}{dx} = \dfrac{d^2y}{dx^2}$

Then

$$\frac{ds}{dx} = \frac{w}{H} \sqrt{1+s^2}$$

$$\frac{ds}{\sqrt{1+s^2}} = \frac{w}{H}\, dx$$

Integrating,

$$\ln\left(s + \sqrt{1+s^2}\right) = \frac{wx}{H} + c_1$$

$$s + \sqrt{1+s^2} = c_2\, e^{\frac{w}{H}x}$$

Bring in the initial condition

$$s(0) = \frac{dy}{dx}(0) = 0$$

$$\Longrightarrow \quad c_2 = 1$$

So

$$s + \sqrt{1+s^2} = e^{\frac{w}{H}x}$$

Solving for s,

$$\frac{dy}{dx} = s = \frac{e^{\frac{w}{H}x} - e^{-\frac{w}{H}x}}{2}$$

So

$$y = \frac{H}{W} \frac{e^{\frac{W}{H}x} + e^{-\frac{W}{H}x}}{2} + C_3$$

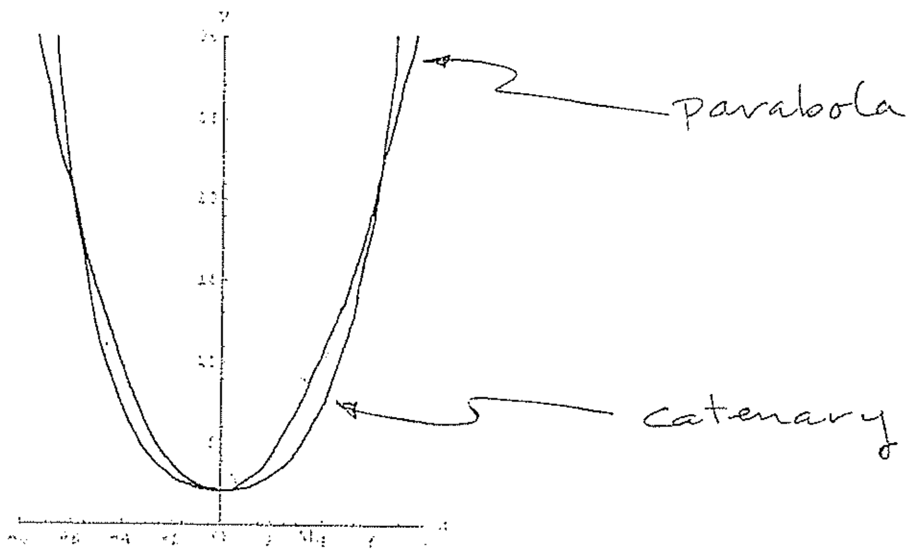
$$= \frac{H}{W} \cosh \frac{Wx}{H} + C_3$$

Choose the axes so that we have the initial condition $y(0) = \frac{H}{W}$.

Then $C_3 = 0$, and

$$y = \frac{H}{W} \cosh \frac{Wx}{H}$$

- the curve is called a <u>catenary</u>, Latin for chain.


parabola
catenary

In the absence of the initial conditions, we get an expression of the form

$$y = \frac{H}{w} \cosh\left(\frac{w}{H} x + c_4\right) + c_3$$

(where $c_4 = \ln c_2$).

The values of the constants might be determined not by initial conditions but by <u>boundary conditions</u> such as

$$y(0) = \frac{H}{w} \quad \& \quad y'\left(10\frac{H}{w}\right) = \sinh 9$$

– i.e., specifications of the value of y or its derivatives for different values of the independent variable x.

We then have

$$y'\left(10\frac{H}{W}\right) = \sinh\left(10 + c_4\right) = \sinh 9$$

$$\Longrightarrow \quad c_4 = -1 \qquad (\sinh \text{ is invertible})$$

... and therefore

$$y(0) = \frac{H}{W}\cosh(-1) + c_3 = \frac{H}{W}$$

$$\Longrightarrow \quad c_3 = \frac{H}{W}\left[1 - \cosh(-1)\right]$$

so

$$y = \frac{H}{W}\left[\cosh\left(\frac{W}{H}x - 1\right) + \left[1 - \cosh(-1)\right]\right]$$

$$= \frac{H}{W}\left[\cosh \frac{W}{H}\left(x - \frac{H}{W}\right) + \left[1 - \cosh(-1)\right]\right]$$

If boundary conditions are specified, the problem of solving the equation + the boundary conditions is called a boundary value problem . . .

. . . if initial conditions are specified, it's an initial value problem.

This course is mainly concerned with initial value problems.

Notes :

- all the examples we've seen so far have been relatively simple: they can be solved by the method of <u>separation of variables</u> – putting different variables on different sides of the equation

  - e.g.

$$\frac{ds}{\sqrt{1+s^2}} = \frac{w}{H}\,dx$$

  and integrating.

- Such equations are special cases of <u>exact</u> differential equations, which can be solved by integration.

- A variety of methods of solving differential equations involve putting them into forms in which they can be solved by integration.

- The last example was (a simple case of) a <u>second-order</u> differential equation — it contained a second derivative:

$$\frac{d^2y}{dx^2} = \frac{w}{H}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

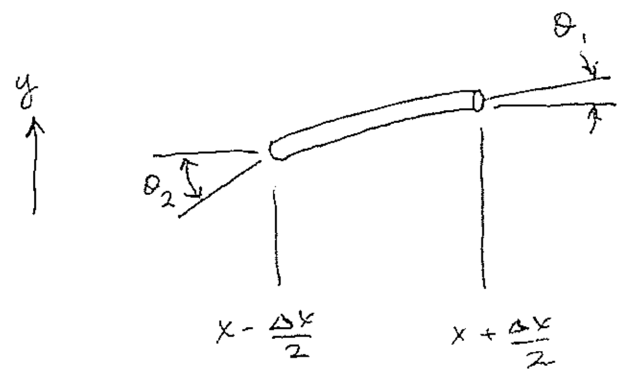
- The other examples were <u>1^{st}-order</u> differential equations.

- This course will only cover <u>ordinary differential equations</u> (ODEs), and not <u>partial differential equations</u> (PDEs), which contain partial derivatives.

PDEs arise when there are multiple independent variables.

For example, the equation of motion of a guitar string requires independent variables for both time and space

- assume uniform tension T (no lengthwise vibration)
- assume mass per unit length = ρ



- For small displacements in the y direction we can write ma = F as follows:

$$\rho \Delta x \frac{\partial^2 y}{\partial t^2} = T \sin\theta_1 - T\sin\theta_2$$

$$\simeq T\tan\theta_1 - T\tan\theta_2$$

$$= T \left.\frac{\partial y}{\partial x}\right|_{x-\frac{\Delta x}{2}} - T \left.\frac{\partial y}{\partial x}\right|_{x+\frac{\Delta x}{2}}$$

Letting Δx become infinitesimal,

$$\rho \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2}$$

$$\Longleftrightarrow \quad \frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho} \frac{\partial^2 y}{\partial x^2}$$

- the <u>wave equation</u> (which also arises in many other applications).

- some solutions:

$$y = A \sin k \left(x - \sqrt{\frac{T}{\rho}}\, t \right)$$

(a wave travelling in the x direction, with speed $\sqrt{\frac{T}{\rho}}$, and angular frequency $k\sqrt{\frac{T}{\rho}}$).

- boundary conditions:

$$y(0,t) = y(L,t) = 0, \quad \text{for all } t$$

(ends of string are fixed).

- Boundary conditions are satisfied by solutions like this (among others):

$$y = A_+ \sin k\left(x - \sqrt{\tfrac{T}{\rho}}\, t\right) + A_- \sin k\left(x + \sqrt{\tfrac{T}{\rho}}\, t\right)$$

- by the first boundary condition ($y(0,t) = 0$),

$$A_+ = A_-$$

(waves of equal amplitude travelling in opposite directions → "standing" wave).

- by the second boundary condition ($y(L,t) = 0$),

$$A_+ \sin k\left(L - \sqrt{\tfrac{T}{\rho}}\, t\right) + A_- \sin k\left(L + \sqrt{\tfrac{T}{\rho}}\, t\right) = 0, \quad \forall t$$

$$\Longrightarrow \quad kL = \pm n\pi, \quad n = 0, 1, 2, \ldots$$

$$\Longleftrightarrow \quad k = \pm \frac{n\pi}{L}, \quad n = 0, 1, 2, \ldots$$

→ waves can only have certain frequencies :

$$\frac{\frac{n\pi}{L}\sqrt{\frac{T}{\rho}}}{2\pi} = \frac{n}{L}\sqrt{\frac{T}{\rho}}$$

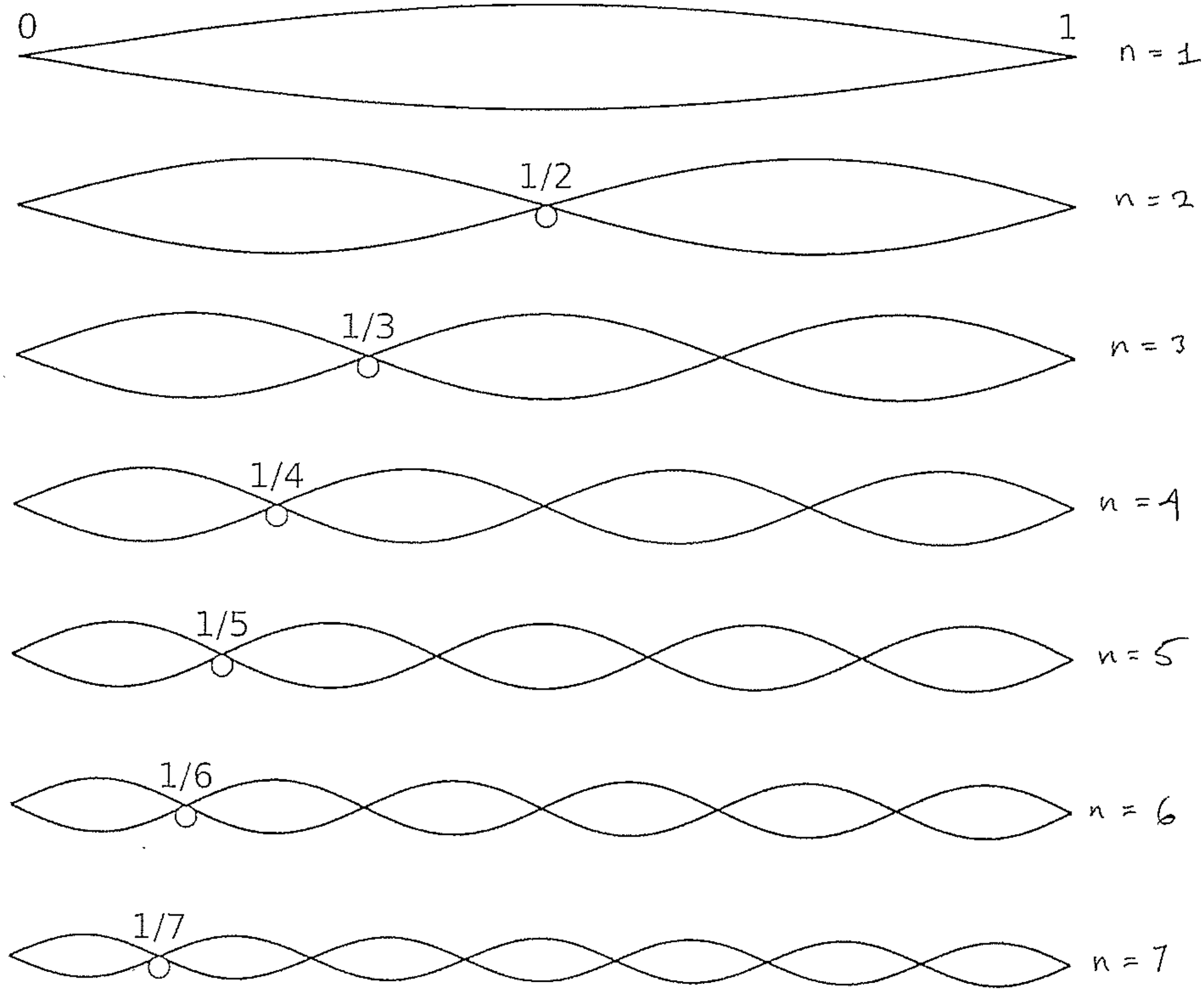
$n = 1$: "fundamental frequency"

$n > 1$: "harmonics"

- This equation is linear, which means that any linear combination of solutions is itself a solution . . .

. . . so the sound of the string may include any combination of the above frequencies.

- But this course will focus on linear ODEs.



Vibrations at fundamental frequency, and first six harmonics