$$u(t) \longrightarrow \boxed{\begin{cases}\dot{x}(t) = f(x(t), u(t)) \\ y(t) = h(x(t))\end{cases}} \longrightarrow y(t)$$

LTI

$$u(t) \longrightarrow \boxed{\begin{cases}\dot{x} = Ax + Bu \\ y = Cx + Du\end{cases}} \longrightarrow y(t)$$

$$\left.\frac{\partial h}{\partial u}\right|_{\substack{x=\bar{x} \\ u=\bar{u}}}$$

1. Obtain linear diff. eqns $\quad u(t) \longrightarrow \boxed{\text{diff. eqns}} \longrightarrow y(t)$

2. LAPLACE TRANSFORM diff eqns $\quad U(s) \longrightarrow \boxed{\text{algebr. eqns}} \longrightarrow Y(s)$

3. To find the response of the system, $y(t)$, to an input $u(t)$

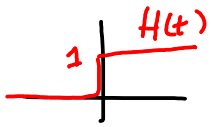
   3a. LT $u(t)$ into $U(s)$

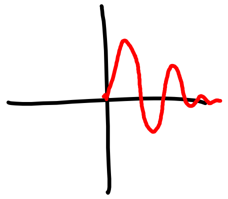
   3b. Compute $Y(s)$ algebraically

   3c. Inverse LT $Y(s)$ into $y(t)$

<u>Recall</u> LT of $f(t)$

$$\mathcal{L}\{f(t)\} = F(s) = \int_{0^-}^{\infty} f(t)\, e^{-st}\, dt$$

$f(t)$	$F(s)$
IMPULSE 	1
STEP 	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$

$f(t)$	$F(s)$
 $e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
$\frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi \omega_n t} \sin\left(\sqrt{1-\xi^2}\,\omega_n t\right)$, $\xi < 1$	$\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$

PROPERTIES OF LT

① LINEARITY $\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$

② TRANSLATION in t $\mathcal{L}\{f(t-\tau)\} = e^{-\tau s} F(s)$

in s $\mathcal{L}\{e^{\alpha t} f(t)\} = F(s-\alpha)$

③ DIFFERENTIATION in t $\mathcal{L}\{\frac{d}{dt} f(t)\} = s F(s) - f(0^-)$

in s $\mathcal{L}\{t f(t)\} = -\frac{dF(s)}{ds}$

④ INTEGRATION $\mathcal{L}\left\{\int_{0^-}^{t} f(\tau) d\tau\right\} = \frac{1}{s} F(s)$

⑤ CONVOLUTION $\mathcal{L}\{f_1(t) * f_2(t)\} = F_1(s) F_2(s)$

$$\int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau$$

6 INITIAL VALUE THEOREM $\quad f(0^+) = \lim_{s \to \infty} s\,F(s)$

FINAL VALUE THEOREM $\quad f(\infty) = \lim_{s \to 0} s\,F(s)$

LONG-TERM BEHAVIOR WITHOUT SOLVING THE DIFF. EQN. DESCRIBING THE SYSTEM

$$u(t) \longrightarrow \boxed{\text{LTI SYSTEM}} \longrightarrow y(t)$$

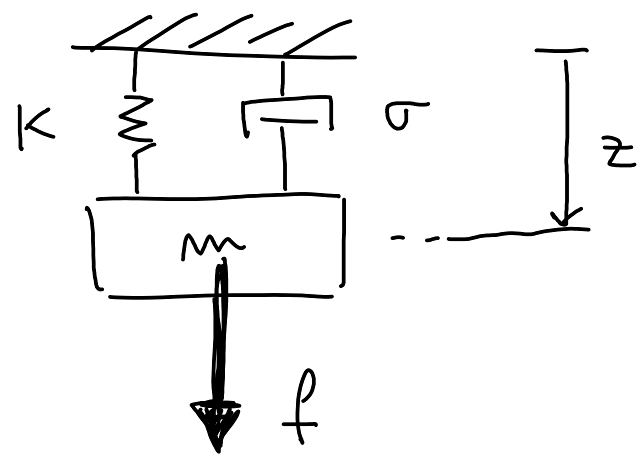
The <u>transfer function</u> (TF) of an LTI system is the ratio $\boxed{\dfrac{Y(s)}{U(s)}}$, when the LT is taken assuming <u>zero initial conditions.</u>

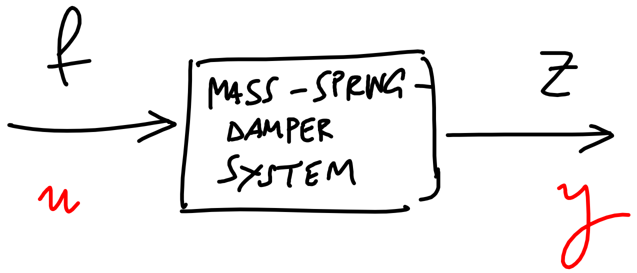
NO STATE

I/O RELATION

$$x(0) = 0$$

EXAMPLE





$$m\ddot{z} + \sigma\dot{z} + kz = f$$

$$m\ddot{y} + \sigma\dot{y} + ky = u \xrightarrow{LT} m\underbrace{s^2 Y(s)}_{\substack{\text{zero}\\ \text{I.C.}}} + \sigma\underbrace{s\,Y(s)}_{''} + kY(s) = U(s)$$

$$\frac{Y(s)}{U(s)} = \frac{1}{ms^2 + \sigma s + k}$$

T.F. $G(s)$ is <u>(real) rational</u> if

$$G(s) = \frac{b_m s^m + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}, \quad a_i, b_i \in \mathbb{R}$$

$G(s)$ is <u>proper</u> $\lim_{s \to \infty} G(s)$ exists in $\mathbb{C}$

(if $G(s)$ is rational : $n \geq m$)

$G(s)$ is <u>strictly proper</u> $\lim_{s \to \infty} G(s) = 0$

(if $G(s)$ is rational : $n > m$)

$p \in \mathbb{C}$ is a <u>pole</u> of $G(s)$ if $\lim_{s \to p} |G(s)| = \infty$

$z \in \mathbb{C}$ is a <u>zero</u> of $G(s)$ if $\lim_{s \to z} G(s) = 0$

<u>$G(s)$ rational</u> & numerator and denominator <u>are coprime</u>

poles of $G(s)$ are roots of denominator

zeros of $G(s)$ // // numerator

Factorization in "zero-pole-gain" form:

$$G(s) = K \frac{(s - z_1) \cdots (s - z_m)}{(s - p_1) \cdots (s - p_n)}$$

GAIN

EXAMPLE

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{ms^2+\sigma s+K}$$

$$= \frac{1}{s^2+3s+2}$$

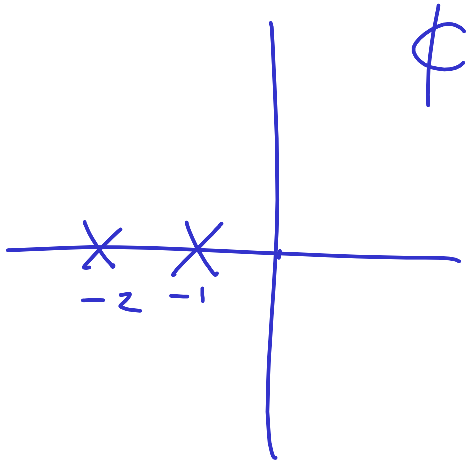
ASSUME

$m = 1$
$\sigma = 3$
$K = 2$

$G(s)$ is rational
proper
strictly proper $\left(\begin{array}{l} n = 2 \\ m = 0 \end{array}\right)$

$$G(s) = \frac{1}{(s+1)(s+2)} \longrightarrow$$

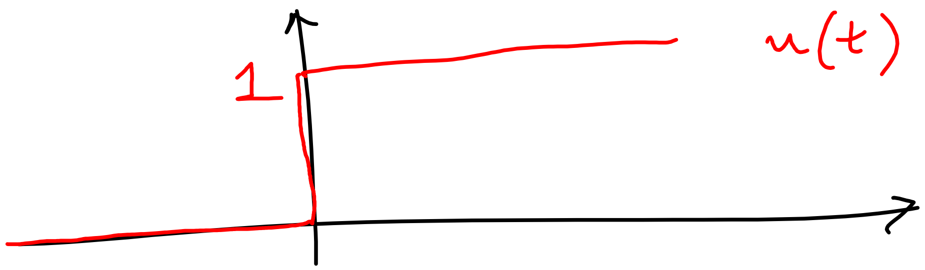
NO zeros
$p_1 = -1 \quad , \quad p_2 = -2$



EXERCISE

$$U(s) \longrightarrow \boxed{\frac{1}{s^2+3s+2}} \longrightarrow Y(s)$$

Compute the response, $y(t)$, of the mass-spring-damper system to a step force input



ⓐ LT $u(t) \longrightarrow U(s)$

ⓑ $Y(s) = G(s) \cdot U(s) \quad \left(= \overbrace{\frac{Y(s)}{U(s)}}^{G(s)} U(s) \right)$ ALGEBRAIC!

ⓒ Inverse LT $Y(s) \longrightarrow y(t)$