University of Waterloo
David R. Cheriton School of Computer Science

# MATH 213 – Advanced Mathematics for Software Engineers
# Final Exam, Spring 2009

August 8, 2009, 7:30-10:00 PM

**Instructor**: Dr. Oleg Michailovich

**Student's name**: ______________________________________________

**Student's ID #**: ______________________________________________

## Instructions:

- This exam has **2** pages.
- **No books and lecture notes are allowed on the exam**. Please, turn off your cell phones, PDAs, etc., and place your bags, backpacks, books, and notes under the table or at the front of the room.
- Please, place your **WATCARD** on the table, and fill out the exam attendance sheet when provided by the proctor after the exam starts.
- Question marks are listed by the question.
- Please, do not separate the pages, and indicate your Student ID at the top of every page.
- Be neat. Poor presentation will be penalized.
- **No questions will be answered during the exam**. If there is an ambiguity, state your assumptions and proceed.
- **No student can leave the exam room in the first 45 minutes or the last 10 minutes**.
- If you finish before the end of the exam and wish to leave, remain seated and raise your hand. A proctor will pick up the exam from you, at which point you may leave.
- When the proctors announce the end of the exam, put down your pens/pencils, close your exam booklet, and remain seated in silence. The proctors will collect the exams, count them, and then announce you may leave.

## Problem №1 (20%)

Given the vectors $\mathbf{u}$ and $\mathbf{v}$ defined as

$$\mathbf{u} = (1, 1, a, -2) \text{ and } \mathbf{v} = (2, b, -1, 1)$$

where $a$ and $b$ are real scalars,

a) determine $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, and the angle $\theta$ between the two vectors;

b) find conditions on $a$ and $b$ under which the vectors $\mathbf{u}$ and $\mathbf{v}$ become orthogonal;

c) prove that the Schwarz inequality $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|$ holds for any values of $a$ and $b$.

## Problem №2 (20%)

Let $\mathbb{S} = \mathbb{R}^3$ and let

$$\mathbf{e}_1 = (1, 0, 2), \ \mathbf{e}_2 = (-1, 1, 0.5), \ \mathbf{e}_3 = (-2, -2.5, 1)$$

be a basis that spans $\mathbb{S}$. Find the best approximation to $\mathbf{u} = (1, -1, 0)$ within $\operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2\}$ and within $\operatorname{span}\{\mathbf{e}_1, \mathbf{e}_3\}$ .

## Problem №3 (20%)

Find a particular solution to the differential equation

$$x'' - 4x' + 4x = f(t),$$

where $f(t)$ is a periodic function defined over one period as

$$f(t) = \begin{cases} 0, & -\pi \leq t < 0, \\ \sin(t), & 0 \leq t \leq \pi. \end{cases}$$

## Problem №4 (20%)

Find the eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$\begin{aligned} & y'' + \lambda y = 0 \\ & y(0) = 0, \ y'(0) + y'(\pi) = 0, \end{aligned}$$

and then expand the function $f(x) = 1$ using the resulting basis of eigenfunctions.

## Problem №5 (20%)

Using the properties of Fourier transform, find the function $y(u)$ that satisfies the following equation

$$\int_{-\infty}^{\infty} \frac{y(u)\, du}{(x-u)^2 + a^2} = \frac{1}{x^2 + b^2}, \tag{1}$$

where $0 < a < b$ are two scalars.
**Hint:** The integral on the left has the form of a convolution integral.

# Table of Fourier Transforms

| | $f(x)$ | $\hat{f}(\omega)$ |
|---|---|---|
| 1. | $\dfrac{1}{x^2+a^2} \quad (a>0)$ | $\dfrac{\pi}{a}e^{-a\lvert\omega\rvert}$ |
| 2. | $H(x)e^{-ax} \quad (\mathrm{Re}\, a>0)$ | $\dfrac{1}{a+i\omega}$ |
| 3. | $H(-x)e^{ax} \quad (\mathrm{Re}\, a>0)$ | $\dfrac{1}{a-i\omega}$ |
| 4. | $e^{-a\lvert x\rvert} \quad (a>0)$ | $\dfrac{2a}{\omega^2+a^2}$ |
| 5. | $e^{-x^2}$ | $\sqrt{\pi}e^{-\omega^2/4}$ |
| 6. | $\dfrac{1}{2a\sqrt{\pi}}e^{-x^2/(2a)^2} \quad (a>0)$ | $e^{-a^2\omega^2}$ |
| 7. | $\dfrac{1}{\sqrt{\lvert x\rvert}}$ | $\sqrt{\dfrac{2\pi}{\lvert\omega\rvert}}$ |
| 8. | $e^{-a\lvert x\rvert/\sqrt{2}}\sin\left(\dfrac{a}{\sqrt{2}}\lvert x\rvert+\dfrac{\pi}{4}\right) \quad (a>0)$ | $\dfrac{2a^3}{\omega^4+a^4}$ |
| 9. | $H(x+a)-H(x-a)$ | $\dfrac{2\sin\omega a}{\omega}$ |

Figure 1: Some common Fourier transforms