University of Waterloo
CS240 Spring 2022
Assignment 1

Due Date: Wednesday, May 18 at 5:00pm

The integrity of the grade you receive in this course is very important to you and the University of Waterloo. As part of every assessment in this course you must read and sign an Academic Integrity Declaration before you start working on the assessment and submit it **before the deadline of May 18** along with your answers to the assignment; i.e. **read, sign and submit A01-AID.txt now or as soon as possible**. The agreement will indicate what you must do to ensure the integrity of your grade. If you are having difficulties with the assignment, course staff are there to help (provided it isn't last minute).

The Academic Integrity Declaration must be signed and submitted on time or the assessment will not be marked.

Please read `https://student.cs.uwaterloo.ca/~cs240/s22/assignments.phtml#guidelines` for guidelines on submission. **Each question must be submitted individually to MarkUs as a PDF** with the corresponding file names: a1q1.pdf, a1q2.pdf, ... , a1q5.pdf . It is a good idea to submit questions as you go so you aren't trying to create several PDF files at the last minute.

Late Policy: Assignments are due at 5:00pm on Wednesday. Students are allowed to submit **one** late assignment, 2 days after the due date on Friday by 5:00pm. Assignments submitted after Friday at 5:00pm or Wednesday at 5:00pm (if you have already used your one late submission) will not be accepted for grading but may be reviewed (by request) for feedback purposes only.

Note: you may assume all logarithms are base 2 logarithms: $\log = \log_2$.

Question 1 [3+3+4=10 marks]

Provide a complete proof of the following statements from first principles (i.e., using the original definitions of order notation) if the statement is true. Otherwise simply state that it is false.

a) $7n^4 - 5n^2 + 6 \in O(n^4)$

Need to show that $7n^4 - 5n^2 + 6 \le cn^4$ for all $n \ge n_0$ where $c > 0$ and $n_0 \ge 0$.

$$
\begin{aligned}
7n^4 - 5n^2 + 6 &\leq 7n^4 + 6 \text{ since } -5n^2 \leq 0 \text{ for } n \geq 0 \\
&\leq 7n^4 + 6n^4 \text{ for } n \geq 1 \\
&= 13n^4
\end{aligned}
$$

Thus, let $c = 13$ and $n_0 = 1$ to show $7n^4 - 5n^2 + 6 \in O(n^4)$.

b) $7n^4 - 5n^2 + 6 \in \Omega(n^4)$

Need to show that $7n^4 - 5n^2 + 6 \geq cn^4$ for all $n \geq n_0$ where $c > 0$ and $n_0 \geq 0$.

$$
\begin{aligned}
7n^4 - 5n^2 + 6 &\geq 7n^4 - 5n^2 \text{ for } n \geq 0 \\
&\geq 7n^4 - 5n^4 \text{ for } n \geq 0 \\
&= 2n^4
\end{aligned}
$$

Thus, let $c = 2$ and $n_0 = 0$ to show $7n^4 - 5n^2 + 6 \in \Omega(n^4)$.

c) $5n^2 + 15 \in o(n^3)$

Let $c > 0$ be given (remember that $0 < c < 1$ is possible).

We want to find an n_0 (possibly based on the given c) such that $5n^2 + 15 < cn^3$ for all $n \geq n_0$. We derive the n_0 as follows:

$$
\begin{aligned}
5n^2 + 15 &< 21n^2 \text{ for } n > 0 \\
&= \frac{21}{n} n^3
\end{aligned}
$$

For a given c, we need to find a corresponding n_0 such that $\frac{21}{n} n^3 \leq cn^3$.
Note: $\frac{21}{n} \leq c$ holds for all $n \geq \frac{21}{c}$ since $n > 0$.

So, for a given c, choosing $n_0 = \frac{21}{c}$ (or the next integer larger than $\frac{21}{c}$) shows $5n^2 + 15 \in o(n^3)$.

Question 2 [3+3+4=10 marks]

For each pair of the following functions, fill in the correct asymptotic notation among Θ, o, and ω in the statement $f(n) \in \sqcup(g(n))$. Provide a brief justification of your answers. In your justification you may use any relationship or technique that is described in class.

a) $f(n) = n^2 + 22n(\log n) + 13$ versus $g(n) = n^2 \log n + 14$

First, we show that $f(n) \in \Theta(n^2)$ and $g(n) \in \Theta(n^2 \log n)$ by taking two limits:

$$\begin{aligned}\lim_{n\to\infty} \frac{f(n)}{n^2} &= \lim_{n\to\infty} \left(1 + \frac{22\log n}{n} + \frac{13}{n^2}\right) \\ &= \lim_{n\to\infty} \left(1 + \frac{c}{n} + \frac{13}{n^2}\right) \text{ (by L'Hôpital's Rule)} \\ &= 1\end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty} \frac{g(n)}{n^2 \log n} &= \lim_{n\to\infty} \left(1 + \frac{14}{n^2 \log n}\right) \\ &= 1\end{aligned}$$

With a final limit we compare n^2 with $n^2 \log n$:

$$\begin{aligned}\lim_{n\to\infty} \frac{n^2}{n^2 \log n} &= \lim_{n\to\infty} \left(\frac{1}{\log n}\right) \\ &= 0\end{aligned}$$

So $f(n) \in \mathrm{o}(g(n))$.

b) $f(n) = \sqrt{n}$ versus $g(n) = (\log n^4)$

We can compare these two functions using the limit test as follows:

$$\begin{aligned}\lim_{n\to\infty} \frac{(\log n)^4}{n^{\frac{1}{2}}} &= \lim_{n\to\infty} \left(\frac{c_1}{c_2(n^{\frac{1}{2}})}\right) \text{ (by L'Hôpital's Rule 4 times)} \\ &= 0\end{aligned}$$

A limit of 0 shows $f(n) \in \omega(g(n))$.
Note: students can also reference the proof we did during lecture that shows this property.

c) $f(n) = 10^n + 99n^{10}$ versus $g(n) = 75^n + 25n^{27}$

We can see $f(n) \in \Theta(10^n)$ and $g(n) \in \Theta(75^n)$ by taking two limits:

$$\begin{aligned}\lim_{n\to\infty} \frac{f(n)}{10^n} &= \lim_{n\to\infty} \left(1 + \frac{99n^{10}}{10^n}\right) \\ &= \lim_{n\to\infty} \left(1 + \frac{c_1}{c_2 10^n}\right) \text{ (by L'Hôpital's Rule 10 times)} \\ &= 1\end{aligned}$$

$$
\begin{aligned}
\lim_{n\to\infty} \frac{g(n)}{75^n} &= \lim_{n\to\infty} \left(1 + \frac{25n^{27}}{75^n}\right) \\
&= \lim_{n\to\infty} \left(1 + \frac{c_1}{c_2 75^n}\right) \text{ (by L'Hôpital's Rule 27 times)} \\
&= 1
\end{aligned}
$$

With a final limit we can show that $10^n \in \mathrm{o}(75^n)$:

$$
\begin{aligned}
\lim_{n\to\infty} \frac{10^n}{75^n} &= \lim_{n\to\infty} \left(\frac{2}{15}\right)^n \\
&= 0 \text{ since } \frac{2}{15} < 1
\end{aligned}
$$

So $f(n) \in \mathrm{o}(g(n))$.

Question 3 [4+4+4+4+4=20 marks]

Prove or disprove each of the following statements. To prove a statement, you should provide a formal proof that is based on the definitions of the order notations. To disprove a statement, you can either provide a counter example and explain it or show that the truth of the statement leads to a contradiction. All functions map positive integers to positive integers.

a) If $f(n) \in O(g(n))$ then $g(n) \in \Omega(f(n))$.

True. We know that there exist some $c_1 > 0, n1_0 > 0$ such that $f(n) \le c_1 g(n)$ for all $n \ge n1_0$. Dividing each side of the equality by c_1 we obtain $\frac{1}{c_1} f(n) \le g(n)$ for all $n \ge n1_0$. Letting $c = 1/c_1$ and $n_0 = n1_0$ shows that $g(n) \in \Omega(f(n))$.

b) There exists $f(n)$ and $g(n)$ such that $f(n) \in o(g(n))$ and $f(n) \in \omega(g(n))$.

False. Assume for contradiction that both are true. Then for any $c > 0$ there exists $n_1(c)$ such that $f(n) < cg(n)$ for $n \ge n_1$. There also exists for any $c > 0$ an $n_2(c)$ such that $f(n) > cg(n)$ for $n \ge n_2$. Fix $n_0 = \max\{n_1(1), n_2(2)\}$. Then for all $n \ge n_0$ we have $f(n) < g(n)$ and $f(n) > 2g(n)$, so $f(n) > f(n)$, which is impossible.

(The following is also acceptable: Since $f(n) \in \omega(g(n))$, we also have $f(n) \in \Omega(g(n))$. We showed in class that this implies $f(n) \notin o(g(n))$.

c) If $f(n) \in O(g(n))$ then $2^{f(n)} \in O(2^{g(n)})$.

False. Let $f(n) = 1000 \log n$ and $g(n) = \log n$. Clearly $f(n) \in O(g(n))$. But $2^{1000 \log n} = n^{1000}$ while $2^{\log n} = n$, so $2^{f(n)}$ is not in $O(2^{g(n)})$.

d) $(\log n)^{\log n} \in O(n^2)$.

False. To see this, observe that

$$(\log n)^{\log n} = (2^{\log \log n})^{\log n} = (2^{\log n})^{\log \log n} = n^{\log \log n}.$$

Set $n_0 = 2^{2^3} = 32$, then $\log \log n_0 = 3$. Therefore for all $n \geq n_0$ we have

$$(\log n)^{\log n} = n^{\log \log n} \geq n^3 \in \omega(n^2).$$

Therefore this function cannot be in $O(n^2)$.

e) $\log n \times 2^{\sin(n^3)} \in O(n)$.

True. Note that no matter what x, we have $\sin(x) \leq 1$ and $\sin(x) \geq -1$. Hence $2^{\sin(n^3)}$ is always between $\frac{1}{2}$ and 2. So the left-hand side is at most $2 \log n$, which from class we know to be in $O(n)$.

Question 4 [2+4=6 marks]

Dr. I. M. Smart has recently invented a new class of functions, denoted $O'(f)$. A function $g(n)$ is in $O'(f)$ if there is a constant $c > 0$ such that $g(n) \leq cf(n)$ for all $n > 0$. Prove the following statements about the relationship between $O(f)$ and $O'(f)$.

$f(n)$ and $g(n)$ are functions that map positive integers to positive integers.

a) Prove that $f(n) \in O'(g(n))$ implies that $f(n) \in O(g(n))$.

b) Prove that $g(n) \in O(f(n))$ implies that $g(n) \in O'(f(n))$.

Part a): Set $n_0 = 1$ in the definition of n_0 and use the same c.

Part b): Suppose that f and g assume positive values only and $g(n) \leq cf(n)$ for all $n \geq n_0$. Let $c_1 = \max_{0<i<n_0} \frac{g(i)}{f(i)}$ and $c_2 = \max(c, c_1)$. Then $g(n) \leq c_2 f(n)$ for all n. Hence $g(n) \in O'(f(n))$.

Question 5 [4+4+5+6(+5)=15(+5) marks]

For parts a), b), c) below, analyze the following pieces of pseudocode and for each of them give a tight (Θ) bound on the running time as a function of n. A formal proof is not required, but you should justify your answer. In all cases, n is a positive integer.

a)
```
for i := 1 to n do
    for j := 1 to i * i do
        for k := 1 to log(n)
            s := s + 1
```

The innermost loop iterates $\log(n)$ times doing a constant c amount of work each time. We can express the runtime of this code as

$$f(n) = \sum_{i=1}^{n} \sum_{j=1}^{i^2} c \log n$$

Simplifying we get

$$\begin{aligned} f(n) &= c \log n \sum_{i=1}^{n} \sum_{j=1}^{i^2} 1 \\ &= c \log n \sum_{i=1}^{n} i^2 \\ &= (c \log n)(n(n+1)(2n+1)/6) \\ &= \frac{1}{6}(c \log n)(2n^3 + 3n + 1) \\ &\in \Theta(n^3 \log n) \end{aligned}$$

b)
```
s = 1
i = 2
while (i < n) do
   for j = 1 to n do
      s = s + 1
   i = i * i
```

The inner for loop runs n times and the body in $\Theta(1)$. The outer while loop will run until $i \geq n$. Since i is squared at each iteration, let k be the number of times i (with initial value 2) is squared before i is no smaller than n for the first time. We have $2^{2^k} \geq n$, which is equivalent to $k \geq \log \log n$. Also, we have $2^{2^{k-1}} < n$, which is equivalent to $k < \log \log n + 1$. Thus, $k = \lfloor \log \log n \rfloor$ and the runtime is in $\Theta(kn) = \Theta(n \log \log n)$.

c)
```
s := -42
i := 1
while i < 5n do
   j := n * n * n
   while j > i do
      s := 2 - s
      j := j - i
   i := i + 5
```

Note: i is initialized to 1 before the loop and the outer loop terminates if $i \geq 5n$. Since i is incremented by 5 in each iteration, the outer loop will iterate $\frac{5n}{5} = n$ times. Let i' be the iterator of a summation expressing the runtime of the outer loop from 0 to $n-1$ (n iterations), then $i = 5i' + 1$.

For the inner loop, j is initialized to n^3 and terminates if $j \leq i$. Since j is decremented by i in each iteration, j will range from n^3 down to i in steps of size i resulting in the inner loop iterating $\frac{n^3-i}{i} = \frac{n^3}{i} - 1$ iterations. If we let j' be the iterator of a summation expressing the runtime of the outer loop from 1 to $\frac{n^3}{i} - 1$ then we must express the upper bound in terms of i' the iterator for the outer summation.

The runtime can then be expressed as

$$f(n) = c_0 + \sum_{i'=1}^{n} \left(c_1 + \sum_{j'=1}^{\frac{n^3}{5i'+1}-1} c_2 \right)$$

where c_0, c_1, and c_2 represent the constant runtime of work before the loop, inside the outer loop and inside the inner loop, repectively. Simplifying we get

$$\begin{aligned} f(n) &= c_0 + \sum_{i'=1}^{n} \left(c_1 + c_2 \sum_{j'=1}^{\frac{n^3}{5i'+1}-1} 1 \right) \\ &= c_0 + \sum_{i'=1}^{n} \left(c_1 + c_2 (\frac{n^3}{5i'+1} - 1) \right) \\ &= c_0 + \sum_{i'=1}^{n} c_1 + \sum_{i'=1}^{n} c_2 \frac{n^3}{5i'+1} - \sum_{i'=1}^{n} c_2 \\ &= c_0 + c_1 n + \frac{c_2 n^3}{5} \sum_{i'=1}^{n} \frac{1}{i' + \frac{1}{5}} - c_2 n \\ &\in \Theta(n^3 \log n) \end{aligned}$$

where we apply the formula for the Harmonic sequence.