University of Waterloo
David R. Cheriton School of Computer Science

MATH 213 – Advanced Mathematics for Software Engineers
Midterm Exam, Spring 2016

June 27, 7:00 – 8:20 PM

Instructor: Dr. Oleg Michailovich

Surname								
Legal Given Name(s)								
UW Student ID Number								

Instructions:

- This exam has **3** pages.
- **No books and lecture notes are allowed on the exam**. Please, turn off your cell phones, PDAs, etc., and place your bags, backpacks, books, and notes under the table or at the front of the room.
- Please, place your **WATCARD** on the table, and fill out the exam attendance sheet when provided by the proctor after the exam starts.
- Question marks are listed by the question.
- Please, do not separate the pages, and indicate your Student ID at the top of every page.
- Be neat. Poor presentation will be penalized.
- **No questions will be answered during the exam**. If there is an ambiguity, state your assumptions and proceed.
- **No student can leave the exam room in the first 45 minutes or the last 15 minutes**.
- If you finish before the end of the exam and wish to leave, remain seated and raise your hand. A proctor will pick up the exam from you, at which point you may leave.
- When the proctors announce the end of the exam, put down your pens/pencils, close your exam booklet, and remain seated in silence. The proctors will collect the exams, count them, and then announce you may leave.

Question 1 (25%)

Show that the equation is exact and obtain its general solution:

$$4\cos 2x \, dx - e^{-5y} dy = 0.$$

Question 2 (25%)

Obtain the general solution of the following system of equations (with unknowns $x(t)$ and $y(t)$):

$$\begin{aligned} x' + y' + x - y &= e^t \\ x' + 2y' + 2x - 2y &= 1 - t. \end{aligned}$$

Question 3 (25%)

Using Laplace Transform, solve $x' - x = f(t)$, where $x(0) = 0$ and

$$f(t) = \begin{cases} 20, & 0 < t < 1 \\ 10, & 1 < t < 2 \\ 0, & t > 2. \end{cases}$$

Question 4 (25%)

Invert the following Laplace transform:

$$F(s) = \ln\left(1 - \frac{a^2}{s^2}\right),$$

where a is a constant. Don't forget that $(\sinh t)' = \cosh t$ and $(\cosh t)' = \sinh t$.

Table of Laplace Transforms

$f(t)$	$\bar{f}(s) = \int_0^\infty f(t)e^{-st}\,dt$

NOTE: s is regarded as real here.

1. 1	$\frac{1}{s} \quad (s > 0)$
2. e^{at}	$\frac{1}{s-a} \quad (s > a)$
3. $\sin at$	$\frac{a}{s^2+a^2} \quad (s > 0)$
4. $\cos at$	$\frac{s}{s^2+a^2} \quad (s > 0)$
5. $\sinh at$	$\frac{a}{s^2-a^2} \quad (s > \lvert a\rvert)$
6. $\cosh at$	$\frac{s}{s^2-a^2} \quad (s > \lvert a\rvert)$
7. $t^n \quad (n = \text{positive integer})$	$\frac{n!}{s^{n+1}} \quad (s > 0)$
8. $t^p \quad (p > -1)$	$\frac{\Gamma(p+1)}{s^{p+1}} \quad (s > 0)$
9. $e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2} \quad (s > a)$
10. $e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2} \quad (s > a)$
11. $t\sin at$	$\frac{2as}{(s^2+a^2)^2} \quad (s > 0)$
12. $t\cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2} \quad (s > 0)$
13. $t\sinh at$	$\frac{2as}{(s^2-a^2)^2} \quad (s > a)$

Solutions

Question 1

We are going to solve:

$$4\cos 2x\,dx - e^{-5y}dy = 0.$$

Observe that $M(x,y) = 4\cos 2x$ and $N(x,y) = e^{-5y}$. In this case

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 0,$$

and so the equation is exact.

We start with

$$\frac{\partial F}{\partial x} = 4\cos 2x \quad \Longrightarrow \quad F(x,y) = \int 4\cos 2x dx = 2\sin 2x + A(y).$$

In this case

$$\frac{\partial F}{\partial y} = A'(y) = -e^{-5y} \quad \Longrightarrow \quad A = \int -e^{-5y}dy = 1/5e^{-5y} + C.$$

Thus, finally, we have

$$F(x,y) = 2\sin 2x + 1/5e^{-5y} = \tilde{C}.$$

Question 2

Obtain the general solution of the following system of equations (with unknowns $x(t)$ and $y(t)$):

$$\begin{aligned} x' + y' + x - y &= e^t \\ x' + 2y' + 2x - 2y &= 1 - t. \end{aligned}$$

Multiply the first equation by 2 and subtract the second equation to get:

$$x' = 2e^t + t - 1 \quad \Longrightarrow \quad x = 2e^t + t^2/2 - t + C.$$

Now the first equation from the second equation to get:

$$y' + x - y = 1 - t - e^t \quad \Longrightarrow y' - y = -3e^t - t^2/2 + 1 - C.$$

The homogeneous solution is given by:

$$y' - y = 0 \quad \Longrightarrow \quad \lambda - 1 = 0 \quad \Longrightarrow \quad \lambda = 1 \quad \Longrightarrow \quad y_h(t) = Ae^t.$$

Next, we are going to find the first particular solution:

$$y' - y = -3e^t.$$

Due to the redundancy, we look for $y_{p_1}(t)$ in the form $y_{p_1}(t) = Bte^t$. Then, substitution of $y_{p_1}(t)$ into the above equation leads to $B = -3$ and thus we have:

$$y_{p_1}(t) = -3te^t.$$

Now, we are going to find the second particular solution:

$$y' - y = -t^2/2.$$

We are looking for $y_{p_2}(t)$ in the form $y_{p_2}(t) = Ct^2 + Dt + E$. Substitution of $y_{p_2}(t)$ into the above equation leads to $C = 1/2$, $D = 1$, and $E = 1$. Thus,

$$y_{p_2}(t) = t^2/2 + t + 1.$$

Finally, we are going to find the third particular solution:

$$y' - y = 1 - C.$$

We are looking for $y_{p_3}(t)$ in the form $y_{p_3}(t) = F$. Substitution of $y_{p_3}(t)$ into the above equation leads to $F = C - 1$. Thus,

$$y_{p_3}(t) = C - 1.$$

Consequently, the final answer is:

$$\begin{aligned} x(t) &= 2e^t + t^2/2 - t + C \\ y(t) &= Ae^t - 3te^t + t^2/2 + t + C. \end{aligned}$$

Question 3

Using Laplace Transform, solve $x' - x = f(t)$, where $x(0) = 0$ and

$$f(t) = \begin{cases} 20, & 0 < t < 1 \\ 10, & 1 < t < 2 \\ 0, & t > 2. \end{cases}$$

Alternatively, the equation can be written as:

$$x' - x = 20\left[1 - H(t-1)\right] + 10\left[H(t-1) - H(t-2)\right] = 20 - 10H(t-1) - 10H(t-2).$$

Thanking Laplace transform on sides of the above equation results in:

$$sX(s) - X(s) = \frac{20}{s} - 10\frac{e^{-s}}{s} - 10\frac{e^{-2s}}{s},$$

and thus

$$X(s) = \frac{20}{s(s-1)} - 10\frac{e^{-s}}{s(s-1)} - 10\frac{e^{-2s}}{s(s-1)}.$$

In the time domain, the above equation is equivalent to

$$\begin{aligned}
x(t) = e^t * [20 - 10H(t-1) - 10H(t-2)] = \\
&= 20\int_0^t e^{\tau} d\tau - 10\int_0^t H(\tau-1)e^{t-\tau}d\tau - 10\int_0^t H(\tau-2)e^{t-\tau}d\tau = \\
&= 20(e^t - 1) - 10H(t-1)\int_1^t e^{t-\tau}d\tau - 10H(t-2)\int_2^t e^{t-\tau}d\tau = \\
&= 20(e^t - 1) - 10H(t-1)(e^{t-1} - 1) - 10H(t-2)(e^{t-2} - 1).
\end{aligned}$$

Question 4

Invert the following Laplace transform:

$$F(s) = \ln\left(1 - \frac{a^2}{s^2}\right)$$

We start with the following observation:

$$\mathcal{L}\{f\} = \int_0^{\infty} f(t)e^{-st}dt = \ln\left(1 - \frac{a^2}{s^2}\right).$$

Then, using the differentiation theorem, we get:

$$\int_0^{\infty} -tf(t)e^{-st}dt = \frac{d}{ds}\left[\ln\left(1 - \frac{a^2}{s^2}\right)\right] = \frac{2a^2}{s}\frac{1}{s^2 - a^2}.$$

Consequently,

$$\{L\}^{-1}\left[\frac{2a^2}{s}\frac{1}{s^2 - a^2}\right] = 2a^2\, 1 * \frac{1}{a}\sinh at = 2a\int_0^t \sinh a\tau d = 2(\cosh at - 1).$$

Thus, we have

$$-tf(t) = 2(\cosh at - 1),$$

and therefore

$$f(t) = \frac{2}{t}(1 - \cosh at).$$