6

Name: [redacted]

I.D.#: [redacted]

Section: [redacted]

(Section 1 MWF 10:30 Mike LaCroix)
(Section 2 MWF 1:30 Bill Cunningham)
(Section 3 MWF 10:30 Dan Younger)

Math 239
Introduction to Combinatorics
Midterm Examination
Winter 2006

Date: February 28 (Tues), 2006
Time: 4:30 - 6:30 p.m.
Instructors: Section 1 MWF 10:30 Mike LaCroix
Section 2 MWF 1:30 Bill Cunningham
Section 3 MWF 10:30 Dan Younger

1. Print your name and student number in the space provided above.
2. This examination has 10 pages, including this cover page. Make sure you have a complete copy.
3. No notes or calculating aids are allowed.
4. Answer questions on these pages in the space indicated. If you need more space, use the back of the previous page.

Question	Marks	Student Mark
#1	6	3
#2	12	4
#3	10	8
#4	13	12
#5	8	8
#6	11	5
Total	60	40

#1. [6 marks = 3 + 3]

[3] (a) Find $[x^n](1-3x^2)^{-2}$ for all integers $n \geq 0$.

$[x^n] \sum_{i\geq 0} \binom{-2}{i} (-1)^i (3)^i x^{2i} = [x^n] \sum_{i\geq 0} \binom{2+i-1}{1} 3^i x^{2i}$

$= [x^n] \sum_i (i+1) \; 3^i x^{2i}$

$= (n+1) 3^{n} x^{2n}$ Shouldn't have x's after extracting coefficients.

We want the coefficient when $2i = n$ (for n even)

1/3

[3] (b) Find $[x^n](1+2x)(1-3x)^{-m}$ for all non-negative integers n, m.

$[x^n] (1+2x) \sum_{i\geq 0} \binom{-m}{i} (-1)^i (3)^i x^i$

$= [x^n] (1+2x) \sum_{i=0}^{n} \binom{m+i-1}{i} 3^i x^i$ ✓

$= [x^n] \sum_{i=0}^{n} \binom{m+i-1}{i} (3^i x^i + 3^i 2 x^{i+1})$

$= \ldots$

2/3

Math 239 NAME: Leah Evans

2. [12 marks = 6 + 6]

Assumption n & m are odd ✗

[6] (a) Where n and m are non-negative integers, find the number of compositions of $2n$ into $2m$ parts, each of which is odd. each part is odd

let $\mathbb{N} = 1, 2, \ldots$ ✗ $w(i) = i$

$m = 2j+1$

let $S = \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} = (\mathbb{N})^{2(2j+1)}$ $\alpha(i) = c_1 + c_2 + \ldots + c_m$

By Product Lemma $\Phi_{\mathbb{N}}(x) = \frac{x}{1-x}$

$$\Phi_S(x) = (\Phi_{\mathbb{N}}(x))^{4j+2} = \left(\frac{1}{1-x}\right)^{4j+2}$$ ✗

$$= \sum \binom{-(4j+2)}{k}(-1)^k x^k = \sum \binom{4j+1+k}{k} x^k$$

$$[x^{2n}]\,\Phi_S(x) = [x^{2n}] \sum \binom{2m-1+k}{2m-1} x^k$$

$$= \begin{cases} \binom{2m-1+2n}{2n} x^{2n} & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$\frac{3}{6}$

[6] **#2**(b) Let a_n be the number of compositions of n, each of whose parts is at least 2. Find the generating function $\sum_{n\geq 0} a_n x^n$ and express it as a ratio of polynomials.

$\mathbb{N} = \{\cancel{1}, 2, 3, 4, 5, \ldots, n\}$ $w(i) = i$

$S = \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} = (\mathbb{N})^{k+1}$ $k \geq 1$

$\alpha(i) = c_1 + c_2 + \cdots c_i$

By Product Lemma

$\Phi_S(x) = \left(\Phi_{\mathbb{N}}(x)\right)^{k+1} = \left(\frac{1}{1-x}\right)^{k+1}$

$= \sum \binom{-(k+1)}{i} (-1)^i x^i = \sum_{i\geq 0} \binom{k+i}{i} x^i$

$\frac{1}{6}$

3. [10 marks = 2 + 4 + 4]

[2] (a) Let $A = \{0, 01, 011\}$ and $B = \{1, 11\}$. For each of AB and BA, determine whether its elements are uniquely created. Justify your answer. These are small enough to list all possibilities.

$AB = \{0,01,011\}\{1,11\}$ $BA = \{1, 11\}\{0, 01, 011\}$

01	10
011 ← not uniquely created	101
011 ←	1011
0111	110
⋮	1101
	11011

so... conclude what?

4/2 ✓

[4] (b) Where A, B are defined in part (a), find the generating function with respect to length for each of A^* and B^*.

$$\Phi_{A^*}(x) = \frac{1}{1-\Phi_A(x)} = \frac{1}{1-(x+x^2+x^3)} = \frac{1}{1-x-x^2-x^3}$$

$$\Phi_{B^*}(x) = \frac{1}{1-\Phi_B(x)} = \frac{1}{1-(x+x^2)} = \frac{1}{1-x-x^2}$$

not U.C.

2/4

[4] (c) Suppose that the sequence b_n satisfies the recurrence

$$b_n = 7b_{n-1} - 10b_{n-2}, \Rightarrow b_n - 7b_{n-1} + 10b_{n-2} = 0$$

with initial conditions $b_0 = -1$, $b_1 = 1$. Find a formula for b_n.

$E(x) = x^2 - 7x + 10$
$= (x-2)(x-5)$

✓

Hence, $b_n = A2^n + B5^n$

$b_0 = A + B = -1$
$b_1 = 2A + 5B = 1$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 2 & 5 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

$A = -2$ $B = 1$

$b_n = -2 \cdot 2^n + 5^n$

$\boxed{b_n = -2^{n+1} + 5^n}$

Check: $b_2 = 7(1) + 10 \cdot = 17$

$b_2 = -(2)^3 + 5^2 = -8 + 25 = 17$ ✓

#4. [13 marks = 5 + 8]

[5] (a) Let R be the set of all binary strings in which each block of 0's has even length. Let S be the set of all binary strings in which each block of 0's has even length, except for the first block of 0's, which may have even or odd length. (Note: Some strings in R and S have no 0's at all.)

(i) Give a decomposition for R, in which each element of R is uniquely created.

$$\boxed{\{0,1\}^* = 1^* \{0\ 1^*\}^*}$$

$$R = 1^* \{00\ 1^*\}^*$$

$\frac{2}{2}$

(ii) Give a decomposition for S, in which each element of S is uniquely created.

$$S = 1^* 0^* \{1(00)^*\}^*$$

$\frac{2}{3}$ not unique

[8] **#4**(b) Let $\Phi_T(x) = \sum_{n\geq 0} a_n x^n$ be the generating function with respect to length of the set

$$T = \{0\}^*(\{100\}\{00\}^* \cup \{110\}\{00\}^*)^*. = \{0\}^*\{(100,110)\}(00)^*$$

Calculate $\Phi_T(x)$ and use it to find a recurrence relation for a_n, with initial conditions.

$\Phi_T(x)$

By Product lemma $= \Phi_{\{0\}^*} \Phi_{\{(10,11)(00)^*\}^*}$

8

$$= \frac{1}{1-\Phi_{\{0\}}} \frac{1}{1-\Phi_{(100,110)}\Phi_{(00)^*}}$$

Sum Lemma

$$= \frac{1}{1-x} \frac{1}{1-(\Phi_{100}+\Phi_{110})\frac{1}{1-\Phi_{(00)}}}$$

$$= \frac{1}{1-x} \frac{1}{1-(x^3+x^3)\frac{1}{1-x^2}}$$

$$= \frac{1}{1-x} \frac{1-x^2}{1-x^2-2x^3}$$

$$= \frac{1+x}{1-x^2-2x^3}$$

Initial conditions

$a_0 = \{\epsilon\} = 1$
$a_1 = \{0\} = 1$
$a_2 = \{00\} = 1$

$$E(x) = 1 - x^2 - 2x^3$$

$$0 = a_n - a_{n-2} - 2a_{n-3}$$

$$\boxed{a_n = 2a_{n-3} + a_{n-2},\ n \geq 3}$$

Check

$a_3 = \{000, 100, 110\} = 3$ ✓

$a_4 = \{0000, 0100, 0110\} = 3$ ✓

← nice

$a_5 = \{00000, 00100, 00110, 10000, 11000\} = 5$ ✓

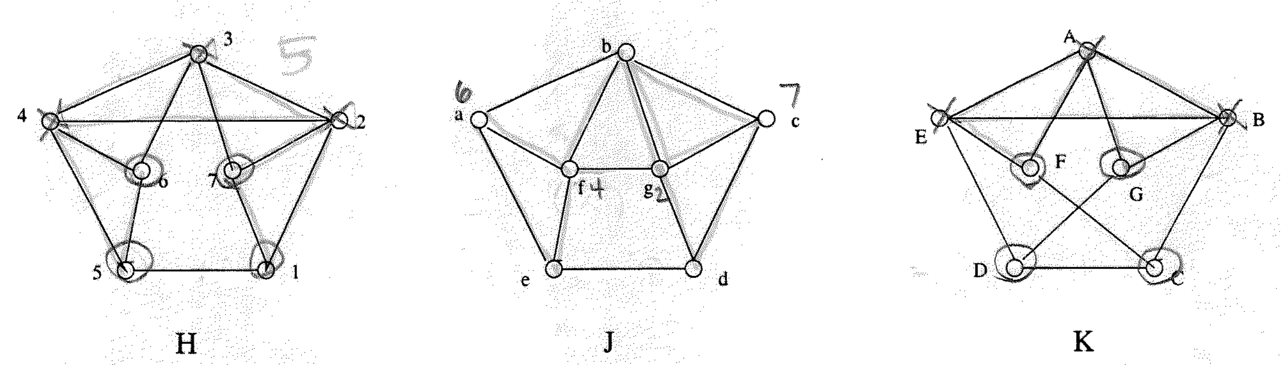


Figure 1: Graphs for Question 6

5. [8 marks = 4 + 4]

Consider the graphs H, J, and K for which drawings are given in Figure 1.

[4] (a) Find two of the graphs that are isomorphic, and give an isomorphism between them.

$h(a) = 6$
$h(b) = 3$
$h(c) = 7$
$h(d) = 1$
$h(e) = 5$
$h(f) = 4$
$h(g) = 2$

Function h(x) preserves adjancency. Hence, H, J are isomorphic.

4/4

[4] (b) Find two of the graphs that are *not* isomorphic, and justify your answer.

If we look at the smallest cycle possible, which of size 3, H has 5 of these cycles and K only has 3.

Hence, there exists no funtion $g(V(H)) = V(K)$ such that will preserve adjancey from H to K.

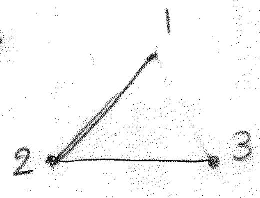
Therefore H & K are not isomorphic.

4/4

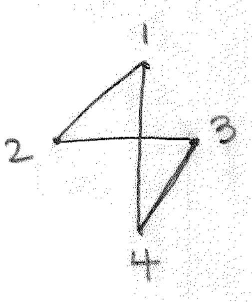
#6. [11 marks = 4 + 3 + 4]

[4] (a) Let n be a positive integer, and let P_n be the graph for which $V(P_n) = \{1, 2, \ldots, n\}$, with vertices u, v adjacent if and only if $u + v$ is a prime number. Prove that P_n is bipartite.

P_3



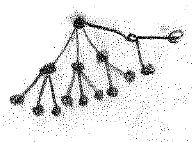
P_4



Since all vertices have distinct labels, starting at 1 & the smallest prime number is 3, $u+v$ will always be odd (cannot be equal to 2, the only even prime number). Hence, we can divide all pairs into even and odd. This bijection? applies to all edges. So P_n is bipartite.

4/4

[3] (b) Let T be a tree having 2 vertices of degree 3 and 3 vertices of degree 4. Prove that T has at least 15 vertices.



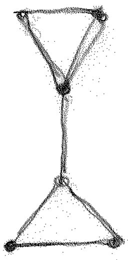
Since T is a tree, it has no cycles by definition. Hence,

$E(T) = \frac{\sum \deg(v)}{2}$.

why? The smallest T can be is having all paths from the parent ? contain only vertices of deg 3 and 4 and 1 vertex containing deg 1. All these trees would be isomorphic to the above picture and would contain 15 vertices.

1/3

[4] # **6**(c) Let G be a graph having a bridge $e = \{u, v\}$. Prove that G has a vertex of odd degree.



$2+2+3+3+2+2$

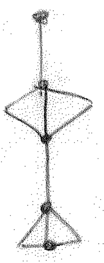
$\frac{6+8}{2} = \frac{14}{2} = 7$

G has p vertice and q edges.

G-e divides G into 2 components G_1 and G_2.

G-e has p vertices and q-1 edges

G_1 and G are connected.



pectively

$|V(G_1)|+|V(G_2)|=p$ and $|E(G_1)|+|E(G_1)|+1 = q$

0/4

If both components are connected