University of Waterloo
David R. Cheriton School of Computer Science

MATH 213 – Advanced Mathematics for Software Engineers
Final Exam, Spring 2016

August 10, 9:00 – 11:30 AM

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Surname								
Legal Given Name(s)								
UW Student ID Number								

Instructions:

- This exam has **4** pages (6 questions and two tables).
- **No books and lecture notes are allowed on the exam**. Please, turn off your cell phones, PDAs, etc., and place your bags, backpacks, books, and notes under the table or at the front of the room.
- Please, place your **WATCARD** on the table, and fill out the exam attendance sheet when provided by the proctor after the exam starts.
- Question marks are listed by the question.
- Please, do not separate the pages, and indicate your Student ID at the top of every page.
- Be neat. Poor presentation will be penalized.
- **No questions will be answered during the exam**. If there is an ambiguity, state your assumptions and proceed.
- **No student can leave the exam room in the first 45 minutes or the last 15 minutes**.
- If you finish before the end of the exam and wish to leave, remain seated and raise your hand. A proctor will pick up the exam from you, at which point you may leave.
- When the proctors announce the end of the exam, put down your pens/pencils, close your exam booklet, and remain seated in silence. The proctors will collect the exams, count them, and then announce you may leave.

Question 1 (15%)

Using the method of separation of variables, find the particular solution of

$$y' = (y^2 - y)\, e^x, \quad y(0) = 2,$$

with $y = y(x)$. The solution should be expressed in explicit form. Note: $\int dy/y = \ln(y) + C$.

Question 2 (20%)

Solve $x' + x = f(t)$ by the Laplace transform, where $x(0) = x_0$ and $f(t)$ is the square form shown below. Note that the square form is assumed to last infinitely long in time, i.e., for all $t \geq 0$. Hint: start with representing $f(t)$ as a linear combination of shifted Heavyside functions.

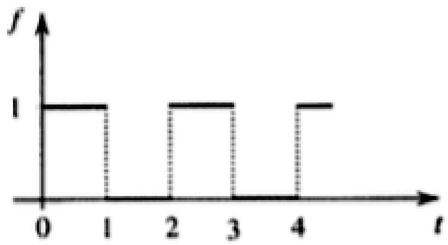

Figure 1: Pertaining to Question 2. Only an initial fragment of $f(t)$ is shown.

Question 3 (15%)

Let $\mathbf{u} = (1, 2, 0, 1)$, $\mathbf{v} = (1, 0, 1, 1)$, and $\mathbf{w} = (2, -1, 1, 1)$. Find scalars α, β, γ and vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ such that $\mathbf{u}_1 = \mathbf{u}$, $\mathbf{u}_2 = \mathbf{u} + \alpha\mathbf{v}$, $\mathbf{u}_3 = \mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w}$ is a nontrivial (i.e., non-zero) *orthogonal* set.

Question 4 (15%)

Let $\mathcal{S}$ be a normed inner product vector space of continuous functions defined on $[-1, 1]$, with the following definition of the inner product

$$\langle u, v \rangle = \int_{-1}^{1} u(x)v(x)dx, \qquad \forall u, v \in \mathcal{S}.$$

Given $u(x) = a + bx + cx^3$ (with a, b, and c being some arbitrary real constants), finds its best approximation in $\mathcal{T} = \mathrm{Span}\{\mathrm{u}_1, \mathrm{u}_2\}$, where $u_1 = 1$ and $u_2 = x^2$.

Question 5 (20%)

Let $f(t)$ be a *periodic* function that is defined over one period as

$$f(t) = \begin{cases} 5t, & 0 \leq t < 1 \\ 10 - 5t, & 1 \leq t < 2 \end{cases}.$$

Find the steady-state solution to

$$x'' + x = f(t),$$

where by "steady state" we mean a solution without the homogeneous part (i.e., discarding $x_h(t)$).

Question 6 (15%)

Compute the inverse Fourier transform for

$$\hat{f}(\omega) = \frac{1}{\omega^2 + i\omega + 2}, \quad \text{where} \quad i = \sqrt{-1}.$$

Sketch the obtained solution.

Table of Laplace Transform pairs

$f(t)$	$\bar{f}(s) = \int_0^\infty f(t)e^{-st}\,dt$
NOTE: s is regarded as real here.	
1. 1	$\frac{1}{s}$ $(s > 0)$
2. e^{at}	$\frac{1}{s-a}$ $(s > a)$
3. $\sin at$	$\frac{a}{s^2+a^2}$ $(s > 0)$
4. $\cos at$	$\frac{s}{s^2+a^2}$ $(s > 0)$
5. $\sinh at$	$\frac{a}{s^2-a^2}$ $(s > \lvert a \rvert)$
6. $\cosh at$	$\frac{s}{s^2-a^2}$ $(s > \lvert a \rvert)$
7. t^n $(n =$ positive integer$)$	$\frac{n!}{s^{n+1}}$ $(s > 0)$
8. t^p $(p > -1)$	$\frac{\Gamma(p+1)}{s^{p+1}}$ $(s > 0)$
9. $e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$ $(s > a)$
10. $e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$ $(s > a)$
11. $t\sin at$	$\frac{2as}{(s^2+a^2)^2}$ $(s > 0)$
12. $t\cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$ $(s > 0)$
13. $t\sinh at$	$\frac{2as}{(s^2-a^2)^2}$ $(s > a)$

Table of Fourier Transform pairs

	$f(x)$	$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$
1.	$\dfrac{1}{x^2+a^2} \quad (a>0)$	$\dfrac{\pi}{a}e^{-a\lvert\omega\rvert}$
2.	$H(x)e^{-ax} \quad (\mathrm{Re}\,a>0)$	$\dfrac{1}{a+i\omega}$
3.	$H(-x)e^{ax} \quad (\mathrm{Re}\,a>0)$	$\dfrac{1}{a-i\omega}$
4.	$e^{-a\lvert x\rvert} \quad (a>0)$	$\dfrac{2a}{\omega^2+a^2}$
5.	e^{-x^2}	$\sqrt{\pi}e^{-\omega^2/4}$
6.	$\dfrac{1}{2a\sqrt{\pi}}e^{-x^2/(2a)^2} \quad (a>0)$	$e^{-a^2\omega^2}$
7.	$\dfrac{1}{\sqrt{\lvert x\rvert}}$	$\sqrt{\dfrac{2\pi}{\lvert\omega\rvert}}$
8.	$e^{-a\lvert x\rvert/\sqrt{2}}\sin\left(\dfrac{a}{\sqrt{2}}\lvert x\rvert+\dfrac{\pi}{4}\right) \quad (a>0)$	$\dfrac{2a^3}{\omega^4+a^4}$
9.	$H(x+a)-H(x-a)$	$\dfrac{2\sin\omega a}{\omega}$
10.	$\delta(x-a)$	$e^{-i\omega a}$
11.	$f(ax+b) \quad (a>0)$	$\dfrac{1}{a}e^{ib\omega/a}\hat{f}\left(\dfrac{\omega}{a}\right)$
12.	$\dfrac{1}{a}e^{-ibx/a}f\left(\dfrac{x}{a}\right) \quad (a>0,\ b\ \text{real})$	$\hat{f}(a\omega+b)$
13.	$f(ax)\cos cx \quad (a>0,\ c\ \text{real})$	$\dfrac{1}{2a}\left[\hat{f}\left(\dfrac{\omega-c}{a}\right)+\hat{f}\left(\dfrac{\omega+c}{a}\right)\right]$
14.	$f(ax)\sin cx \quad (a>0,\ c\ \text{real})$	$\dfrac{1}{2ai}\left[\hat{f}\left(\dfrac{\omega-c}{a}\right)-\hat{f}\left(\dfrac{\omega+c}{a}\right)\right]$
15.	$f(x+c)+f(x-c) \quad (c\ \text{real})$	$2\hat{f}(\omega)\cos\omega c$

Solutions

Question 1

To find the particular solution to

$$y' = (y^2 - y)\, e^x, \quad y(0) = 2,$$

we first separate the variables and integrate to obtain

$$\int \frac{dy}{y^2 - y} = \int e^x dx.$$

Using the method of partial fractions, the integral on the left-hand side can be expressed as

$$\int \frac{dy}{y^2 - y} = \int \frac{dy}{y(y-1)} = -\int \frac{dy}{y} + \int \frac{dy}{y-1},$$

and, therefore, we obtain

$$\begin{aligned} \int \frac{dy}{y-1} - \int \frac{dy}{y} &= \int e^x dx, \\ \ln(y-1) - \ln(y) &= e^x + C, \\ \ln\left(\frac{y-1}{y}\right) &= e^x + C, \end{aligned}$$

where C is an integration constant. To find its value, we use the initial condition $y(0) = 2$. Specifically

$$\ln\left(\frac{2-1}{2}\right) = e^0 + C,$$

which (after some algebra) leads to $C = -\ln(2e)$. Consequently,

$$\ln\left(\frac{y-1}{y}\right) = e^x - \ln(2e).$$

To express the solution in explicit form, we "merge" the two logarithms and exponentiate to obtain

$$\begin{aligned} \ln\left(2e\frac{y-1}{y}\right) &= e^x, \\ \ln\left(2e - \frac{2e}{y}\right) &= e^x, \\ 2e - \frac{2e}{y} &= e^{e^x}. \end{aligned}$$

Thus,

$$y(x) = \frac{2e}{2e - e^{e^x}},$$

which obviously satisfies $y(0) = 2$.

Question 2

To solve

$$x' + x = f(t), \quad x(0) = x_0,$$

with the given $f(t)$, we first observe that the latter can be expressed as

$$f(t) = 1 - H(t-1) + H(t-2) - H(t-3) + ... = \sum_{n=0}^{\infty} (-1)^n H(t-n).$$

Then, applying the Laplace transform to the both sides of the differential equation, we obtain

$$s\bar{x}(s) - x_0 + \bar{x}(s) = \bar{f}(s),$$

which gives us

$$\bar{x}(s) = \frac{x_0}{s+1} + \frac{1}{s+1}\,\bar{f}(s),$$

which, in the time domain, is equivalent to

$$\begin{aligned} x(t) = x_0 e^{-t} + e^{-t} * f(t) &= x_0 e^{-t} + \int_0^t e^{-(t-\tau)} \sum_{n=0}^{\infty} (-1)^n H(\tau - n) d\tau = ... \\ &= x_0 e^{-t} + e^{-t} \int_0^t e^{\tau} \sum_{n=0}^{\infty} (-1)^n H(\tau - n) d\tau = ... \\ &= x_0 e^{-t} + e^{-t} \sum_{n=0}^{\infty} (-1)^n \int_0^t e^{\tau} H(\tau - n) d\tau = ... \\ &= x_0 e^{-t} + e^{-t} \sum_{n=0}^{\infty} (-1)^n H(t-n) \int_n^t e^{\tau} d\tau = ... \\ &= x_0 e^{-t} + e^{-t} \sum_{n=0}^{\infty} (-1)^n H(t-n)(e^t - e^n) = ... \\ &= x_0 e^{-t} + \sum_{n=0}^{\infty} (-1)^n H(t-n)(1 - e^{n-t}). \end{aligned}$$

Question 3

Starting with $\mathbf{u} = (1, 2, 0, 1)$, $\mathbf{v} = (1, 0, 1, 1)$, and $\mathbf{w} = (2, -1, 1, 1)$, we first compute

$$\mathbf{u}\cdot\mathbf{u} = 6, \quad \mathbf{v}\cdot\mathbf{v} = 3, \quad \mathbf{w}\cdot\mathbf{w} = 7, \quad \mathbf{u}\cdot\mathbf{v} = 2, \quad \mathbf{u}\cdot\mathbf{w} = 1, \quad \mathbf{v}\cdot\mathbf{w} = 4.$$

Then, the orthogonality of $\mathbf{u}_1$ and $\mathbf{u}_2$ suggests that

$$\mathbf{u}_1 \cdot \mathbf{u}_2 = \mathbf{u} \cdot (\mathbf{u} + \alpha\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + \alpha\,\mathbf{u}\cdot\mathbf{v} = 6 + 2\,\alpha = 0,$$

which yields $\alpha = -3$. Next the orthogonality of $\mathbf{u}_1$ and $\mathbf{u}_3$ as well as of $\mathbf{u}_2$ and $\mathbf{u}_3$ suggests that

$$\begin{aligned} \mathbf{u}_1 \cdot \mathbf{u}_3 &= \mathbf{u} \cdot (\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w}) = 6 + 2\beta + \gamma = 0, \\ \mathbf{u}_2 \cdot \mathbf{u}_3 &= (\mathbf{u} - 3\mathbf{v}) \cdot (\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w}) = -7\beta - 11\gamma = 0. \end{aligned}$$

The above system of equations in β and γ results in $\beta = -22/5$ and $\gamma = 14/5$. Consequently,

$$
\begin{aligned}
\mathbf{u}_1 &= \mathbf{u} = (1, 2, 0, 1), \\
\mathbf{u}_2 &= \mathbf{u} - 3\mathbf{v} = (-2, 2, -3, -2), \\
\mathbf{u}_3 &= \mathbf{u} - 22/5\mathbf{v} + 14/5\mathbf{w} = (11/5, -4/5, -8/5, -3/5).
\end{aligned}
$$

Question 4

Long solution: One can start with converting the basis $\{u_1, u_2\}$ into an orthonormal one $\{\tilde{e}_1, \tilde{e}_2\}$ using the Gram-Schmidt process. In particular,

$$
\tilde{e}_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{\int_{-1}^{1} 1 \cdot 1 \, dx}} = 1/\sqrt{2},
$$

which is a constant function with its value equal to $1/\sqrt{2}$. To find $\tilde{e}_2$, we first compute e_2 as

$$
e_2 = u_2 - \langle u_2, \tilde{e}_1 \rangle \tilde{e}_1 = x^2 - \left(\int_{-1}^{1} \frac{1}{\sqrt{2}} x^2 \, dx \right) \frac{1}{\sqrt{2}} = x^2 - 1/3.
$$

Subsequently,

$$
\tilde{e}_2 = \frac{e_2}{\|e_2\|} = \frac{x^2 - 1/3}{\sqrt{\int_{-1}^{1} (x^2 - 1/3)^2 \, dx}} = \sqrt{\frac{45}{8}} (x^2 - 1/3).
$$

Now, the orthogonal projection of $u(x) = a + bx + cx^3$ onto $\mathcal{T} = \mathrm{Span}\{u_1, u_2\} = \mathrm{Span}\{\tilde{e}_1, \tilde{e}_2\}$ (or, equivalently, the best approximation of u in $\mathcal{T}$) is give by

$$
\mathrm{Proj}_{\mathcal{T}}\{u\} = c_1 \tilde{e}_1 + c_2 \tilde{e}_2,
$$

where

$$
c_1 = \langle u, \tilde{e}_1 \rangle = \int_{-1}^{1} \frac{1}{\sqrt{2}} (a + bx + cx^3) \, dx = \frac{a}{\sqrt{2}} \int_{-1}^{1} dx = \sqrt{2} a
$$

and

$$
c_2 = \langle u, \tilde{e}_2 \rangle = \int_{-1}^{1} \sqrt{\frac{45}{8}} (x^2 - 1/3)(a + bx + cx^3) \, dx = a \sqrt{\frac{45}{8}} \int_{-1}^{1} (x^2 - 1/3) \, dx = 0.
$$

Consequently, we obtain

$$
\mathrm{Proj}_{\mathcal{T}}\{u\} = c_1 \tilde{e}_1 + c_2 \tilde{e}_2 = \sqrt{2} a \frac{1}{\sqrt{2}} + 0 = a.
$$

Short solution: Observe that neither x nor x^3 (which are odd functions on $[-1, 1]$) can be expressed as a linear combination of 1 and x^2 (which are even functions on $[-1, 1]$). Thus, in $u(x) = a + bx + cx^3$, it is only the first term, *viz.* a, that can be expressed as a linear superposition of 1 and x^2. Therefore, the closest vector in $\mathcal{T}$ to u is a.

Question 5

We first note that the fundamental period of $f(t)$ is $T = 2l = 2$, and therefore $l = 1$. Also, due to the periodicity of $f(t)$, we can perform our analysis over the interval $[-1, 1]$. Since $f(t)$ is even, its Fourier series has the form given by

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos \pi n t,$$

where

$$a_0 = \frac{1}{2l} \int_{-l}^{l} f(t)\, dt = \int_0^1 5t\, dt = \frac{5}{2}$$

and

$$\begin{aligned} a_n &= \frac{1}{l} \int_{-l}^{l} f(t) \cos \frac{\pi n t}{l} dt = 2 \int_0^1 5t \cos \pi n t\, dt = \frac{10}{\pi^2 n^2} \cos \pi n t \Big|_0^1 = ... \\ &= \frac{10}{\pi^2 n^2} (\cos \pi n - 1) = \begin{cases} \frac{20}{\pi^2 n^2}, & n = 1, 3, 5, ... \\ 0, & n = 2, 4, 6, ... \end{cases} \end{aligned}$$

Thus, we have

$$f(t) = \frac{5}{2} + \sum_{n=1,3,5,...} \left(-\frac{20}{\pi^2 n^2} \right) \cos \pi n t.$$

Next, based on the principle of superposition, we need to find particular solutions for $x'' + x = 5/2$ and $x'' + x = \cos \pi n t$. In the first case, the generating set is $\{1, 0, 0, ...\}$, and thus $x_{p,1} = A$. Plugging the solution back into $x'' + x = 5/2$ and equating the left and right sides results in $x_{p,1} = A = 5/2$. In the second case, the generating set is $\{\cos \pi n t\ \sin \pi n t\}$, and therefore $x_{p,2}$ has the form of

$$x_{p,2}(t) = B \cos \pi n t + C \sin \pi n t.$$

Plugging this expression back into $x'' + x = \cos \pi n t$ and equating the left and right sides yields $B = 1/(1 - \pi^2 n^2)$, and hence

$$x_{p,2}(t) = \frac{1}{1 - \pi^2 n^2} \cos \pi n t.$$

Finally, using the principle of superposition, we conclude

$$x(t) = \frac{5}{2} - \frac{20}{\pi^2} \sum_{n=1,3,5,...} \frac{1}{n^2 (1 - \pi^2 n^2)} \cos \pi n t.$$

Question 6

First, we note that the roots of $\omega^2 + i\omega + 2 = 0$ are

$$\omega_{1,2} = \frac{-i \pm \sqrt{-1 - 8}}{2} = \{-2i, i\}.$$

So, we have

$$\begin{aligned}
\mathcal{F}^{-1}\left\{\frac{1}{\omega^2+i\omega+2}\right\} = \mathcal{F}^{-1}\left\{\frac{1}{(\omega-i)(\omega+2i)}\right\} &= \mathcal{F}^{-1}\left\{-\frac{1/3i}{(\omega-i)}+\frac{1/3i}{(\omega+2i)}\right\} = \ldots \\
&= \frac{1}{3}\mathcal{F}^{-1}\left\{\frac{1}{1+i\omega}\right\}+\frac{1}{3}\mathcal{F}^{-1}\left\{\frac{1}{2-i\omega}\right\} = \ldots \\
&= \frac{1}{3}H(x)e^{-x}+\frac{1}{3}H(-x)e^{2x}.
\end{aligned}$$