Introduction to Feedback Control

Updated: November 20, 2023

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Analysis of Feedback Control Systems

Routh-Hurwitz Criterion

October 27, 2023

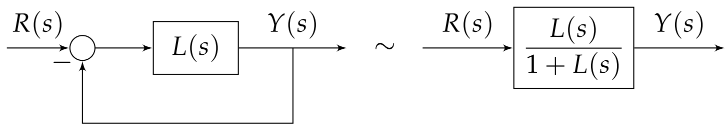


Figure 1: Transfer function of a closed-loop system.

To assess the stability of a closed-loop system (Fig. 1) we need to determine its poles, i.e. the roots of the polynomial

$$1 + L(s) = 0. \tag{1}$$

Consider the polynomial

$$\pi(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0. \tag{2}$$

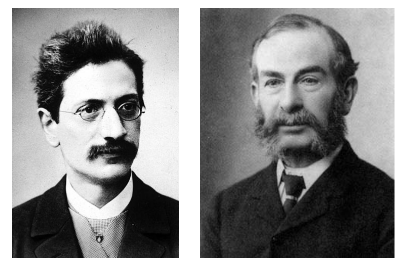


Figure 2: Adolf Hurwitz (1859–1919, left), German mathematician. His doctoral advisor was Felix Klein, who devised the *Klein bottle*. Edward Routh (1831–1907, right), English mathematician. He was born in Quebec.

Definition 1 *$\pi(s)$ is Hurwitz if all its roots have negative real part.*

Suppose $\pi(s)$ has r real roots, $\lambda_1, \ldots, \lambda_r$, and p complex conjugate pairs of roots, $\mu_1, \bar{\mu}_1, \ldots, \mu_p, \bar{\mu}_p$. Then, we can write

$$\pi(s) = \underbrace{(s-\lambda_1)\ldots(s-\lambda_r)}_{(\star)}\underbrace{(s-\mu_1)(s-\bar{\mu}_1)\ldots(s-\mu_p)(s-\bar{\mu}_p)}_{(\star\star)}. \tag{3}$$

$\pi(s)$ being Hurwitz means that $\lambda_i < 0,\ i = 1, \ldots, r$ and $\Re(\mu_i) < 0,\ i = 1, \ldots, p$. In such case, $\pi(s)$ has positive coefficients. This can be seen by expanding the polynomials $(\star)$ and $(\star\star)$ in (3) and realizing that both have positive coefficients, hence their product does too. Thus, if $\pi(s)$ is Hurwitz, necessarily its coefficients $a_i > 0,\ \forall i$.

Example 1

$s^4 + 3s^3 - 2s^2 + 5s + 6$	*Not Hurwitz as* $a_2 < 0$
$s^3 + 4s + 6$	*Not Hurwitz as* $a_2 = 0$
$s^3 + 5s^2 + 9s + 1$	*Don't know*

A NECESSARY AND SUFFICIENT CONDITION for a polynomial to be Hurwitz is provided by Routh's algorithm and the Routh-Hurwitz criterion. The first step of Routh's algorithm consists of building the following table—the *Routh table*—starting from the polynomial $\pi(s)$ in (2):

Step 1 of Routh's algorithm

$$
\begin{array}{c|ccc}
s^n & 1 & a_{n-2} & a_{n-4} \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} \\
s^{n-2} & r_{2,0} & r_{2,1} & r_{2,2} \\
s^{n-3} & r_{3,0} & r_{3,1} & r_{3,2} \\
\vdots & & & \\
s^2 & r_{n-2,0} & r_{n-2,1} & \\
s^1 & r_{n-1,0} & & \\
s^0 & r_{n,0} & &
\end{array}
$$

where

$$
\begin{aligned}
r_{2,0} &= -\frac{1}{a_{n-1}} \begin{vmatrix} 1 & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}, \qquad r_{2,1} = -\frac{1}{a_{n-1}} \begin{vmatrix} 1 & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}, \qquad \dots \\
r_{3,0} &= -\frac{1}{r_{2,0}} \begin{vmatrix} a_{n-1} & a_{n-3} \\ r_{2,0} & r_{2,1} \end{vmatrix}, \qquad r_{3,1} = -\frac{1}{r_{2,0}} \begin{vmatrix} a_{n-1} & a_{n-5} \\ r_{2,0} & r_{2,2} \end{vmatrix}, \qquad \dots,
\end{aligned}
\tag{4}
$$

continuing along each row until a 0 appears, and terminating if a 0 appears in the first column.

The second step of Routh's algorithm consists in applying the following criterion.

Step 2 of Routh's algorithm

Theorem 1 (Routh-Hurwitz criterion)

(i) $\pi(s)$ *is Hurwitz* $\Leftrightarrow$ *All elements in the first column of the Routh table have the same sign*

(ii) If the first column of the Routh table has no 0*'s, then*

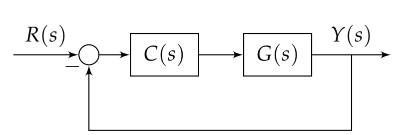
(a) # sign changes = # roots with positive real part

(b) $\nexists$ *roots on the imaginary axis*

Using Routh's algorithm, we can predict whether a polynomial is Hurwitz without explicitly computing its roots. This is convenient to ensure the stability of a system, in particular of a closed-loop system, whose poles are the roots of $1 + L(s)$.

Example 2 (P-control design using Routh's algorithm) *We are given a system with transfer function*

$$
G(s) = \frac{1}{s^4 + 6s^3 + 11s^2 + 6s}. \tag{5}
$$



The system is not stable and we would like to design a controller $C(s) = K$ *for some* $K \in \mathbb{R}$ *in order to stabilize it. The transfer function of the closed-loop system is*

$$
\frac{Y(s)}{R(s)} = \frac{K}{s^4 + 6s^3 + 11s^2 + 6s + K}. \tag{6}
$$

The Routh table is

s^4	1	11	K
s^3	6	6	0
s^2	10	K	0
s^1	$6-\frac{3}{5}K$	0	0
s^0	K		

For the closed-loop system to be stable, the Routh-Hurwitz criterion says that we need $6-\frac{3}{5}K>0$ *and* $K>0$*, i.e.* $0<K<10$.

Nyquist Plot

October 30, 2023

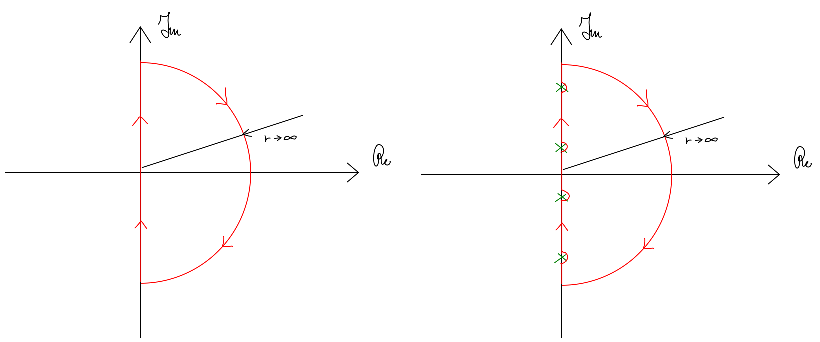


Figure 3: Nyquist contour (left) and its modified version to account for poles on the imaginary axis.

Figure 4: Harry Nyquist (1889–1976), Swedish-American physicist. Nyquist is also a programming language for sound synthesis.

Definition 2 *The Nyquist plot of a transfer function* $L(s)$ *is the image of the Nyquist contour (Fig. 3, on the left).*

If $L(s)$ has poles on the imaginary axis, then we modify the Nyquist contour to avoid these poles by going around them on infinitesimal semicircles on the right half plane (Fig. 3, on the right).

Note 1

(i) $L(-j\omega)=\overline{L(j\omega)}$

(ii) If $L(s)$ *is strictly proper, then* $L(\infty)=0$*, i.e. the whole semicircle of infinite radius is mapped to 0*

Example 3

$$L(s)=\frac{\mu}{1+\tau s},\quad \mu>0,\tau>0. \tag{7}$$

Theorem 2 (Nyquist criterion) *The closed-loop system in Fig. 1 is stable if and only if* $N=P$*, where*

- *P is the number of poles of* $L(s)$ *with positive real part*
- *N is the number of loops that the Nyquist plot of* $L(s)$ *makes around the point -1 in the complex plane (positive if counterclockwise, negative if clockwise); N is undefined if the Nyquist plot passes through -1*

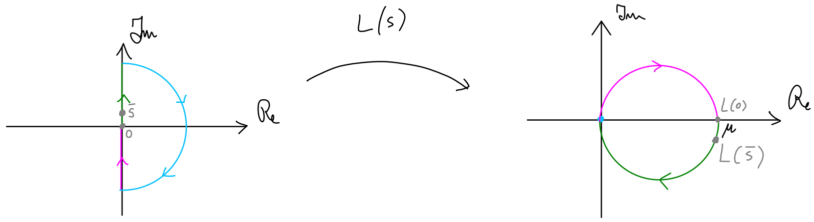


Figure 5: Nyquist plot of the transfer function (7).

Note 2

(i) If N is undefined, the closed-loop can be stable or unstable

(ii) If N is well-defined and $N \neq P$, then the closed-loop system is unstable and it has $P - N$ poles with positive real part

Example 8 (Continued) *$P = 0$ and $N = 0$. By the Nyquist criterion, the negative feedback interconnection of (7) is stable.*

Exercise 1 *Applying the Nyquist criterion, determine the stability property of the closed-loop system in Fig. 1, where $L(s) = \frac{\mu}{1+\tau s}$, where $\mu < 0$ and $\tau > 0$.*

Corollary 1 (to Theorem 2) *Given a stable $L(s)$, the following are sufficient conditions for the closed-loop system in Fig. 1 to be stable.*

- $|L(j\omega)| < 1 \; \forall\omega$
- $|\angle L(j\omega)| < 180° \; \forall\omega$

Stability Margins

November 1, 2023

Assume we designed a controller $C(s)$ for a system $G(s)$ so that the Nyquist plot of $L(s) = C(s)G(s)$ is the one in Fig. 6. The further the Nyquist plot from the point -1, the higher the safety margin with respect to perturbations of $L(s)$ (coming, for instance, from unmodeled dynamics). This margin can be conveniently decomposed into two stability margins which can be read directly from the Bode plots of $L(s)$.

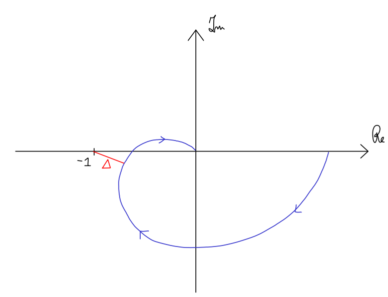


Figure 6: Δ is a stability margin.

Assume $L(s)$ has positive steady-state gain, no unstable poles, and that its Nyquist plot intersects the negative real axis once, on the right of -1. Let ω_π be the frequency such that $\angle L(j\omega_\pi) = -180°$. The gain margin, k_m, is defined as follows (Fig. 7, on the left):

$$k_m = \frac{1}{|L(j\omega_\pi)|}, \tag{8}$$

and it represents the maximum multiplicative factor on the gain of $L(s)$ at ω_π that the system can tolerate before becoming unstable.

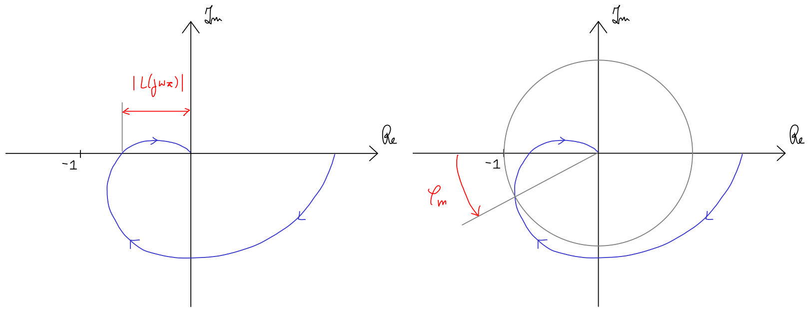


Figure 7: Gain margin, $k_m = \frac{1}{|L(j\omega_\pi)|}$, on the left, and phase margin $\varphi_m = 180° - |\angle L(j\omega_c)|$, on the right.

Assume the Nyquist plot of $L(s)$ crosses the unit circle only once, from outside to inside. Let ω_c be the frequency such that $|L(j\omega_c)| = 1$—ω_c is the *crossover* frequency. The phase margin, φ_m, is defined as follows (Fig. 7, on the right):

$$\varphi_m = 180° - |\angle L(j\omega_c)|, \tag{9}$$

and it represents the maximum (negative) phase shift of $L(s)$ at ω_c that the system can tolerate before becoming unstable.

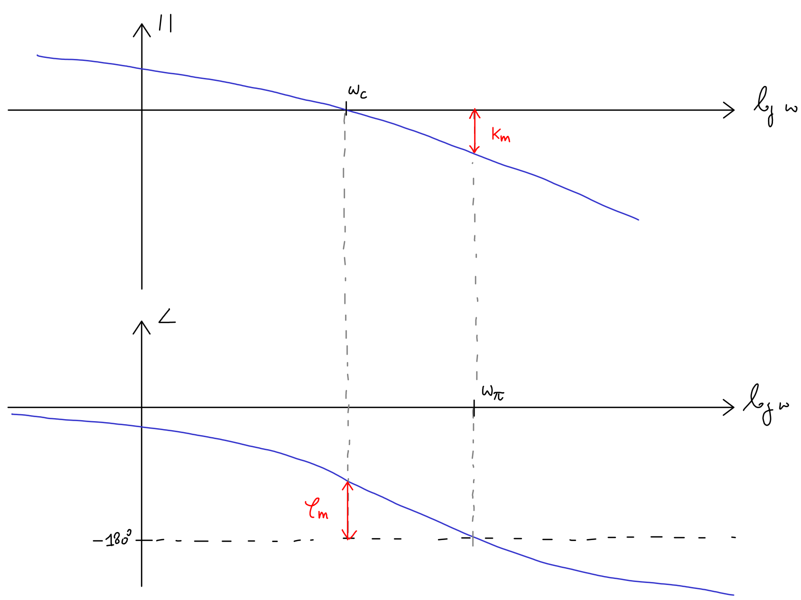


Figure 8: Reading k_m and φ_m on the Bode plots in correspondence of ω_π and ω_c, respectively.

As anticipated, the gain and phase margins can be read off from the Bode plots of $L(s)$, as shown in Fig. 8.

Positive gain (in decibels) and phase margins of $L(s)$ ensure that the closed-loop system is stable.

Note 3 *Just like for the Nyquist criterion, using the gain and phase margins we are able to predict the stability of the closed-loop system by evaluating metrics defined on the open-loop system.*

Controller Synthesis

Loop Shaping

November 3–6, 2023

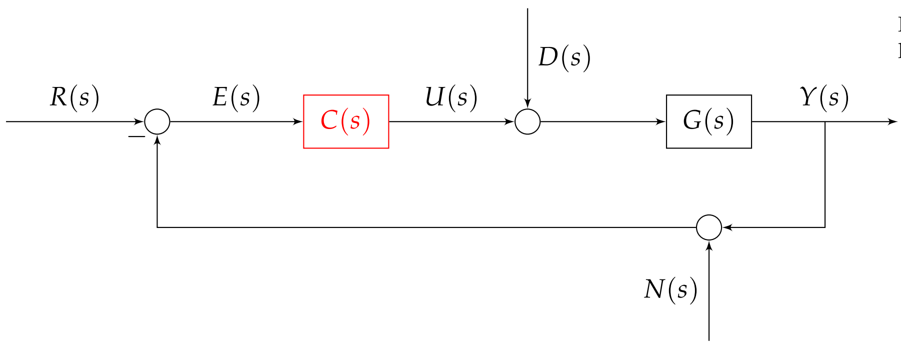


Figure 9: Negative feedback control loop.

The goal of controller synthesis is to design the controller transfer function $C(s)$, in order to make the closed-loop system, modeled by the transfer function

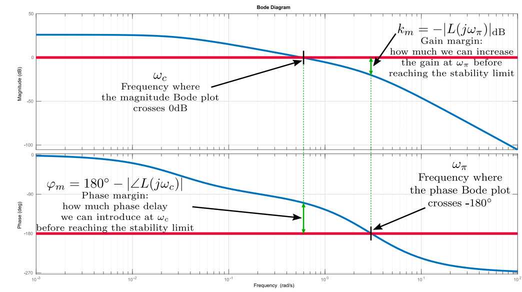
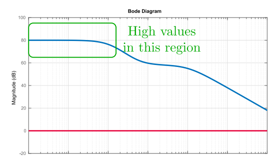
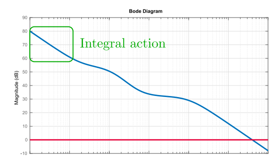
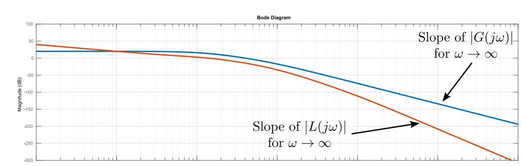
$$F(s) = \frac{Y(s)}{R(s)} = \frac{L(s)}{1 + L(s)} = \frac{C(s)G(s)}{1 + C(s)G(s)}, \tag{10}$$

satisfy desired specifications.

Specifications are typically defined in the time domain. Examples include rise time, overshoot, and steady-state value of the output signal $y(t)$ in response to a step reference signal $r(t)$. Therefore, the control design process can be broken down in the following steps:

- Step 1: Turn specifications from the time domain behavior of $F(s)$ to the frequency domain behavior of $L(s)$
- Step 2: Design the transfer function of the controller $C(s)$ so that $L(s) = C(s)G(s)$ has the desired frequency domain behavior

Step 1 is summarized in the following table, where the time domain specifications of $F(s)$ are listed next to the corresponding frequency domain specification of $L(s)$. The latter are specified as constraints on the Bode plot of $L(s)$.

Time domain specification of $F(s)$	Frequency domain specification of $L(s)$	Graphical representation on the Bode plot of $L(s)$
Stability	• Positive gain margin $k_m > 0$ • Positive phase margin $\varphi_m > 0$ • No cancelations in the computation of $L(s) = C(s)G(s)$	
Robust stability	• The higher k_m and φ_m, the more robust we can expect our design to be • Upper bound on ω_c: a time delay of τ contributes to $-\omega_c\tau$ phase shift at ω_c	
Static performance $\lim_{s\to 0} s\frac{F(s)}{s} = F(0) = \frac{\mu}{s^\rho+\mu}$ We want $F(0)$ as close to 1 as possible So, the higher μ the better, or $\rho > 0$ (μ and ρ are the steady-state gain and the number of poles at the origin, respectively, of $L(s)$)	• High magnitude at low frequencies • Or, $\|L(j\omega)\| \xrightarrow{\omega\to 0} \infty$ ($F(0) = 1$, i.e. zero steady-state error)	 
Dynamic performance Tracking fast reference trajectories, not just regulating to constant references	• Lower bound on ω_c (as $\|F(j\omega)\| \approx 1$ for $\omega < \omega_c$, while $\omega > \omega_c$ are attenuated)	
Disturbance rejection $\frac{Y(s)}{D(s)} = \frac{G(s)}{1+C(s)G(s)}$ must attenuate frequencies characterizing the disturbance (typically low frequencies)	• $\|L(j\omega)\| \gg 1$ for the range of ω characterizing the disturbance (typically low frequencies) • Or, equivalently, lower bound on ω_c	Same as "Static performance"
Noise attenuation $\frac{Y(s)}{N(s)} = -\frac{C(s)G(s)}{1+C(s)G(s)}$ must attenuate frequencies characterizing the noise (typically high frequencies)	• $\|L(j\omega)\| \ll 1$ for the range of ω characterizing the noise (typically high frequencies) • Or, equivalently, upper bound on ω_c	
Realizability of the controller $C(s)$ must be proper	• Slope of $\|L(j\omega)\| \leq$ slope of $\|G(j\omega)\|$ for $\omega \to \infty$	

Example 4 *Consider a system modeled by the following transfer function*

$$G(s) = \frac{10}{(1+10s)(1+5s)(1+s)}. \tag{11}$$

Design a controller $C(s)$ such that the closed-loop system has the following specifications:

- *The steady-state error in response to a unit step reference is $|e_\infty| \leq 0.1$*
- *The crossover frequency is $\omega_c \geq 0.2$*
- *The phase margin is $\varphi_m \geq 60°$*

Let us start by separating the controller into its static and dynamic components, $C_1(s)$ and $C_2(s)$:

$$C(s) = \underbrace{\frac{\mu}{s^\rho}}_{C_1(s)} \underbrace{\frac{\prod_i (1+T_i s) \prod_i \left(1 + \frac{2\xi_i}{\alpha_{n,i}} s + \frac{s^2}{\alpha_{n,i}^2}\right)}{\prod_i (1+\tau_i s) \prod_i \left(1 + \frac{2\zeta_i}{\omega_{n,i}} s + \frac{s^2}{\omega_{n,i}^2}\right)}}_{C_2(s)}. \tag{12}$$

In a first attempt to satisfy all control specifications, $C_1(s)$ can be designed to control the steady-state error, while $C_2(s)$ can be used to achieve the desired crossover frequency and phase margin.

Regarding $C_1(s)$, let us choose $\rho = 0$. To ensure $|e_\infty| \leq 0.1$ in response to a unit step, we need y_∞ to be between 0.9 and 1.1. y_∞ is given by the following expression:

$$y_\infty = \frac{C_1(0)C_2(0)G(0)}{1 + \underbrace{C_1(0)}_{\mu}\underbrace{C_2(0)}_{1}\underbrace{G(0)}_{10}} = \frac{10\mu}{1+10\mu}, \tag{13}$$

therefore $\mu \geq 0.9$ will satisfy the specification on the steady-state error. Let us chose $C_1(s) = 1$.

Regarding $C_2(s)$, we can proceed as follows:

(i) Define a desired open-loop transfer function, $L^\star(s)$, that has same steady-state gain as $C_1(s)G(s)$ and fulfills the desired specifications on crossover frequency and phase margin

(ii) Let $C_2(s) = \frac{L^\star(s)}{C_1(s)G(s)}$, so that $L(s) = C_1(s)C_2(s)G(s) = L^\star(s)$ and $C_2(0) = 1$, i.e. $C_2(s)$ does not change the steady-state behavior of the closed-loop system

Let

$$L^\star(s) = \frac{10}{(1+\frac{s}{0.03})(1+\frac{s}{3})^2}, \tag{14}$$

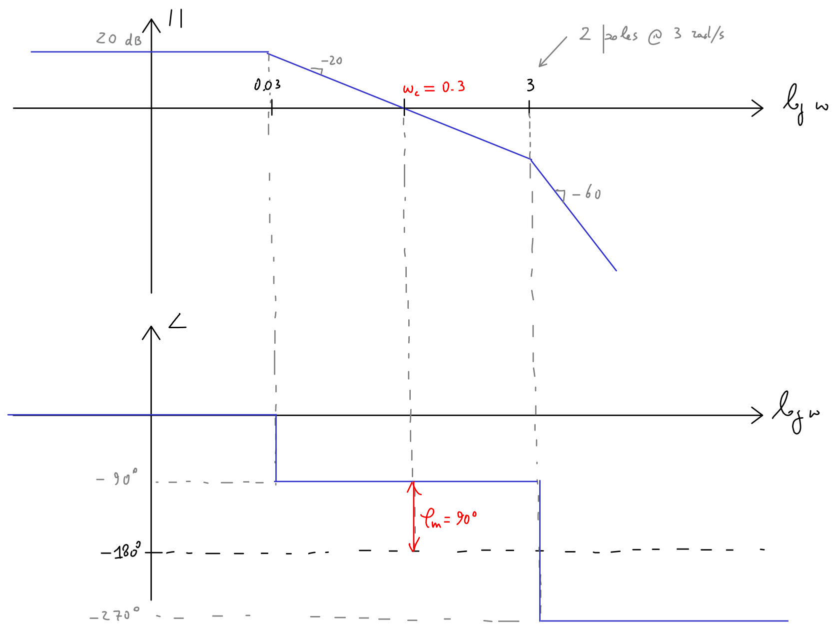
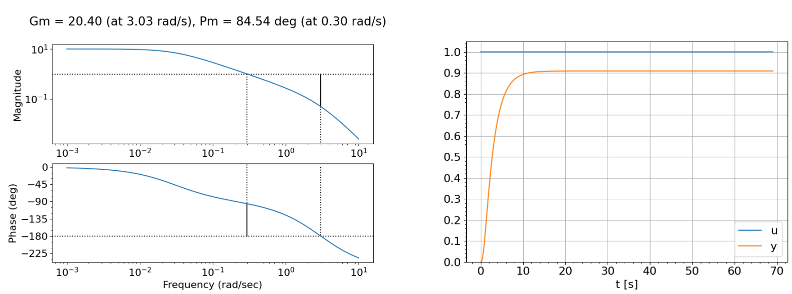


Figure 10: Asymptotic Bode plots of $L^{\star}(s)$.

so that $L^{\star}(0) = C_1(0)G(0) = 10$, $\omega_c \approx 0.3$, $\varphi_m \approx 90°$, *and the controller is realizable*[1] *(see Fig. 10).* $C_2(s)$ *is computed as in step (ii) above and evaluates to*

$$C_2(s) = \frac{(1+10s)(1+5s)(1+s)}{(1+\frac{s}{0.03})(1+\frac{s}{3})^2}. \tag{15}$$

Figure 11 shows that the designed controller results in $L(s)$ *to have the*



(a) Bode plots of the open-loop system, $L(s)$.

(b) Step response of the closed-loop system, $F(s)$.

Figure 11: Results of the control design: the system fulfills all desired specifications.

prescribed crossover frequency and phase margin, and that the closed-loop system achieves a steady-state error less than 0.1 *in response to a step reference input.*

[1] The relative degree—difference between the degree of the denominator and that of the numerator—of $L^{\star}(s)$ is no less than the one of $G(s)$.

Integral Control

November 8, 2023

The main purpose of an integral controller

$$C(s) = \frac{\mu}{s} \tag{16}$$

is to obtain zero steady-state error, i.e. to have the measured output precisely track the reference signal. The output of the integral controller integrates the error between the reference signal and the measured output. Whenever an integral action is introduced, one needs to pay attention to the phase Bode plot of $L(s)$, since a pole at the origin shifts the phase down by $90°$ for all frequencies, which could result in reducing the phase margin of rendering it negative.

Exercise 2 *Consider a system modeled by the following transfer function*

$$G(s) = \frac{10}{(1+10s)(1+5s)(1+s)}. \tag{17}$$

Design a controller $C(s)$ such that the closed-loop system has the following specifications:

- *The steady-state error in response to a unit step reference is $|e_\infty| = 0$*
- *The crossover frequency is $\omega_c \geq 0.2$*
- *The phase margin is $\varphi_m \geq 60°$*

Lead-lag Compensators

November 10, 2023

A way to increase the phase margin without losing static performance consists in using a phase-lead compensator, i.e. a controller with the following transfer function:

$$C(s) = \mu \frac{1+Ts}{1+\alpha Ts}, \tag{18}$$

where $\mu > 0$, $T > 0$, and $0 < \alpha < 1$. The Bode plots of the lead compensator are reported in Fig. 12, where significant phase anticipation (lead) can be seen between the zero and the pole of the controller, at the expense of some amplification at high frequencies.

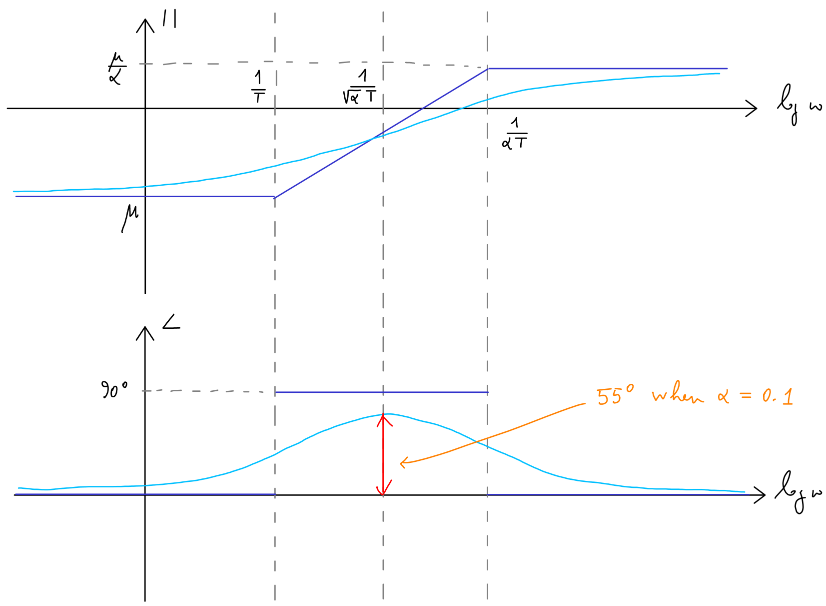


Figure 12: Bode plots of the lead compensator.

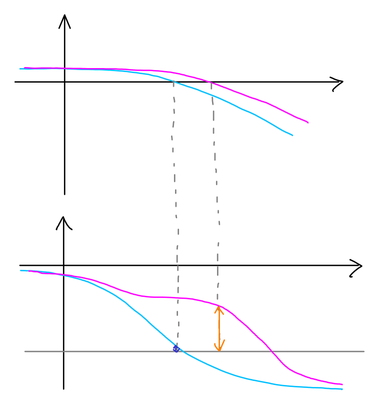


Figure 13: Effect of a lead compensator on the phase margin (without in cyan, with in magenta).

Note 4

(i) *The gain at high frequencies introduced by the lead compensator is* $\frac{\mu}{\alpha}$

(ii) $\alpha = 0.1$ *is a good compromise between having a decent phase lead* ($\approx 55°$) *and not too much amplification at high frequencies*

(iii) *Choosing* $\frac{1}{\sqrt{\alpha}T} \approx \omega_c$ *leads to an increase of the phase margin*

(iv) *For* $\alpha = 0$*, the phase-lead controller*

$$C(s) = \underbrace{\mu}_{\text{P}} + \underbrace{\mu Ts}_{\text{D}} \tag{19}$$

is a proportional-derivative (PD) controller

A WAY TO IMPROVE STATIC PERFORMANCE without losing stability margins consists in using a phase-lag compensator, i.e. a controller with the following transfer function:

$$C(s) = \mu \frac{1 + Ts}{1 + \alpha Ts}, \tag{20}$$

where $\mu > 0$, $T > 0$, and $\alpha > 1$. The Bode plots of the lag compensator are reported in Fig. 14, where a high steady-state gain can be achieved, at the expense of a few degrees of phase margin.

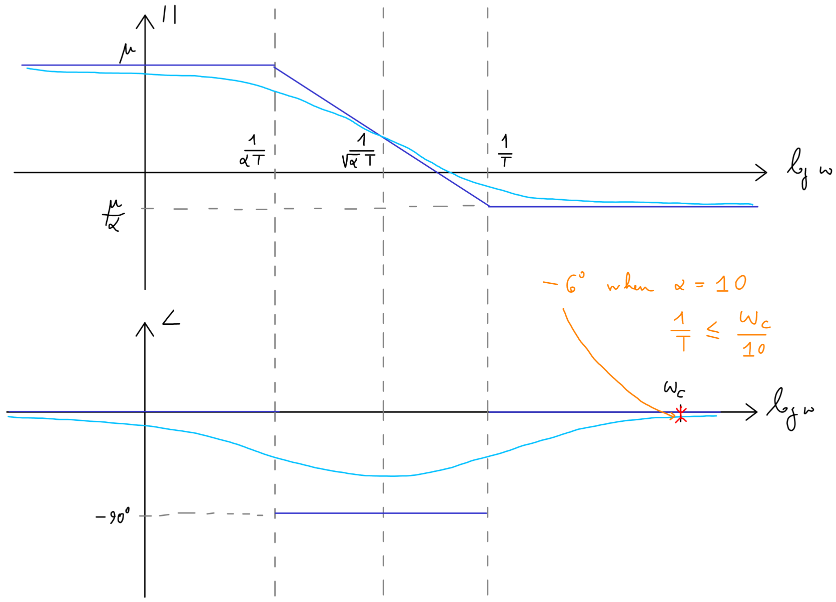


Figure 14: Bode plots of the lag compensator.

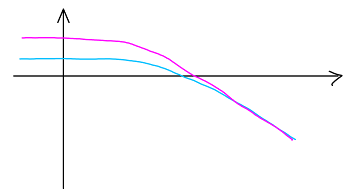


Figure 15: Effect of a lag compensator with $\mu = \alpha$ on the steady-state gain (without in cyan, with in magenta).

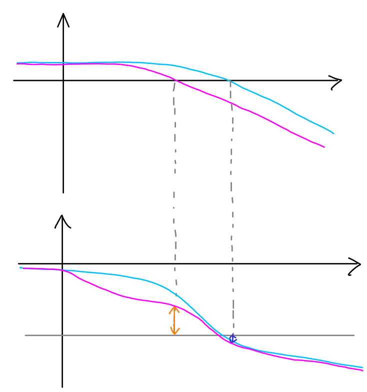


Figure 16: Effect of a lag compensator with $\mu = 1$ on the phase margin (without in cyan, with in magenta).

Note 5

(i) *$\alpha = 10$ is a good compromise between having a decent steady-state gain increase without introducing too much phase lag*

(ii) *Choosing $\frac{1}{T} \leq \frac{\omega_c}{10}$ leads to only $\approx 6^\circ$ phase margin decrease*

(iii) *$\mu = \alpha$ results in a compensator that does not alter the behavior of the system at high frequencies*

(iv) *A lag compensator with $\mu = 1$ can be employed to reduce the crossover frequency in order to increase the phase margin*

(v) *For $\alpha \to \infty$, with a corresponding increase of the steady-state gain μ s.t. the ratio $\frac{\mu}{\alpha} = c$ is constant, the phase-lag controller*

$$C(s) = \underbrace{c}_{\text{P}} + \underbrace{\frac{c}{T}\frac{1}{s}}_{\text{I}} \tag{21}$$

is a proportional-integral (PI) controller

PID Controllers

November 13, 2023

A Proportional-Integral-Derivative (PID) controller produces a control action $u(t)$ proportional to the error $e(t)$, as well as to its integral and derivative, i.e.:

$$u(t) = K_p e(t) + K_I \int_0^t e(\tau)\mathrm{d}\tau + K_D \frac{\mathrm{d}e}{\mathrm{d}t}(t), \tag{22}$$

where $K_P, K_I, K_D > 0$. The transfer function of an ideal PID controller is:

$$C_{\text{PID}}(s) = K_P + \frac{K_I}{s} + K_D s. \tag{23}$$

Another common expression of the PID controller transfer function is:

$$C_{\text{PID}}(s) = K_P \left(1 + \frac{1}{T_I s} + T_D s\right), \tag{24}$$

given in terms of the *integration time*, $T_I = K_P/K_I$, and the *derivative time*, $T_D = K_D/K_P$.

The integral action, $\frac{K_I}{s}$, results in zero steady-state error and perfect disturbance rejection of the closed-loop system, thanks to the pole at the origin (infinite gain for $\omega \to 0$). The derivative action, $K_D s$, results in a faster response of the closed-loop system, thanks to the introduced phase lead (infinite gain for $\omega \to \infty$).

THE PRESENCE OF THE DERIVATIVE ACTION, however, makes $C_{\text{PID}}(s)$ not realizable, because improper. In practice, a pole at high frequency is introduced in order to make the controller realizable:

$$C_{\text{PID}}(s) = K_P \left(1 + \frac{1}{T_I s} + \frac{T_D s}{1 + \frac{T_D}{N} s}\right), \tag{25}$$

where N is chosen so that the high-frequency pole at $-N/T_D$ is beyond the bandwidth of the control task.

Note 6

(i) *PID controllers are used in more than* 90% *of industrial applications*

(ii) *Practical considerations in applications of PID controllers include*

- *Limitation of derivative action (not to spike in response to step reference signals)*
- *Anti-windup of the integral action in presence of input saturation (not to integrate the error when the input saturates)*

(iii) *There exist several methods to tune gains of PID controllers (e.g. open-loop and closed-loop Ziegler & Nichols methods)*

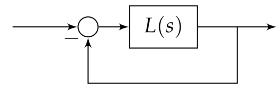
Root Locus

November 15, 2023

Consider an open-loop transfer function

$$L(s) = \mu \frac{N_0(s)}{D(s)}, \tag{26}$$

where $N_0(s)$ and $D(s)$ are polynomials of degree m and n, respectively.



Definition 3 *The root locus for loop gain is the set of all points of the complex plane which are roots of* $1 + L(s)$ *for some* $\mu \in (-\infty, 0) \cup (0, \infty)$.

Basic rules to draw the root locus for $\mu > 0$ are the following:

1. The locus has n branches, one per pole
2. The locus is symmetric with respect to the real axis
3. The branches of the locus start at the poles of $L(s)$
4. For $\mu \to \infty$

$$\begin{cases} m \text{ branches} & \to \quad \text{zeros of } L(s) \\ n - m \text{ branches} & \to \quad \infty \end{cases}$$

5. The asymptotes of the root locus meet on the real axis at $x_a = \frac{1}{n-m}\left(\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i\right)$—$p_i$ and z_i being the poles and zeros of $L(s)$—and have slope $\psi_a = \frac{2k+1}{n-m}\pi$, $k = 0, \ldots, n-m-1$
6. The locus includes all points of the real axis on the left of an odd number of poles/zeros

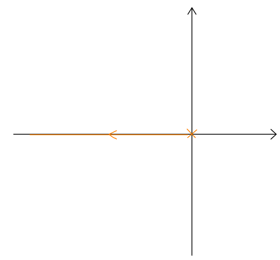


Figure 17: Root locus of $L(s) = \frac{\mu}{s}$.

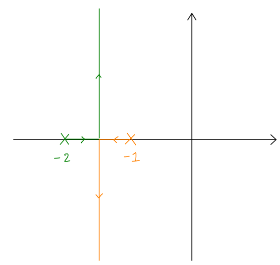


Figure 18: Root locus of $L(s) = \frac{\mu}{(s+1)(s+2)}$.

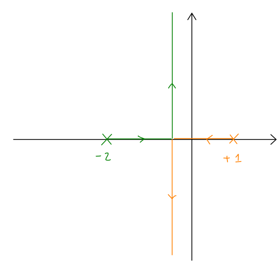


Figure 19: Root locus of $L(s) = \frac{\mu}{(s-1)(s+2)}$.

Example 5 *The root locus of* $L(s) = \frac{\mu}{s}$ *is shown in Fig. 17. Increasing the gain* μ *makes the closed-loop system faster and faster. The close-loop system is stable for all* $\mu > 0$.

Example 6 *The root locus of* $L(s) = \frac{\mu}{(s+1)(s+2)}$ *is shown in Fig. 18. Increasing the gain* μ *makes the closed-loop system faster up to the point where the poles meet on the real axis, after which the damping of the pair of complex conjugate poles starts to decrease. The close-loop system is stable for all* $\mu > 0$.

Example 7 *The root locus of* $L(s) = \frac{\mu}{(s-1)(s+2)}$ *is shown in Fig. 19. A high enough gain (*$\mu > 2$*) stabilizes the closed-loop system, bringing the unstable pole to the left half-plane.*

Pole Placement

November 17, 2023

The root locus for loop gain shows how the poles of the closed-loop move on the complex plane as a function of the gain of the open-loop transfer function. The position of the dominant poles of the closed-loop system determine its performance. Assuming the closed-loop system has a pair of complex conjugate dominant poles, the following expressions for important performance metrics hold:

- O.S.% $= 100e^{-\zeta\pi\sqrt{1-\zeta^2}}$

- Peak time $= \frac{1}{\omega_n\sqrt{1-\zeta^2}}$
- Oscillation frequency $= \omega_n\sqrt{1-\zeta^2}$
- Settling time $= -\frac{\ln 0.01\varepsilon}{\zeta\omega_n} = \frac{\ln 0.01\varepsilon}{\sigma}$
- Phase margin $\approx 100\zeta$
- Crossover frequency $\approx \omega_n$

where ζ, ω_n, and $\sigma = -\zeta\omega_n$ are damping, natural frequency, and real part of the pair of complex conjugate poles. In summary:

- The damping of the dominant poles, ζ, controls the overshoot, the peak time, the frequency of oscillations, the settling time, and the phase margin
- The natural frequency of the dominant poles, ω_n, controls the peak time, the frequency of oscillations, the settling time, and the crossover frequency
- The real part of the dominant poles, $\sigma = \zeta\omega_n$, controls the settling time

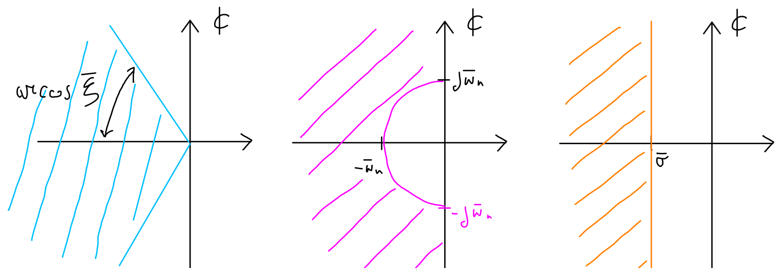


Figure 20: Regions of the complex plane where $\zeta \geq \bar{\zeta}$ (left), $\omega_n \geq \bar{\omega}_n$ (middle), $\sigma \leq \bar{\sigma}$ (right).

Figure 20 shows regions of the complex plane characterized by values of damping greater than a given threshold $\bar{\zeta}$, values of the natural frequency greater than $\bar{\omega}_n$, and values of the real part less then $\bar{\sigma}$.

State Feedback Control

November 20, 2023

There are cases where a static feedback of the output is not sufficient to stabilize a system. In such cases, assuming we have access to the system state, a static state feedback might be sufficient to stabilize a system.

In order to build a state feedback controller, we need a state space representation of a transfer function. Given

$$G(s) = \frac{b_{n-1}s^{n-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}, \tag{27}$$

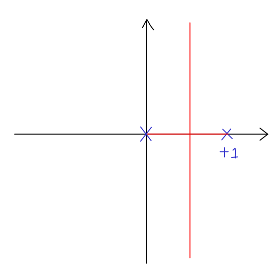


Figure 21: Root locus of $L(s) = \frac{\mu}{s(s-1)}$. There is no μ such that the closed-loop system is stable.

its *controllable canonical state space form* is defined as follows:

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & 1 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} b_0 & b_1 & \dots & b_{n-1} \end{bmatrix} x. \end{cases} \tag{28}$$

Example 8 *Consider the system*

$$G(s) = \frac{1}{s(s-1)}, \tag{29}$$

whose root locus is shown in Fig. 21. Its state space representation in controllable canonical form is

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x. \end{cases} \tag{30}$$

The poles of G (and the eigenvalues of the A matrix of its state space representation) are 0 *and* 1*. The system is unstable.*

A static state feedback controller is defined as follows:

$$u = -Kx, \tag{31}$$

where $K^T \in \mathbb{R}^n$. Controlling a system

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{32}$$

with the controller (31) yields

$$\begin{cases} \dot{x} = (A - BK)x \\ y = Cx. \end{cases} \tag{33}$$

Therefore, the state-feedback-controlled closed-loop system behaves according to the eigenvalues of the matrix $A - BK$.

Theorem 3 (Pole placement via state feedback) *Given a system in controllable canonical form and n complex numbers $\lambda_1^\star, \dots, \lambda_n^\star$ (either real or pairs of complex conjugate numbers), there exists $K^T \in \mathbb{R}^n$, such that the eigenvalues of $A - BK$ are $\lambda_1^\star, \dots, \lambda_n^\star$.*

Theorem 3 can be proved constructively as follows. Given the expression of the matrices A and B in (28), and let $K = [K_1, K_2, \ldots, K_n]$. The eigenvalues of the matrix $A - BK$ are the roots of its characteristic polynomial:

$$\begin{aligned} &|\lambda I - (A - BK)| \\ =&\lambda^n + (a_{n-1} + K_n)\lambda^{n-1} + \ldots + (a_1 + K_2)\lambda + (a_0 + K_1). \end{aligned} \tag{34}$$

Given $\lambda_1^\star, \ldots, \lambda_n^\star$, the desired characteristic polynomial of $A - BK$ is

$$P(\lambda) = (\lambda - \lambda_1^\star) \ldots (\lambda - \lambda_n^\star) = \lambda^n + c_{n-1}\lambda^{n-1} + \ldots + c_1\lambda + c_0, \tag{35}$$

for some $c_0, c_1, \ldots, c_{n-1}$. Therefore, choosing K such that

$$\begin{cases} a_{n-1} + K_n = c_{n-1} \\ \vdots \\ a_1 + K_2 = c_1 \\ a_0 + K_1 = c_0, \end{cases} \tag{36}$$

results in a closed-loop system with eigenvalues at $\lambda_1^\star, \ldots, \lambda_n^\star$.