Out: Sunday, April 9, 2023 (at long last)
Due: Whenever (bare-bone solutions will be posted in a few days)
This is an optional problem set for practice purposes only.

Problem 1

A discrete-time period signal $x[n]$ is real-valued and has a fundamental period of $N = 5$. The non-zero Fourier coefficients for $x[n]$ are:
$a_0 = 1$, $a_2 = e^{j\pi/4}$, $a_4 = 2e^{j\pi/3}$.
Express $x[n]$ in the form
$x[n] = A_0 + \Sigma_{k=1}^{\infty} A_k sin(\omega_k\, n + \phi_k)$.

Problem 2

Suppose we are given the following information about a signal $x[n]$:

1. $x[n]$ is a real and even signal.
2. $x[n]$ has period $N = 10$ and Fourier coefficients a_k.
3. $a_{11} = 5$.
4. $\Sigma_{n=0}^{9} |x[n]|^2 = 500$.

Find $x[n]$.

Problem 3

Let $x(t)$ be a signal with Fourier transform $\hat{x}(\omega)$. Suppose we are given the following information:

1. $x(t)$ is a real.
2. $x(t) = 0$ for $t \leq 0$.
3. $\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{R}e\left\{\hat{x}(\omega)\right\} e^{j\omega t} d\omega = |t|e^{-|t|}$.

Find a closed-form expression for $x(t)$.

Problem 4

a) Calculate the Fourier transform of $x(t) = e^{-|t|}$.
b) Using the result from part a) and the appropriate Fourier transform properties, find the Fourier transform of $te^{-|t|}$.
c) Use the results from part b), along with the duality property, to find the Fourier transform of $\frac{4t}{(1+t^2)^2}$.

Problem 5

Consider a continuous-time ideal band-pass filter whose frequency response is $H(j\omega) = 1$ when $\omega_c \leq |\omega| \leq 3\omega$ and $H(j\omega) = 0$ otherwise.
a) If $h(t)$ is the impulse response of this filter, determine the function $g(t)$ such that $h(t) = \left(\frac{sin\,\omega_c t}{\pi t}\right) g(t)$.
b) As ω_c is increased, does the impulse response of the filter get more concentrated or less concentrated about the origin?

Problem 6

Let $x(t)$ be a real-valued signal for which $\hat{x}(\omega) = 0$ when $|\omega| > 2,000\pi$. Amplitude modulation is performed to produce the signal $g(t) = x(t)sin(2,000\pi t)$.
A demodulation system is proposed, in which $g(t)$ is the input signal and inside which the input signal is first multiplied by $cos(2000\pi t)$ and the resulting signal is then sent through a low-pass filter with cut-off frequency $2,000\pi$ and passband gain of 2. Determine the signal $y(t)$ that is the output of this demodulation system (the signal after the low-pass filter) for $g(t)$ being the input.

Problem 7 (extra)

We recently discussed the concept of two-dimensional complex exponentials, two-dimensional Fourier transforms, and their applications to image processing. We also mentioned that a lens can act as a Fourier transform machine: When a light emitting image is placed in the focal plane of a lens, the lens will produce a Fourier transform of the image in the focal plane on the other side (which can be detected if a screen or an image sensor is placed in that plane). The original image can be recovered by performing a second Fourier transform with another lens placed so that the distance between the two lenses is equal to the sum of their focal lengths (the image will be reformed in the focal plane on the other side of the second lens). Propose and conceptually explain how you would implement in this system the following:

a) a high-pass filter for the spatial frequency components of the original image.

b) a low-pass filter for the spatial frequency components of the original image.