Part II:

Signals and Systems

The first part of the course looked at differential equations from a traditional perspective, such as a mathematician or physicist might usually take.

We assumed a given "forcing function," $f(t)$, and found the corresponding solution $y(t)$.

In this part of the course we'll think of $f(t)$ as an input and $y(t)$ as an output of a "system." We'll be interested in understanding how this system responds to a broad range of inputs, not just a particular $f(t)$.

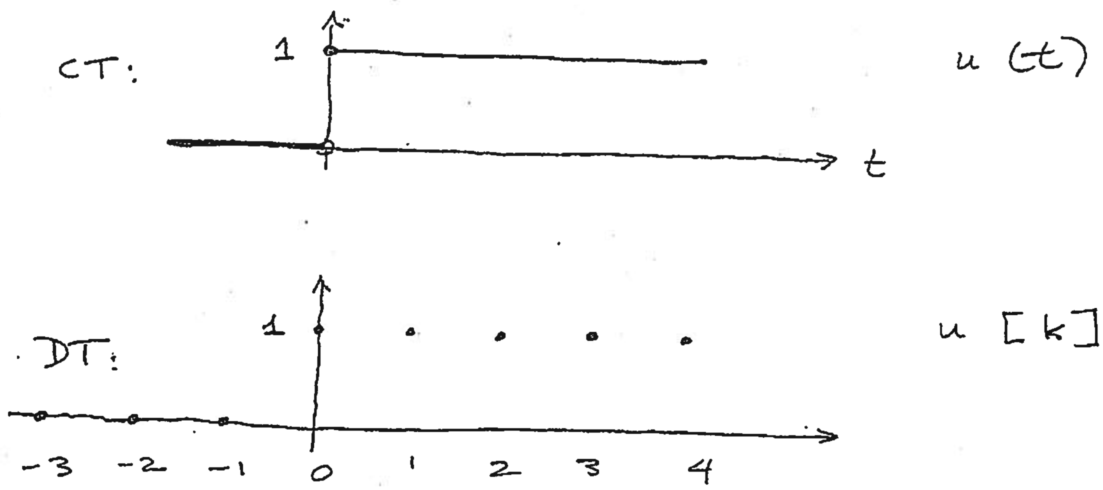
This approach is more typical of engineering — for example, control, signal-processing or communications engineering.

Intro (ctd):

To model systems mathematically, we need to make a few definitions:

- signal: a real- or complex-valued function of a real variable t
    - t will usually represent time, though sometimes a different variable is preferable
        - e.g., crankshaft angle in engine control
    - if the domain of the signal is $\mathbb{R}$ or one of its intervals, the signal is continuous-time (CT)
        - e.g., most "physical" signals
    - if the domain is a discrete set like $\mathbb{Z}$ or $\mathbb{N}$ the signal is discrete-time (DT)
        - e.g.,
            - monthly bank balance
            - value of a variable in a computer program.
            - sampled version of a continuous-time signal

A simple signal that is often used as a reference in control systems is a <u>unit step</u>:



- <u>system</u> :

  - informally, a device or process whereby certain "input" signals determine certain "output" signals.

  - mathematically, a mapping (function) from a class $\mathcal{F}$ of input signals to a class $\mathcal{Y}$ of output signals

Notation:

$$\mathcal{F} \xrightarrow{S} \mathcal{Y}$$

$$y(t) = (Sf)(t)$$

$$y = Sf$$

Output $y(\cdot) = (Sf)(\cdot)$ is called the system's <u>response</u> to input $f(\cdot)$.

Properties of systems:

1. CT, DT & hybrid
   - if the input and output classes are of CT signals then the system is continuous - time (CT)
     - e.g. most models of physical systems
   - if the input and output classes are of DT signals then the system is discrete - time (DT)
     - e.g. digital hardware
   - in a hybrid system, the signals classes are of different kinds

     e.g. A/D converter

     D/A "

A differential equation may represent a CT system, provided that for any signal in the input class, there is a unique signal in the output class that satisfies the equation.

DT systems are often represented by <u>difference equations</u>. Technically, these are equations involving DT signals, say $y[\cdot]$ and $f[\cdot]$, and their <u>differences</u>, e.g.

$$\nabla y[k] = y[k] - y[k-1] \quad \text{(1st diff.)}$$
$$\nabla^2 y[k] = \nabla y[k] - \nabla y[k-1] \quad \text{(2nd ..)}$$
$$\vdots$$
$$\nabla^n y[k] = \nabla^{n-1} y[k] - \nabla^{n-1} y[k-1] \quad \text{(nth ..)}$$

<u>initial conditions</u> give the values of the differences of $y[\,]$ at some "starting time"

It is more common to write a <u>recurrence equation</u>, e.g.

$$y[k] + a_1 y[k-1] + a_2 y[k-2] + \ldots + a_n y[k-n]$$
$$= b_0 f[k] + b_1 f[k-1] + \ldots + b_m f[k-m]$$

and to specify values of $y[\,]$ at a number of different time points. We still commonly use the terms "difference equation" and "initial conditions" in this case.

Properties of systems

2: Memoryless vs. dynamic

- In a memoryless system, the instantaneous output value $y(t)$ depends only on the input value $f(t)$

  - e.g. ideal amplifier:

$$V_{out}(t) = K\, V_{in}(t)$$

- A system that is not memoryless is dynamic.

  - e.g. mechanical system:

$$M\ddot{y}(t) = f(t), \qquad f(t)=0,\ \forall t \le \bar{t}, \quad \dot{y}(\bar{t}) = y(\bar{t}) = 0$$

$$\Rightarrow \quad y(t) = \frac{1}{M}\int_{-\infty}^{t}\left[\int_{-\infty}^{\tau} f(\theta)\,d\theta\right]d\tau$$

This system is dynamic because of mechanical inertia.

Most interesting control problems involve dynamic "plants."

3. Causality

S is <u>causal</u> if $y(t) = (Sf)(t)$ depends only on

$$\{ f(\tau) : \tau \leq t \}$$

— i.e. only on prior (& present) values of the input

In other words, if $f_1(\tau) = f_2(\tau)$, $\forall \tau \leq t$, and $y_1 = Sf_1$ & $y_2 = Sf_2$, then

$$y_1(\tau) = y_2(\tau) \quad \forall \tau \leq t$$

examples:

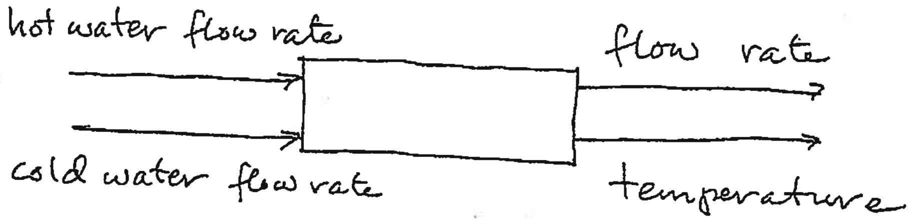
- memoryless systems — causal
- $y[k] = f[k+2]$ — noncausal

- Real-time controllers are causal, but much (off-line) signal processing involves noncausal systems.

4. <u>Multivariable / scalar</u>

- <u>multivariable</u> - system with multiple inputs & outputs

  - e.g., shower



- <u>scalar</u>, or single-input, single-output

  - as the name suggests.

Multivariable systems pose special problems for control.

5. <u>Linearity</u>

- if the input is a linear combination of input signals, then the output is a linear combination (of the same form) of their respective responses.

- more precisely, $\forall c_1, c_2 \in \mathbb{R}$, $\forall f_1, f_2 \in \mathcal{F}$,

$$S(c_1 f_1 + c_2 f_2) = c_1 S(f_1) + c_2 S(f_2)$$

<u>examples</u>:

$$M\ddot{y}(t) = f(t) \quad [f(t)=0, \forall t \le \bar{t}, \dot{y}(\bar{t}) = y(\bar{t}) = 0]$$

- yes

$$y(t) = f(t) + 1$$

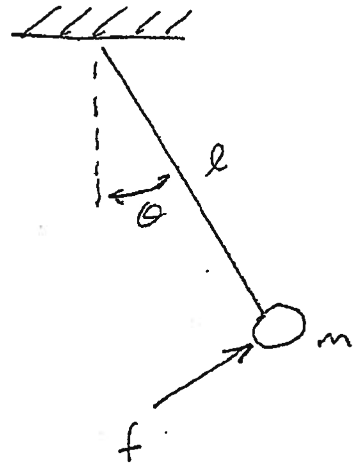
- No

- Linearity greatly simplifies mathematical analysis.

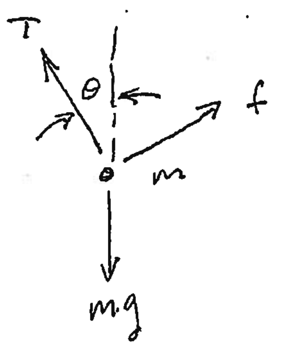
- Among physical systems, nonlinearity is common

→ We often approximate the operation of a nonlinear system about an "operating point" with a "linearized" model.

example:



- free-body diagram



- Newton's law:

$$m l^2 \ddot{\theta} = f l - mgl \underbrace{\sin\theta}_{\text{nonlinear}}$$

- for sufficiently small θ,

$$m l^2 = f l - mgl\theta$$

- linear

6. <u>Time - invariance</u>

Roughly speaking, a system is <u>time - invariant</u> if its behaviour doesn't change with time ...

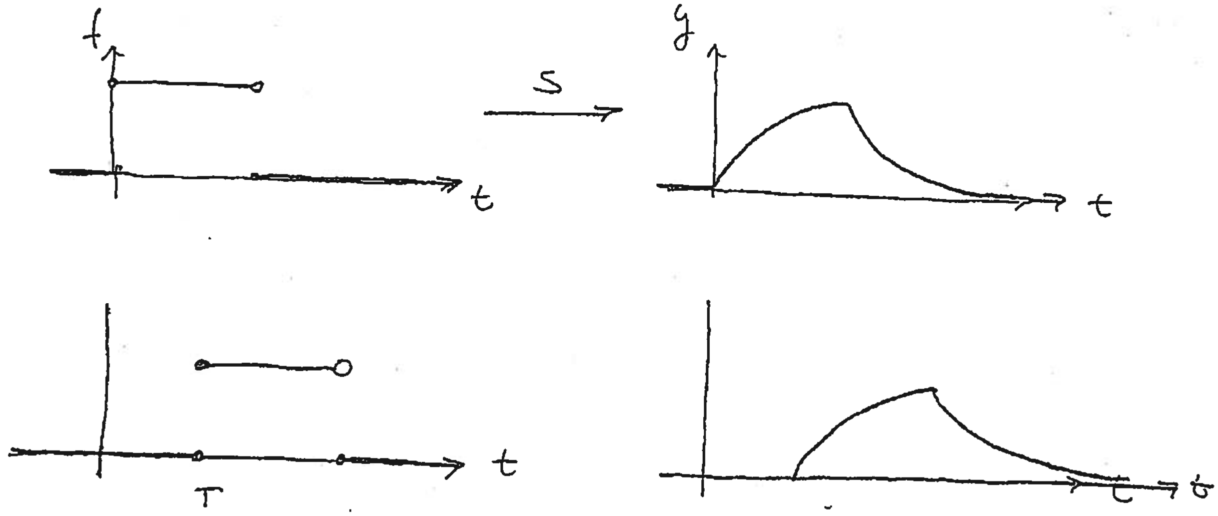
... mathematically, if

$$f(t) \xrightarrow{S} y(t),$$

then

$$f(t-T) \xrightarrow{S} y(t-T)$$

Picture:



Examples

a. $M\ddot{y}(t) = f(t)$, $\quad f(t) = 0,\ \forall t \le t_0,$

$\dot{y}(t_0^-) = y(t_0^-) = 0$

$$\Rightarrow \quad y(t) = \frac{1}{M}\int_{-\infty}^{t}\int_{-\infty}^{\tau} f(\theta)\, d\theta$$

Now replace $f(t)$ with $\tilde{f}(t) = f(t-T)$.
The corresponding response is

$$\tilde{y}(t) = \frac{1}{M}\int_{-\infty}^{t}\left[\int_{-\infty}^{\tau} f(\theta - T)\, d\theta\right] d\tau$$

$$= \frac{1}{M}\int_{-\infty}^{t}\left[\int_{-\infty}^{\tau - T} f(\theta)\, d\theta\right] d\tau \quad \text{(change of variable)}$$

$$= \frac{1}{M}\int_{-\infty}^{t-T}\left[\int_{-\infty}^{\tau} f(\theta)\, d\theta\right] d\tau \quad (\quad " \quad)$$

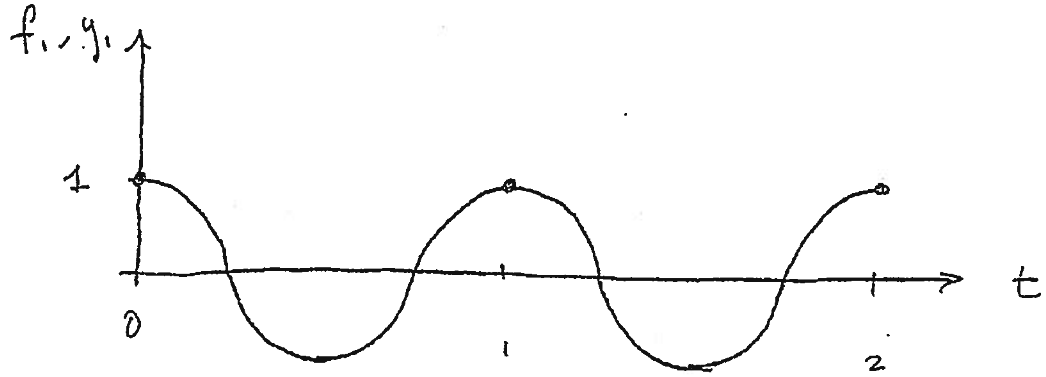
$$= y(t-T)$$

$\rightarrow$ time-invariant

b. "Ideal sampler"

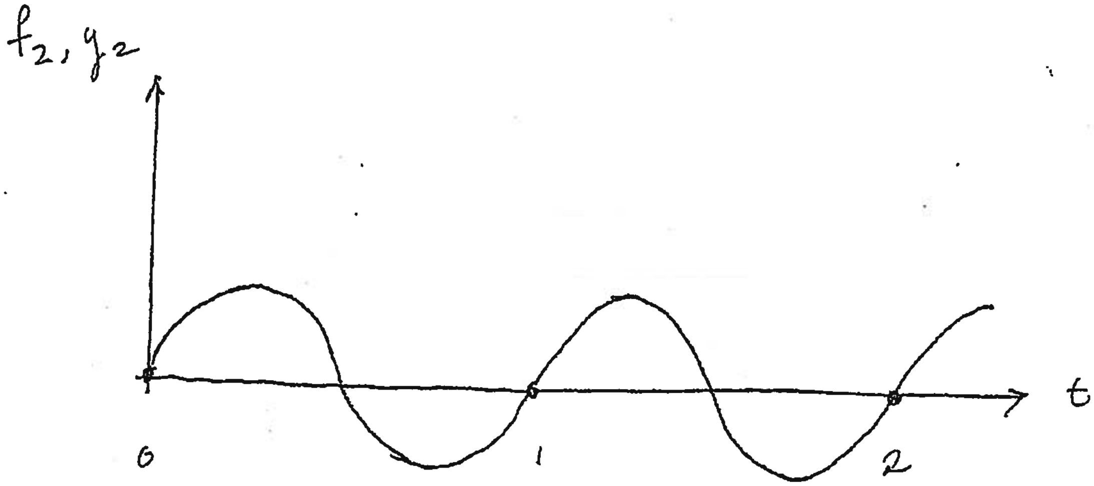
$$y[k] = f(k), \; k \in \mathbb{Z}$$

$$f_1(t) = \cos 2\pi t \xrightarrow{S} y_1[k] = 1, \; \forall k \in \mathbb{Z}$$



$$f_2(t) = f_1(t - \tfrac{1}{4})$$
$$= \cos(2\pi(t - \tfrac{1}{4}))$$
$$= \cos(2\pi t - \tfrac{\pi}{2})$$
$$= \sin(2\pi t)$$

$$\xrightarrow{S} \quad y_2[k] = 0, \; \forall k \in \mathbb{Z}$$



$\longrightarrow$ NOT time-invariant.

Linear, Time-Invariant (LTI) systems

We'll focus on linear, time-invariant (LTI) systems, which are often modelled using linear, constant-coefficient ODEs.

We'll begin by establishing a couple of fundamental properties of LTI systems:

1. Impulse response and convolution

2. Responses to exponential inputs

1. Impulse response and convolution

It's simplest to start by looking at this context in the discrete-time setting - say, where $t \in \mathbb{Z}$.

DT "impulse":

$$\delta[t] = \begin{cases} 1, & t = 0 \\ 0, & \text{otherwise.} \end{cases}$$

This impulse function — which, unlike the Dirac delta function, is a true function — also has a "sifting property":

$$f[t] = \sum_{\tau=-\infty}^{\infty} f[\tau]\,\delta[\tau - t]$$

$$= \sum_{\tau=-\infty}^{\infty} f[\tau]\,\delta[t-\tau]$$

$\rightarrow$ we can interpret a signal $f[t]$ as a weighted sum of impulses.

Suppose we wish to find the zero-state response to a "one-sided" input $f[t]$:

$$f[t] = \sum_{\tau=0}^{\infty} f[\tau]\, \delta[t-\tau].$$

- Let the zero-state response to a DT impulse be $h[t]$ – we call this the <u>impulse response</u>.

- By time-invariance, the response to a shifted impulse $\delta[t-\tau]$ is $h[t-\tau]$.

- By linearity, the response to

$$f[t] = \sum_{\tau=0}^{\infty} f[\tau]\, \delta[t-\tau]$$

is

$$y[t] = \sum_{\tau=0}^{\infty} f[\tau]\, h[t-\tau]$$

- this is just the (DT) <u>convolution</u> of the input with the impulse response.

- Impulse response & convolution (CT)

The property

$$f(t_0) = \int_{0^-}^{\infty} f(\tau)\, \delta(\tau - t_0)\, d\tau$$

(for a "well-behaved" one-sided function $f(\cdot)$) is called the "sifting" or "sampling" property of the impulse $\delta(\cdot)$.

Dropping the subscript, we write

$$f(t) = \int_{0^-}^{\infty} f(\tau)\, \delta(\tau - t)\, d\tau$$

$$= \int_{0^-}^{\infty} f(\tau)\, \delta(t - \tau)\, d\tau$$

Note: this represents $f(t)$ as a superposition of impulses — specifically, as a train of impulses $\delta(t-\tau)$, each of which "arrives" at a different value of $t = \tau$, and is weighted by $f(\tau)$.

- Given an <u>LTI system</u> whose response to an impulse $\delta(t)$ is $h(t)$, by time-invariance, its response to $\delta(t-\tau)$ is $h(t-\tau)$ —— so, by linearity,

$$y(t) = \int_{0^-}^{\infty} f(\tau)\, h(t-\tau)\, d\tau = \int_{0^-}^{\infty} h(\tau)\, f(t-\tau)\, d\tau$$

- the response is the <u>convolution</u> of the input $f(\cdot)$ with the impulse response $h(\cdot)$.

- This key property of LTI systems reduces the analysis of their time-domain responses to convolution . . .

. . . for another key property, let's consider the response of an LTI system to an <u>exponential</u> input.

2. Suppose that S is LTI, and that, for some $s \in \mathbb{C}$,

$$e^{st} \xrightarrow{S} y(t).$$

Then, by time invariance, for any $T \in \mathbb{R}$,

$$e^{s(t-T)} \xrightarrow{S} y(t-T).$$

... but $e^{s(t-T)} = e^{-sT} e^{st}$, so by linearity,

$$e^{s(t-T)} = e^{-sT} e^{st} \xrightarrow{S} e^{-sT} y(t) = y(t-T)$$

Since this holds for any $T \in \mathbb{R}$, we can, in particular, set $T = t$, for any given $t \in \mathbb{R}$: then

$$e^{-st} \; y(t) = y(0)$$

$$\Longleftrightarrow \qquad y(t) = y(0)\, e^{st}$$

... so the response is just the input e^{st}, multiplied by a constant, $y(0)$.

What's the value of this constant?

By the convolution integral,

$$y(t) = \int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, f(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau$$

$$= e^{st} \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau$$

We'll call the function $H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$ the <u>transfer function</u> of the system...

... so to find its response to an exponential input e^{st}, we simply multiply this input by $H(s)$:

$$y(t) = H(s)\, e^{st}$$

We motivated the use of the Laplace transform for solving ODEs by noting that differentiating e^{st} amounts to simply multiplying it by s . . .

. . . now, we find that convolving e^{st} with $h(t)$ amounts to simply multiplying it by $H(s)$.

In either case, it makes sense to use the Laplace transform to express other signals as weighted sums of exponentials.

In the case of convolution, we know what happens:

$$y(t) = (h * f)(t)$$

$$\Longleftrightarrow Y(s) = H(s)\, F(s)$$

So, if we work in the Laplace domain, we just have to multiply.

The transfer function

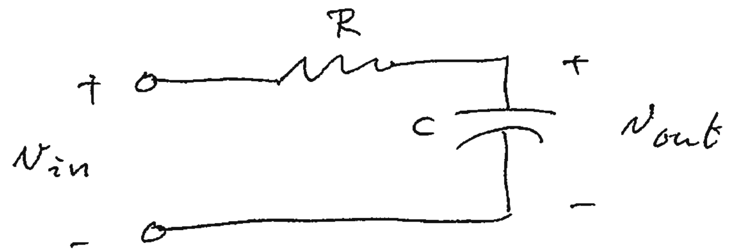
We'll call the transform $H(s)$ of the impulse response $h(t)$ the <u>transfer function</u> of our LTI system.

<u>The transfer function will be our main model for LTI systems</u>.

To find the transfer function, we don't necessarily have to find the impulse response first.

We've already seen that we can find the transfer function by taking Laplace transforms of both sides of a constant-coefficient linear ODE, setting all initial conditions to zero ...

Example: RC circuit



What is the transfer function relating the input to the output?

Differential equation:

$$RC \frac{d v_{out}}{dt} + v_{out} = v_{in}$$

Taking Laplace transforms:

$$RC \left[s V_{out}(s) - v_{out}(0^-) \right] + V_{out}(s) = V_{in}(s)$$

Now, the transfer function is the transform of the impulse response, and the impulse response is a zero-state response — so set the initial condition to zero.

$$[sRC + 1]\, V_{out}(s) = V_{in}(s)$$

$$\Longleftrightarrow \quad V_{out}(s) = \frac{1}{sRC + 1} V_{in}(s)$$

So the transfer function is

$$H(s) = \frac{1}{sRC + 1}$$

- This is called a first-order transfer function, because it has only one pole.