University of Waterloo
David R. Cheriton School of Computer Science

MATH 213 – Advanced Mathematics for Software Engineers
Midterm Exam, Spring 2009

June 25, 6:30–8:30 PM

Instructor: Dr. Oleg Michailovich

Student's name: ____

Student's ID #: ____

Instructions:

- This exam has **1** page.
- **No books and lecture notes are allowed on the exam**. Please, turn off your cell phones, PDAs, etc., and place your bags, backpacks, books, and notes under the table or at the front of the room.
- Please, place your **WATCARD** on the table, and fill out the exam attendance sheet when provided by the proctor after the exam starts.
- Question marks are listed by the question.
- Please, do not separate the pages, and indicate your Student ID at the top of every page.
- Be neat. Poor presentation will be penalized.
- **No questions will be answered during the exam**. If there is an ambiguity, state your assumptions and proceed.
- **No student can leave the exam room in the first 45 minutes or the last 10 minutes**.
- If you finish before the end of the exam and wish to leave, remain seated and raise your hand. A proctor will pick up the exam from you, at which point you may leave.
- When the proctors announce the end of the exam, put down your pens/pencils, close your exam booklet, and remain seated in silence. The proctors will collect the exams, count them, and then announce you may leave.

Question 1 (20%)

Using the method of integrating factor, find the general solution of the following equation.

$$(3x^2 \sinh 3y - 2x)\, dx + 3x^3 \cosh 3y\, dy = 0.$$

Question 2 (20%)

Using the method of undetermined coefficients, find the general solution of the following equation.

$$y'' - 4y = 5\left(\sinh 2x + x\right).$$

Note: $\sinh(\theta) = 0.5\left(e^{\theta} - e^{-\theta}\right)$.

Question 3 (20%)

Find the Laplace transform of the following function.

$$f(t) = t^2\left(2H(t-1) - H(t-2)\right).$$

Note: $\mathcal{L}\{t^n\} = n! s^{-(n+1)}$ and $0! = 1$.

Question 4 (20%)

Invert the following Laplace transform

$$F(s) = \frac{e^{-s}}{(s+1)\left(1 - e^{-2s}\right)}.$$

Note: $(1 - e^{-Ts})^{-1} = \sum_{n=0}^{\infty} e^{-nTs}$ and $\mathcal{L}\{H(t)\} = \mathcal{L}\{u(t)\} = 1/s$.

Question 5 (20%)

Find the general solution for $x(t)$ on $0 =< t < \infty$ using the method of Laplace transformation.

$$2x'' - x' = \delta(t-1) - \delta(t-2), \quad x(0) = x'(0) = 0.$$

Note: $f(t) * H(t) = \int_0^t f(\tau) d\tau$.

Question # 1

Solve by the method of integrating factor:

$$(3x^2 \sinh 3y - 2x)\,dx + 3x^3 \cosh 3y\,dy = 0$$

Solution: The above equation is a non-linear equation, which cannot be expressed in the form

$$y' + p(x)\,y = q(x).$$

Consequently, the formula

$$y(x) = e^{-\int p(x)dx}\left(\int e^{\int p(x)dx} q(x)\,dx + C\right)$$

is not applicable in this case.

However, it is easy to see that the equation is exact. Thus:

$$\underbrace{(3x^2 \sinh 3y - 2x)}_{M = \frac{\partial F}{\partial x}}\,dx + \underbrace{3x^3 \cosh 3y}_{N = \frac{\partial F}{\partial y}}\,dy = 0$$

$$\frac{\partial M}{\partial y} = 9x^2 \cosh 3y \quad \text{and} \quad \frac{\partial N}{\partial x} = 9x^2 \cosh 3y$$

and therefore: $\quad M_y = N_x\,!$

$$\frac{\partial F}{\partial x} = 3x^2 \sinh 3y - 2x$$

$$\Rightarrow \quad F(x,y) = x^3 \sinh 3y - x^2 + A(y)$$

Substitute into:

$$\frac{\partial F}{\partial y} = N(x,y) = 3x^3 \cosh 3y + A'(y)$$

$$\Rightarrow A'(y) = 0 \Rightarrow A(y) = C$$

Finally:

$$x^3 \sinh 3y - x^2 = C$$

Question #2

Solve by the method of undetermined coefficients:

$$y'' - 4y = 5\sinh 2x + 5x$$

Solution: 1) Solve first $\quad y'' - 4y = 0$

$$\lambda^2 - 4 = 0 \Rightarrow \lambda_{1,2} = \pm 2$$

Thus $\quad y_h(x) = Ae^{2x} + Be^{-2x}$

2) The function $\sinh 2x = \dfrac{e^{2x} - e^{-2x}}{2}$ generates the set $\{e^{2x}, e^{-2x}\}$ which is dependent on $y_h(x)$.

Thus we assume:

$$y_{p1}(x) = Cxe^{2x} + Dxe^{-2x}$$

Substituting $y_{p1}(x)$ into $y'' - 4y = 5\sinh 2x$ leads to:

$$C \cdot 4e^{2x} - D \cdot 4e^{-2x} = 5\sinh 2x = \tfrac{5}{2}e^{2x} - \tfrac{5}{2}e^{-2x}$$

Therefore: $\quad C = D = \frac{5}{8}$, and:

$$y_{p1}(x) = \tfrac{5}{8}xe^{2x} + \tfrac{5}{8}xe^{-2x} = \tfrac{5}{4}x\cosh 2x$$

3) The function x generates the set $\{x, 1\}$, and hence

$$y_{p2}(x) = E \cdot x + F$$

Substituting $y_{p2}(x)$ into $y'' - 4y = 5x$ leads to:

$$-4Ex - F = 5x \Rightarrow E = -\tfrac{5}{4},\ F = 0$$

$$\Rightarrow \quad y_{p2}(x) = -\tfrac{5}{4}x$$

Finally:

$$y(x) = Ae^{2x} + Be^{-2x} + \tfrac{5}{4}x\cosh 2x - \tfrac{5}{4}x$$

Question #3

Find $\mathcal{F}(s)$ of $f(t) = t^2[2\mathcal{H}(t-1) - \mathcal{H}(t-2)]$

Solution: First note that

$$t^2 = t^2 - 2t + 1 + 2t - 1 = (t-1)^2 + 2(t-1) + 1$$

and

$$t^2 = t^2 - 4t + 4 + 4t - 4 = (t-2)^2 + 4(t-2) + 4$$

Thus:

$$f(t) = 2\left[(t-1)^2 + 2(t-1) + 1\right]\cdot\mathcal{H}(t-1) - \left[(t-2)^2 + 4(t-2) + 4\right]\cdot\mathcal{H}(t-2)$$

Consequently:

$$\mathcal{L}\{f(t)\} = \mathcal{F}(s) = \left(\frac{4}{s^3} + \frac{4}{s^2} + \frac{2}{s}\right)e^{-s} - \left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right)e^{-2s}$$

Question #4

$$\mathcal{F}(s) = \frac{e^{-s}}{(s+1)(1-e^{-2s})} = \frac{e^{-s}}{s+1}\sum_{n=0}^{\infty} e^{-2ns} = \sum_{n=0}^{\infty} \frac{e^{-(2n+1)s}}{s+1}$$

Hence:

$$f(t) = \mathcal{L}^{-1}\{\mathcal{F}(s)\} = \sum_{n=0}^{\infty} e^{-(t-2n-1)} \cdot \mathcal{H}(t-2n-1).$$

Question # 5

Solve by the method of Laplace transform:

$$2x'' - x' = \delta(t-1) - \delta(t-2)$$

$$x(0) = x'(0) = 0$$

Solution: By applying the Laplace transform:

$$2s^2 X(s) - sX(s) = \mathcal{L}\{\delta(t-1)\} - \mathcal{L}\{\delta(t-2)\}$$

$$X(s) = \frac{\mathcal{L}\{\delta(t-1)\}}{s(s\cdot 2 - 1)} - \frac{\mathcal{L}\{\delta(t-2)\}}{s(2s-1)}$$

Note that:

$$\frac{1}{s(2s-1)} = \frac{2}{2s-1} - \frac{1}{s} = \frac{1}{-0.5} - \frac{1}{s}$$

Moreover:

$$\mathcal{L}^{-1}\left\{\frac{1}{s(2s-1)}\right\} = u(t)e^{t/2} - u(t) = u(t)(e^{t/2} - 1)$$

Consequently:

$$x(t) = \left[u(t)(e^{t/2} - 1)\right] * \delta(t-1) - \left[u(t)(e^{t/2} - 1)\right] * \delta(t-2)$$

Finally:

$$x(t) = u(t-1)\cdot\left(e^{\frac{t-1}{2}} - 1\right) - u(t-2)\cdot\left(e^{\frac{t-2}{2}} - 1\right)$$