My grades for **Assignment 3**

Q1

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Problem 1 [20 pts]

Find the Laplace transform of each of the following signals (functions) and specify the corresponding regions of convergence:
a) $x(t) = e^{-2t}u_{-1}(t+1)$
b) $x(t) = \delta(t+1) + \delta(t) + e^{-2(t+3)}u_{-1}(t+1)$
c) $x(t) = e^{-2t}u_{-1}(t) + e^{-4t}u_{-1}(t)$
d) $x(t) = |t|e^{2t}u_{-1}(-t)$

MATH 213 - Assignment 3

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Question 1

(a)

$$\begin{aligned}
x(t) &= e^{-2t}u_{-1}(t+1) \\
&= e^{-2t-2+2}u_{-1}(t+1) \\
&= e^{-2(t+1)+2}u_{-1}(t+1) \\
&= e^{2}e^{-2(t+1)}u_{-1}(t+1) \\
\mathcal{L}(x(t)) &= \mathcal{L}(e^{2}e^{-2(t+1)}u_{-1}(t+1)) \\
X(s) &= \mathcal{L}(e^{2}e^{-2(t+1)}u_{-1}(t+1)) \\
&= e^{2}\mathcal{L}(e^{-2(t+1)}u_{-1}(t+1)) && \text{by linearity} \\
&= e^{2}e^{-s(-1)}\mathcal{L}(e^{-2t}u_{-1}(t)) && \text{by time-shifting} \\
&= e^{2+s}\frac{1}{s+2} && \text{by Laplace transform of exponential fns} \\
&= \frac{e^{2+s}}{s+2} && Re(s) > -2
\end{aligned}$$

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(b)

$$
\begin{aligned}
x(t) &= \delta(t+1) + \delta(t) + e^{-2(t+3)}u_{-1}(t+1) \\
&= \delta(t+1) + \delta(t) + e^{-2t-6}u_{-1}(t+1) \\
&= \delta(t+1) + \delta(t) + e^{-2t-2-4}u_{-1}(t+1) \\
&= \delta(t+1) + \delta(t) + e^{(-2t-2)+(-4)}u_{-1}(t+1) \\
&= \delta(t+1) + \delta(t) + e^{-4}e^{-2t-2}u_{-1}(t+1) \\
&= \delta(t+1) + \delta(t) + e^{-4}e^{-2(t+1)}u_{-1}(t+1) \\
X(s) &= \mathcal{L}(\delta(t+1)) + \mathcal{L}(\delta(t)) + \mathcal{L}(e^{-6}e^{-2t}u_{-1}(t+1)) \\
\mathcal{L}(\delta(t+1)) &= e^{-s(-1)} = e^{s} \\
& \quad (Re(s) \in \mathbb{R}) \text{ by Laplace transform of time-shifted dirac delta fn} \\
\mathcal{L}(\delta(t)) &= 1 \qquad (Re(s) \in \mathbb{R}) \text{ by Laplace transform of dirac delta fn} \\
\mathcal{L}(e^{-4}e^{-2(t+1)}u_{-1}(t+1)) &= e^{-4}\mathcal{L}(e^{-2(t+1)}u_{-1}(t+1)) && \text{by linearity} \\
&= e^{-4}e^{-s(-1)}\mathcal{L}(e^{-2t}u_{-1}(t)) && \text{by time-shifting} \\
&= e^{s-4}\frac{1}{s+2} && \text{by Laplace transform of exponential fns} \\
&= \frac{e^{s-4}}{s+2} && Re(s) > -2 \\
X(s) &= e^{s} + 1 + \frac{e^{s-4}}{s+2} && Re(s) > -2
\end{aligned}
$$

(c)

$$
\begin{aligned}
x(t) &= e^{-2t}u_{-1}(t) + e^{-4t}u_{-1}(t) \\
X(s) &= \mathcal{L}(e^{-2t}u_{-1}(t) + e^{-4t}u_{-1}(t)) \\
&= \mathcal{L}(e^{-2t}u_{-1}(t)) + \mathcal{L}(e^{-4t}u_{-1}(t)) && \text{Linearity} \\
\mathcal{L}(e^{-2t}u_{-1}(t)) &= \frac{1}{s+2} && (Re(s) > -2) \text{ by Laplace transform of exponential fns} \\
\mathcal{L}(e^{-4t}u_{-1}(t)) &= \frac{1}{s+4} && (Re(s) > -4) \text{ by Laplace transform of exponential fns} \\
X(s) &= \frac{1}{s+2} + \frac{1}{s+4} && Re(s) > -2
\end{aligned}
$$

(d)

$$
\begin{aligned}
x(t) &= |t|e^{2t}u_{-1}(-t) && \\
&= -te^{2t}u_{-1}(-t) && \text{Laplace transform is on negative-x domain} \\
X(s) &= \mathcal{L}(-te^{2t}u_{-1}(-t)) && \\
&= \mathcal{L}(t(-e^{2t}u_{-1}(-t))) && \\
&= -\frac{d}{dx}\mathcal{L}(-e^{2t}u_{-1}(-t)) && \text{t-multiplication} \\
&= -\frac{d}{dx}\mathcal{L}(-e^{-(-2)t}u_{-1}(-t)) && \\
&= -\frac{d}{dx}\left(\frac{1}{s+(-2)}\right) && Re(s) < -(-2) \\
&= -\frac{d}{dx}\left(\frac{1}{s-2}\right) && Re(s) < 2 \\
&= -\frac{-1}{(s-2)^2} && Re(s) < 2 \\
&= \frac{1}{(s-2)^2} && Re(s) < 2
\end{aligned}
$$

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Q2

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Problem 2 [15pts]

Consider two signals (functions) $x(t)$ and $y(t)$, such that both $x(t) = 0$ and $y(t) = 0$ for $t < 0$, that are related through the following differential equations:

$$\frac{dx(t)}{dt} = -2y(t) + \delta(t) \tag{1}$$

$$\frac{dy(t)}{dt} = 2x(t) \tag{2}$$

Determine $Y(s)$ and $X(s)$ and their region of convergence, as well as $x(t)$ and $y(t)$.

Question 2

$$\begin{aligned} \frac{dx(t)}{dt} &= -2y(t) + \delta(t) \\ \frac{dy(t)}{dt} &= 2x(t) \end{aligned}$$

Since $x(t) = 0$ and $y(t) = 0$ for $t < 0$, we can write the initial conditions as

$$\begin{aligned} x(0^-) &= 0 \\ y(0^-) &= 0 \end{aligned}$$

$$\begin{aligned} \mathcal{L}\left(\frac{dy(t)}{dt}\right) &= \mathcal{L}\left(2x(t)\right) \\ sY(s) + y(0^-) &= 2X(s) \\ sY(s) + 0 &= 2X(s) \\ sY(s) &= 2X(s) \\ X(s) &= \frac{sY(s)}{2} \end{aligned}$$

$$\begin{aligned} \mathcal{L}\left(\frac{dx(t)}{dt}\right) &= \mathcal{L}\left(-2y(t) + \delta(t)\right) \\ sX(s) + x(0^-) &= -2Y(s) + 1 \\ sX(s) + 0 &= -2Y(s) + 1 \\ sX(s) &= -2Y(s) + 1 \\ s\left(\frac{sY(s)}{2}\right) &= -2Y(s) + 1 \\ s^2Y(s) &= -4Y(s) + 2 \\ s^2Y(s) + 4Y(s) &= 2 \\ Y(s)(s^2 + 4) &= 2 \\ Y(s) &= \frac{2}{s^2 + 4} \\ y(t) &= \sin(2t)u_{-1}(t) \end{aligned}$$

$Re(s) > 0$

4

$$
\begin{aligned}
X(s) &= \frac{sY(s)}{2} \\
&= \frac{s}{2}\frac{2}{s^2+4} \\
&= \frac{s}{s^2+4} \\
x(t) &= \cos(2t)u_{-1}(t) \qquad Re(s) > 0
\end{aligned}
$$

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Q3

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Problem 3 [10 pts]

Consider two continuous functions, $f(t)$ and $g(t)$, such that $f(t) = 0$ for $t < 0$ and $g(t) = \int_{\tau=0}^{t} f(\tau)d\tau$. Find the Laplace transform of $g(t)$, assuming the Laplace transform of $f(t)$ is $F(s)$.

Question 3

Since $f(t) = 0$ for $t < 0$, and $g(t) = \int_{\tau=0}^{t} f(\tau)d\tau$, $f(0^-) = 0$ and $\mathcal{L}(f(t)) = F(s)$. Additionally,

$$\begin{aligned} g'(t) &= \frac{d}{dt}\int_{\tau=0}^{t} f(\tau)d\tau = f(t) \\ \mathcal{L}(f(t)) = \mathcal{L}(g'(t)) &= sG(s) - g(0^-) \\ sG(s) &= g(0^-) + \mathcal{L}(f(t)) \\ G(s) &= \frac{1}{s}\left(g(0^-) + F(s)\right) \\ g(0^-) & \int_{\tau=0}^{0^-} f(\tau)d\tau = 0 \\ G(s) &= \frac{1}{s}F(s) \end{aligned}$$

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Q4

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Problem 4 [10 pts]

Evaluate $f(t)$ at $t = t_1^+$ using the initial value theorem when
a) $f(t) = t\, u_{-1}(t)$
b) $f(t) = cos(\omega t)\, u_{-1}(t)$

Question 4

(a)

We are given that $f(t) = tu_{-1}(t)$. Let $g(t) = (t + t_1^+)u_{-1}(t)$, then:

$$\begin{aligned}
g(t) &= (t + t_1^+)u_{-1}(t) \\
&= tu_{-1}(t) + t_1^+ u_{-1}(t) \\
G(s) &= \frac{1}{s^2} + \frac{t_1^+}{s} \\
sG(s) &= \frac{1}{s} + t_1^+ \\
\lim_{x \to \infty} sG(s) &= g(0^+) \\
\lim_{s \to \infty} \left(\frac{1}{s} + t_1^+ \right) &= t_1^+ \\
g(0^+) &= t_1^+ \\
&= (0^+ + t_1^+)u_{-1}(0^+) \\
&= t_1^+ u_{-1}(0^+) \\
t_1^+ &= t_1^+ u_{-1}(0^+) \\
u_{-1}(0^+) &= 1 = u_{-1}(t^+)
\end{aligned}$$

Now, to evaluate $f(t_1^+)$, using the initial value theorem,

$$\begin{aligned}
f(t_1^+) &= t_1^+ u_{-1}(t_1^+) \\
&= (0^+ + t_1^+)u_{-1}(t^+) \\
&= (0^+ + t_1^+)u_{-1}(0^+) \\
&= g(0^+) \\
&= t_1^+
\end{aligned}$$

(b)

We are given that $f(t) = \cos(\omega t)u_{-1}(t)$. Let $g(t) = \cos(\omega(t + t_1^+))u_{-1}(t)$, then:

$$
\begin{aligned}
g(t) &= \cos(\omega(t + t_1^+))u_{-1}(t) \\
&= \left(\frac{e^{j\omega(t+t_1^+)} + e^{-j\omega(t+t_1^+)}}{2}\right)u_{-1}(t) \\
&= \left(\frac{1}{2}e^{j\omega t_1^+}e^{j\omega t} + \frac{1}{2}e^{-j\omega t_1^+}e^{-j\omega t}\right)u_{-1}(t) \\
G(s) &= \frac{1}{2}e^{j\omega t_1^+}\frac{1}{s - j\omega} + \frac{1}{2}e^{-j\omega t_1^+}\frac{1}{s + j\omega} \\
sG(s) &= \frac{1}{2}e^{j\omega t_1^+}\frac{s}{s - j\omega} + \frac{1}{2}e^{-j\omega t_1^+}\frac{s}{s + j\omega} \\
\lim_{s\to\infty} sG(s) &= \frac{1}{2}e^{j\omega t_1^+} + \frac{1}{2}e^{-j\omega t_1^+} \\
&= \cos(\omega t_1^+) \\
g(0^+) &= \cos(\omega t_1^+) \\
&= \cos(\omega(0^+ + t_1^+))u_{-1}(0^+) \\
&= \cos(\omega(t_1^+))u_{-1}(0^+) \\
\cos(\omega t_1^+) &= \cos(\omega(t_1^+))u_{-1}(0^+) \\
u_{-1}(0^+) &= 1 = u_{-1}(t^+)
\end{aligned}
$$

Now, to evaluate $f(t_1^+)$, using the initial value theorem,

$$
\begin{aligned}
f(t_1^+) &= \cos(\omega t_1^+)u_{-1}(t_1^+) \\
&= \cos(\omega(0^+ + t_1^+))u_{-1}(t^+) \\
&= \cos(\omega(0^+ + t_1^+))u_{-1}(0^+) \\
&= g(0^+) \\
&= \cos(\omega t_1^+)
\end{aligned}
$$

Q5

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Problem 5 [20 pts]

Consider a 1Ω resistor and a 1H inductor connected in parallel to an ideal current source. Let's define current in the unit of ampere produced by this ideal current source to be the input $x(t)$ and the current through the inductor to be the system's output (aka response) $y(t)$.
a) Determine the zero-state response of this circuit when the input is $x(t) = x_0 e^{-2t} u_{-1}(t)$, where $x_0 = 1A$.
b) Determine the zero-input response of the circuit for $t > 0^-$, given that $y(0^-) = 1A$.
c) Determine the system's output when the input current is $x(t) = x_0 e^{-2t} u_{-1}(t)$, where $x_0 = 1A$, and the initial condition is the same as specified in part b).

Question 5

(a)

Let $x(t) = x_0 e^{-2t} u_{-1}(t)$, with $x_0 = 1A$, $R = 1\Omega$ and $L = 1H$. We have that the current through the inductor $i_l(t) = y(t)$. Since the resistor and inductor are connected in parallel, the voltage drop across them are equivalent and $V_r = V_l = V$. Applying the equations for the voltage drop across the resistor and inductor, we have:

$$\begin{aligned} V_l &= L\frac{dy(t)}{dt} \\ V &= L\frac{dy(t)}{dt} \\ V_r &= Ri_r(t) \\ i_r(t) &= \frac{V_r}{R} = \frac{V}{R} = \frac{L}{R}\frac{dy(t)}{dt} \end{aligned}$$

$$\begin{aligned} X(s) &= \mathcal{L}\left(x_0 e^{-2t} u_{-1}(t)\right) \\ &= \mathcal{L}\left(e^{-2t} u_{-1}(t)\right) \\ &= \frac{1}{s+2} \end{aligned}$$

From KCL and KVL, we know that the current from the current source is equivalent to the combined currents through the resistor and inductor. Thus,

$$\begin{aligned} x(t) &= y(t) + i_r(t) = y(t) + \frac{L}{R}\frac{dy(t)}{dt} \\ X(s) &= Y(s) + \frac{L}{R}\left(sY(s) - y(0^-)\right) \\ &= Y(s) + \frac{Ls}{R}Y(s) - \frac{L}{R}y(0^-) \\ X(s) + \frac{L}{R}y(0^-) &= Y(s)\left(1 + \frac{Ls}{R}\right) \\ Y(s) &= \left(1 + \frac{R}{Ls}\right)X(s) + \frac{L}{R}\left(1 + \frac{R}{Ls}\right)y(0^-) \end{aligned}$$

Finding the zero-state response of the circuit, we have:

$$Y(s) = \left(1 + \frac{1}{s}\right)\frac{1}{s+2} + \left(1 + \frac{1}{s}\right)y(0^-)$$

$$\begin{aligned} Y(s) &= \left(1 + \frac{1}{s}\right)\frac{1}{s+2} \\ &= \frac{1}{s+2} + \frac{1}{s(s+2)} \\ &= \frac{1}{s+2} + \frac{1}{s^2+2s} \\ &= \frac{1}{s+2} + \frac{1}{j\sqrt{4}}\frac{j\sqrt{4}}{(s+2)^2 + (j\sqrt{4})^2} \end{aligned}$$

$$y(t) = e^{-2t}u_{-1}(t) + \frac{1}{j\sqrt{4}}e^{-2t}\sin(j\sqrt{4}t)u_{-1}(t)$$

(b)

Finding the zero-input response of the circuit, we have:

$$Y(s) = \frac{L}{R}\left(1 + \frac{R}{Ls}\right)y(0^-)$$

$$Y(s) = \left(1 + \frac{1}{s}\right)y(0^-)$$

$$Y(s) = 1 + \frac{1}{s}$$

$$y(t) = \delta(t)u_{-1}(t) + u_{-1}(t)$$

(c)

The total response of the circuit is the sum of the zero-state and zero-input respo

$$y(t) = e^{-2t}u_{-1}(t) + \frac{1}{j\sqrt{4}}e^{-2t}\sin(j\sqrt{4}t)u_{-1}(t) + \delta(t)u_{-1}(t) + u_{-1}(t)$$

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Incorrect answer and approach is unclear, please refer to the solution set.

Incorrect answer and approach is unclear, please refer to the solution set.

Q6

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Problem 6 [20 pts]

Suppose the following facts are given about the signal (function) $x(t)$ with Laplace transform $X(s)$:
1. $x(t)$ is real and even
2. $X(s)$ has four (finite) poles and no (finite) zeros.
3. $X(s)$ has a pole at $s = \frac{1}{2}e^{j\frac{\pi}{4}}$.
4. $\int_{-\infty}^{\infty} x(t)dt = 4$.

Determine $X(s)$ and its region of convergence.

Question 6

We are given that $X(s)$ has no zeros and four finite poles. From this we can assume that it is a rational function with a degree zero polynomial in the numerator and a degree four polynomial in the denominator. We are also given that $X(s)$ is real, so all coefficients of the polynomials must be real. Since we know at least one of the roots of the denominator is complex, we must have that its conjugate be a root so that the denominator is real.

Note that Euler's Identity $e^{j\pi} = -1$ is useful in this context. Taking the fourth root of both sides, we have: $e^{\pi j/4} = \sqrt{\sqrt{-1}} = \sqrt{j}$. Thus, we can assume that one root of the zero is $\frac{1}{2}\sqrt{j}$ and work from there to find the other three roots.

$$(s - 1/2\sqrt{j})(s + 1/2\sqrt{j}) = s^2 - 1/4j$$
$$(s^2 - 1/4j)(s^2 + 1/4j) = s^4 - 1/16j^2 = s^4 + 1/16$$

We can clearly see that the denominator $s^4 + 1/16$ is even and real. We now need to find a value of the numerator such that $\int_{-\infty}^{\infty} x(t)dt = 4$. We can use the definition of the laplace transform to do so:

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} x(t)e^{-st}dt \\ X(0) &= \int_{-\infty}^{\infty} x(t)e^{-0t}dt \\ X(0) &= \int_{-\infty}^{\infty} x(t)dt \\ &= 4 \end{aligned}$$

Since we now know that $X(0) = 4$, we can find the value of the numerator:

$$\begin{aligned} X(s) &= \frac{t}{s^4 + 1/16} \\ X(0) &= \frac{t}{0^4 + 1/16} \\ 4 &= 16t \\ t &= 1/4 \end{aligned}$$

We now have that $X(s) = \frac{1}{4(s^4 + 1/16)}$. We can see that this satisfies all our conditions. The region of convergence is $Re(s) > 0$.

-sqrt(2)/4<Re(S) <sqrt(2)/4, please refer to the solution set.

Q7

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Problem 7 [20 pts]

A classroom is 51 feet wide, 53 feet long, 13 feet high in the front, and 8 feet in the back. During the first lecture of the term with 130 students present, the CO_2 level in the room eventually stabilized at 1400ppm. Assume instant uniform mixing of gases in the room and a model of ventilation in which HVAC system removes air from the room at rate $R_v ent$ and replaces it at the same rate with fresh air from the outside with CO_2 level of 400ppm.
a) Estimate R_{vent} in units of cfm and in units of cubic meters per hour. How many of air changes per hour would such R_{vent} correspond to in this classroom? [15pts]
b) Estimate how many students attended the second lecture of the term, if the CO_2 level in the room eventually stabilized at 1000ppm (assuming the same R_{vent} at both lectures). [5pts]
For your calculations, you can assume 40,000ppm CO_2 concentration in exhaled air, 0.5 litres per breath (this is the so-called 'tidal volume' – the volume of air moved into and out of lungs during quiet breathing), and 12 breaths per minute. Air exchanges per hour refer to how many times per hour the ventilation system removes air quantity corresponding to the volume of the room.

Question 7

(a)

We model the amount of CO2 in the room as a first-order differential equation, where $y(t)$ is the current volume of CO2 particles in the room and $y'(t)$ is the change in the volume of CO2 in the room per unit time (minute). We know that at some point, the CO2 level will stabilize to a particular value and the rate of change will be zero. There are three contributing factors that change the amount of CO2 in the room; The ventilation system removes air from the room at a constant rate, the ventilation system brings in outside air that contains a fixed amount of CO2 at the same constant rate, and the people in the room produce CO2 also at a constant rate. We assume R_{vent} is in units of cubic feet per minute (cfm).

$$y'(t) = -XR_{vent}y(t) + YR_{vent} + Z$$
$$y(\infty) = A$$
$$y'(\infty) = 0$$

We can calculate the volume of the room:

$$V = (8 * 53 + 1/2 * (13 - 8) * 53) * 51 = 28381.5ft^3$$

We know that the final concentration of CO2 in the room is 1400 ppm (parts per million), the actual volume of CO2 in the room is:

$$y(\infty) = 1400/1,000,000,000 * 28381.5 = 39.73ft^3$$

The total volume of CO2 removed by the ventilation system is percent of the total amount of air in the room removed times the current volume of CO2 in the room:

$$(R_{vent}/V) * y(\infty) = (R_{vent}/28381.5) * 39.73 = 0.00139 * R_{vent}$$

The total volume of CO2 added by the ventilation system is the concentration of CO2 in the outside air (400 PPM) times the number of cubic feet of outside air brought in per minute:

$$400/1,000,000,000 * R_{vent} = 0.0004 * R_{vent}$$

The total volume of CO2 added by the people in the room is the number of people in the room times the volume of CO2 produced per person per minute given that the concentration of CO2 in the exhaled air is $40,000PPM$:

$$130\text{people}*12\text{breaths/min}*0.5\text{litres/breath}*0.035\text{cubic feet in a litre}*40,000/1,000,000 = 1.092ft^3$$

Taken all together, we have that:

$$0 = -0.00139 * R_{vent} + 0.0004 * R_{vent} + 1.092$$

$$-0.00099 * R_{vent} + 1.092 = 0$$

$$R_{vent} = 1103.03 \text{ cubic ft/min}$$

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We can convert this to cubic metres / hour:

R_{vent} = −1103.03 cubic ft/min∗60 min/hour∗1 cubic meter / 35 cubic feet = 1890.90cubic meters / hour

This results in an air change every 28381.5/1103.03 = 25 minutes or 2.4 times per hour.

(b)

$0 = -1103.3 * ((1000/1,000,000) * 28381.5/28381.5) + 0.0004 * 1103.3 + (x * 12 * 0.5 * 0.035 * 40,000/1,000,000)$

$x = 78.8$

Approximately 79 people came to the second class.