Midterm Example

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1 Consider the following system:

$$\dot{x}_1(t) = -2x_1(t) + 3u(t)$$
$$\dot{x}_2(t) = -x_2(t) + u(t)$$
$$y(t) = 3x_1(t) + 5x_2(t).$$

a Compute the output response of the system to a unit impulse input.

$$\dot{x} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} x + \underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{B} u(t)$$

$$y = \underbrace{\begin{bmatrix} 3 & 5 \end{bmatrix}}_{C} x \qquad D = 0$$

$$\dot{x} = Ax + Bu$$
$$sX = AX + BU$$
$$(sI - A)X = BU$$
$$X(s) = (sI-A)^{-1} B\, U(s)$$
$$Y(s) = C X(s)$$

$$\Rightarrow Y(s) = C(sI-A)^{-1} B\, U(s) \quad U(s) \to 1$$

$$= \begin{bmatrix} 3 & 5 \end{bmatrix} \underbrace{\begin{bmatrix} s+2 & 0 \\ 0 & s+1 \end{bmatrix}^{-1}} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\hookrightarrow \frac{1}{(s+2)(s+1)} \begin{bmatrix} s+1 & 0 \\ 0 & s+2 \end{bmatrix} = \begin{bmatrix} \frac{1}{s+2} & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 5 \end{bmatrix} \begin{bmatrix} \frac{1}{s+2} & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 5 \end{bmatrix} \begin{bmatrix} \frac{3}{s+2} \\ \frac{1}{s+1} \end{bmatrix}$$

$$Y(s) = \frac{9}{s+2} + \frac{5}{s+1}$$

$$\Rightarrow y(t) = 9e^{-2t} + 5e^{-t}$$

b Compute the state transition matrix.

$$\hookrightarrow e^{At} = e^{\begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} t} = \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-t} \end{bmatrix}$$

2 Consider the following system:

$$G(s) = \frac{\mu(1+Ts)}{1+\tau s},$$

and assume it is asymptotically stable. $\longrightarrow Re\{\lambda_i(A)\} < 0$

a Determine its step response.

$$Y(s) = G(s)U(s) = \frac{\mu(1+Ts)}{(1+\tau s)s} \overset{PFD}{=} \frac{a}{s} + \frac{b}{1+\tau s} = \frac{\mu}{s} + \boxed{\frac{\mu(T-\tau)}{1+\tau s}} \rightarrow = \frac{\frac{\mu}{\tau}(T-\tau)}{s+\frac{1}{\tau}}$$

$U(s) \to 1/s$

$$\mu(1+Ts) = a(1+\tau s) + bs$$

$$s=0: \quad \mu = a$$

$$s = -\frac{1}{\tau}: \quad \mu\left(1 - \frac{T}{\tau}\right) = -\frac{b}{\tau}$$

$$\mu(T-\tau) = b$$

$$\Downarrow$$

$$\boxed{y(t) = \mu H(t) + \frac{\mu(T-\tau)}{\tau} e^{-t/\tau}}$$

$$\hookrightarrow = \mu\left(H(t) + \frac{T-\tau}{\tau} e^{-t/\tau}\right)$$

b Compute its settling time when $T \neq 0$ and compare it with the case $T = 0$.

When $T=0$, system is 1st order $\rightarrow G(s) = \frac{\mu}{1+\tau s}$ settling time $\approx 4\tau$

If $T \neq 0$: $0.02 = \left|\frac{T-\tau}{\tau}\right| e^{-t/\tau}$

$$\ln\left(\left|\frac{0.02\tau}{T-\tau}\right|\right) = -t/\tau$$

$$-\tau \ln\left(\left|\frac{0.02\tau}{T-\tau}\right|\right) = t$$

3

a Discuss the stability properties of the following system:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

A.S. ?

$$\begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = 0$$

$$\lambda^2 + 1 = 0$$

$\lambda = \pm j \longrightarrow$ Not A.S. $\Rightarrow$ Not exponentially stable

BIBO Stable?

$$G(s) = C(sI - A)^{-1}B + D$$
$$= C \frac{\operatorname{adj}(sI-A)}{\det(sI-A)} B + D$$
$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{\begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix}}{s^2+1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$= \frac{-1}{s^2+1}$$

Poles: $s = \pm j \rightarrow$ Not BIBO stable

b Discuss the stability properties of the following system:

$$G(s) = \frac{1}{s^2+1}$$

Not BIBO stable

$\Rightarrow$ Not A.S.

Is it stable? $\longrightarrow$ No! $\rightarrow$ see c

c Compute the response of the system in (1) to the input signal $u(t) = \cos t$.

$$\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = -\frac{2s}{(s^2+1)^2}$$

$$U(s) = \frac{s}{s^2+1}$$

$$Y(s) = G(s)U(s) = \frac{1}{s^2+1}\left(\frac{s}{s^2+1}\right) = \frac{s}{(s^2+1)^2}$$

Note: $t^n f(t) \xleftrightarrow{\text{Laplace}} (-1)^n F^{n}(s)$

$$\frac{s}{s^2+1}\left(\frac{1}{s^2+1}\right) = \frac{s}{(s^2+1)^2}$$

$t^n f(t) \Leftrightarrow (-1)^n F^n(s)$

$$= t \cdot \mathcal{L}^{-1}\left\{\frac{1/2}{s^2+1}\right\} = \tfrac{1}{2} t \sin(t)$$

d Discuss the relation between your answers in **b** and **c**.

Notice input is bounded in c, but output isn't $\Rightarrow$ unstable!