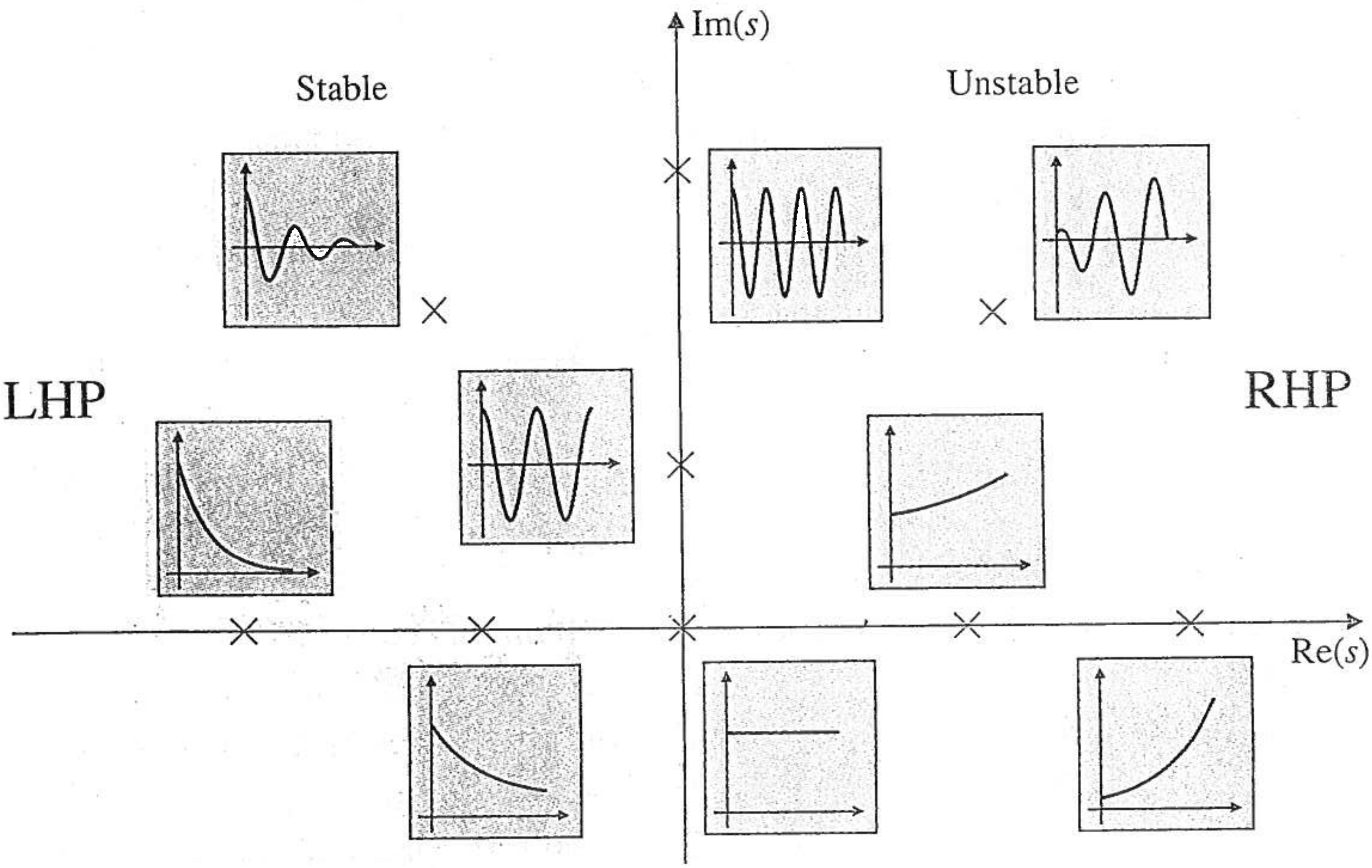
A summary of the relationship between pole locations and impulse response:



- from Franklin, Powell, and Emami-Naeini, <u>Feedback Control of Dynamic Systems</u>, 7th ed.

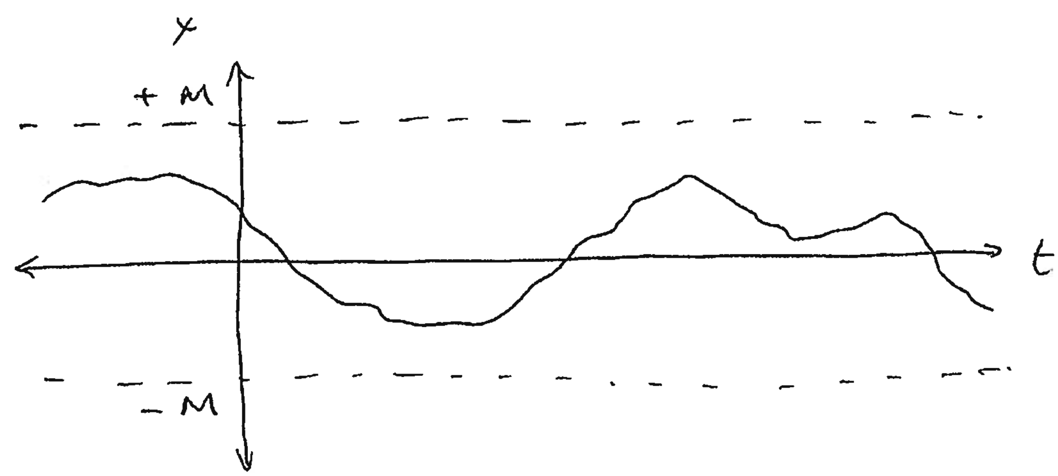
In all cases, poles that lie to the right of the imaginary axis give rise to growing exponentials in the time domain . . .

For this reason, we say that a rational transfer function is <u>stable</u> if all of its poles lie strictly to the left of the imaginary axis — that is, if they all have real parts that are negative.

A useful notion of stability of an LTI system rests on the notion of boundedness of a signal:

A signal $x(t)$ is bounded if there exists a real number M such that

$$|x(t)| \leq M, \quad \forall t$$

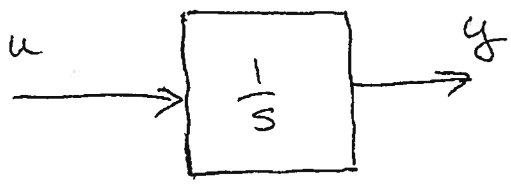


A SISO LTI system with a rational transfer function is bounded-input, bounded-output (BIBO) stable if its zero-state response is bounded whenever its input is.

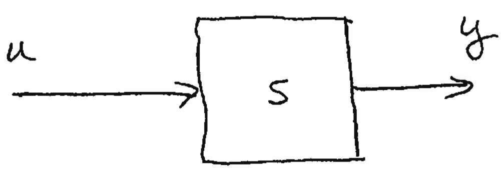
- <u>example</u>: if their transfer functions are stable, then our standard first- and second-order systems are BIBO stable.

- <u>non-examples</u>:

  - integrator:



  - differentiator:



(Consider their step responses.)

<u>Theorem</u>: A SISO LTI system with rational transfer function is BIBO stable if and only if its transfer function is both stable and proper.

<u>Proof</u>:

(If) If the transfer function $H(s)$ is stable and proper, it can be decomposed into a constant a_0 plus terms of the form

$$\frac{a_{ik}}{(s-p_i)^k}, \quad Re(p_i) < 0$$

Such a term has an inverse Laplace transform of the form

$$t^k e^{p_i t} u_{-1}(t)$$

If the input is a bounded signal $u(t)$, with $|u(t)| \leq M$, then the output is a sum of convolutions of the form

$$\int_{0^-}^{\infty} u(t-\tau)\, a_0\, \delta(\tau)\, d\tau$$

$$\leq M a_0$$

and

$$\int_{0^-}^{\infty} u(t-\tau)\, \tau^k e^{p_i \tau}\, d\tau$$

$$\leq M \int_{0^-}^{\infty} \tau^k e^{p_i \tau}\, d\tau$$

Each of the convolutions, and therefore their sum, is bounded.

(Only if):

Suppose that the transfer function $H(s)$ is unstable.

Then $H(s)$ has a pole p with

$Re(p) \geq 0$.

If $Re(p) > 0$, then the step response includes an increasing exponential term.

If $p = 0$, then the step response includes a ramp.

If $p = j\omega$, for some $\omega \in \mathbb{R}$, then let the input $u(t) = e^{j\omega t}$, then the output contains a term

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-j\omega)^2}\right\}$$

$$= t\, e^{j\omega t}$$

It follows that if $H(s)$ is unstable, the system is not BIBO stable.

Suppose now that $H(s)$ is stable but improper. Then

$$H(s) = Q(s) + \frac{N(s)}{D(s)}$$

where Q, N and D are polynomials (Q nonconstant and D nonzero) and $N(s)/D(s)$ is strictly proper.

If $u(t) = u_{-1}(t)$, then the output $y(t)$ contains a term

$$\mathcal{L}^{-1}\left\{ Q(s)\frac{1}{s} \right\}$$

which includes a unit impulse $\delta(t)$ (and possibly "derivatives" of unit impulses). It follows that $y(t)$ is unbounded.

$\square$