Bode plots

1 Factorization of the transfer function

In order to sketch Bode plots of a transfer function, it is convenient to consider the following factorization:

$$G(s) = \frac{\mu \prod_i (1 + T_i s) \prod_i \left(1 + \frac{2\xi_i}{\alpha_{n,i}} s + \frac{s^2}{\alpha_{n,i}^2}\right)}{s^\rho \prod_i (1 + \tau_i s) \prod_i \left(1 + \frac{2\zeta_i}{\omega_{n,i}} s + \frac{s^2}{\omega_{n,i}^2}\right)}, \tag{1}$$

where

- μ is the gain
- $\rho = \#$ of poles at the origin if $\rho > 0$, or $\rho =$ -# of zeros at the origin if $\rho < 0$; if $\rho = 0$ there are no poles/zeros at the origin
- T_i, τ_i are the time constants of the real zeros and poles, located at $-1/T_i$ and $-1/\tau_i$, respectively
- ξ_i, ζ_i, with $|\xi_i| < 1, |\zeta_i| < 1$, are the damping of the complex conjugate zeros and poles, respectively (see Fig. 1)
- $\alpha_{n,i}, \omega_{n,i}$ are the natural frequencies of the complex conjugate zeros and poles, respectively (see Fig. 1)

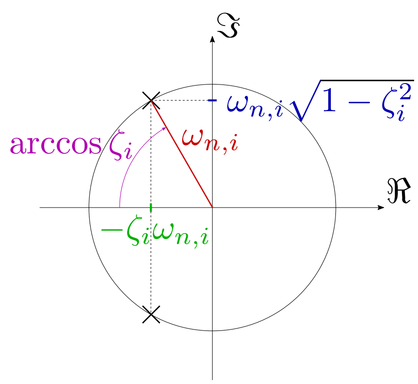


Figure 1: Parameters of the complex conjugate poles located at $-\zeta_i\omega_{n,i} \pm j\omega_{n,i}\sqrt{1-\zeta_i^2}$. Parameters of the complex conjugate zeros $-\xi_i\alpha_{n,i} \pm j\alpha_{n,i}\sqrt{1-\xi_i^2}$ are defined similarly.

2 Magnitude and phase plots

The Bode plots consist of (i) the magnitude, $|G(j\omega)|$, and (ii) the phase, $\angle G(j\omega)$, of the complex number $G(j\omega) = e^{j\angle G(j\omega)}$ as a function of the frequency ω.

A logarithmic scale is used for the axis of ω. This way, for frequencies $\omega_1, \omega_2, \omega_3, \omega_4$ such that $\omega_1/\omega_2 = \omega_3/\omega_4$, the distance on the axis between ω_1 and ω_2 is equal to the one between ω_3 and ω_4. In particular, if $\omega_1/\omega_2 = 10$ we say that ω_1 and ω_2 are a *decade* apart.

The magnitude is represented in decibels (dB) defined as follows:

$$\left|G(j\omega)\right|_{dB} = 20\log\left|G(j\omega)\right|,$$

where the base of the log is 10. A linear scale for the magnitude expressed in decibels, $\left|G(j\omega)\right|_{dB}$—i.e. a logarithmic scale for the magnitude $|G(j\omega)|$—is used on the axis of the first Bode plot. $\left|G(j\omega)\right|_{dB} > 0$, $\left|G(j\omega)\right|_{dB} = 0$, $\left|G(j\omega)\right|_{dB} < 0$ correspond to $|G(j\omega)| > 1$, $|G(j\omega)| = 1$, and $|G(j\omega)| < 1$. The phase is represented in linear scale, typically in degrees.

Thanks to the factorization of $G(s)$ in (1), the magnitude of $G(j\omega)$ in dB can be evaluated as follows:

$$\begin{aligned}
\left|G(j\omega)\right|_{dB} = \;& 20\log|\mu| \\
& -20\rho\log|j\omega| \\
& +\sum_i 20\log|1+j\omega T_i| \\
& +\sum_i 20\log\left|1+\frac{2j\xi_i\omega}{\alpha_{n,i}}-\frac{\omega^2}{\alpha_{n,i}^2}\right| \\
& -\sum_i 20\log|1+j\omega\tau_i| \\
& -\sum_i 20\log\left|1+\frac{2j\zeta_i\omega}{\omega_{n,i}}-\frac{\omega^2}{\omega_{n,i}^2}\right|,
\end{aligned} \tag{2}$$

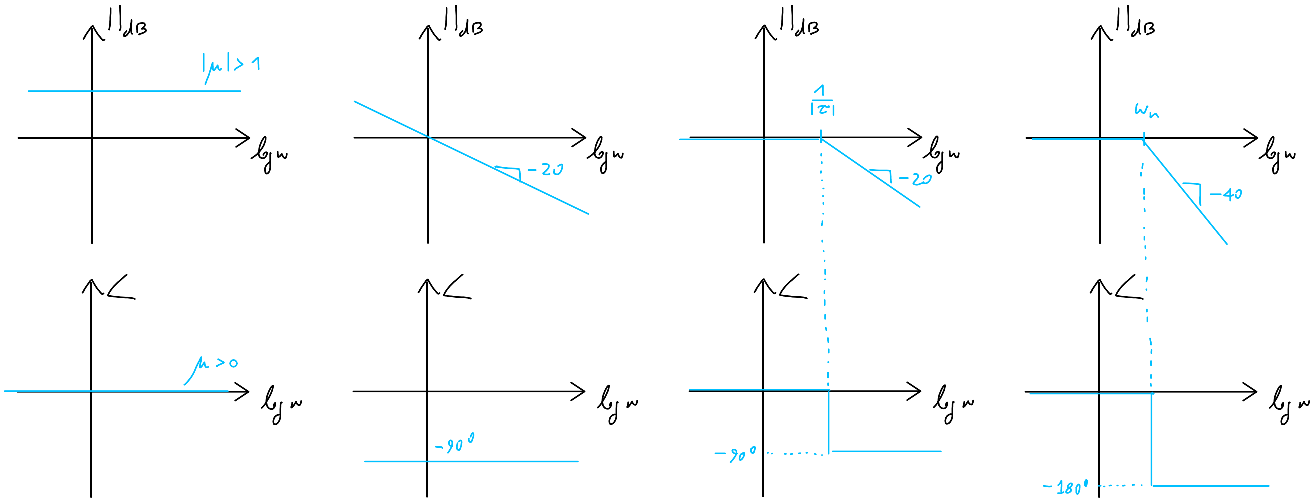
while the phase of $G(j\omega)$ is given by

$$\begin{aligned}
\angle G(j\omega) = \;& \angle\mu \\
& -\rho\angle(j\omega) \\
& +\sum_i \angle\left(1+j\omega T_i\right) \\
& +\sum_i \angle\left(1+\frac{2j\xi_i\omega}{\alpha_{n,i}}-\frac{\omega^2}{\alpha_{n,i}^2}\right) \\
& -\sum_i \angle\left(1+j\omega\tau_i\right) \\
& -\sum_i \angle\left(1+\frac{2j\zeta_i\omega}{\omega_{n,i}}-\frac{\omega^2}{\omega_{n,i}^2}\right).
\end{aligned} \tag{3}$$

From (2) and (3), it is clear that the Bode plots of $G(s)$ can be obtained by sketching the Bode plots of the factors appearing in (1) separately, and then summing them up. It is then sufficient to consider the Bode plots of the following transfer functions:

$$\begin{aligned}
G_a(s) &= \mu \\
G_b(s) &= \frac{1}{s} \\
G_c(s) &= \frac{1}{1+\tau s} \\
G_d(s) &= \frac{1}{1+\frac{2\zeta}{\omega_n}s+\frac{s^2}{\omega_n^2}}.
\end{aligned} \tag{4}$$

The asymptotic (hand-drawn in Fig. 2) and exact (generated using a computer program, Figures 3–6) Bode plots of these factors are reported in the following.



(a) $G_a(s) = \mu$ (b) $G_b(s) = \frac{1}{s}$ (c) $G_c(s) = \frac{1}{1+\tau s}$ (d) $G_d(s) = \frac{1}{1+\frac{2\zeta}{\omega_n}s+\frac{s^2}{\omega_n^2}}$

Figure 2: Approximate Bode plots for the transfer functions in (4).

3 Example

Consider the transfer function

$$G(s) = \frac{100s(1 + 0.1s)}{(1 + 16s + 100s^2)(1 + 0.01s)}. \tag{5}$$

The factors to consider in order to sketch the Bode plots are the following:

$$\begin{aligned} G_1(s) &= 100 \\ G_2(s) &= s \\ G_3(s) &= 1 + 0.1s \\ G_4(s) &= \frac{1}{1 + 0.01s} \\ G_5(s) &= \frac{1}{1 + 16s + 100s^2}. \end{aligned} \tag{6}$$

$G_1(s)$ is of type $G_a(s)$ defined in (4), the plots of $G_2(s)$ are symmetric with respect to the axis of ω to those of a transfer function of type $G_b(s)$, the plots of $G_3(s)$ are symmetric to those of $G_c(s)$, $G_4(s)$ is of type $G_c(s)$, and $G_5(s)$ is of type $G_d(s)$. The Bode plots of $G(s)$ constructed summing up the contributions of the ones of $G_1(s), \ldots, G_5(s)$ are shown in Figures 7 and 8, sketched by hand and using a computer program, respectively.

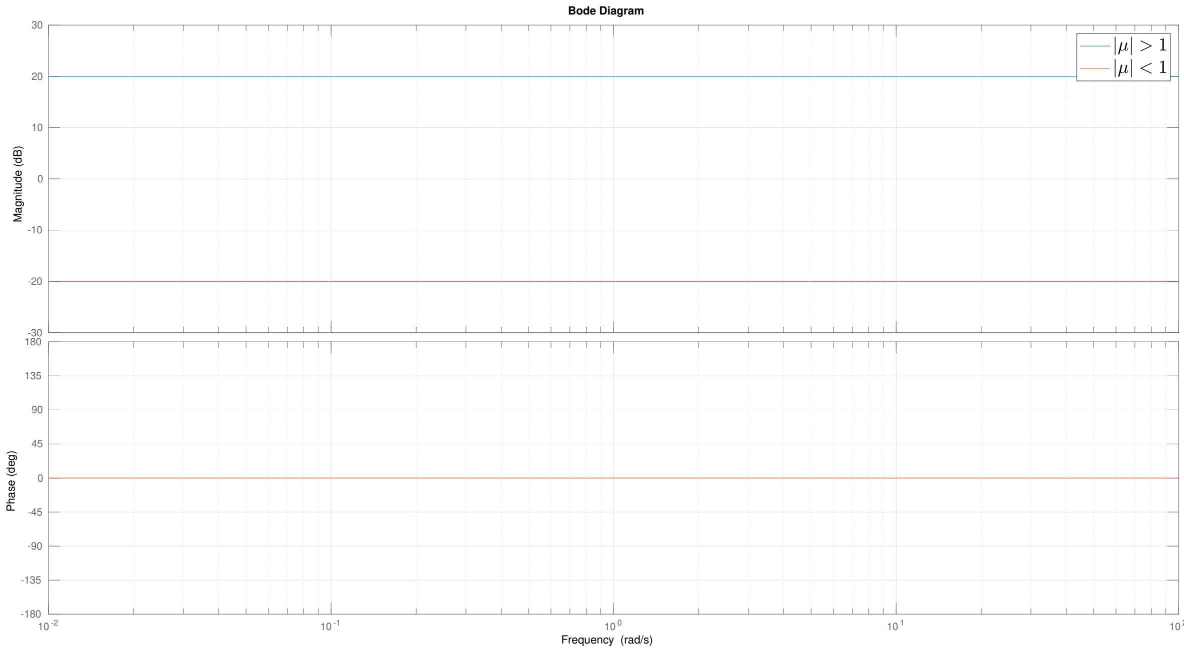


Figure 3: Bode plots of $G_a(s)$. If $\mu < 0$, we define $\angle\mu = -180^\circ$.

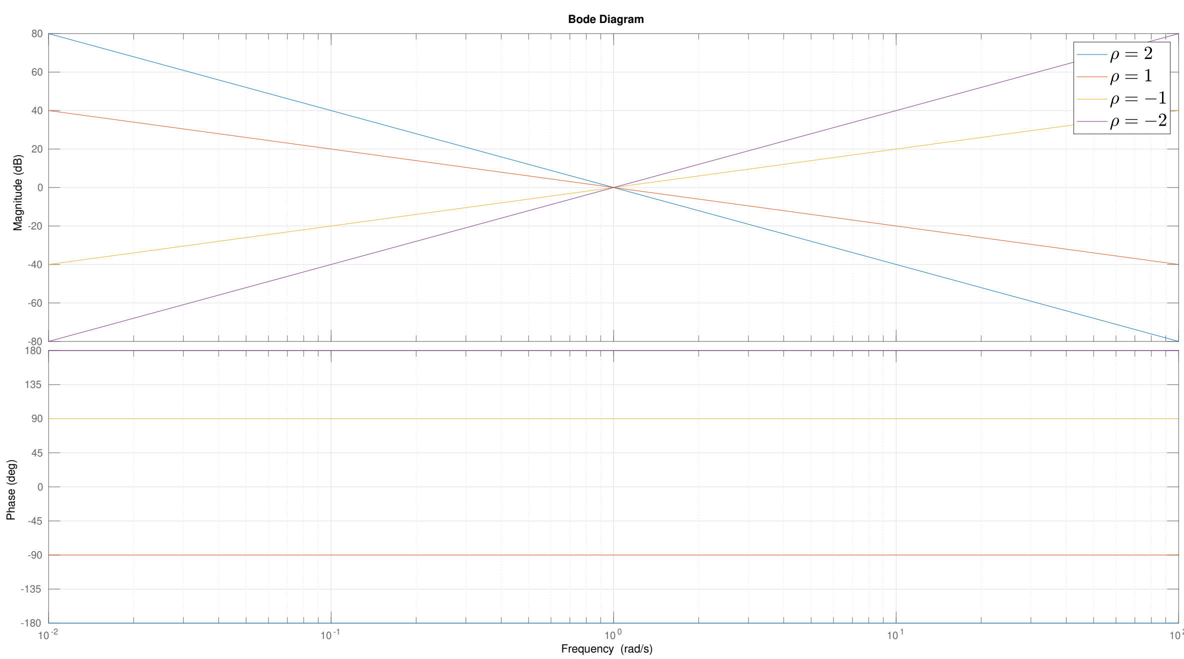


Figure 4: Bode plots of $G_b(s)$. Consider the transfer function $\frac{1}{s^\rho}$: for $\rho = 1$ (resp. $\rho = -1$), the magnitude decreases (resp. increases) by 20 dB/decade; for $\rho = 2$ (resp. $\rho = -2$), the decrease (resp. increase) is 40 dB/decade.

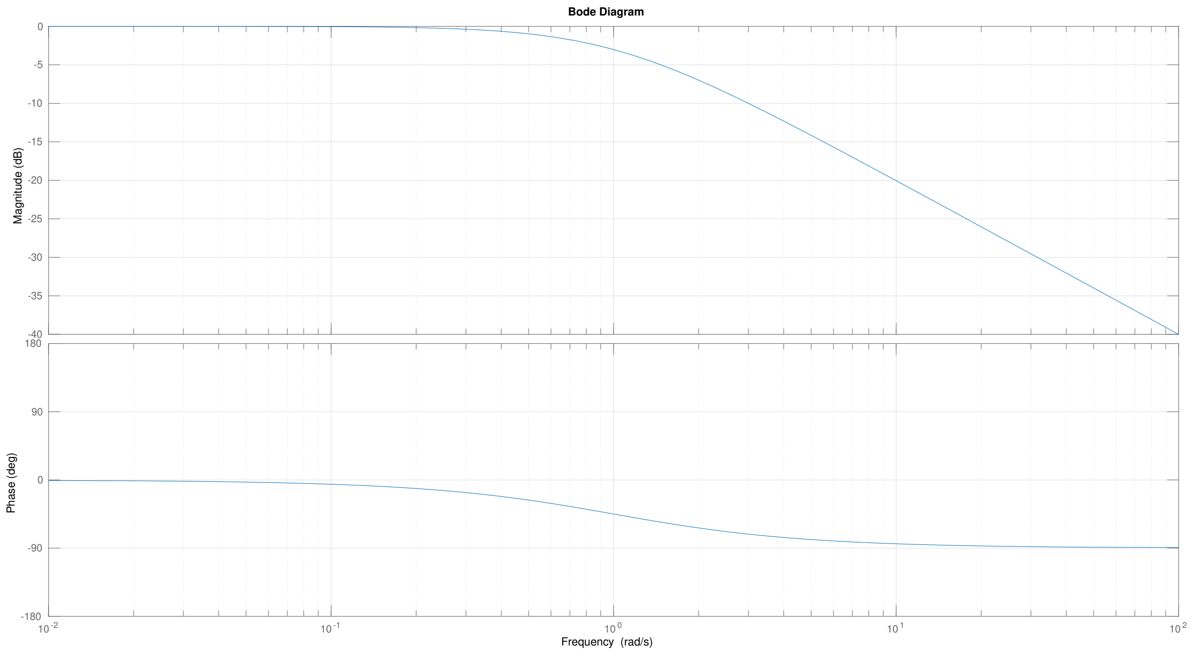


Figure 5: Bode plots of $G_c(s)$. The axis of frequencies is normalized by $1/|\tau|$, so that the value 10^0 corresponds to $\omega = 1/|\tau|$, while the value 10^2 corresponds to $\omega = 100/|\tau|$.

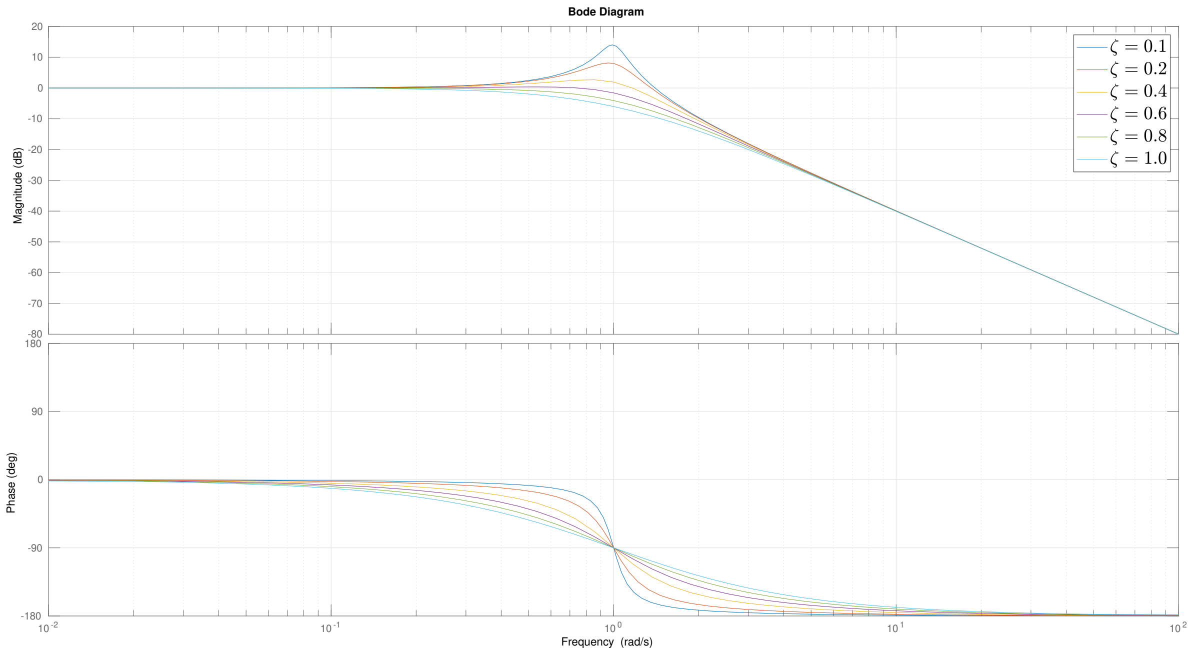


Figure 6: Bode plots of $G_d(s)$. The axis of frequencies is normalized by ω_n, so that the value 10^0 corresponds to $\omega = \omega_n$, while the value 10^2 corresponds to $\omega = 100\omega_n$.

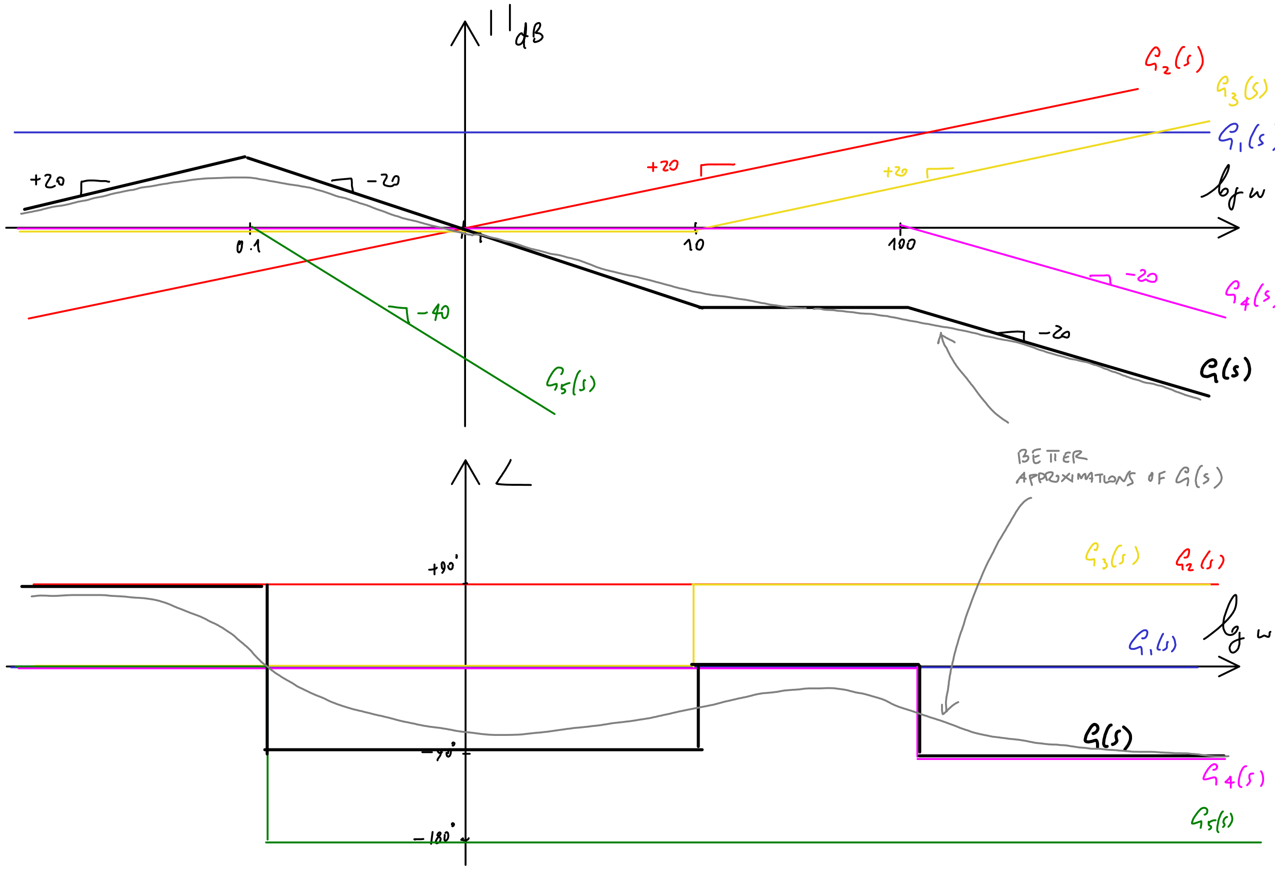


Figure 7: Bode plots of $G(s)$ in (5) sketched from the approximate Bode plots of $G_1(s), \ldots, G_5(s)$ in (6).

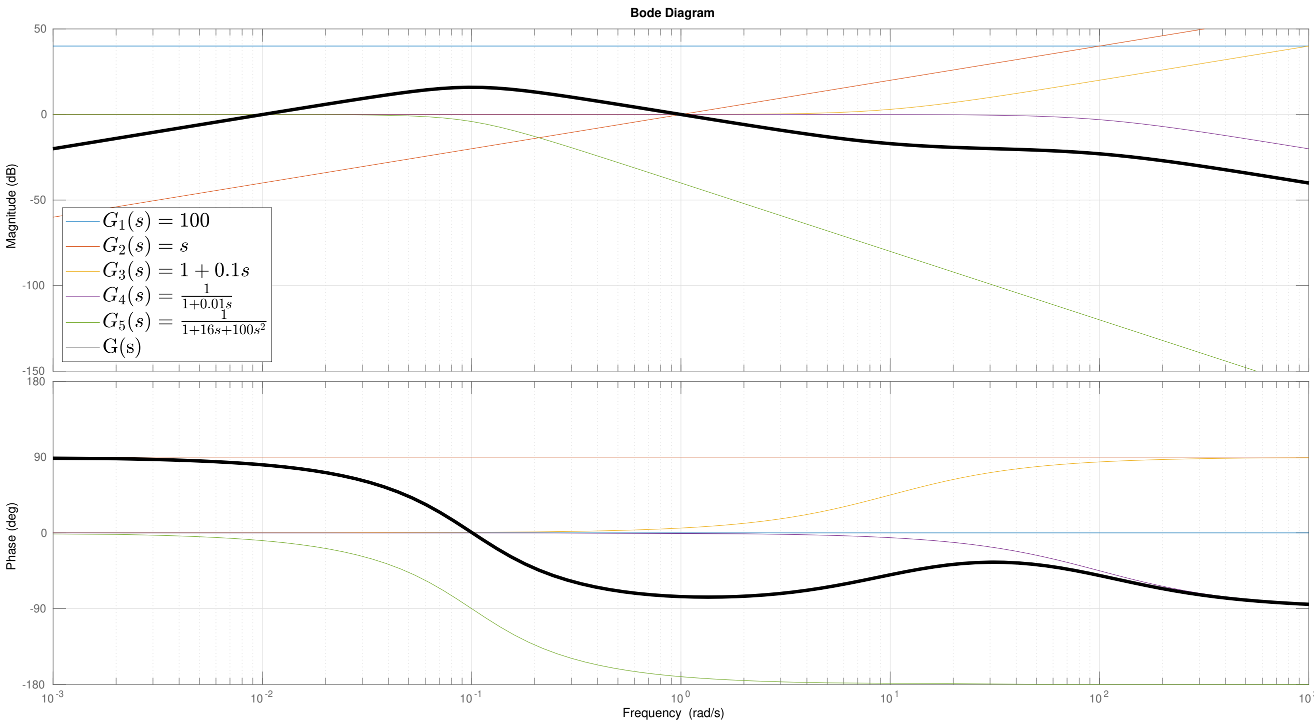


Figure 8: Bode plots of $G(s)$ in (5) and of $G_1(s), \ldots, G_5(s)$ in (6).