My grades for Assignment 2

Q1

10 / 10

1 Laplace transform of cos [10pts]

Following the approach shown in the lecture for $sin(\omega t)u_{-1}(t)$, calculate the Laplace transform of $f(t) = cos(\omega t)u_{-1}(t)$.

MATH 213 - Assignment 2

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1 Question 1

$$
\begin{aligned}
f(t) &= \cos(wt)u_{-1}(t) \\
&= \frac{1}{2}\left(e^{jwt} + e^{-jwt}\right)u_{-1}(t) \\
&= \frac{e^{jwt}}{2}u_{-1}(t) + \frac{e^{-jwt}}{2}u_{-1}(t) \\
F(t) &= \int_{-\infty}^{\infty} f(t)e^{-st}dt \\
&= \frac{1}{2}\left(\int_{-\infty}^{\infty} e^{j\omega t}e^{-st}dt + \int_{-\infty}^{\infty} e^{-j\omega t}e^{-st}dt\right) \\
&= \frac{1}{2}\left(\int_{-\infty}^{\infty} e^{j\omega t-st}dt + \int_{-\infty}^{\infty} e^{-j\omega t-st}dt\right) \\
&= \frac{1}{2}\left(\int_{-\infty}^{\infty} e^{-(s-j\omega)t}dt + \int_{-\infty}^{\infty} e^{-(j\omega+s)t}dt\right) \\
&= \frac{1}{2}\left(\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right) \\
&= \frac{1}{2}\left(\frac{2s}{s^2+\omega^2}\right) \\
&= \frac{s}{s^2+\omega^2}
\end{aligned}
$$

1

Q2

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2 Fun with unit step functions [20 pts]

Function x(t) can be described as follows: $x(t) = 0$ when $t < -1$; $x(t) = 1$ when $-1 \leq t < 0$; $x(t) = 2$ when $0 \leq t < 1$; $x(t) = 1$ when $1 \leq t < 2$; and $x(t) = 0$ when $2 \leq t$.
a) Sketch a plot of $x(t)$ and find an expression of $x(t)$ as an appropriate combination of unit step functions.
b) Sketch a plot of $x(-t)$ and find an expression of $x(-t)$ as an appropriate combination of unit step functions.
c) Sketch a plot of $x(\frac{t}{2})$ and find an expression of $x(\frac{t}{2})$ as an appropriate combination of unit step functions.

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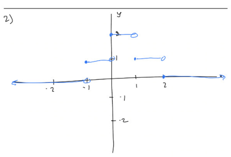
2 Question 2

2.1 (a)

$$x(t) = \begin{cases} 0 & t < -1 \\ 1 & -1 \leq t < 0 \\ 2 & 0 \leq t < 1 \\ 1 & 1 \leq t < 2 \\ 0 & 2 \leq t \end{cases}$$

$$\begin{aligned} x(t) &= 0 + (1-0)u_{-1}(t - -1) + (2-1)u_{-1}(t-0) + (1-2)u_{-1}(t-1) + (0-1)u_{-1}(t-2) \\ &= u_{-1}(t+1) + u_{-1}(t) - u_{-1}(t-1) - u_{-1}(t-2) \end{aligned}$$

2)

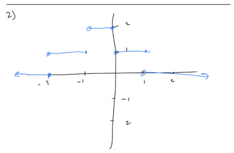


2.2 (b)

$$x(t) = \begin{cases} 0 & t \leq -2 \\ 1 & -2 < t \leq -1 \\ 2 & -1 < t \leq 0 \\ 1 & 0 < t \leq 1 \\ 0 & 1 < t \end{cases}$$

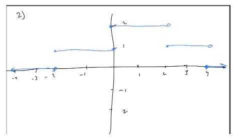
$$\begin{aligned} x(-t) &= u_{-1}(-(t-1)) + u_{-1}(-t) - u_{-1}(-(x+1)) - u_{-1}(-(x-2)) \\ &= u_{-1}(-t+1) + u_{-1}(-t) - u_{-1}(-x-1) - u_{-1}(-x+2) \\ &= u_{-1}(1-t) - u_{-1}(t) - u_{-1}(-x-1) - u_{-1}(2-x) \end{aligned}$$

2



2.3 (c)

$$x(t/2) = u_{-1}(t+2) + u_{-1}(t) - u_{-1}(t-2) - u_{-1}(t-4)$$



3

Q3

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3 RLC circuit [20 pts]

A resistor R ($v = iR$) is connected in series with an inductor L ($v = L\frac{di}{dt}$) and this pair is connected in parallel with a capacitor C ($i = C\frac{dv}{dt}$) . The leads on the capacitor are then connected across the terminals of an ideal voltage source V, which is then instantaneously turned off (disconnected from the circuit) at time t=0.
a) Write down the second order ODE that will describe how the current $i(t)$ through the three components will evolve in time and specify what the initial conditions are.
b) Using the 'hand-wavy' approach introduced in lecture 5 to solve the mass-spring-damper system, write down the condition for the values of R, L, and C that will lead to an oscillating $i(t)$.

3 Question 3

3.1 (a)

We can derive the voltage of the capacitor from $i(t) = C\frac{dv}{dt}$:

$$
\begin{aligned}
i(t) &= C\frac{dv}{dt} \\
dv &= \frac{1}{C}i(t)dt \\
\int dv &= \frac{1}{C}\int i(t)dt \\
V_c(t) &= \frac{1}{C}\int_0^t i(t)dt
\end{aligned}
$$

We can use KVL around the Capacitor-Resistor-Inductor loop to find a second-order differential equation for the current $i(t)$.

$$
\begin{aligned}
V_c + V_r + V_L &= 0 \\
\frac{1}{C}\int_0^t i(t)dt + Ri(t) + L\frac{di}{dt} &= 0 \\
\frac{d}{dt}\left(\frac{1}{C}\int_0^t i(t)dt + Ri(t) + L\frac{di}{dt}\right) &= \frac{d}{dt}0 \\
L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i(t) &= 0
\end{aligned}
$$

The initial conditions are at $t = 0$, so the voltage source has just been disconnected from the capacitor. The capacitor is initially charged to V_0, and there is no current in the circuit yet and no current passing through any of the resistor, capacitor, inductor. The voltage across the resistor and inductor is 0 since there is no current passing through either component.

3.2 (b)

We use the hand-wavy approach to model the differential equation controlling the current using the function $i(t) = Ae^{st}$.

$$
\begin{aligned}
Ls^2Ae^{st} + RsAe^{st} + \frac{1}{C}Ae^{st} &= 0 \\
Ae^{st}\left(Ls^2 + Rs + \frac{1}{C}\right) &= 0 \\
Ae^{st}\left(s^2 + \frac{R}{L}s + \frac{1}{CL}\right) &= 0
\end{aligned}
$$

4

If we factor out the exponentials to find the zeros of the following quadratic $s^2 + \frac{R}{L}s + \frac{1}{CL} = 0$,

$$\begin{aligned} s &= \frac{-R}{2L} \pm \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}} \\ &= \frac{-R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \end{aligned}$$

We have two solutions from the quadratic equation, one positive and one negative, and so we can model the solutions as: $i(t) = Ae^{s_1 t}Be^{s_2 t}$. To make notation easier, lets represent $k = \frac{R}{2L}$ and $\omega = \sqrt{k^2 - \frac{1}{CL}}$. Then we have that if we assume that $w < 0$, then $s = -k \pm jw$. We will show that we want $w < 0$ in a moment so that our system oscillates.

If we substitute the solutions for s into our exponential approximation, we get

$$\begin{aligned} i(t) &= Ae^{s_1 t}Be^{s_2 t} \\ i(t) &= Ae^{(-k+jw)t}Be^{(-k-jw)t} \\ i(t) &= Ae^{-kt}e^{jwt}Be^{-kt}e^{-jwt} \\ i(t) &= e^{-kt}\left(Ae^{jwt}Be^{-jwt}\right) \\ i(t) &\approx e^{-kt}\left(\sin\omega t + \cos\omega t\right) \end{aligned}$$

We can see now that if we have that $w < 0$, then the current in the circuit will oscillate (since $\sin\omega t$ and $\cos\omega t$ oscillate). The e^{-kt} term is the damping and exponential decay term of the current and will control how fast the oscillating current decays to zero. The conditions for the values of R, L, and C that need to be satisfies to have oscillation are:

$$\begin{aligned} w &< 0 \\ \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} &< 0 \\ \left(\frac{R}{2L}\right)^2 - \frac{1}{LC} &< 0 \\ \left(\frac{R}{2L}\right)^2 &< \frac{1}{LC} \\ \frac{R^2}{4L^2} &< \frac{1}{LC} \\ \frac{R^2}{4L} &< \frac{1}{C} \end{aligned}$$

Q4

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4 Partial fractions [10 pts]

Find the inverse Laplace transform of $F(s) = \frac{2(s+2)}{s^2+7s+12}$, where $\mathcal{R}e\{s\} > -3$.

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4 Question 4

$$
\begin{aligned}
F(s) &= \frac{2(s+2)}{s^2+7s+12} \\
&= \frac{2(s+2)}{(s+4)(s+3)} \\
&= \frac{A}{s+4} + \frac{B}{s+3} \qquad \text{(Partial fraction decomposition gives us A = 4, B = -2)} \\
&= \frac{4}{s+4} + \frac{-2}{s+3} \\
\mathcal{L}^{-1}(F(s)) &= \mathcal{L}^{-1}\left(\frac{4}{s+4}\right) + \mathcal{L}^{-1}\left(\frac{-2}{s+3}\right) \\
&= \begin{cases} 4e^{-4t} - 2e^{-3t} & , t \geq -3 \\ 0 & , t < -3 \end{cases}
\end{aligned}
$$

Q5

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5 Mystery constants [20 pts]

Let $g(t) = x(t) + \alpha x(-t)$, where $x(t) = \beta e^{-t} u_{-1}(t)$,
and assume the Laplace transform of $g(t)$ is $G(s) = \frac{s}{s^2-1}$, $-1 < \mathcal{R}e\{s\} < 1$.
Find the values of the constants α and β.

5 Question 5

$$
\begin{aligned}
G(t) &= \frac{s}{s^2-1} = \frac{s}{(s+1)(s-1)} \\
&= \frac{A}{s+1} + \frac{B}{s-1} \\
&= \frac{1/2}{s+1} + \frac{1/2}{s-1} && \text{(By partial fraction decomposition)} \\
x(t) &= \beta e^{-t}u_{-1}(t) \\
g(t) &= x(t) + \alpha x(-t) \\
&= \beta e^{-t}u_{-1}(t) + \alpha\beta e^{t}u_{-1}(-t) \\
G(t) &= \mathcal{L}\{g(t)\} \\
&= \int_{-\infty}^{\infty} g(t)e^{-st}dt \\
&= \int_{-\infty}^{\infty} (\beta e^{-t}u_{-1}(t) + \alpha\beta e^{t}u_{-1}(-t))e^{-st}dt \\
&= \int_{-\infty}^{\infty} \beta e^{-t}u_{-1}(t)e^{-st}dt + \int_{-\infty}^{\infty} \alpha\beta e^{t}u_{-1}(-t)e^{-st}dt \\
&= \int_{0}^{\infty} \beta e^{-t}e^{-st}dt + \int_{-\infty}^{0} \alpha\beta e^{t}e^{-st}dt \\
&= \int_{0}^{\infty} \beta e^{-(1+s)t}dt + \int_{-\infty}^{0} \alpha\beta e^{(1-s)t}dt \\
&= \beta\frac{-1}{1+s}e^{-(1+s)t}\Big|_{0}^{\infty} + \alpha\beta\frac{1}{1-s}e^{(1-s)t}\Big|_{-\infty}^{0} \\
&= \beta\frac{1}{1+s} + \alpha\beta\frac{1}{1-s} \\
&= \frac{\beta}{1+s} + \frac{\alpha\beta}{1-s} \\
&= \frac{\beta}{s+1} + \frac{-\alpha\beta}{s-1} \\
G(t) &= \frac{\beta}{s+1} + \frac{-\alpha\beta}{s-1} = \frac{1/2}{s+1} + \frac{1/2}{s-1}
\end{aligned}
$$

From here we can see that the values of α and β are given by $\alpha = -1$ and $\beta = 1/2$.

Q6

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6 Lasing threshold [20pts]

The word 'laser' is actually an acronym that stands for "Light Amplification by the Stimulated Emission of Radiation". Practically, a basic laser is implemented by placing a 'gain medium' between two parallel mirrors spaced at some distance apart that form an 'optical resonator' (aka 'cavity'). The gain medium can be viewed as a bunch of atoms, which are pumped into excited state at a constant rate by some external mechanism (such as electric current in semiconductor lasers or a plasma discharge in gas lasers). This is somewhat similar to the carbon dating problem from HW#1, where ^{14}C was created at a constant rate by cosmic radiation. Here, the s-orbital could be the ground state that the atoms are normally in and the a p-orbital could be the excited state, if you want to connect this to your chemistry class. Once in the excited state, an atom can emit light into free space or into the resonator. If we ignore the effects of the cavity and of the pump, the number of atoms in the excited state N_{ex} will decay exponentially with time (just like ^{14}C in an artifact): $\frac{dN_{ex}}{dt} = -\Gamma_0 N_{ex}$

The optical resonator will add another exponential decay term of the form $-\Gamma_{cav} N_{ex}(p+1)$, where p is the number of photons present inside the cavity. In other words, the decay of the atoms of the gain medium from their excited state is sped up by the cavity and the presence of photons. The pump will add atoms in excited state at a rate R_{pump}. At the same time, if we ignore the effects of the gain medium, the number of photons inside the cavity will also decay exponentially, $\frac{dp}{dt} = -\kappa p$, where κ is determined by the reflectivity of the mirrors forming the cavity. The gain medium will cause new photons to appear in the cavity at a rate given by $\Gamma_{cav} N_{ex}(p+1)$. This term describes the stimulated emission process in the laser – each atom decay out of the excited state will add a photon into the cavity and that more photons in the cavity will lead to faster atom decay (when $p = 0$, the system is in a spontaneous emission regime). The system is thus described by two coupled differential equations:

$$\frac{dN_{ex}}{dt} = R_{pump} - \Gamma_0 N_{ex} - \Gamma_{cav} N_{ex}(p+1) \quad (1)$$

$$\frac{dp}{dt} = -\kappa p + \Gamma_{cav} N_{ex}(p+1) \quad (2)$$

The lasing threshold is then defined as a steady state for p, with $p = 1$ (notice that defining p to be in steady state will force N_{ex} into steady state as well). In this state the stimulated emission into the cavity dominates (over spontaneous emission into the cavity) and the system produces 'laser light'. Find the expression for the minimum value of R_{pump} that will result in the system being at or above the lasing threshold for a system with properties given by κ and $\beta = \frac{\Gamma_{cav}}{\Gamma_{cav}+\Gamma_0}$. Show that for small β the threshold $R_{pump} \approx \frac{\kappa}{2\beta}$.

6 Question 6

We are told that both differential equations are in steady-state, and so we can set the right-hand side of each equation to zero. We are also told that $p = 1$ and we can further substitute this into the equations to simplify.

$$\begin{aligned}
\beta &= \frac{\Gamma_{\text{cov}}}{\Gamma_{\text{cov}} + \Gamma_0} \\
\Gamma_{\text{cov}} + \Gamma_0 &= \frac{\Gamma_{\text{cov}}}{\beta} \\
\frac{dp}{dt} &= 0 \\
&= -kp + \Gamma_{\text{cov}} N_{ex}(p+1) \\
k &= 2\Gamma_{\text{cov}} N_{ex} \\
\frac{k}{2} &= \Gamma_{\text{cov}} N_{ex} \\
\frac{dN_{ex}}{dt} &= R_{\text{pump}} - \Gamma_0 N_{ex} - \Gamma_{\text{cov}} N_{ex}(p+1) = 0 \\
0 &= R_{\text{pump}} - \Gamma_0 N_{ex} - 2\Gamma_{\text{cov}} N_{ex} \\
&= R_{\text{pump}} - N_{ex}\left(\Gamma_0 + \Gamma_{\text{cov}} + \Gamma_{\text{cov}}\right) \\
R_{\text{pump}} &= N_{ex}\left(\Gamma_0 + \Gamma_{\text{cov}} + \Gamma_{\text{cov}}\right) \\
&= N_{ex}\left(\frac{\Gamma_{\text{cov}}}{\beta} + \Gamma_{\text{cov}}\right) \\
&= N_{ex}\Gamma_{\text{cov}}\left(\frac{1}{\beta} + 1\right) \\
R_{\text{pump}} &= \frac{k}{2}\left(\frac{1}{\beta} + 1\right)
\end{aligned}$$

For small values of β, we can approximate $\frac{1}{\beta} + 1 \approx \frac{1}{\beta}$ and so we can get: $R_{\text{pump}} \approx \frac{k}{2\beta}$.