Notes on the power or energy of signals

Average power

Let's start with a real-valued signal $f(t)$. Inspired by voltages across, and currents through, resistances, we think of the instantaneous power of $f(t)$ as (proportional to) $f(t)^2$. Hence, if $f(t)$ is periodic with period T, we define its *average power* to be, for any t_0,

$$\frac{1}{T}\int_{t_0}^{t_0+T} f(t)^2 dt \ .$$

That is, the average power is the integral of the power over any period – the *energy* in one period – divided by the length of the period.

Average power of real sinusoidal signals

Suppose that $f(t)$ is sinusoidal. Using double-angle formulas,

$$\sin\theta\sin\phi = \frac{1}{2}[\cos(\theta-\phi) - \cos(\theta+\phi)] \ .$$

so

$$\sin^2\theta = \frac{1}{2}[1-\cos 2\theta] \ .$$

The average power in a real sinusoid with amplitude A is therefore

$$\frac{1}{T}\int_{t_0}^{t_0+T} A^2\sin^2\frac{2\pi}{T}t dt = \frac{1}{T}\int_{t_0}^{t_0+T}\frac{1}{2}A^2[1-\cos\frac{4\pi}{T}t]dt = \frac{T}{T}\frac{1}{2}A^2 = \frac{1}{2}A^2 \ .$$

[In order to treat sinusoidal signals more like d.c. signals when computing power, instead of specifying the amplitude A we sometimes use the "root mean square," or RMS, value, $\frac{1}{\sqrt{2}}A$; the average power is simply the square of the RMS value.]

Average power of frequency components in Fourier series

Recall that, when we derived Parseval's Theorem, we interpreted the integral

$$\frac{1}{T}\int_{-T/2}^{T/2}|f(t)|^2 dt$$

as the average power of the complex-valued signal $f(t)$. If we let $f(t)$ be one of the complex-exponential terms in the Fourier series, we find

$$\frac{1}{T}\int_{-T/2}^{T/2}|c_n e^{j\frac{2\pi}{T}nt}|^2 dt = \frac{1}{T}\int_{-T/2}^{T/2} c_n e^{j\frac{2\pi n}{T}t}c_n^* e^{-j\frac{2\pi n}{T}t}dt = \frac{1}{T}\int_{-T/2}^{T/2} c_n c_n^* dt = |c_n|^2 \ .$$

To relate this result to real sinusoids, consider that

$$\begin{aligned} A\sin(\frac{2\pi}{T}t) &= \frac{Ae^{j\frac{2\pi}{T}t} - Ae^{-j\frac{2\pi}{T}t}}{2j} \\ &= \frac{A}{2j}e^{j\frac{2\pi}{T}t} - \frac{A}{2j}e^{-j\frac{2\pi}{T}t} \ . \end{aligned}$$

The average power of the signal on the left-hand side is $A^2/2$; the average power of each of the mutually-orthogonal signals on the right-hand side is $A^2/4$.

Real sinusoidal form of the Fourier series

For further comparison, consider the Fourier series of a real-valued signal with period T. When $f(t)$ is real, we have $c_{-n} = c_n^*$, so,

$$\begin{aligned}
c_n e^{j\frac{2\pi n}{T}t} + c_{-n} e^{-j\frac{2\pi n}{T}t} &= 2\mathrm{Re}\{c_n e^{j\frac{2\pi n}{T}t}\} \\
&= 2\mathrm{Re}\{|c_n| e^{j(\frac{2\pi n}{T}t + \angle c_n)}\} \\
&= 2|c_n| \cos(\frac{2\pi n}{T}t + \angle c_n) \\
&= 2|c_n| \{\cos(\frac{2\pi n}{T}t)\cos(\angle c_n) - \sin(\frac{2\pi n}{T}t)\sin(\angle c_n)\} \\
&= 2\{\mathrm{Re}\{c_n\}\cos(\frac{2\pi n}{T}t) - \mathrm{Im}\{c_n\}\sin(\frac{2\pi n}{T}t)\} \\
&= a_n \cos(\frac{2\pi n}{T}t) + b_n \sin(\frac{2\pi n}{T}t)
\end{aligned}$$

where $a_n := 2\mathrm{Re}\{c_n\}$ and $b_n := -2\mathrm{Im}\{c_n\}$. We therefore have

$$\sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t} = a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{2\pi n}{T}t) + \sum_{n=1}^{\infty} b_n \sin(\frac{2\pi n}{T}t) \ .$$

The right-hand side is called the real sinusoidal form of the Fourier series, often used for real-valued $f(t)$. In fact, it follows from the definitions of the a_n and b_n that, for $n > 0$,

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(\frac{2\pi n}{T}t)dt \quad \& \quad b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(\frac{2\pi n}{T}t)dt \ ,$$

so the real sinusoidal form can be computed directly in this way.

The average power in each of the cosine terms is $\frac{1}{2}a_n^2$, and in the sine terms $\frac{1}{2}b_n^2$. The average powers of these orthogonal terms are additive: we already know that the average power of the sum of these terms is

$$|c_n|^2 + |c_{-n}|^2 = 2|c_n|^2 = 2[(\mathrm{Re}\{c_n\})^2 + (\mathrm{Im}\{c_n\})^2] = \frac{1}{2}[a_n^2 + b_n^2] \ .$$

The form of Parseval's Theorem for the real sinusoidal Fourier series is therefore

$$\frac{1}{T}\int_{-T/2}^{T/2} (f(t))^2 dt = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 + \frac{1}{2}\sum_{n=1}^{\infty} b_n^2 \ .$$

The right-hand side is just the sum of the average powers of the respective terms of the Fourier series.