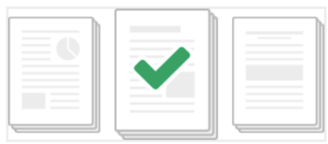
Assignment 7

Due: Thursday, April 6, 2023 11:59 pm (Eastern Daylight Saving Time)



Thanks for your submission!

Your assignment has been received and is waiting to be graded.

Review your submission

Q1 (20 points)

2 pages submitted

Problem 1 [20 pts]

Recall, as discussed in class and in a former assignment, a system may or may not be

1) Memoryless
2) Time invariant
3) Linear
4) Causal
5) Stable

Determine if each of the continuous-time systems described below have or do not have these properties (make sure to show formal proofs for all of your answers). In each case, $x(t)$ denotes the system input and $y(t)$ the system output.

a) $y(t) = x(t-5) + 2x(t)$

b) $y(t) = \int_{-\pi/11}^{5t} \frac{\cos x dx}{1+\sqrt{\sin 3x}}$

c) $y(t) = k \int_0^{2\pi} \lambda(\theta)\sqrt{a^2 \sin^2\theta + b^2\cos^2\theta}\ d\theta$

d) $y(t) = k\rho(t) \int_0^{2\pi}\int_0^L \frac{(a^2-d^2\cos^2(\theta))}{((a^2-d^2\cos^2(\theta))^2+b^2d^2\sin^2(\theta))^{3/2}} dx d\theta$

Hint: In part d) you may assume you are working with a charged ovaloid [1] in 3D space

1: What's an ovaloid?

MATH 213 - Assignment 7

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Question 1

(a)

$$y(t) = x(t-5) + 2x(t)$$

Memoryless: A system is memoryless if the output at time t depends only on the input at the same time t. In this case, the output depends on $x(t-5)$ and $x(t)$, so the system is not memoryless.

Time Invariant: A system is time-invariant if a time shift in the input results in the same time shift in the output. Let $x_1(t) = x(t-\tau)$, then:

$$y_1(t) = x_1(t-5) + 2x_1(t) = x(t-\tau-5) + 2x(t-\tau)$$

Now, let $y_2(t) = y(t-\tau)$, then:

$$y_2(t) = y(t-\tau) = x(t-\tau-5) + 2x(t-\tau)$$

The two formulations are equivalent and the system is time-invariant.

Linear:

$$\begin{aligned} x(t) &= \alpha x_1(t) + \beta x_2(t) \\ y(t) &= x(t-5) + 2x(t) \\ &= (\alpha x_1(t-5) + \beta x_2(t-5)) + 2(\alpha x_1(t) + \beta x_2(t)) \\ &= \alpha(x_1(t-5) + 2x_1(t)) + \beta(x_2(t-5) + 2x_2(t)) \end{aligned}$$

Since the system satisfies both superposition and homogeneity, it is linear.

Causal: A system is causal if the output at time t depends only on the input at times t or earlier. In this case, the output depends on $x(t-5)$ and $x(t)$, so the system is causal.

Stable: A system is stable if the output is bounded for all bounded inputs. In this case, for any bounded input $x(t)$, the output $y(t)$ will also be bounded since it is a linear combination of $x(t-5)$ and $x(t)$. Therefore, the system is stable.

(b)

x in the integral is the variable of integration and not the input. The system is memoryless (as it does not depend on the value of the input), time-invariant (as it does not depend on the time at which the input is given), not linear (as it doesn't use the value of the input), causal (as it does not depend on the value of the system at any time). The system doesn't depend on the value of the input being bounded and so it is stable if the value of the integral is stable, which it is.

(c)

The output of the system does not depend on the input and so the system is memoryless (as it does not depend on the value of the input at any time), time-invariant (as it does not depend on the time at which the input is given), not linear (as it does not satisfy superposition since it does not use the value of the input), causal (as it does not depend on the value of the input at any time). The system is stable iff the integral of $\lambda(\theta)$ wrt θ is bounded as we can clearly see that the part of the system in the square root is bounded over the integral.

(d)

x in the integral is the variable of integration and not the input. The system is memoryless (as it does not depend on the value of the input), time-invariant (as it does not depend on the time at which the input is given), not linear (as it doesn't use the value of the input), causal (as it does not depend on the value of the system at any time). The system doesn't depend on the value of the input being bounded and so it is stable if the the value of $\rho(t)$ is bounded since the rest of the system is constants and an integral which refers to the surface area of a ovaloid (which is bounded).

Q2 (20 points)

Not submitted

Problem 2 [20 pts]

You and your best friend have decided to play the classic child's game Tug of War! However, this time, to breathe new life into a timeless game you have decided to introduce some new mechanics.

Instead of rope have decided to use a special sticky 1D tensile constraint sloopy™. Sloopy™ has a very unique elasticity to it, for this question you may assume it can be approximated as a dampened harmonic oscillator (k spring constant and damping constant b).

Now, that's not enough, you want to turn this into the game of minds, not just strength! You have decided to cut the sloopy™ in half, and attach an end of each half to a large egg you found. You will be playing tug of war with an egg binding the two ends of the sloopy™ together. The egg will eventually break in half from the tension, so you have decided that the *loser* of the game will be whoevers side the egg lands on (At the time of breakage, depending on the instantaneous, acceleration, velocity, and position, if the egg lands on your side you lose!). The egg breaks when the sum of the forces pointing in opposite directions is equal to the parameter B.

Assume that neither you or your friend move at all, and neither of you let go of the sloopy™ (you're both stronger than that!). Your friend has great stamina and dexterity, so his strategy will be that he pulls his sloopy™ towards him with force $\cos(15t)$ at any given time t from the start of the game. You, on the other hand are great with bruteforce (or so your professors say) so you will pull the egg towards you with linearly increasing force αt where α is a positive constant.

a) Model the horizontal position $x(t)$, velocity $v_x(t)$ and acceleration $a_x(t)$ of the egg as a differential equation with the tugging of you and your friend as a system input.

b) Use Laplace transformations to solve for $x(t)$, determine the time of breakage.

c) For what values of α will you win the game? [2]

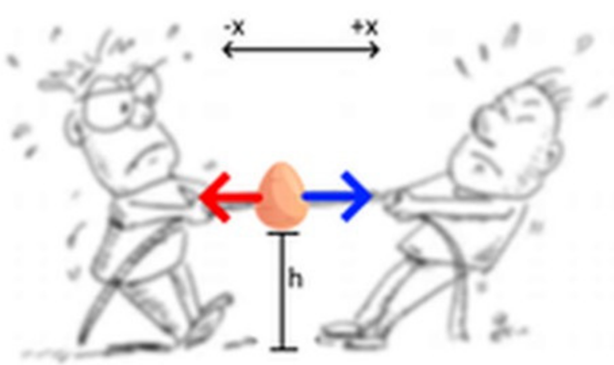

Figure 1: You and your friend playing Tug of War with an egg!

2: Remember, don't lose the game!

Q3 (20 points)

Not submitted

Problem 3 [20 pts]

We recently discussed the concept of two-dimensional complex exponentials, two-dimensional Fourier transforms, and their applications to image processing. We also mentioned that a lens can act as a Fourier transform machine.

When a light emitting image is placed in the focal plane of a lens, the lens will produce a Fourier transform of the image in the focal plane on the other side (which can be detected if a screen or an image sensor is placed in that plane). The original image can be recovered by performing a second Fourier transform with another lens placed so that the distance between the two lenses is equal to the sum of their focal lengths (the image will be reformed in the focal plane on the other side of the second lens). Propose and conceptually explain how you would implement in this system the following:

a) [8 pts] a high-pass filter for the spatial frequency components of the original image.

b) [12 pts] a low-pass filter for the spatial frequency components of the original image.

Q4 (20 points)

Not submitted

Problem 4. Relationship to Quantum Mechanics [20 pts]

The Schrödinger equation is a linear partial differential equation that governs the wave function of a quantum-mechanical system. The equation gives the evolution over time of a wave function, which is the quantum-mechanical characterization of an isolated physical system. the wave function allows for the derivation of the measurable properties of the particles in the system, such as energy, position, and momentum. The wave function for can be obtained by solving the Schrödinger equation.

The time-independent Schrödinger [3] equation describes stationary states, or states where the wave functions form standing waves. It allows for the analysis of certain quantum-mechanical systems using the techniques described in the lectures, which is important as many other systems are unsolvable analytically.

This is *fundamental* in quantum mechanics and its applications (e.g. quantum well lasers).

It is also equivalent to solving the following equation:

$$\hat{H}|\Psi\rangle = E|\Psi\rangle$$

In a quantum-mechanical system, the energy of the system is described using the Hamiltonian of the system $\hat{H}$, which is simply the sum of kinetic energies $\hat{T}$ and potential energies $\hat{V}$. In the most simple quantum-mechanical system, we consider a 'box' that occupies a finite region $[0, L]$ in 1-dimensional space. The wavefunction then describes the probabilities for the particle's position in a probability distribution. In particular, $\int_{-\infty}^{\infty} \psi^2(x)dx = 1$. The potential energy operator $\hat{V}$ in a 1-dimensional LTI system is given as a function of the position $V(x)$. The energies of the system are such that the potential energy $V(x)$ is 0 for the region inside the box, and infinite otherwise, to keep the particle inside the box. We imagine a moving particle in this box and consider its behaviour using the time-independent Schrödinger equation. The kinetic energy operator $\hat{T}$ in a 1-dimensional LTI system for a single particle of mass m is given by $-\frac{\hbar^2}{2m}$, where $\hbar$ is the reduced Planck Constant where $\hbar = \frac{h}{2\pi}$. The measurable properties of the particle are given by its wavefunction, which is a time-independent function of position $\psi(x)$. The energy E is simply a real number in this system. The time-independent Schrödinger equation for the 'particle in a box' quantum-mechanical system is given by the differential equations:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x) \tag{1}$$

$$\psi(x) = A\sin(kx) + B\cos(kx) \tag{2}$$

Quantum well lasers are laser diodes consisting of a 'sandwich' of semiconductor 'well' materials. The layers of the semiconductor sandwich are thin enough for quantum confinement effects to be observed. The quantum well behaviour can be represented by a 'particle in a box' model, but have exponentially decaying 'walls' for $x < 0$ and $x > L$ in practice. The well laser relies on the interaction between light and electrons as you may have seen in your chemistry class, involving the movement of a single electron between different energy states. Again, these equations are unsolvable through analytical methods in general, but solvable for simple LTI quantum-mechaincal systems, such as the 'particle in a box' system. Find the values of constants A, B, and k wavefunction for the 'particle in a box' system.

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