University of Waterloo
CS 341 Fall 2022
Written Assignment 2

Due Date: Monday, Oct 17 at 11:59pm to Crowdmark
All work submitted must be the student's own.

Question 1 [12 marks] Recursion Tree

Solve the following recurrence relations using the recursion tree method. Express your solution in terms of a Θ bound on $T(n)$. Show your work clearly.

- Draw the final tree showing at least 4 levels (including the root and leaves).
- Give a mathematical expression for the sum of work in the recursion tree identifying the work done in the base cases and the recursive cases (leave this as a summation) - an induction proof is not required.
- Simplify the expression (show your work) to give a closed form and derive a Θ bound on $T(n)$.

Note: You may use the Master Theorem to verify your result.

a) You may assume that n is a power of 3.

$$T(n) = \begin{cases} 4, & n = 1, \\ 5T(n/3) + n\sqrt{n}, & n > 1. \end{cases}$$

b) You don't have to show all nodes on the third level but show enough that we know what you are doing.

$$T(n) = \begin{cases} 2, & n \leq 1, \\ 6T(\frac{3}{7}n) + n^2, & n > 1. \end{cases}$$

Question 2 [12 marks] Lucky Guess

Use induction to verify the following recurrence with the corresponding guess:

a) $T(n) = 3T(\lfloor n/3 \rfloor) + 2n$ for $n > 2$ and $T(n) = 1$ for $n \leq 2$
Guess: $T(n) = O(n \log n)$.

b) $T(n) = 3T(\lfloor n/3 \rfloor) + 10$ for $n > 2$ and $T(n) = 2$ for $n \leq 2$
Guess: T(n) = O(n)

Clearly indicate the following components: Basis, Induction Hypothesis, Induction Step and Concluding Statement. You should also clearly label where you are using the induction hypothesis in the induction step.

Question 3 [12 marks] Dynamic Programming

Consider the expense of renting a car for a one-way trip across Canada. Car rental agencies allow one-way car rentals but charge a fee if they are returned to a location other than the one it is rented from. Also, the farther away the rental car is returned, the larger the fee.

Suppose there are n cities (in order) along your one-way trip (city n is your destination) each with a car rental agency. You are given a rental cost array $C(i, j)$ = the cost of renting a car from city i and returning it in city j where $1 \leq i, j \leq n$

- $C(i, i) = 0$ for $1 \leq i \leq n$
- $C(i, j) = \infty$ for $i > j$

a) Define a recurrence relation for a dynamic programming algorithm that determines the value of the cheapest car rental scheme that gets you from city 1 to n.

b) Describe the table used in this dynamic programming algorithm as follows:

- State the dimension of the table.
- State the meaning of each entry in the table
- State the table initialization.
- State the order in which the table will be filled in.

c) Write out the algorithm (in pseudocode) to solve this dynamic programming problem and state how to obtain the final value.

d) Write out the algorithm (in pseudocode) with brief description to determine the optimal sequence of cities in which to rent cars from to get from city 1 to n.

Question 4 [12 marks] Dynamic Programming

We are given n items, where item i has value v_i and weight w_i such that v_i and w_i are positive integers. Suppose we have two knapsacks, each can store up to a weight of W and want to select items to store in the two knapsacks, maximizing the total value of the selected items.

Equivalently, we want to select disjoint subsets $A, B \in \{1, \ldots, n\}$ to maximize:

$$\sum_{k \in A} v_k + \sum_{k \in B} v_k$$

subject to the constraints

- $\sum_{k \in A} w_k \leq W$ and
- $\sum_{k \in B} w_k \leq W$.

Design a dynamic programming algorithm that solves the problem in $O(nW^2)$ time by providing the following:

a) Give a definition of the subproblems.

b) Provide the base cases.

c) Derive and justify the recursive formula.

d) Write pseudocode for computing the optimal value.

e) Write pseudocode for outputting the optimal subsets.

f) Give an analysis of the running time and space usage.

Suggestion: Use a 3-dimensional table ...