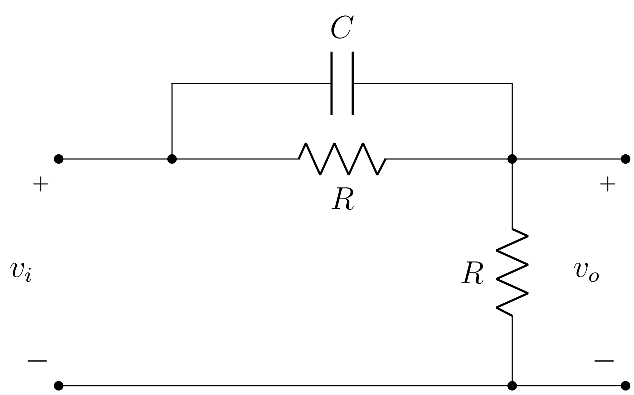
Linear, Constant-Coefficient ODEs

A *linear* differential equation has the form

$$\frac{d^n}{dt^n}y(t) + a_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}}y(t) + \ \ldots \ + a_0(t)y(t)$$
$$= b_m(t)\frac{d^m}{dt^m}f(t) + b_{m-1}(t)\frac{d^{m-1}}{dt^{m-1}}f(t) + \ \ldots \ + b_0(t)f(t) \ .$$

Here, we have included a "forcing term" $f(t)$: in the standard approach to differential equations, this would be assumed given. The solution of the differential equation would consist in finding a $y(t)$ that satisfies the equation (and any specified boundary conditions) for the given $f(t)$. For our purposes, it is convenient to allow $f(t)$ and $y(t)$ to be complex-valued.

Example Consider the following circuit:



Summing the currents flowing out of the node above the right-hand resistor, assuming that none is being drawn to the right,

$$\frac{1}{R}v_o(t) + \frac{1}{R}(v_o(t) - v_i(t)) + C\frac{d}{dt}(v_o(t) - v_i(t)) = 0 \ ;$$

and regrouping terms,

$$\frac{d}{dt}v_o(t) + \frac{2}{RC}v_o(t) = \frac{d}{dt}v_i(t) + \frac{1}{RC}v_i(t) \ .$$

If we suppose that $v_i(t)$ is an applied “input” voltage, it could play the role of $f(t)$, while $v_o(t)$ would represent a response to that stimulus, corresponding to $y(t)$. $\square$

Example A mass-spring-damper system, driven by an applied force $f(t)$, might be modelled by the equation

$$m\frac{d^2}{dt^2}y(t) + b\frac{d}{dt}y(t) + ky(t) = f(t)$$

where m is the mass, b is the coefficient of a friction that is proportional to speed, and k is a spring constant. Dividing by m, we find an equation of the above form. $\square$

In the theory of differential equations, it is typical to assume that $f(t)$, and therefore the whole right-hand side of the equation, is given, and then to solve for $y(t)$.

Such linear equations have the important property of respecting the **principle of superposition**. If $y(t) = y_1(t)$ is a solution of the equation for some $f(t) = f_1(t)$, and $y(t) = y_2(t)$ is a solution for some $f(t) = f_2(t)$, then, for any complex constants $\mathbb{c}_1$ and $\mathbb{c}_2$, $y(t) = \mathbb{c}_1 y_1(t) + \mathbb{c}_2 y_2(t)$ solves the equation for $f(t) = \mathbb{c}_1 f_1(t) + \mathbb{c}_2 f_2(t)$.[1]

Constant coefficients

One of the few cases that admit analytical solution is that in which the coefficients $a_i(\cdot)$ and $b_j(\cdot)$ are constant. In that case, every term in the equation consists of some dependent variable, or one of its derivatives, multiplied by a constant coefficient.[2]

The above circuit model and mass-spring-damper are examples.

[1]Mathematicians think of this property as the conjunction of *homogeneity* and *additivity*.

[2]While the class of linear, constant-coefficient ODEs is, mathematically, quite a special case, it is very useful for applications. In particular, nonlinear differential equations can often be suitably approximated, within limited ‘operating regimes,’ by linear equations obtained from Taylor-series expansions.

It's convenient to think of $\frac{d}{dt}$ as a differentiation operator, denoted by D. So,

$$\begin{aligned} Dy(t) &= \frac{d}{dt}y(t) \ , \\ D^2y(t) &= \frac{d^2}{dt^2}y(t) \ , \\ &\text{etc.} \end{aligned}$$

In this way, we can consider the two sides of an equation with constant coefficients to contain polynomials in the differentiation operator:

$$Q(D)y(t) = P(D)f(t) \ .$$

Example Using the differentiation operator, we could represent the differential equation for the circuit example like this:

$$\left(D + \frac{2}{RC}\right) v_o(t) = \left(D + \frac{1}{RC}\right) v_i(t) \ ; \ \text{or,}$$

$$\left(D + \frac{2}{RC}\right) y(t) = \left(D + \frac{1}{RC}\right) f(t) \ .$$

Here, $Q(D)$ is $D + \frac{2}{RC}$ and $P(D)$ is $D + \frac{1}{RC}$.

In the case of the mass-spring-damper, the equation can be written

$$\left(D^2 + \frac{b}{m}D + \frac{k}{m}\right) y(t) = \frac{1}{m}f(t) \ .$$

Here, $Q(D) = \left(D^2 + \frac{b}{m}D + \frac{k}{m}\right)$ and $P(D)$ is the constant polynomial $\frac{1}{m}$. □

In the classical approach to differential equations, the typical procedure is to solve first for $\tilde{y}(t)$ in the modified equation

$$Q(D)\tilde{y}(t) = f(t) \ .$$

Given such a $\tilde{y}(t)$, we have

$$\begin{aligned} Q(D)P(D)\ \tilde{y}(t) &= P(D)Q(D)\ \tilde{y}(t) \\ &= P(D)f(t)\ , \end{aligned}$$

so $y(t) = P(D)\tilde{y}(t)$ solves the original equation. So $P(D)$ plays a relatively insignificant role in the classical approach; it's taken out of the picture at the outset, and only reinserted at the end.

On the other hand, in engineering problems, $f(t)$ may not be given; rather, the aim may be to determine in some sense whether there exists some $f(t)$ that gives rise to a response $y(t)$ of some desired form, and if so, how to find and generate such a forcing term. For such questions, the form of $P(D)$ may be critical.[3] So, the classical approach to differential equations has very significant blind spots, from an engineering perspective. We'll therefore follow an alternative method, based on the *Laplace transform*. That method will lead naturally into the "signals-and-systems" part of the course, where we won't just study the response $y(t)$ to a specific $f(t)$, but the general relationship between input and output "signals."

To motivate the Laplace transform, recall the defining characteristic of the exponential function – namely, that it satisfies our very first example of a differential equation:

$$\frac{d}{dt}\ e^{at} = a\ e^{at}\ .$$

The above equation suggests that solving differential equations would be easy if we only had to consider exponential functions. The differentiation of e^{at} reduces to simple multiplication (by a): the left side of the equation features calculus; the right side mere algebra. The Laplace transform exploits this property by decomposing a broad class of functions into

[3]This will be seen, in particular, in SE380, where you will 'design' forcing functions $f(t)$, especially using feedback, with a view to achieving a desirable $y(t)$.

linear combinations of exponentials e^{st} (generally, uncountably infinite linear combinations), where $s \in \mathbb{C}$.

Because our differential equations are linear, and their coefficients constant, use of the Laplace transform will effectively reduce the solution of differential equations to that of linear algebraic equations, in the variable s. Moreover, it will reduce the solution of systems of coupled differential equations to that of systems of coupled linear algebraic equations.

Our use of the Laplace transform will be (partially) justified by the following

Fact: If $m \leq n$ and $f(t)$ is continuous on an interval $a \leq t \leq b$ then there exists a solution $y(t)$ satisfying the above differential equation with constant coefficients and also the initial conditions for $a \leq t_0 \leq b$:

$$y(t_0) = p_0, \dot{y}(t_0) = p_1 \ \ldots y^(n-1)(t_0) = p_{n-1}$$

where $p_0, p_1, \ \ldots \ , p_{n-1} \in \mathbb{C}$ are constants. Moreover, this solution is unique. ∎

Following standard practice, we will in fact apply the Laplace transform in the case where $f(t)$ is piecewise continuous, and even in cases of "generalized functions" or "distributions," examples of which will be seen shortly. Existence and uniqueness will still apply.

Piecewise-continuous functions: A function is *piecewise-continuous*, in a given interval, if it has only a finite number of discontinuities in that interval, and at each such discontinuity, both its left- and its right-hand limits exist.

That is, for any point of discontinuity $t_0 \in \mathbb{R}$, $\lim_{t \uparrow t_0} f(t)$ and $\lim_{t \downarrow t_0} f(t)$ both exist.

If $f(t)$ is piecewise-continuous in every interval of finite length, we shall simply call it *piecewise-continuous*. □

The Laplace transform

The Laplace transform (Pierre-Simon Laplace, 1749 - 1827) can be interpreted as a means of representing a function as a weighted sum of exponentials. The transform is itself the weighting function:

$$F(s) := \mathcal{L}\{f(t)\} := \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt$$

- the transform is defined only if the above integral converges.

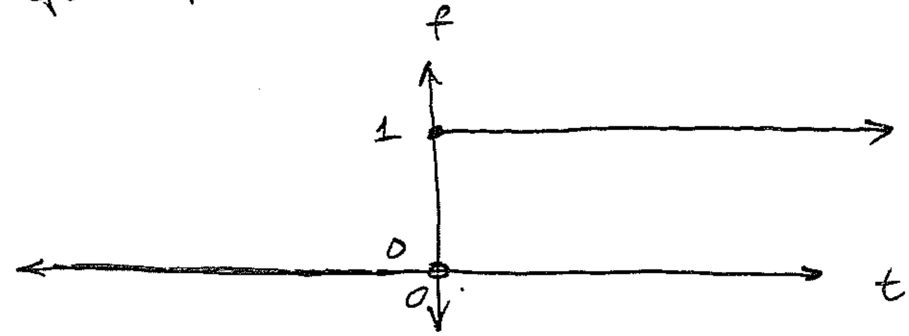
To ensure convergence, we shall suppose that, for some real α, the integral

$$\int_{-\infty}^{\infty} |f(t)|\, e^{-\alpha t}\, dt$$

converges.

Example:

Suppose $f(t)$ is the unit step function:



Then its transform is

$$F(s) := \mathcal{L}\{f(t)\} := \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt$$

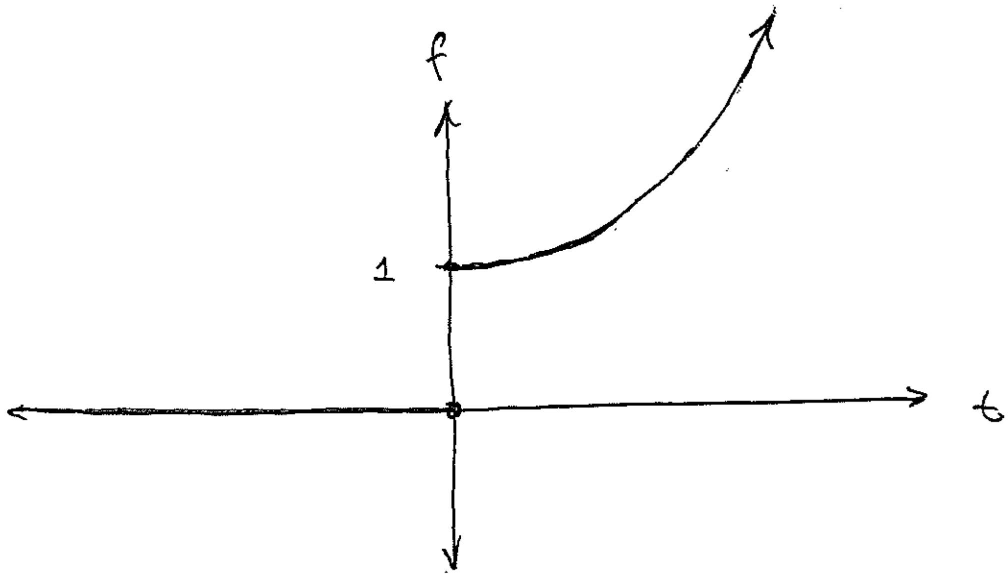
$$= \int_0^{\infty} e^{-st}\, dt$$

$$= \frac{1}{-s} e^{-st} \Big|_0^{\infty}$$

$$= \begin{cases} \frac{1}{s}, & \text{if } \mathrm{Re}(s) > 0 \\ \infty, & \text{otherwise} \end{cases}$$

Example:

$$f(t) = \begin{cases} e^{\alpha t}, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$



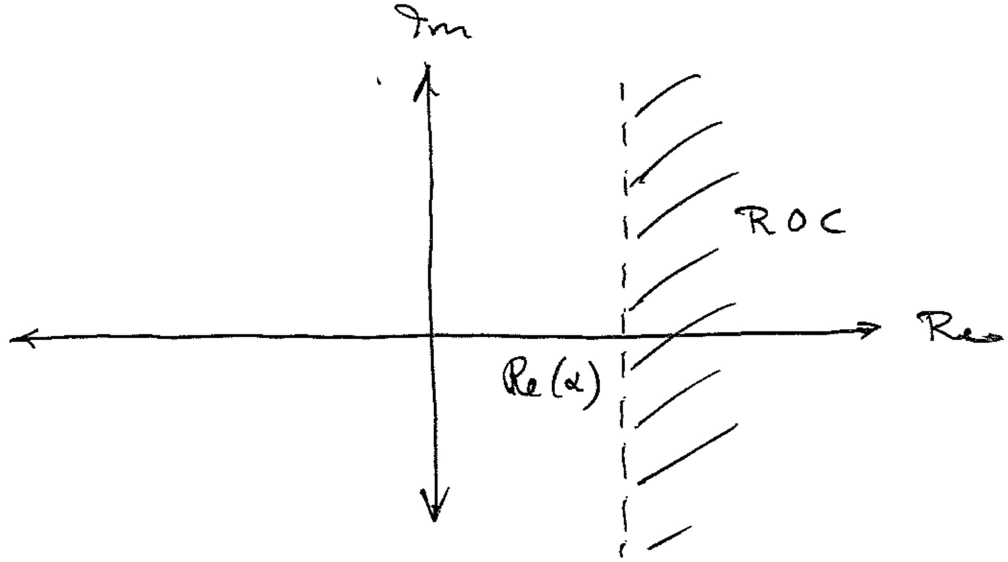
$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-st} \, dt = \int_0^{\infty} e^{\alpha t} e^{-st} \, dt$$

$$= \int_0^{\infty} e^{-(s-\alpha)t} \, dt$$

$$= \frac{-1}{s-\alpha} e^{-(s-\alpha)t} \Big|_0^{\infty}$$

Hence,

$$F(s) = \frac{1}{s-\alpha}, \text{ provided } \mathrm{Re}(s) > \mathrm{Re}(\alpha)$$



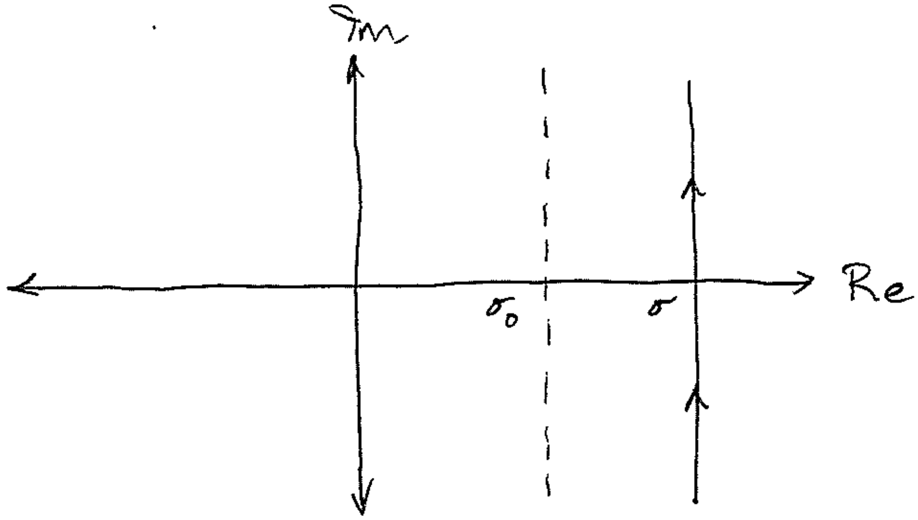
- Both of these examples illustrate how the factor e^{-st} can make the integral converge (in some cases) for sufficiently large Re(s).

The transform can be inverted by means of the following <u>inversion integral</u>:

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s)\, e^{st}\, ds$$

- This is a "contour integral" on the complex plane...

... the contour of integration, the straight line $\mathrm{Re}(s) = \sigma$, must lie within the ROC of $F(s)$:



- As a practical matter, we won't compute this integral ...

... instead, we'll apply table look-up — after a suitable partial-fractions decomposition.

- But, given that an integral is just a sum, the inversion formula shows that $f(t)$ is a sum of exponentials e^{st}, weighted by $F(s)$.

For this reason, it will greatly simplify the solution of linear ODES with constant coefficients.

Example:

Suppose $F(s) = \dfrac{1}{s(s+10)}$

(for $\mathrm{Re}(s) > 10$).

- partial fractions:

$$\frac{1}{s(s+10)} = \frac{1/10}{s} + \frac{-\frac{1}{10}}{s+10}$$

- therefore, the inverse transform of $F(s)$ is

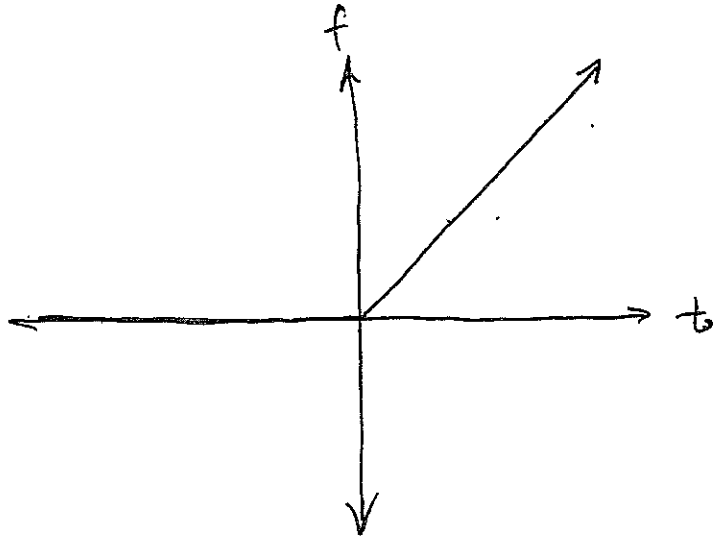
$$\mathcal{L}^{-1}\left\{\frac{1}{s(s+10)}\right\} = \mathcal{L}^{-1}\left\{\frac{1/10}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{-1/10}{s+10}\right\}$$

(by linearity of $\mathcal{L}\{\cdot\}$)

$$= \begin{cases} \frac{1}{10}\left[1 - e^{-10t}\right], & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Example:

Suppose $f(t) = \begin{cases} t, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$



Then

$$F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt = \int_{0}^{\infty} t\, e^{-st}\, dt$$

- Use integration by parts, with

$$u = t \quad , \quad v = \left(\frac{-1}{s}\right) e^{-st}$$

Thus,

$$\int_0^\infty t\, e^{-st}\, dt = \left(\frac{-1}{s}\right) t\, e^{-st}\Big|_0^\infty - \int_0^\infty \left(\frac{-1}{s}\right) e^{-st}\, dt$$

$$= 0 + \frac{1}{s^2}, \text{ provided } \mathrm{Re}(s) > 0$$

So $F(s) = \dfrac{1}{s^2}$.

We will mainly be interested in "one-sided" functions ...

... that is, in functions $f(t)$ that have the value 0 for $t < 0$, ..

We'll therefore mainly use the 'one-sided' Laplace transform:

$$F(s) := \mathcal{L}\{f(t)\} := \int_{0^-}^{\infty} f(t)\, e^{-st}\, dt$$

<u>Notation</u>: Let the unit-step function of our first example be denoted $u_{-1}(t)$ (because its transform is s^{-1}).

- This will facilitate the definition of functions that have the value 0 when $t < 0$:

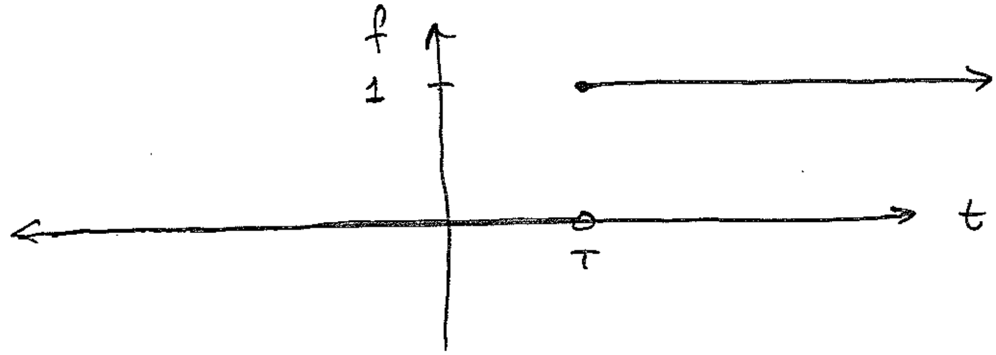
- Ex.

$$\text{Let} \quad f(t) = \begin{cases} 0, & t < 0 \\ t, & \text{otherwise.} \end{cases}$$

$$\text{Then} \quad f(t) = t\,u_{-1}(t), \quad \forall t$$

Example

$$f(t) = u_{-1}(t-T)$$



$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st}\,dt = \int_{T}^{\infty} e^{-st}\,dt$$

$$= \left.\frac{-e^{-st}}{s}\right|_{T}^{\infty}$$

$$= e^{-sT}\frac{1}{s},$$

provided $\mathrm{Re}(s) > 0$

Example:

$$f(t) = (\sin \omega t)\, u_{-1}(t)$$

$$= \frac{e^{j\omega t} - e^{-j\omega t}}{2j} \cdot u_{-1}(t)$$

$$= \frac{1}{2j} e^{j\omega t} u_{-1}(t) - \frac{1}{2j} e^{-j\omega t} u_{-1}(t)$$

So

$$F(s) = \frac{1}{2j} \frac{1}{s - j\omega} - \frac{1}{2j} \frac{1}{s + j\omega}$$

$$(\mathrm{Re}(s) > 0)$$

$$= \frac{1}{2j} \frac{2j\omega}{s^2 + \omega^2}$$

$$= \frac{\omega}{s^2 + \omega^2}$$

Key properties:

1. Linearity

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$$

(the ROC is the intersection of those of $F(s)$ and $G(s)$).

$$\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha f(t) + \beta g(t)$$

2. Time-scaling: for $c > 0$,

$$\mathcal{L}\{f(ct)\} = \int_{\infty}^{\infty} f(ct)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} f(\tau)\, e^{-s\frac{\tau}{c}}\, \frac{1}{c}\, d\tau$$

$$= \frac{1}{c} F\left(\frac{s}{c}\right)$$

Example:

By a previous example,

$$\mathcal{L}\{(\sin t)\, u_{-1}(t)\} = \frac{1}{s^2+1} .$$

Hence, for $\omega > 0$,

$$\mathcal{L}\{(\sin \omega t)\, u_{-1}(t)\} = \frac{1}{\omega} \frac{1}{\left(\frac{s}{\omega}\right)^2 + 1}$$

$$= \frac{1}{\omega} \frac{\omega^2}{s^2+1}$$

$$= \frac{\omega}{s^2+1} ,$$

as we have already calculated.

3. <u>Exponential modulation</u>

$$\mathcal{L}\{e^{\alpha t} f(t)\} = \int_{-\infty}^{\infty} f(t)\, e^{-(s-\alpha)t}\, dt$$

$$= F(s-\alpha)$$

Example:

$$\mathcal{L}\{e^{\alpha t} u_{-1}(t)\} = \frac{1}{s-\alpha}$$

(as we have already calculated).

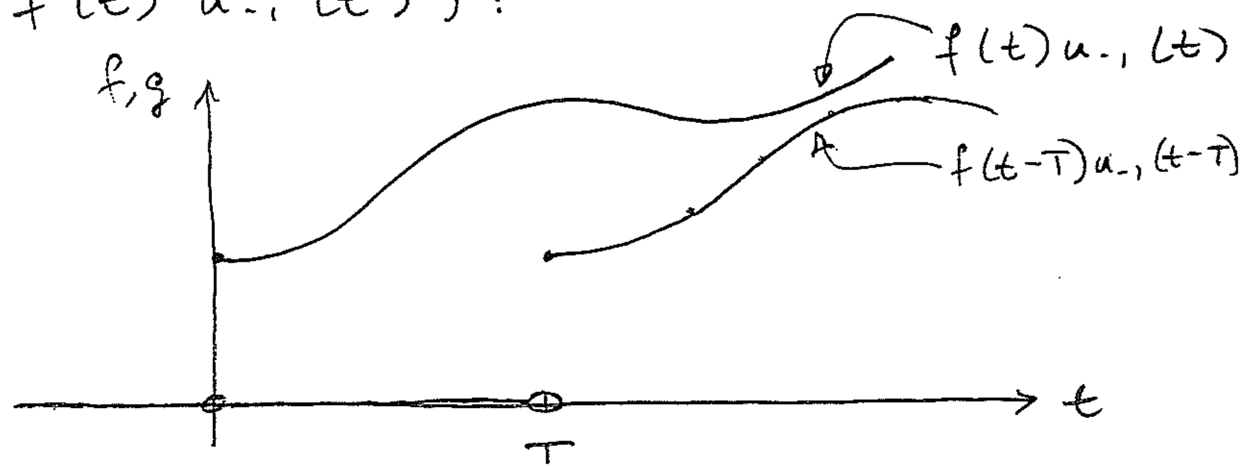
4. Time - shifting

Suppose $F(s) = \mathcal{L}\{f(t)\, u_{-1}(t)\}$

(that is, $F(s)$ is the one-sided transform of $f(t)$).

Let $g(t) = f(t-T) \cdot u_{-1}(t-T)$

(that is, a 'delayed' version of $f(t)\, u_{-1}(t)$):



Then

$$G(s) = \int_{-\infty}^{\infty} g(t) e^{-st}\, dt = \int_{-\infty}^{\infty} f(t-T) u_{-1}(t-T) e^{-st}\, dt$$

$$= \int_{T}^{\infty} f(t-T) e^{-st}\, dt$$

Then

$$G(s) = \int_{-\infty}^{\infty} g(t) \cdot e^{-st} \, dt$$

$$= \int_{-\infty}^{\infty} f(t-T) \, u_{-1}(t-T) \cdot e^{-st} \, dt$$

$$= \int_{T^-}^{\infty} f(t-T) \, e^{-st} \, dt$$

$$= \int_{0^-}^{\infty} f(\tau) \, e^{-s(\tau+T)} \, d\tau$$

$$= e^{-sT} \int_{0^-}^{\infty} f(\tau) \, e^{-s\tau} \, d\tau$$

$$= e^{-sT} \, F(s)$$

5. <u>Multiplication by t</u>:

$$\mathcal{L}\{t \cdot f(t)\} = \int_{-\infty}^{\infty} t \cdot f(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} t\, e^{-st} f(t)\, dt$$

$$= \int_{-\infty}^{\infty} -\frac{\partial}{\partial s} e^{-st} f(t)\, dt$$

$$= -\frac{d}{ds} \int_{-\infty}^{\infty} e^{-st} f(t)\, dt$$

(this can be shown to follow from absolute convergence)

$$= -\frac{d}{ds} F(s)$$

Example:

We've already seen that

$$\mathcal{L}\{t \cdot u_{-1}(t)\} = \frac{1}{s^2}$$

$$= -\frac{d}{ds}\frac{1}{s}$$

$$= -\frac{d}{ds}\mathcal{L}\{u_{-1}(t)\}$$

It follows from the above property that

$$\mathcal{L}\{t^{n-1} \cdot u_{-1}(t)\} = (-1)^{n-1}\frac{d^{n-1}}{ds^{n-1}}\frac{1}{s}$$

$$= (-1)^{n-1}(-1)^{n-1}\frac{(n-1)!}{s^n}$$

$$= \frac{(n-1)!}{s^n}$$

6. Differentiation / Integration

Suppose that there exists a real α such that

$$\int_{0^-}^{\infty} |f(t)| e^{-\alpha t}\, dt$$

converges, and that there exists a function $f'(t)$ such that, for $t \geq 0$,

$$f(t) = f(0^-) + \int_{0^-}^{t} f'(\tau)\, d\tau$$

and there exists a real β such that

$$\int_{0^-}^{\infty} |f'(t)| e^{-\beta t}\, dt$$

converges. Then both $f(\cdot)$ and $f'(\cdot)$ must have Laplace transforms.
(One-sided Laplace transforms, at least.)

$$\int_{0^-}^{\infty} f(t)e^{-st}\,dt = \int_{0^-}^{\infty}\left[f(0^-) + \int_{0^-}^{t} f'(\tau)\,d\tau\right]e^{-st}\,dt$$

$$= \frac{1}{s}f(0^-) + \int_{0^-}^{\infty}\int_{0^-}^{t} f'(\tau)\,d\tau\, e^{-st}\,dt$$

Integrating by parts,

$$\int_{0^-}^{\infty}\int_{0^-}^{t} f'(\tau)\,d\tau\, e^{-st}\,dt$$

$$= \left(\frac{-1}{s}\right)\int_{0^-}^{t} f'(\tau)\,d\tau\, e^{-st}\Bigg|_{0^-}^{\infty} + \frac{1}{s}\int_{0^-}^{\infty} f'(t)e^{-st}\,dt$$

The second term on the right is $\frac{1}{s}\mathcal{L}\{f'(t)\}$; the first term is zero . . .

To see why, recall that, for some real β — and hence, for some real $\beta > 0$ — the integral

$$\int_{0^-}^{\infty} |f'(t)| e^{-\beta t} \, dt$$

converges. Now,

$$\left| \int_{0^-}^{t} f'(\tau) \, d\tau \right| = \left| \int_{0^-}^{t} f'(\tau) e^{\beta\tau} e^{-\beta\tau} \, d\tau \right|$$

$$\leq e^{\beta t} \left| \int_{0^-}^{t} f'(\tau) e^{-\beta\tau} \, d\tau \right|$$

$$\leq e^{\beta t} \int_{0^-}^{\infty} |f'(\tau)| e^{-\beta\tau} \, d\tau$$

Hence,

$$\lim_{t \to \infty} \int_{0^-}^{t} f'(\tau) \, d\tau \cdot e^{-st} = 0,$$

provided $\operatorname{Re}(s) > \beta$.

Going back to our first equation, we therefore have, for $\text{Re}(s) > \beta$,

$$F(s) = \frac{1}{s} f(0^-) + \frac{1}{s} \mathcal{L}\{f'(t)\},$$

or

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0^-)$$

This property gives us
a powerful tool for solving
differential equations —
by converting them into
algebraic equations.

Example:

$$\dot{y} + y = t + e^{t}, \quad y(0^-) = 1$$

$$\Longleftrightarrow \quad sY(s) - y(0^-) + Y(s) = \frac{1}{s^2} + \frac{1}{s-1}$$

$$\Longleftrightarrow \quad (s+1)\,Y(s) = 1 + \frac{1}{s^2} + \frac{1}{s-1}$$

$$\Longleftrightarrow \quad Y(s) = \frac{1}{s+1} + \frac{1}{s^2(s+1)} + \frac{1}{(s+1)(s-1)}$$

$$= \frac{3/2}{s+1} + \frac{1}{s^2} - \frac{1}{s} + \frac{1/2}{s-1}$$

(by partial fractions)

$$\Longleftrightarrow \quad y(t) = \frac{3}{2}e^{-t} + t - 1 + \frac{1}{2}e^{t}$$

Owing to the previous property, we'll be solving ODEs by doing algebra in the Laplace domain.

Algebra is based on the operations of addition and multiplication. By linearity, we know that addition in the Laplace domain corresponds to addition in the time domain. What about multiplication?

It turns out that the counterpart of multiplication is an operation called <u>convolution</u>. Naturally, convolution therefore bears an important relationship to ODEs.

Given two functions $f(t)$ and $g(t)$, their <u>convolution</u> is the function

$$(f*g)(t) = \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau$$

Consider integrating instead with respect to $u = t - \tau$ $\iff$ $\tau = t - u$. We have

$$(f*g)(t) = \int_{+\infty}^{-\infty} f(t-u)\, g(u)\, (-du)$$

$$= \int_{-\infty}^{\infty} g(u)\, f(t-u)\, du$$

$$= (g*f)(t)$$

So convolution is commutative.

When we compute $f * g = g * f$, we say that we are <u>convolving</u> the functions.

<u>If the functions being convolved are one-sided</u>, we can simplify the integral:

$$(f*g)(t) = \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau$$

$$= \int_{0}^{\infty} f(\tau)\, g(t-\tau)\, d\tau \quad (\because f \text{ is 1-sided})$$

$$= \int_{0}^{t} f(\tau)\, g(t-\tau)\, d\tau \quad (\because g \text{ is 1-sided})$$

In this case, the convolution is also a one-sided function.

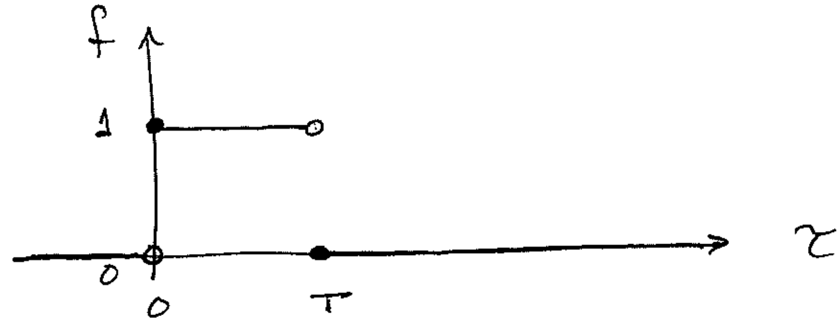
Example:

Suppose $f(t) = \begin{cases} 1, & 0 \le t < T \\ 0, & \text{otherwise} \end{cases}$

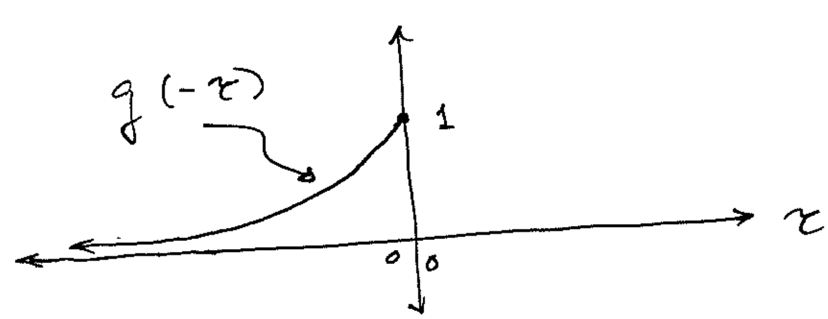
and $g(t) = e^{-t} u_{-1}(t)$. Then

$$f * g\,(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

It's easy to see that $f(\tau)$ has this graph:

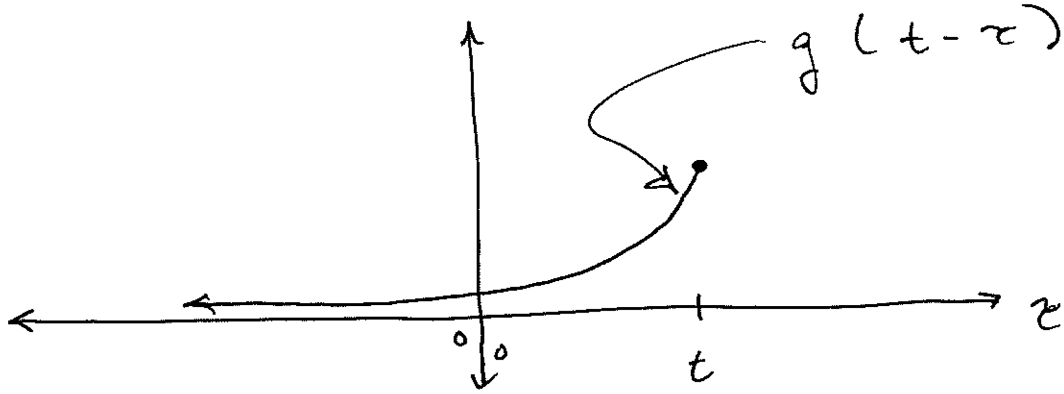


To plot $g(t-\tau)$, consider first $g(-\tau)$:

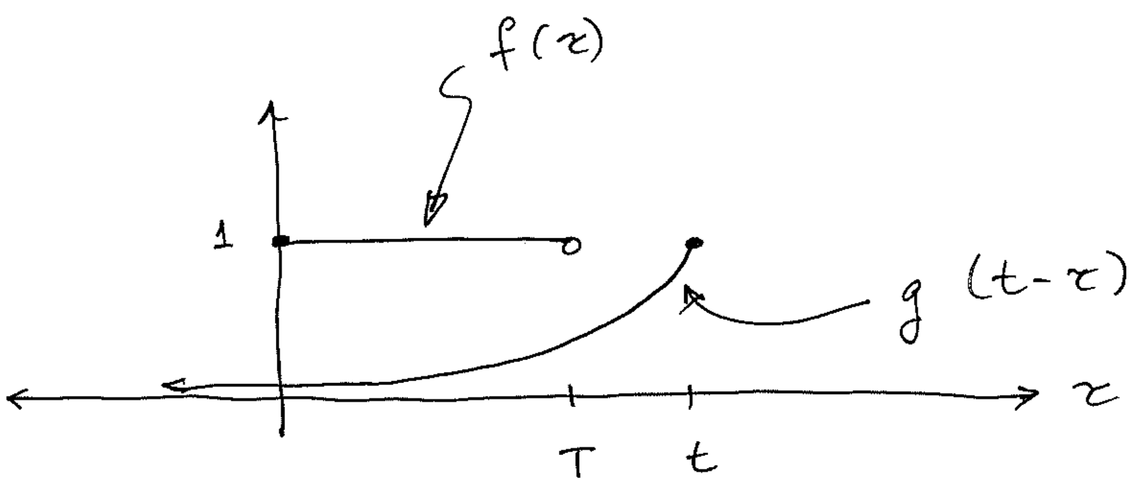


So $g(t-\tau) = g(-(\tau - t))$

looks the same, but shifted right by an interval t:



To find the convolution, multiply the two functions and integrate from 0 to t:



It can be seen from the picture that, if $0 \le t < T$, the value of the convolution is the area under the curve $e^{-\tau}$ from $\tau = 0$ to $\tau = t$.

That is, for $0 \le t < T$,

$$(f * g)(t) = \int_0^t e^{-\tau}\, d\tau$$

$$= -e^{-\tau}\Big|_0^t$$

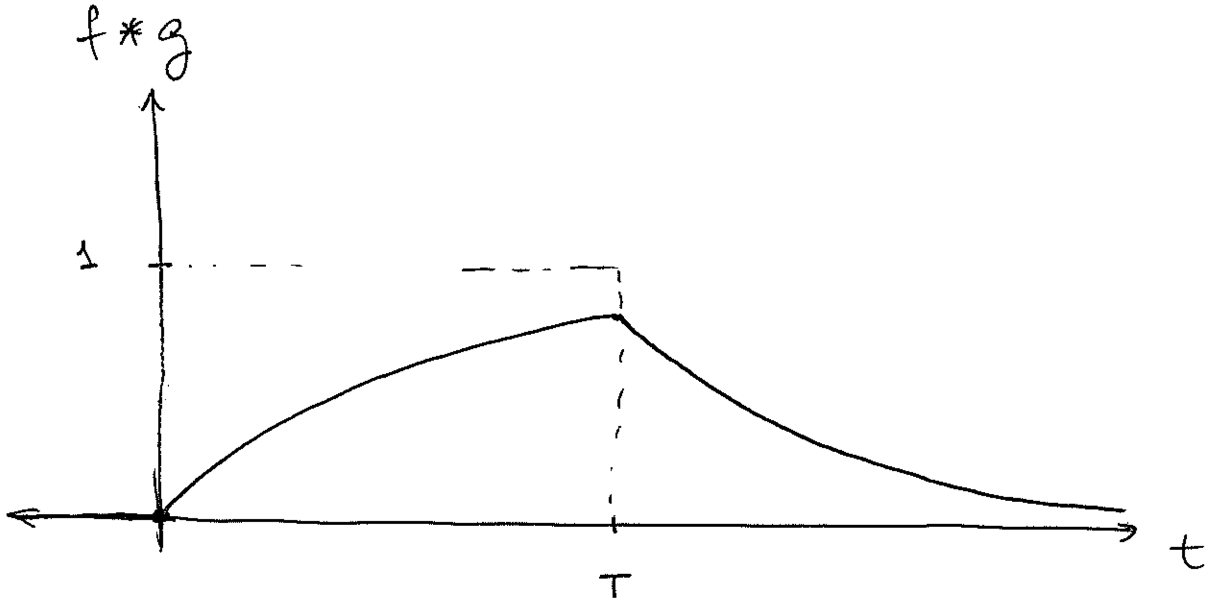
$$= 1 - e^{-t}$$

For $t \geq T$, $(f * g)(t)$ is the area under $e^{-\tau}$ between $\tau = t - T$ and $\tau = t$:

$$(f * g)(t) = \int_{t-T}^{t} e^{-\tau}\, d\tau$$

$$= -e^{-\tau}\Big|_{t-T}^{t} = e^{T-t} - e^{-t}$$

So the plot of $f*g\,(t)$ looks like this:



We'll discuss convolution further a little later. For now, let's continue establishing the properties of the Laplace transform.

7. Convolution

Suppose that, for some real α and β, the integrals

$$\int_{-\infty}^{\infty} |f(t)| e^{-\alpha t}\, dt \quad \& \quad \int_{-\infty}^{\infty} |g(t)| e^{-\beta t}\, dt$$

converge.

We'll show that this means that $(f*g)(t)$ has a Laplace transform, and that

$$\mathcal{L}\{(f*g)(t)\} = F(s)\,G(s)$$

Suppose that $\gamma \geq \alpha, \beta$, and consider the product of the two convergent integrals:

$$\int_{-\infty}^{\infty} |f(t)| e^{-\gamma t} \, dt \int_{-\infty}^{\infty} |g(\tau)| e^{-\gamma \tau} \, d\tau$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(t)| e^{-\gamma t} |g(\tau)| e^{-\gamma \tau} \, dt \, d\tau$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(u-\tau)| e^{-\gamma(u-\tau)} |g(\tau)| e^{-\gamma \tau} \, du \, d\tau$$

$(u = t + \tau)$.

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(u-\tau)| e^{-\gamma(u-\tau)} |g(\tau)| e^{-\gamma \tau} \, d\tau \, du$$

(by "Fubini's Theorem")

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(u-\tau)| |g(\tau)| \, d\tau \, e^{-\gamma u} \, du$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(u-\tau)\, g(\tau)| \, d\tau \; e^{-\gamma u} \, du$$

$$\geq \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} f(u-\tau)\, g(\tau) \, d\tau \right| e^{-\gamma u} \, du$$

$$= \int_{-\infty}^{\infty} |(f * g)(u)| \, e^{-\gamma u} \, du$$

So the last integral converges, and $f * g$ has a Laplace transform.

Now,

$$\mathcal{L}\{(f*g)(t)\} = \int_{-\infty}^{\infty} (f*g)(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t-\tau)\, g(\tau)\, d\tau\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t-\tau)\, g(\tau)\, e^{-st}\, dt\, d\tau$$

(by the previous proof & "Fubini's Th$^{\underline{m}}$")

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u)\, g(\tau)\, e^{-s(u+\tau)}\, du\, d\tau$$

$$= \int_{-\infty}^{\infty} f(u)\, e^{-su}\, du \int_{-\infty}^{\infty} g(\tau)\, e^{-s\tau}\, d\tau$$

$$= F(s)\, G(s)$$

If $f(\cdot)$ is piecewise-continuous and $\int_{0^-}^{\infty} |f(t)|\, e^{-\alpha t}\, dt$ converges for some real α, then

What does this limit mean?

- think of the real part of s as tending to $+\infty$.

- There's a different way of defining the "limit at infinity" of a function of a complex variable, but it doesn't apply, for example, to complex exponentials.

- Depending on the direction in which s "goes to infinity" on the complex plane, e^{-st} may not have a limit.

- It's said to have an "essential singularity" at ∞.

Proof of the initial-value theorem:

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} s\int_{0^-}^{\infty} f(t)\, e^{-st}\, dt.$$

Let

$$s\int_{0^-}^{\infty} f(t)e^{-st}dt = s\int_{0^-}^{\epsilon} f(t)e^{-st}dt + s\int_{\epsilon}^{\infty} f(t)e^{-st}dt,$$

for some $\epsilon > 0$.

Then

$$s\int_{\cancel{\infty}\,\epsilon}^{\infty} f(t)e^{-st}dt \leq s\int_{\epsilon}^{\infty} |f(t)|e^{-\alpha t}\left|e^{-(s-\alpha)t}\right| dt$$

$$\leq s\left|e^{-(s-\alpha)\epsilon}\right| \int_{\epsilon}^{\infty} |f(t)|e^{-\alpha t} dt$$

(when $\mathrm{Re}(s)$ is sufficiently large, because $\epsilon > 0$).

- The above term vanishes when we take the limit as $\mathrm{Re}(s) \to \infty$ (by convergence of the integral).
- Therefore, $\lim_{s\to\infty} s\int_{0^-}^{\epsilon} f(t)e^{-st}dt$ must be independent of ϵ.
- As ϵ approaches 0, $f(t)$ behaves like $f(0^+)$ over the interval $[0, \epsilon]$, so

$$s\int_{0^-}^{\epsilon} f(t)e^{-st}dt \text{ behaves like } \cancel{s}\, f(0^+)\,\frac{1-e^{-s\epsilon}}{\cancel{s}}.$$

- Taking the limit as $\mathrm{Re}(s) \to \infty$, we get $f(0^+)$.

$\square$

Examples:

① $F(s) = e^{-sT}\frac{1}{s^2}$, $T > 0$

$\lim_{s\to\infty} sF(s) = 0$

CHECK: $f(t) = \mathcal{L}^{-1}\{F(s)\} = (t-T)\,u_{-1}(t-T)$

$f(0^+) = \lim_{t\downarrow 0} f(t) = 0$ ✓

② $F(s) = \frac{1}{s}$

$\lim_{s\to\infty} sF(s) = 1$

CHECK: $f(t) = u_{-1}(t)$, so $f(0^+) = 1$ ✓

③ $F(s) = \frac{s}{s^2+\omega^2}$

$\lim_{s\to\infty} sF(s) = 1$

CHECK: $f(t) = \cos\omega t$,

$f(0^+) = \lim_{t\downarrow 0} \cos\omega t = 1$ ✓

Rational functions

Many of the Laplace transforms that we've seen so far take the form of <u>rational functions</u> — that is, functions represented as ratios of polynomials:

- e.g. $F(s) \; \dfrac{s}{s^2 + 2s + 2}$

Just as with rational numbers, common factors in the numerator and denominator cancel out:

$\dfrac{s \quad (s+3)}{(s^2 + 2s + 2)(s+3)}$ is considered equivalent to $\dfrac{s}{s^2 + 2s + 2}$ —

just as $\frac{2}{4}$ is equivalent to $\frac{1}{2}$.

Moreover, just as the rational numbers extend the integers to a <u>field</u>, so the rational functions extend the polynomials to a field (by ensuring that every element has a multiplicative inverse).

Indeed, all of the transforms that we have seen consist of rational functions in s, possibly multiplied by exponentials in s.

The roots of the numerator of a rational function are called the function's (finite) <u>zeros</u>; the roots of the denominator are called its (finite) <u>poles</u>.

The function $\dfrac{s}{s^2+2s+2}$ has one finite zero at $s=0$, and two finite poles at $s=-1 \pm j$.

This function is also said to have a <u>zero at infinity</u>, because, in the theory of complex analysis, it tends to zero as s tends to infinity. The reciprocal of the function is said to have a pole at infinity.

If we don't specify which type of pole or zero we're speaking of, assume we're referring to finite ones.

A rational function is <u>proper</u> if the degree of its numerator is less than or equal to that of its denominator; it is <u>strictly proper</u> if the degree of the numerator is strictly less than that of the denominator.

- ex.

$$\frac{s}{s^2 + 2s + 2}$$

is not only proper but strictly proper.

- Strictly proper functions have zeros at infinity.

Exercise

Write your own proof of the initial-value theorem, for the special case where $F(s)$ is a proper rational function.

If $F(s)$ is a proper rational function — possibly multiplied by a complex exponential in s — then we can prove a "final-value theorem" ...

9. The final-value theorem

Let $F(s)$ be a proper rational function, all of whose poles have real parts that are strictly negative, with the possible exception of a single pole at $s=0$. (alternatively, $F(s)$ may consist of the product of such a rational function with a complex exponential e^{sT}) Then

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Moreover, if the poles of the rational function do not satisfy the above condition, then the limit

$$\lim_{t\to\infty} f(t)$$

does not exist.

When it applies, this result lets us calculate the "final value," $\lim_{t \to \infty} f(t)$ without inverting the transform $F(s)$.

What does the right-hand limit mean?

A function $G(z)$ of a complex variable z is said to have a limit $\lambda \in \mathbb{C}$ as z approaches $z_0 \in \mathbb{C}$ if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$| G(z) - \lambda | < \epsilon$$

whenever $| z - z_0 | < \delta$.

Example:

$$F(s) = \frac{s}{s^2 + \omega^2}$$

$$\lim_{s \to 0} sF(s) = 0$$

Check:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \cos\omega t$$

$$\lim_{t \to \infty} f(t) = \lim_{t \to \infty} \cos\omega t$$

– which doesn't exist

But the conditions of the FVT aren't satisfied — $F(s)$ has poles on the imaginary line.

Example: $$F(s) = \frac{10}{5s+1} \cdot \frac{1}{s}$$

(satisfies the conditions).

$$\lim_{s \to 0} sF(s) = \lim_{s \to 0} \cancel{s} \frac{10}{5s+1} \cdot \frac{1}{\cancel{s}} = 10$$

Proof of FVT:

First note that an easy case analysis shows that if the conditions on $F(s)$ are not satisfied, then $\lim_{t\to\infty} f(t)$ does not exist.

Now suppose that the conditions are satisfied. Then $F(s)$ can be decomposed into a sum of terms

$\frac{A}{s}$ and $\frac{B_{ik}}{(s-p_i)^k}$, where the p_i are poles of $F(s)$ that lie to the left of the imaginary axis.

It follows that $\lim_{t\to\infty} f(t) = A$.

But what is the value of A?

- By the "Heaviside cover-up,"

$$A = \lim_{s\to 0} sF(s)$$

$\square$

Summary of Laplace-Transform Properties

Property	Time domain	Laplace domain
1. linearity	$\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$
2. time-scaling	$f(ct)$	$\frac{1}{c}F(\frac{s}{c})$
3. exponential modulation	$e^{\alpha t}f(t)$	$F(s-\alpha)$
4. time-shifting	$f(t-T)u_{-1}(t-T)$	$e^{-sT}F(s)$
5. t-multiplication	$tf(t)$	$-\frac{\mathrm{d}}{\mathrm{d}s}F(s)$
6. differentiation/integration	$f'(t)$	$sF(s) - f(0^-)$
7. convolution	$(f * g)(t)$	$F(s)G(s)$
8. initial-value theorem	$f(0^+)$	$\lim_{s\to\infty} sF(s)$ *
9. final-value theorem	$\lim_{t\to\infty} f(t)$	$\lim_{s\to 0} sF(s)$ **

* where the real part of s goes to infinity.

** provided that all poles of F(s) have negative real parts, with the possible exception of a single pole at the origin.

Linear ODEs with constant coefficients: conclusion

The Laplace-transform method extends straightforwardly to systems of coupled linear ODEs with constant coefficients. So, in our standard equation, the dependent variables can be vector-valued and the coefficients can be matrices:

$$\begin{aligned}\frac{d^n}{dt^n}y(t) + a_{n-1}\frac{d^{n-1}}{dt^{n-1}}y(t) + \ &\ldots\ + a_0 y(t) \\ &= b_m \frac{d^m}{dt^m}f(t) + b_{m-1}\frac{d^{m-1}}{dt^{m-1}}f(t) + \ \ldots\ + b_0 f(t)\ .\end{aligned}$$

Provided that $m \le n$ and, as shown above, the leading coefficient on the left-hand side is the identity matrix, there is no problem with existence or uniqueness of solutions. Therefore, solution amounts to transforming the system to one of coupled algebraic equations, solving, and transforming back to the time domain.

The methods we've been using are originally due to Oliver Heaviside, and comprised his "operational calculus," originally proposed without rigorous theoretical foundations. The Laplace transform furnishes suitable foundations, but notice that some of the details of the framework didn't really enter into our solution methods. In particular, as long as all of the relevant unilateral transforms exist, we don't need to worry about their exact regions of convergence; as long as the real part of s is sufficiently large, everything will converge. Likewise, as long as the real part of s is sufficiently large, the inverses of any polynomials (including matrix polynomials) will exist.

All of the Laplace-transform properties are potentially useful in solving differential equations, but the importance of property 6, on differentiation and integration, stands out. Up to this point, in order to ensure the existence of a suitable function $f'(t)$, we have restricted attention to the case where all but the highest-order derivatives of dependent variables are continuous, and the highest-order derivative is piecewise-continuous. Under those conditions, $f'(t)$ is just an ordinary derivative.

But Heaviside also introduced a "generalized function," commonly called a unit impulse, that plays the role of $f'(t)$ in the case where $f(t)$ is a unit step. We'll introduce the unit impulse, or Dirac delta function, in the next instalment of notes. It will play an important role in the conceptual and theoretical underpinnings of the "systems and signals" part of the course.

The systems-and-signals approach differs from typical differential-equations methods primarily in the sense that the "forcing term," which we'll typically call the "input" of a dynamical system, is not given. In an engineering context, we're often interested in finding some $f(t)$ that yields a response $y(t)$ with some particular desired properties: we wish to do some design, whereas the typical aim of solving differential equations is analysis.

Consequently, when we solve a differential equation using the Laplace transform, we'll find an algebraic equation relating the transform $Y(s)$ of the "output" to the transform $F(s)$ of the input signal. That exercise will give us some insight into the way a whole range of possible input signals relate to their corresponding outputs, helping us design inputs that give rise to desirable responses.

Example As we said earlier, a mass-spring-damper system, driven by an applied force $f(t)$, might be modelled by the equation

$$m\frac{d^2}{dt^2}y(t) + b\frac{d}{dt}y(t) + ky(t) = f(t)$$

where m is the mass, b is the coefficient of a friction that is proportional to speed, and k is a spring constant. Similar models are used to design real engineering systems, such as automotive suspension systems, and other vibration-damping mechanisms.

Applying the one-sided Laplace transform (and using a standard "dot" notation for time derivatives), we find

$$
\begin{aligned}
& ms\left[s[sY(s) - y(0^-)] - \dot{y}(0^-)\right] + b[sY(s) - y(0^-)] + kY(s) = F(s) \\
\Longleftrightarrow\ & [ms^2 + bs + k]Y(s) = F(s) + [ms + b]y(0^-) + m\dot{y}(0^-) \\
\Longleftrightarrow\ & Y(s) = \frac{1}{ms^2 + bs + k}F(s) + \frac{[ms + b]y(0^-) + m\dot{y}(0^-)}{ms^2 + bs + k}
\end{aligned}
$$

□

In the last line of the example, we have separated the right-hand side into a term that depends on the "input" $f(t)$, and another that depends on the initial conditions. The first is the transform of the so-called *zero-state response*, and the second that of the *zero-input response*. Here we're essentially applying the principle of superposition, to find the "total response" to the input and the initial conditions, by superimposing the responses for the cases where the initial conditions are zero, and where the input is zero-valued.

Both of these responses are important, but we tend to pay particular attention to the zero-state response, which captures the relationship between the input $f(t)$ and the output, or response, $y(t)$. To find the zero-state response, we effectively assume that all dependent variables, and derivatives thereof, are zero-valued for all negative t. In particular, we effectively assume that the input signal is multiplied by the unit step.

This is another, specific reason to come up with a function $f'(t)$ that, according to the formulation of our differentiation rule, can play the role of the "derivative" of the unit step. That will be the subject of the next section.

The unit-impulse, or Dirac delta function

Mechanical impulses

The term "impulse" originates in classical mechanics. Consider that, if p stands for the momentum of a particle, and f for the net force on that particle, then Newton's second law can be written

$$f = \dot{p} \ .$$

Solving this simple linear ODE with constant coefficients, we find, for $t \geq 0$,

$$\begin{aligned} p(t) &= \int_{0^-}^{t} f(\tau) d\tau + p(0^-) \\ \Longleftrightarrow p(t) - p(0^-) &= \int_{0^-}^{t} f(\tau) d\tau \ . \end{aligned}$$

In other words, the change in momentum over the interval from 0 to t is given by the integral of the applied force over that interval. The value of that integral is called the *impulse*.

Note that the exact shape of the function $f(t)$ is immaterial. So, in a sense, is the length of the interval from 0 to t. All that matters is the "area under the curve." Infinitely many different force profiles can give rise to the same impulse; and the same impulse can, in principle, be applied over any finite time interval. As long as the impulse is the same, the change in momentum will be the same.

The idea behind the unit-impulse function is to idealize the application of the impulse so that the length of that interval becomes vanishingly small. It models an infinitely "swift kick" to the system, giving rise to a change of momentum that is instantaneous – a step change.

A mathematical idealization

Intuitively, a unit impulse can be thought of as a pulse of arbitrary shape – rectangular, triangular, or whatever you wish – whose integral equals one. The "width" of this pulse is allowed to approach zero, subject to the constraint that the "area under the curve" remains equal to one. In other words, as the width goes to zero, the "height" must go to infinity, so as to respect the constraint on the integral. An impulse can therefore be thought of as an infinitely narrow, but infinitely high, pulse.

More formally, a unit impulse $\delta(t)$ is defined so that the "integral"

$$\int_a^b \delta(t) = 1 \ ,$$

whenever $a < 0$ and $b > 0$. It is defined solely in terms of its antiderivative, the unit step function:

$$\int_a^b \delta(t) = u_{-1}(b) - u_{-1}(a) \ .$$

More generally, its true defining property is what is commonly known as the *sifting property*. Let $f(t)$ be a "suitably well-behaved function." Then

$$\int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau = f(t) \ .$$

Consider that the impulse $\delta(t-\tau)$ can be considered to vanish whenever $t \neq \tau$. Therefore, the value of $f(\tau)$ for any τ different from t is irrelevant to the above integral. But if, over the vanishingly small "width" of the pulse $\delta(t-\tau)$, the value of $f(\tau)$ can be considered to be constant, and therefore equal to $f(t)$, then the value of the integral will be

$$\int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau = \int_{-\infty}^{\infty} f(t)\delta(t-\tau)d\tau = f(t)\int_{-\infty}^{\infty} \delta(t-\tau)d\tau = f(t) \ .$$

Thus, the effect of performing the above integral is to "pick off" the value of $f(\tau)$ at the instant $t = \tau$ when the unit impulse occurs. All other values of the function $f(\tau)$ are irrelevant; hence the term "sifting property."

The Laplace transform of $\delta(t)$

While the unit impulse is often referred to using the name of the theoretical physicist Paul Dirac, it appears to have been used first by Heaviside. In order to employ it within our more rigorous framework, we should compute its Laplace transform,

$$\begin{aligned}\mathscr{L}\{\delta(t)\} &= \int_{0^-}^{t} \delta(t)e^{-st}dt \\ &= e^{s0} \\ &= 1\ .\end{aligned}$$

Note that here we have simply applied the sifting property to the definition of the Laplace transform. So $\delta(t)$ is the "function" whose Laplace transform is unity.

Example In our mass-spring-damper example, we found the following relationship between the transform of the displacement (as a function of time), the transform of the force, and the initial conditions,

$$Y(s) = \frac{1}{ms^2+bs+k}F(s) + \frac{[ms+b]y(0^-) + m\dot{y}(0^-)}{ms^2+bs+k}\ .$$

The transform of the "zero-state response" of the system is therefore

$$Y(s) = \frac{1}{ms^2+bs+k}F(s)\ .$$

What is the significance of the quantity that multiplies $F(s)$? Consider the case where the applied force $f(t)$ is an "infinitely swift kick" – namely, a unit impulse, that delivers a finite impulse over a vanishingly small interval. Then

$$Y(s) = \frac{1}{ms^2+bs+k}F(s) = \frac{1}{ms^2+bs+k}\ .$$

The quantity in question is therefore the Laplace transform of the system's response to an input of the form of a unit impulse. It is the transform of the system's *impulse response*. □

Given the properties of the Laplace transform, what does this tell us about the system's response to an arbitrary (but Laplace-transformable) input $f(t)$? The answer to that question exemplifies what is probably the most fundamental concept in the theory of linear, "time-invariant" systems. It will be covered in the next chapter, as soon as "signals," and "systems," and some of their fundamental properties, have been defined.