Outline of Solutions

1. Divide the square into two triangles by drawing a line segment connecting P_0 and P_2. We are given that clv is between f_0 and f_2. First consider triangle $P_0P_2P_3$. There are 2 possible cases.

   Case (i): $f_3 < clv$

   If $clv < f_2$, then the contour cuts the edge P_2P_3, and it does not cut the edge P_0P_3 since both f_0 and f_3 are smaller than clv. If $clv > f_2$, then $f_3 < clv < f_0$ and hence the contour cuts the edge P_0P_3, and it does not cut the edge P_2P_3 since both f_2 and f_3 are smaller than clv.

   Case (ii): $f_3 > clv$

   Similar to case (i), if $clv > f_2$, then the contour cuts the edge P_2P_3, and it does not cut the edge P_0P_3 since both f_0 and f_3 are bigger than clv. If $clv < f_2$, then $f_0 < clv < f_3$ and hence the contour cuts the edge P_0P_3, and it does not cut the edge P_2P_3 since both f_2 and f_3 are bigger than clv.

   The same argument applies to triangle $P_0P_1P_2$. In conclusion, the contour cuts exactly two edges of the square.

2. (a)

```
t = 1:7;
x = [6.0 9.5 ... 1.0];
y = [1.0 2.5 ... 6.0];
repSx = csape(t,x,'variational');
repSy = csape(t,y,'variational');
```

   (b) Let $Sx(3.5) = a$. Then `a = ppval(repSx,3.5)`. Let $Sy(3.5) = b$. Then `b = ppval(repSy,3.5)`.

3. (a) $x_1 = 0$, and $Sp(0) = y_1$.

$$Sp(0) = 2b_1 \Rightarrow 2b_1 = y_1.$$

   (b) $x_2 = 2$. The values of $Sp(2)$ from the left and right are respectively:

$$\begin{aligned} Sp(2)_{\text{left}} &= a_1\frac{8}{12} + 2c_1 \\ Sp(2)_{\text{right}} &= a_1\frac{1}{6} + b_2. \end{aligned}$$

   $Sp(x)$ continuous at $x = 2$ implies

$$\begin{aligned} a_1\frac{2}{3} + 2c_1 &= a_1\frac{1}{6} + b_2 \\ a_1(\frac{2}{3} - \frac{1}{6}) + 2c_1 - b_2 &= 0. \end{aligned}$$

4. (Need not assume the base to be a certain number such as 10.) In floating point systems, the distance/spacing between floating numbers is the (absolute) floating point representation error, i.e.

$$\text{distance} \approx |fl(x) - x|.$$

On the other hand, the relative error is approximately the machine precision E:

$$\frac{|fl(x) - x|}{|x|} \approx E.$$

From $x = 300$ and spacing 10^{-4}, we can estimate that $E \approx 10^{-4}/300$. Now,

$$\begin{aligned} \text{distance between numbers near } 0.03 &= |0.03|E \approx 10^{-8}, \\ \text{distance between numbers near } 30000 &= |30000|E \approx 10^{-2}. \end{aligned}$$

5. (a) Let $A^{(i)}$ be the sparse matrix of size x_i. Define the right-hand side $b^{(i)}$ a vector of all 1's and length x_i. Generate y_i=flop counts of Gaussian elimination for solving $A^{(i)}z^{(i)} = b^{(i)}$ by:

   $$\begin{aligned} &\text{flops}(0); \\ &A^{(i)} \backslash b^{(i)}; \\ &y_i = \text{flops}; \end{aligned}$$

   Fit the function $f(x) = c_1 + c_2 x + c_3 x^{1.5}$ to the data (x_i, y_i), $i = 1, \ldots, m$. Find the best fit coefficients (c_1, c_2, c_3) by solving the normal equations in part (b). The leading coefficient of $flops(GE)$ is then given by c_3.

   (b) Let TSE be the total squared error function. The normal equations are obtained from setting:

   $$\frac{\partial TSE}{\partial c_i} = 0 \qquad i = 1, 2, 3.$$

   (c)

```
% Form normal equations
x = [x1; x2; ... xm];
y = [y1; y2; ... ym];
e = ones(m,1);
X = [e x x.^(1.5)];
A = X'*X;
b = X'*y;

% Solve normal equations
c = A\b;

% Evaluate f(xi) and plot (xi,f(xi))
f = X*c;
plot(x,f);
```

6. (a) $A = LU \rightarrow$ flops$=\frac{2}{3}n^3$.
Each forward + back solve $\rightarrow$ flops$=2n^2$.
Since there are n right-hand sides, there are n forward + back solves $\rightarrow$ flops$=2n^3$.
Total flops $= \frac{2}{3}n^3 + 2n^3 = \frac{8}{3}n^3$.

(b) $A^2 x = b \rightarrow LULUx = b$. We solve for x by performing an ordered sequence of forward and back solves:

$$\begin{aligned} Ly_1 &= b \\ Uy_2 &= y_1 \\ Ly_3 &= y_2 \\ Ux &= y_3 \end{aligned}$$

(c) $A = LU \rightarrow$ flops$=\frac{2}{3}n^3$.
2 forward solves $\rightarrow$ flops$=2n^2$.
2 back solves $\rightarrow$ flops$=2n^2$.
Total flops $= \frac{2}{3}n^3 + 4n^2 \approx \frac{2}{3}n^3$.