University of Waterloo
David R. Cheriton School of Computer Science

MATH 213 – Advanced Mathematics for Software Engineers
Final Exam, Spring 2008

August 16, 2008, 7:30-10:00 PM

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Student's name: ____

Student's ID #: ____

Instructions:

- This exam has **4** pages.
- **No books and lecture notes are allowed on the exam**. Please, turn off your cell phones, PDAs, etc., and place your bags, backpacks, books, and notes under the table or at the front of the room.
- Please, place your **WATCARD** on the table, and fill out the exam attendance sheet when provided by the proctor after the exam starts.
- Question marks are listed by the question.
- Please, do not separate the pages, and indicate your Student ID at the top of every page.
- Be neat. Poor presentation will be penalized.
- **No questions will be answered during the exam**. If there is an ambiguity, state your assumptions and proceed.
- **No student can leave the exam room in the first 45 minutes or the last 10 minutes**.
- If you finish before the end of the exam and wish to leave, remain seated and raise your hand. A proctor will pick up the exam from you, at which point you may leave.
- When the proctors announce the end of the exam, put down your pens/pencils, close your exam booklet, and remain seated in silence. The proctors will collect the exams, count them, and then announce you may leave.

Problem №1 (20%)

Let $\mathcal{S}$ be the vector space of all continuous functions defined over $[-1, 1]$, i.e., $\mathcal{S} = C([-1, 1])$. We convert $\mathcal{S}$ into a normed inner product space by endowing it with the standard inner (dot) product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)\, g(x)\, dx, \qquad \forall f, g \in \mathcal{S}.$$

and corresponding natural norm

$$\|f\| = \sqrt{\int_{-1}^{1} |f(x)|^2\, dx}.$$

Find the orthogonal projection of the constant function $f(x) = 1$ onto $\operatorname{span}\{1 - |x|,\ x^2\}$.

Problem №2 (20%)

If a steady electric current i flows through a resistor of resistance R, the power delivered is equal to $i^2 R$. In many applications i is not a constant, but a periodic function of the time t. In such cases one defines the *average power* as

$$\text{average power} = \frac{1}{T} \int_{-T/2}^{T/2} i^2(t)\, R\, dt,$$

where T is the fundamental period of $i(t)$. Expressing the latter as a Fourier series,

$$i(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2n\pi t}{T} + b_n \sin \frac{2n\pi t}{T} \right),$$

find an expression for the average power in terms of R, a_0, a_n, and b_n.

Problem №3 (20%)

Find a *particular* solution to the second-order differential equation,

$$x'' + 2\, x' + x = f(t),$$

where $f(x)$ is a 2π-periodic function (and hence expandable in a Fourier series) which is given over one period as

$$f(t) = t/\pi, \qquad t \in [-\pi, \pi). \tag{1}$$

Problem №4 (20%)

Expand the function

$$f(x) = \begin{cases} 2\,x, & 0 \le x < \pi/2 \\ 2\pi - 2\,x, & \pi/2 \le x < \pi \end{cases}$$

in terms of the eigenfunctions of the Sturm-Liouville problem given by

$$y'' + \lambda\, y' = 0, \qquad y'(0) = 0,\ y'(\pi) = 0.$$

Problem №5 (20%)

Evaluate the following inverse Fourier transforms

a)

$$F^{-1}\left\{\frac{9}{2\omega + \jmath}\right\},$$

b)

$$F^{-1}\left\{e^{-\omega^2+4\omega}\right\}.$$

Appendix D

Table of Fourier Transforms

	$f(x)$	$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\,dx$
1.	$\dfrac{1}{x^2+a^2} \quad (a > 0)$	$\dfrac{\pi}{a} e^{-a\lvert\omega\rvert}$
2.	$H(x)e^{-ax} \quad (\operatorname{Re} a > 0)$	$\dfrac{1}{a+i\omega}$
3.	$H(-x)e^{ax} \quad (\operatorname{Re} a > 0)$	$\dfrac{1}{a-i\omega}$
4.	$e^{-a\lvert x\rvert} \quad (a > 0)$	$\dfrac{2a}{\omega^2+a^2}$
5.	e^{-x^2}	$\sqrt{\pi} e^{-\omega^2/4}$
6.	$\dfrac{1}{2a\sqrt{\pi}} e^{-x^2/(2a)^2} \quad (a > 0)$	$e^{-a^2\omega^2}$
7.	$\dfrac{1}{\sqrt{\lvert x\rvert}}$	$\sqrt{\dfrac{2\pi}{\lvert\omega\rvert}}$
8.	$e^{-a\lvert x\rvert/\sqrt{2}} \sin\left(\dfrac{a}{\sqrt{2}}\lvert x\rvert + \dfrac{\pi}{4}\right) \quad (a > 0)$	$\dfrac{2a^3}{\omega^4+a^4}$
9.	$H(x+a) - H(x-a)$	$\dfrac{2\sin\omega a}{\omega}$
10.	$\delta(x-a)$	$e^{-i\omega a}$
11.	$f(ax+b) \quad (a > 0)$	$\dfrac{1}{a} e^{ib\omega/a} \hat{f}\left(\dfrac{\omega}{a}\right)$
12.	$\dfrac{1}{a} e^{-ibx/a} f\left(\dfrac{x}{a}\right) \quad (a > 0,\ b \text{ real})$	$\hat{f}(a\omega + b)$
13.	$f(ax)\cos cx \quad (a > 0,\ c \text{ real})$	$\dfrac{1}{2a}\left[\hat{f}\left(\dfrac{\omega-c}{a}\right) + \hat{f}\left(\dfrac{\omega+c}{a}\right)\right]$
14.	$f(ax)\sin cx \quad (a > 0,\ c \text{ real})$	$\dfrac{1}{2ai}\left[\hat{f}\left(\dfrac{\omega-c}{a}\right) - \hat{f}\left(\dfrac{\omega+c}{a}\right)\right]$
15.	$f(x+c) + f(x-c) \quad (c \text{ real})$	$2\hat{f}(\omega)\cos\omega c$

Problem # 1

We are given $e_1(x) = 1 - |x|$ and $e_2(x) = x^2$

First we compute $\|e_1\|$;

$$\|e_1\|^2 = \int_{-1}^{1} (1-|x|)^2 dx = 2\cdot \int_0^1 (1-|x|)^2 dx =$$

$$= 2\int_0^1 (1-2x+x^2)\,dx = 2\cdot x\Big|_0^1 - 2x^2\Big|_0^1 + \frac{2}{3}x^3\Big|_0^1 = \frac{2}{3}$$

Therefore, $\|e_1\| = \sqrt{\frac{2}{3}}$ and hence $\hat{e}_1 = \sqrt{\frac{3}{2}}(1-|x|)$.

Next, we perform Gram-Schmidt :

$$\langle \hat{e}_1, e_2 \rangle = \int_{-1}^{1} \sqrt{\frac{3}{2}}(1-|x|)\cdot x^2 dx = 2\sqrt{\frac{3}{2}}\int_0^1 (x^2 - x^3)\,dx =$$

$$= \sqrt{6}\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{\sqrt{6}}{12}$$

$$\tilde{e}_2 = e_2 - \langle \hat{e}_1, e_2 \rangle \hat{e}_1 = x^2 - \frac{\sqrt{6}}{12}\cdot\sqrt{\frac{3}{2}}(1-|x|) = x^2 + \frac{1}{4}|x| - \frac{1}{4}$$

Since $\hat{e}_2 = \tilde{e}_2 / \|e_2\|$, we compute next:

$$\|\tilde{e}_2\|^2 = \int_{-1}^{1}\left(x^2 + \frac{1}{4}|x| - \frac{1}{4}\right)^2 dx = 2\int_0^1 \left(x^2 + \frac{1}{4}x - \frac{1}{4}\right)^2 dx =$$

$$= 2\cdot\int_0^1 \left(x^4 + \frac{1}{16}x^2 + \frac{1}{16} + 2\cdot\frac{1}{4}x^3 - 2\cdot\frac{1}{4}x^2 - 2\cdot\frac{1}{16}x\right) dx =$$

$$= 2\cdot\left[\frac{x^5}{5} + \frac{1}{48}x^3 + \frac{1}{16}x + \frac{1}{8}x^4 - \frac{1}{6}x^3 - \frac{1}{16}x^2\right]_0^1 =$$

$$= 2\cdot\left[\frac{1}{5} + \frac{1}{48} + \frac{1}{8} - \frac{1}{6}\right] = \frac{43}{120}$$

Thus, $\hat{e}_2 = \sqrt{120/43}\cdot(x^2 + |x|/4 - 1/4)$.

The last step is to compute:

$$\langle \hat{e}_1, f\rangle = \int_{-1}^{1} \hat{e}_1 \cdot 1\, dx = 2\int_0^1 \sqrt{\tfrac{3}{2}}\,(1-|x|)\, dx =$$

$$= 2\sqrt{\tfrac{3}{2}} \cdot \int_0^1 (1-x)\, dx = 2\sqrt{\tfrac{3}{2}}\left[x - \frac{x^2}{2}\right]_0^1 = \sqrt{\tfrac{3}{2}}.$$

$$\langle \hat{e}_2, f\rangle = \int_{-1}^{1} \hat{e}_2 \cdot 1\, dx = 2\int_0^1 \sqrt{\tfrac{120}{43}}\left(x^2 + \tfrac{1}{4}x - \tfrac{1}{4}\right) dx =$$

$$= 2 \cdot \sqrt{\tfrac{120}{43}}\left[\frac{x^3}{3} + \frac{x^2}{8} - \frac{1}{4}x\right]_0^1 = 2 \cdot \sqrt{\tfrac{120}{43}} \cdot \frac{5}{24} =$$

$$= \frac{5}{12} \cdot \sqrt{\tfrac{120}{43}}$$

Finally:

$$\text{Proj}\{f\} = \langle \hat{e}_1, f\rangle \cdot \hat{e}_1 + \langle \hat{e}_2, f\rangle \hat{e}_2 =$$

$$= \sqrt{\tfrac{3}{2}} \cdot \sqrt{\tfrac{3}{2}}\,(1-|x|) + \frac{5}{12}\sqrt{\tfrac{120}{43}} \cdot \sqrt{\tfrac{120}{43}}\left(x^2 + |x|/4 - 1/4\right) =$$

$$= \frac{3}{2}(1-|x|) + \frac{50}{43}\left(x^2 + |x|/4 - 1/4\right).$$

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Problem # 2

$$i(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2n\pi t}{T} + b_n \sin\frac{2n\pi t}{T}\right)$$

$$\text{average power} = \frac{1}{T}\int_{-T/2}^{T/2} i(t) \cdot R \cdot dt$$

Since $\sin \frac{\;}{T}\Big|_{n=0} = 0$, one can write:

$$i(t) = \sum_{n=0}^{\infty} \left(a_n \cdot \cos \frac{\pi n t}{\ell} + b_n \cdot \sin \frac{\pi n t}{\ell} \right),$$

where $\ell = T/2$.

Then:

$$i^2(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_n \cdot a_k \cos \frac{\pi n t}{\ell} \cos \frac{\pi k t}{\ell} + $$
$$+ \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_n b_k \cos \frac{\pi n t}{\ell} \sin \frac{\pi k t}{\ell} + $$
$$+ \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} b_n b_k \sin \frac{\pi n t}{\ell} \sin \frac{\pi k t}{\ell}.$$

Now:

$$\text{average power} = \frac{R}{T} \int_{-\ell}^{\ell} i^2(t)\, dt$$

Using the orthogonality formulas:

$$\int_{-\ell}^{\ell} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_n a_k \cos \frac{\pi n t}{\ell} \cos \frac{\pi k t}{\ell}\, dt = $$
$$= a_0^2 \cdot (2\ell) + \sum_{n=1}^{\infty} a_n^2 \cdot \ell$$

$$\int_{-\ell}^{\ell} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_n b_k \cos \frac{\pi n t}{\ell} \sin \frac{\pi k t}{\ell}\, dt = 0, \ \forall n, k.$$

$$\int_{-\ell}^{\ell} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} b_n \cdot b_k \cdot \sin \frac{\pi n t}{\ell} \sin \frac{\pi k t}{\ell} = $$
$$= \sum_{n=1}^{\infty} b_n^2 \cdot \ell$$

$$\text{average power} = \frac{R}{T}\cdot\left(2\ell a_0^2 + \ell\sum_{n=1}^{\infty} a_n^2 + \ell\sum_{n=1}^{\infty} b_n^2\right) =$$

$$= R\cdot\left(a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 + \frac{1}{2}\sum_{n=1}^{\infty} b_n^2\right).$$

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Problem #3

First, we find the Fourier series representation of $f(t) = t/\pi$, $t\in[-\pi,\pi)$.

Since $f(t)$ is odd over $[-\pi,\pi)$:

$$a_0 = 0,\quad a_n = 0,\ \forall n.$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \frac{t}{\pi}\sin\frac{\pi n t}{\pi}\,dt = \frac{2}{\pi^2}\int_0^{\pi} t\sin nt\,dt =$$

$$= \frac{2}{\pi^2}\left(-\frac{t}{n}\cdot\cos nt\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nt\,dt\right) =$$

$$= \frac{2}{\pi^2}\left(-\frac{\pi}{n}\cos n\pi + \cancel{\frac{1}{n^2}\sin nt\Big|_0^{\pi}}^{\,0}\right) = \frac{2}{\pi n}(-1)^{n+1}$$

Thus:
$$f(t) = \sum_{n=1}^{\infty}\frac{2}{\pi n}(-1)^{n+1}\sin nt.$$

We need to solve:
$$x'' + 2x' + x = \sum_{n=1}^{\infty}\frac{2}{\pi n}(-1)^{n+1}\cdot\sin nt$$

Let us find first a particular solution to:
$$x'' + 2x' + x = \sin nt$$

We look for a particular solution in the form:

$$x_p(t) = A_n \cos nt + B_n \sin nt$$

$$x_p''(t) = -n^2 A_n \cos nt - n^2 B_n \sin nt$$

$$2x_p'(t) = (-n A_n \sin nt + n \cdot B_n \cos nt) \cdot 2$$

So, the substitution leads us to:

$$\begin{cases} -n^2 A_n + 2n B_n + A_n = 0 \\ -n^2 B_n - 2n A_n + B_n = 1 \end{cases}$$

$$\begin{pmatrix} (1-n^2) & 2n \\ -2n & (1-n^2) \end{pmatrix} \cdot \begin{pmatrix} A_n \\ B_n \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow A_n = \frac{-2n}{(1+n^2)^2}, \quad B_n = \frac{1-n^2}{(1+n^2)^2}.$$

Therefore:

$$x_p(t) = \frac{-2n}{(1+n^2)^2} \cos nt + \frac{1-n^2}{(1+n^2)^2} \sin nt$$

Finally, by the superposition principle, we have a particular solution of the original equation given by:

$$\sum_{n=1}^{\infty} \frac{2}{\pi n} (-1)^{n+1} \left[\frac{-2n}{(1+n^2)^2} \cos nt + \right.$$

$$\left. + \frac{1-n^2}{(1+n^2)^2} \sin nt \right].$$

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Problem # 4

The Sturm-Liouville problem

$$y'' + \lambda y = 0 \quad , \quad y'(0) = 0, \; y'(\pi) = 0$$

has a general solution given by

$$y(t) = \begin{cases} A \cos\sqrt{\lambda}\cdot t + B \sin\sqrt{\lambda}\cdot t & , \lambda \neq 0 \\ C\cdot t + D & , \lambda = 0 \end{cases}$$

Consider $\lambda = 0$ first: $(C\cdot t + D)'\big|_{t=0} = 0 \Rightarrow C = 0$

Therefore $\lambda = 0$ is an eigenvalue with $\phi_0 = 1$.

Next, assume $\lambda \neq 0$:

$$\left(-A\sqrt{\lambda}\sin\sqrt{\lambda}t + B\sqrt{\lambda}\cos\sqrt{\lambda}t\right)\Big|_{t=0} =$$

$$= \left(B\sqrt{\lambda}\cos\sqrt{\lambda}t\right)\Big|_{t=0} = B\sqrt{\lambda} = 0$$

$$\Rightarrow B = 0.$$

From the second condition:

$$-A\sqrt{\lambda}\sin\sqrt{\lambda}\pi = 0$$

A cannot be equal to 0 since this would give us a trivial solution. Assume, for concreteness, $A = 1$.

Then we have:

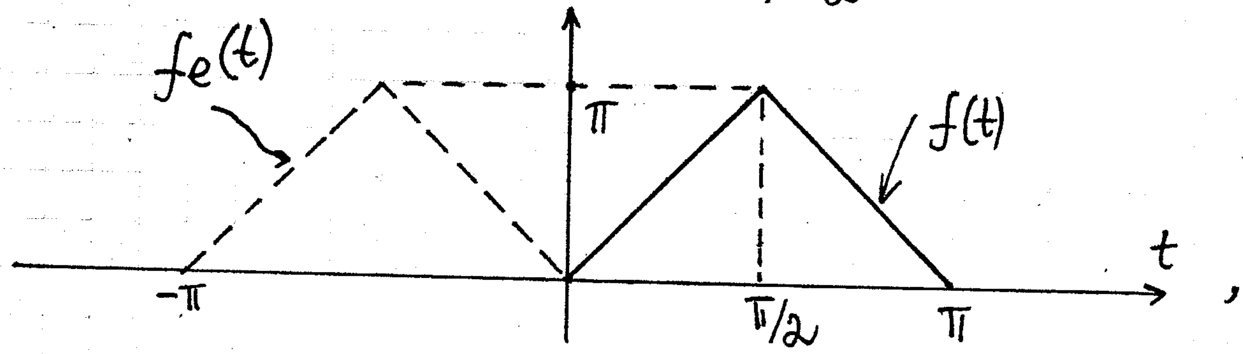
$$\sin\sqrt{\lambda}\pi = 0$$

$$\Rightarrow \sqrt{\lambda}\pi = \pi\cdot n \Rightarrow \lambda_n = n^2$$

In summary, the eigen-system is:

$$\begin{cases} \lambda_0 = 0, & \phi_0 = 1 \\ \lambda_n = n^2, & \phi_n = \cos nt \end{cases}$$

Now, given $f(t) = \begin{cases} 2t, & 0 \le t < \frac{\pi}{2} \\ 2\pi - 2t, & \frac{\pi}{2} \le t < \pi \end{cases}$ ⑦



we need to periodize $f(t)$ as shown in the figure above. The periodization is necessary since $\phi_n(t)$ are 2π-periodic and they are orthogonal with respect to the inner product given by:

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t)\, g(t)\, dt.$$

After the periodization both $f_e(t)$ and $\phi_n(t)$ are even and 2π-periodic. The orthogonal expansion of $f(t)$ in terms of ϕ_n is defined as

$$f(t) = \sum_{n=0}^{\infty} c_n \phi_n(t), \quad \text{where } c_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}.$$

Thus, first we compute:

$$\langle f_e, \phi_0 \rangle = \int_{-\pi}^{\pi} f_e(t) \cdot 1\, dt = \pi^2$$

$$\langle f_e, \phi_n \rangle = \int_{-\pi}^{\pi} f_e(t) \cos nt\, dt = 2 \int_{0}^{\pi} f(t) \cos nt\, dt =$$

$$= 2\int_0^{\pi/2} 2t\cos nt\,dt + 2\cdot\int_{\pi/2}^{\pi}(2\pi-2t)\cos nt\,dt =$$

$$= 4\,\frac{t}{n}\sin nt\Big|_0^{\pi/2} - 4\cdot\int_0^{\pi/2}\frac{1}{n}\sin nt\,dt + 4\pi\int_{\pi/2}^{\pi}\cos nt\,dt -$$

$$-4\,\frac{t}{n}\sin nt\Big|_{\pi/2}^{\pi} + 4\int_{\frac{\pi}{2}}^{\pi}\frac{1}{n}\sin nt\,dt =$$

$$= \frac{2\pi}{n}\cdot\sin\frac{\pi n}{2} + \frac{4}{n^2}\cos nt\Big|_0^{\pi/2} + \frac{4\pi}{n}\sin nt\Big|_{\pi/2}^{\pi} +$$

$$+ \frac{2\pi}{n}\sin\frac{\pi n}{2} - \frac{4}{n^2}\cos nt\Big|_{\pi/2}^{\pi} =$$

$$= \cancel{\frac{4\pi}{n}\sin\frac{\pi n}{2}} + \frac{4}{n^2}\cos\frac{\pi n}{2} - \frac{4}{n^2} - \cancel{\frac{4\pi}{n}\sin\frac{\pi n}{2}} -$$

$$-\frac{4}{n^2}\cos n\pi + \frac{4}{n^2}\cos\frac{\pi n}{2} =$$

$$= \frac{8}{n^2}\cos\frac{\pi n}{2} - \frac{4}{n^2}\left(1+(-1)^n\right).$$

Next,

$$\langle \phi_n, \phi_n\rangle = \int_{-\pi}^{\pi}\cos nt\cdot\cos nt\,dt = \pi$$

$$\langle \phi_0, \phi_0\rangle = \int_{-\pi}^{\pi} 1\,dt = 2\pi.$$

Finally:

$$f(t) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left[\frac{8}{\pi n^2}\cos\frac{\pi n}{2} - \frac{4}{\pi n^2}(1+(-1)^n)\right]\times$$

$$\times\cos nt.$$

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Problem #5

I)

$$\mathcal{F}^{-1}\left\{\frac{9}{2\omega+i}\right\} = \mathcal{F}^{-1}\left\{\frac{-\frac{9}{2}i}{\frac{1}{2}-\omega i}\right\} =$$

$$= -\frac{9}{2}i\,\mathcal{F}^{-1}\left\{\frac{1}{\frac{1}{2}-\omega i}\right\} = -\frac{9}{2}i\,H(-x)\,e^{\frac{1}{2}x}.$$

II)

$$\mathcal{F}^{-1}\left\{e^{-\omega^2+4\omega}\right\} = \mathcal{F}^{-1}\left\{e^{-(\omega-2)^2+4}\right\} = e^4\cdot\mathcal{F}^{-1}\left\{e^{-(\omega-2)^2}\right\} =$$

$$= e^4\cdot\frac{1}{2\sqrt{\pi}}\cdot e^{-x^2/4}\cdot e^{i2x}.$$

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