University of Waterloo
David R. Cheriton School of Computer Science

MATH 213 – Advanced Mathematics for Software Engineers
Midterm Exam, Spring 2008

June 26, 2008, 6:00-8:00 PM

Instructor: Dr. Oleg Michailovich

Student's name: ____

Student's ID #: ____

Instructions:

- This exam has **6** pages.
- **No books and lecture notes are allowed on the exam**. Please, turn off your cell phones, PDAs, etc., and place your bags, backpacks, books, and notes under the table or at the front of the room.
- Please, place your **WATCARD** on the table, and fill out the exam attendance sheet when provided by the proctor after the exam starts.
- Question marks are listed by the question.
- Please, do not separate the pages, and indicate your Student ID at the top of every page.
- Be neat. Poor presentation will be penalized.
- **No questions will be answered during the exam**. If there is an ambiguity, state your assumptions and proceed.
- **No student can leave the exam room in the first 45 minutes or the last 10 minutes**.
- If you finish before the end of the exam and wish to leave, remain seated and raise your hand. A proctor will pick up the exam from you, at which point you may leave.
- When the proctors announce the end of the exam, put down your pens/pencils, close your exam booklet, and remain seated in silence. The proctors will collect the exams, count them, and then announce you may leave.

Problem №1 (24%)

Bring the equations to the form $y' + p(x)\,y = q(x)$ and find the general solutions. The answer may be left in implicit form, rather than in explicit form, if necessary. **<u>Hint:</u>** Remember that which variable is the independent variable and which is the dependent variable is a matter of viewpoint, and one can change one's viewpoint. In these problems, consider whether it might be better to regard x as a function of y, and recall from the calculus that $dy/dx = 1/(dx/dy)$.

a)
$$\frac{dy}{dx} = \frac{1}{x + 3e^y};$$

b)
$$\frac{dy}{dx} = \frac{1}{6x + y^2};$$

c)
$$(6y^2 - x)\frac{dy}{dx} - y = 0;$$

d)
$$(y^2 \sin y + x)\frac{dy}{dx} = y.$$

Problem №2 (26%)

Consider a particle of mass m, carrying an electrical charge q, and moving in a uniform magnetic field of strength B. The equations of motion of the particle (assuming the field is in the positive z direction) are

$$\begin{cases} mx'' &= qBy', \\ my'' &= -qBx', \\ mz'' &= 0, \end{cases}$$

where $x(t)$, $y(t)$, $z(t)$ are the x, y, z displacements of the particle as a function of the time t.

Find the general solution of the above system of equations. How many *independent* arbitrary constants of integration are there?

Hint: To simplify the system, one can integrate its equations once with respect to t. Then, denoting $\alpha = qB/m$, one can reduce the original system to

$$\begin{cases} x' - \alpha y &= E, \\ y' + \alpha x &= G, \\ z &= I\,t + H, \end{cases}$$

where E, G, I, and H are some arbitrary integration constants.

Problem №3 (20%)

Solve $x'' - x = f(t)$, where $x(0) = x'(0) = 0$, by the method of Laplace transform for

$$f(t) = \begin{cases} t, & 0 < t < 2, \\ 2, & t > 2. \end{cases}$$

<u>Hint:</u> You may find useful the fact that $(\cosh t)' = \sinh t$ and $(\sinh t)' = \cosh t$.

Problem №4 (20%)

Using the formula for the Laplace transform of periodic functions shows that indeed

a)
$$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1};$$

b)
$$\mathcal{L}\{\cos t\} = \frac{s}{s^2+1}.$$

<u>Hint:</u> You may find useful the following integrals

- $$\int e^{-st}\sin t\,dt = -\frac{e^{-st}}{s^2+1}\cos t - \frac{s\,e^{-st}}{s^2+1}\sin t + C;$$
- $$\int e^{-st}\cos t\,dt = -\frac{s\,e^{-st}}{s^2+1}\cos t + \frac{e^{-st}}{s^2+1}\sin t + C.$$

Table of Laplace Transforms

$f(t)$	$\bar{f}(s) = \int_0^\infty f(t)e^{-st}\,dt$

NOTE: s is regarded as real here.

1.	1	$\frac{1}{s} \quad (s > 0)$
2.	e^{at}	$\frac{1}{s-a} \quad (s > a)$
3.	$\sin at$	$\frac{a}{s^2+a^2} \quad (s > 0)$
4.	$\cos at$	$\frac{s}{s^2+a^2} \quad (s > 0)$
5.	$\sinh at$	$\frac{a}{s^2-a^2} \quad (s > \lvert a \rvert)$
6.	$\cosh at$	$\frac{s}{s^2-a^2} \quad (s > \lvert a \rvert)$
7.	$t^n \quad (n = \text{positive integer})$	$\frac{n!}{s^{n+1}} \quad (s > 0)$
8.	$t^p \quad (p > -1)$	$\frac{\Gamma(p+1)}{s^{p+1}} \quad (s > 0)$
9.	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2} \quad (s > a)$
10.	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2} \quad (s > a)$
11.	$t\sin at$	$\frac{2as}{(s^2+a^2)^2} \quad (s > 0)$
12.	$t\cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2} \quad (s > 0)$
13.	$t\sinh at$	$\frac{2as}{(s^2-a^2)^2} \quad (s > a)$

Problem #1

a) $$\frac{dy}{dx} = \frac{1}{x+3e^y}$$

Equivalently: $$\frac{dx}{dy} = x+3e^y$$

$$\Rightarrow \quad \frac{dx}{dy} - x = 3e^y$$

Finally: $$x' + p(y)\cdot x = q(y)$$

where $x = x(y)$, $p(y) = -1$, $q(y) = 3e^y$

The general solution is given by:

$$x(y) = e^{-\int p(y)dy}\left(\int e^{\int p(y)dy} \cdot q(y)dy + C \right)$$

Specifically:

$$x(y) = e^y\left(\int e^{-y} \cdot 3e^y dy + C \right)$$

$$\Rightarrow \quad x(y) = e^y(3y + C).$$

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b) $$\frac{dy}{dx} = \frac{1}{6x+y^2} \quad \Rightarrow \quad \frac{dx}{dy} = 6x + y^2$$

$$\Rightarrow \quad x' + p(y)x = q(y)$$

where $x = x(y)$, $p(y) = -6$, $q(y) = y^2$

$$\Rightarrow \quad x(y) = e^{6y}\left(\int e^{-6y} \cdot y^2 dy + C \right).$$

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c) $$(6y^2-x)\,\frac{dy}{dx}-y=0$$

$$\Rightarrow\quad \frac{dy}{dx}=\frac{y}{6y^2-x}$$

$$\Rightarrow\quad \frac{dx}{dy}=\frac{6y^2-x}{y}=6y-\frac{1}{y}\cdot x$$

$$\Rightarrow\quad x'+p(y)\,x=q(y)$$

where $x=x(y)$, $p(y)=\frac{1}{y}$, $q=6y$

Then:

$$x(y)=e^{-\int\frac{1}{y}dy}\left(\int e^{\int\frac{1}{y}dy}\,6y\,dy+C\right)$$

$$\Rightarrow\quad x(y)=e^{-\ln y}\left(\int e^{\ln y}\cdot 6y\,dy+C\right)$$

$$x(y)=\frac{1}{y}\left(6\int y^2\,dy+C\right)$$

$$x(y)=\frac{1}{y}\left(2y^3+C\right)=2y^2+C/y$$

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d) $$(y^2\sin y+x)\,\frac{dy}{dx}=y$$

$$\Rightarrow\quad \frac{dx}{dy}=y\sin y+\frac{1}{y}\cdot x$$

$$\Rightarrow\quad x'+p(y)\,x=q(y)$$

where $x=x(y)$, $p(y)=-\frac{1}{y}$, $q=y\sin y$

$$x(y)=e^{\ln y}\left(\int e^{-\ln y}\,y\sin y\,dy+C\right)$$

$$x(y)=y\left(\int\sin y\,dy+C\right)$$

$$x(y)=y\left(-\cos y+C\right)=-y\cos y+C\cdot y.$$

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Problem #2.

(1) $\mathcal{D}[x] - \alpha y = E$ where E, G, H, I are

(2) $\alpha x + \mathcal{D}[y] = G$ constants

(3) $z = H + I \cdot t$

Using elimination on the first two of the above equations results in:

$x'' + \alpha^2 x = \alpha G$ so $x(t) = J \sin\alpha t + K \cos\alpha t + G/\alpha$ (4)

$y'' + \alpha^2 y = -\alpha E$ so $y(t) = M \sin\alpha t + N \cos\alpha t - E/\alpha$ (5)

To determine any relations among the integration constants put (4) and (5) into (1) or (2), say (1):

$$\alpha J \cos\alpha t - \alpha K \sin\alpha t - \alpha M \sin\alpha t - \alpha N \cos\alpha t + \cancel{E} = \cancel{E}$$

Thus: $N = J$ and $M = -K$. Finally:

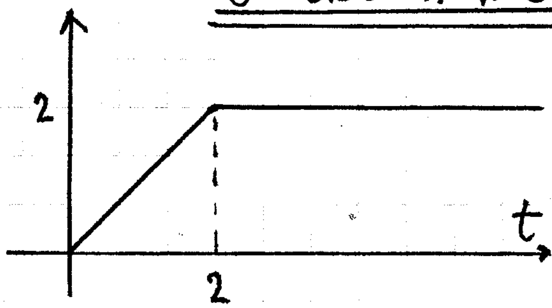
$$x(t) = J \sin\alpha t + K \cos\alpha t + G/\alpha$$

$$y(t) = -K \sin\alpha t + J \cos\alpha t - E/\alpha$$

$$z(t) = H + I \cdot t$$

where J, K, G, E, H, I are 6 arbitrary integration constants.

Problem #3

$f(t) =$  $= t\left[1-\mathcal{H}(t-2)\right] + 2\mathcal{H}(t-2) =$

$= t - (t-2)\mathcal{H}(t-2)$

$$\Rightarrow \quad \mathcal{F}(s) = \mathcal{L}\{f\} = \frac{1}{s^2} - \frac{e^{-2s}}{s^2}$$

Now:

$$x'' - x = f(t) \quad , \quad x'(0) = x(0) = 0$$

$$s^2 \mathcal{X}(s) - \mathcal{X}(s) = \mathcal{F}(s)$$

Hence:
$$\mathcal{X}(s) = \frac{1}{s^2(s^2-1)} - \frac{e^{-2s}}{s^2(s^2-1)}$$

First:
$$\mathcal{L}^{-1}\left\{ \frac{1}{s^2} \cdot \frac{1}{s^2-1} \right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} =$$

$$= t * \sinh t = \int_0^t \tau \cdot \sinh(t-\tau)\, d\tau =$$

$$= -\tau\cosh(t-\tau)\Big|_0^t + \int_0^t \cosh(t-\tau)\, d\tau = -t + \left[-\sin(t-\tau)\right]_0^t =$$

$$= \sinh t - t$$

Thus:
$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2-1)}\right\} = \sinh t - t$$

Therefore:
$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2(s^2-1)}\right\} = \left(\sinh(t-2) - (t-2)\right)\mathcal{H}(t-2)$$

Finally:
$$x(t) = \sinh t - t +$$
$$+ (t-2)\mathcal{H}(t-2) - \sinh(t-2)\mathcal{H}(t-2).$$

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Problem #4

a) $\sin t$ is periodic with period 2π. Hence:

$$\mathcal{L}\{\sin t\} = \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} \sin t \, e^{-st} \, dt =$$

$$= \frac{1}{1-e^{-2\pi s}} \cdot \frac{1-e^{-2\pi s}}{s^2+1} =$$

$$= \frac{1}{s^2+1}.$$

b) $$\mathcal{L}\{\cos t\} = \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} \cos t \, e^{-st} \, dt =$$

$$= \frac{1}{1-e^{-2\pi s}} \cdot \frac{s-se^{-2\pi s}}{s^2+1} =$$

$$= \frac{s}{s^2+1}.$$

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