SE380 Practice Final

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Question 1.

Consider system modeled by the following transfer function

$$G(s) = \frac{1}{(s-1)(s+1)}$$

a) Use Routh-Hurwitz criteron to design a PD controller to stablize the system.

Our controller will be of the form

$$C(s) = K_p + K_d s \text{ where } K_d, K_p \in \mathbb{R}$$

The closed loop transfer function is of the form

$$\frac{CG}{1+CG} = \frac{(K_p + K_d s)\frac{1}{(s-1)(s+1)}}{1 + (K_p + K_d s)\frac{1}{(s-1)(s+1)}} = \frac{K_p + K_d s}{s^2 - 1 + K_p + K_d s}$$

We thus want roots of the polynomial $s^2 + K_d s + K_p - 1$ to have strictly negative real parts.

Routh table:
1 $K_p - 1$
K_d 0
$\frac{-(K_p-1)K_d}{K_d}$

Thus, for stability, we need $K_d > 0$ and $\frac{-(K_p-1)K_d}{K_d} > 0 \implies K_p > 1$.

NOTE: one should note that to properly stabilize the system, we need to know that there are no unstable zero-pole cancellations in $G(s)$

b) Now, we'll use pole placement to stabilize the system.

The system described by $G(s) = \frac{1}{(s-1)(s+1)} = \frac{1}{s^2-1}$ can be represented in controllable canonical form:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

where $x \in \mathbb{R}^2, u \in \mathbb{R}$

Let $u = -Kx$, where $K \in M_{1\times 2}(\mathbb{C})$

Without loss of generality,

$$K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

for some $k_1, k_2 \in \mathbb{C}$ (we will pick these later)

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x - \begin{bmatrix} 0 \\ 1 \end{bmatrix} Kx = \begin{bmatrix} 0 & 1 \\ 1-k_1 & -k_2 \end{bmatrix} x$$

The system is stable if and only if the eigenvalues of $\begin{bmatrix} 0 & 1 \\ 1-k_1 & -k_2 \end{bmatrix}$ have strictly negative real parts.

The eigenvalues of $\begin{bmatrix} 0 & 1 \\ 1-k_1 & -k_2 \end{bmatrix}$ satisfy

$$\left| \begin{bmatrix} 0 & 1 \\ 1-k_1 & -k_2 \end{bmatrix} - \lambda I \right| = \left| \begin{bmatrix} -\lambda & 1 \\ 1-k_1 & -k_2-\lambda \end{bmatrix} \right| = 0$$

$$0 = -\lambda(-k_2-\lambda) - (1-k_1) = \lambda^2 + k_2\lambda + k_1 - 1$$

Thus,

$$\lambda = \frac{-k_2 \pm \sqrt{k_2^2 - 4(k_1-1)}}{2}$$

Now, we need to pick k_1 and k_2 such that $Re(\lambda)$ is always negative. An easy way to do this is $k_2 = 1, k_1 = 2$.

A controller of $u = -\begin{bmatrix} 2 & 1 \end{bmatrix} x$ will stabilize the system.

This control input is a scaler ($\begin{bmatrix} 2 & 1 \end{bmatrix}$ has dimensions 1×2 and x has dimensions 2×1).

Question 2.

These are just bode plots, use matlab to check your work :)

Question 3.

$$\dot{x} = x^2 - xu - 2u$$
$$y = x^3 + u^3$$

a) $\bar{u} = 1$ Let's first find the equilibrium conditions.

$$\dot{x} = 0, u = 1 \implies 0 = x^2 - x - 2$$

Thus,

$$x = \frac{1 \pm \sqrt{1 - 4(-2)}}{2} = \frac{1 \pm 3}{2} \therefore \bar{x} = -1 \text{ or } 2$$

$$\frac{\partial \dot{x}}{\partial x} = 2x - u$$
$$\frac{\partial \dot{x}}{\partial u} = -x - 2$$
$$\frac{\partial y}{\partial x} = 3x^2$$
$$\frac{\partial y}{\partial u} = 3u^2$$

Linearized systems are of the form:

$$\delta\dot{x} = \frac{\partial \dot{x}}{\partial x}|_{x=\bar{x},u=\bar{u}}\delta x + \frac{\partial \dot{x}}{\partial u}_{x=\bar{x},u=\bar{u}}\delta u$$
$$\delta y = \frac{\partial y}{\partial x}|_{x=\bar{x},u=\bar{u}}\delta x + \frac{\partial y}{\partial u}_{x=\bar{x},u=\bar{u}}\delta u$$

Where $\delta x = x - \bar{x}, \delta u = u - \bar{u}, \delta\dot{x} = \dot{x} - (\bar{x}^2 - \bar{x}\bar{u} - 2\bar{u})$ and $\delta y = y - (\bar{x}^3 + \bar{u}^3)$
Because there are two options for $\bar{x}$ that are in equilibrium, we make two linearized systems. First, system α where $\bar{x} = -1$ and $\bar{u} = 1$.

$$\frac{\partial \dot{x}}{\partial x}|_{x=\bar{x},u=\bar{u}} = 2(-1) - 1 = -3$$
$$\frac{\partial \dot{x}}{\partial u}|_{x=\bar{x},u=\bar{u}} = -(-1) - 2 = -1$$
$$\frac{\partial y}{\partial x}|_{x=\bar{x},u=\bar{u}} = 3$$
$$\frac{\partial u}{\partial u}|_{x=\bar{x},u=\bar{u}} = 3$$

$$\delta\dot{x} = -3\delta x - 1\delta u$$
$$\delta y = 3\delta x + 3\delta u$$

Second, system β where $\bar{x} = 2$ and $\bar{u} = 1$

$$\frac{\partial \dot{x}}{\partial x}|_{x=\bar{x},u=\bar{u}} = 2(2) - 1 = 3$$

$$\frac{\partial \dot{x}}{\partial u}|_{x=\bar{x},u=\bar{u}} = -(2) - 2 = -4$$
$$\frac{\partial y}{\partial x}|_{x=\bar{x},u=\bar{u}} = 3(2)^2 = 12$$
$$\frac{\partial u}{\partial u}|_{x=\bar{x},u=\bar{u}} = 3$$

$$\delta\dot{x} = 3\delta x - 4\delta u$$
$$\delta y = 12\delta x + 3\delta u$$

b) First, system α.
$A = -3$. The eigenvalue of A is λ such that $-3x = \lambda x$, clearly $\lambda = -3$ so the eigenvalue of A has strictly negative real parts. Thus, it is asymptotically stable, which implies BIBO and exponential stability (due to linearity).
Second, system β:
$A = 3$. The eignevalue of A is 3, which does not have strictly negative real parts, so the system is unstable which means it is not asymptotically or exponentially stable.
We can check for BIBO stability by finding the transfer function:

$$G_\beta(s) = 12(s-3)^{-1}4 + 3 = \frac{48}{s-3} + 3 = \frac{48 + 3s - 9}{s-3} = \frac{3s-39}{s-3}$$

which is not bibo stable due to the poles on the right half of the complex plane.

c) Finding the steady state response to a sinusodial.

We can't find the response to system β because it is unstable. For system α

$$\delta u = 0.25\sin(0.25t)$$

Note that the transfer function from the input to the output of system α is

$$G(s) = 3(s-(-3))^{-1}(-1) + 3 = \frac{-3}{s+3} + 3$$
$$G(s) = \frac{-3}{s+3} + 3 = \frac{-3+3s+9}{s+3} = \frac{3s+6}{s+3}$$

So, $\delta y(t) = 0.25|G(0.25j)|\sin(0.25t + \angle G(0.25j))$

$$G(0.25j) = \frac{3(0.25j)+6}{(0.25j)+3} = 3\frac{0.25j+2}{0.25j+3} = 3\frac{8+j}{12+j} = 3\frac{(8+j)(12-j)}{(12+j)(12-j)} = 3\frac{(8+j)(12-j)}{(12+j)(12-j)}$$
$$G(0.25j) = 3\frac{96+4j+1}{145} = \frac{3}{145}(97+4j)$$
$$|G(0.25j)| = |\frac{3}{145}(97+4j)| = \frac{3}{145}\sqrt{97^2+4^2}$$
$$\angle G(0.25j) = \arctan(\frac{4}{97})$$

So,
$\delta y(t) = 0.25(\frac{3}{145}\sqrt{97^2+4^2})\sin(0.25t + \arctan(\frac{4}{97}))$