Interest rate models - Mathematical models used to describe and predict how interest rates change over time

Short rate - The instantaneous interest rate at a specific time, t.
Eg if $r(t) = i\%$. This means that the instantaneous rate at t is $i\%$.

Forward rate - The interest rate agreed today over a future time period from t to T
Eg. $f(t,T) = i\%$ means from the interest rate agreed today for the period t to T is $i\%$.
but $f(t;T_1,T_2)$ means the interest rate agreed at t over a period T_1 to T_2.

They don't have to be the interest rates "agreed" on as they can be the interest rates implied or expected by the model used.

Notes from Chapter 23 "Arbitrage Theory in Continuous Time" on Short Rate Models

- The price of a zero coupon bond at maturity can be given by $P(t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)\,ds} \mid \mathcal{F}_t\right]$ where $\mathbb{Q}$ is the risk-neutral measure

- We let $P(t,T) = F(t, r(t); T) = F^T(t, r(t))$

- $r(t)$ follows a stochastic process $dr(t) = \mu(t, r(t))\, dt + \sigma(t, r(t))\, dW_t$

- Since the above is true, F^T is such that it satisfies the term structure equation. (found using Ito's lemma and No Arbitrage Theorem)

$$\frac{\partial F^T}{\partial t} + (\mu - \lambda\sigma)\frac{\partial F^T}{\partial r} + \frac{\sigma^2}{2} \cdot \frac{\partial^2 F^T}{\partial r^2} = rF^T$$

(this is under the real world measure, $\mathbb{P}$)

subject to the boundary condition $F^T(T, r) = P(T, T) = 1$
(the bond has value 1 at maturity)

- $\mu(t, r)$: drift of $r(t)$
- $\sigma(t, r)$: volatility of $r(t)$
- $\lambda(t, r)$: market price of risk
- $\mu - \lambda\sigma$: risk neutral drift of r_t

- The form of $r(t)$ changes depending on which short rate model we apply, hence so does the PDE to be satisfied.

* Review this *

- Bond prices are not uniquely determined by the P-dynamics of the short rate r because pricing is done under the risk-neutral measure $\mathbb{Q}$, not the real world measure $\mathbb{P}$. We are unable to create a replicating portfolio for the bonds because there aren't enough assets to hedge every source of risk. The only traded asset is the bank account that earns the stochastic short rate, r.

Chapter 24 "Martingale Models for the Short Rate"

Martingale Modeling - Specifying σ and μ directly under the martingale measure $\mathbb{Q}$ rather than under the objective probability measure $\mathbb{P}$.

The most popular models on how to specify the $\mathbb{Q}$-dynamics of r are:

Vasiček: $dr = (b - ar)\,dt + \sigma\,dW$, $(a > 0)$

Cox-Ingersoll-Ross (CIR): $dr = a(b - r)\,dt + \sigma\sqrt{r}\,dW$

Dothan: $dr = ar\,dt + \sigma r\,dW$

Black-Derman-Toy: $dr = \theta(t) r\,dt + \sigma(t) r\,dW$

Ho-Lee: $dr = \theta(t)\,dt + \sigma\,dW$

* Hull-White (Extended Vasiček): $dr = (\theta(t) - a(t)r)\,dt + \sigma(t)\,dW$, $(a(t) > 0)$

* Hull-White (Extended CIR): $dr = (\theta(t) - a(t)r)\,dt + \sigma(t)\sqrt{r}\,dW$, $(a(t) > 0)$

To obtain information about $\mathbb{Q}$-drift parameters, we collect price information from the market by inverting the yield curve.

- Start by choosing a model and denoting the parameter vector by α and the $\mathbb{Q}$-dynamics of r are:

$$dr(t) = \mu(t, r(t); \alpha)\,dt + \sigma(t, r(t); \alpha)\,dW(t)$$

- Then solve the term structure equation to arrive at a solution $p(t, T; \alpha) = F^T(t, r; \alpha)$

- Collect information from the bond market (observe $p(0,T)$ $\forall$ T) and denote the empirical term structure $(p^*(0,T); T \geq 0)$

- Choose α in such a way that our theoretical curve $(p(0,T;\alpha); T \geq 0)$ fits the empirical curve $(p^*(0,T;\alpha); T \geq 0)$ sufficiently. This gives our estimated parameter vector α^*.

- We fill α^* into μ and σ, giving us our martingale measure, $\mathbb{Q}$.

- The price process is given by $G(t, r(t))$ where G solves the term structure equations with μ^* and σ^*.

Affine Term Structures

A short rate model is called affine if the bond price takes the form $p(t,T) = e^{(A(t,T) - B(t,T) \cdot r(t))}$. So, the logarithm of the bond price is a linear function of $r(t)$ and $A(t,T)$, $B(t,T)$ are deterministic functions of time.

From the models listed above, Dothan and Black-Derman-Toy do not have an ATS, but the rest do. This means that all other models' bond prices can be expressed as

$$p(t,T) = e^{(A(t,T) + B(t,T) r(t))}.$$

We know that under the risk neutral measure,

$$p(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)\, ds\right) \mid \mathcal{F}_t\right]$$

So we want to know for which model this equals $e^{A(t,T)+B(t,T)r(t)}$

We can see that for models where drift and volatility are linear functions of $r(t)$, then this is sometimes possible.

This is because if $r(t)$ is affine then $\int_t^T r(s)\,ds$ has known distribution (because $r(t)$ has known distribution).

$$r(t) \text{ is affine} \implies \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)\,ds} \,\middle|\, r(t)\right] = e^{A(t,T) - B(t,T)\cdot r(t)}$$

In Vasiček and CIR, the distributions of $r(t)$ are affine and known (Gaussian and non-central chi square) so the integral $\int_t^T r(s)\,ds$ can be solved analytically leading to bond prices in the form $e^{A(t,T) + B(t,T) r(t)}$. But in Black-Derman-Toy and Dothan, the volatility depends non-linearly on $r(t)$, meaning $r(t)$ cannot be affine. This complicates the distribution of $r(t)$ and means the integral $\int_t^T r(s)\,ds$ is very difficult or sometimes impossible to solve analytically.

General Result:

If the drift and volatility of a short-rate model have the form

$$\mu(t,r) = \alpha(t)\cdot r + \beta(t)$$
$$\sigma(t,r) = \gamma(t)\cdot r + \delta(t),$$

then the functions $A(t,T)$, $B(t,T)$ from $p(t,T) = e^{A(t,T) + B(t,T)\cdot r(t)}$ satisfy the following ODEs

- $B_t(t,T) + \alpha(t) B(t,T) - \frac{1}{2}\gamma(t) B^2(t,T) = -1$

  with boundary condition $B(T,T) = 0$

- $A_t(t,T) = \beta(t)B(t,T) - \frac{1}{2}\delta(t)B^2(t,T)$

with boundary condition $A(T,T)=0$

The Hull-White Model (Extended Vasiček)

The Q-dynamics of the short-rate in the standard Hull-White model are given by $dr = (\theta(t) - ar)\,dt + \sigma\,dW(t)$, where a and σ are constants chosen in order to obtain a nice volatility structure, and $\theta(t)$ is a deterministic function of time chosen in order to fit the theoretical bond prices $\{p(0,T);\ T>0\}$ to the empirical bond curve $\{p^*(0,T);\ T>0\}$ (inverting the yield curve)

We can see that $\mu(t,r) = \theta(t) - ar$ and $\sigma(t,r) = \sigma$.

We know the term-structure PDE under the risk neutral measure, Q is

$$\frac{\partial F}{\partial t} + \mu(t,r)\frac{\partial F}{\partial r} + \frac{\sigma^2(t,r)}{2}\frac{\partial^2 F}{\partial r^2} = rF$$

where $F = e^{A(t,T)+B(t,T)\cdot r(t)}$ (from ATS)

$$\frac{\partial F}{\partial t} = (A_t - B_t\cdot r)\cdot F, \quad \frac{\partial F}{\partial r} = -B(t,T)\cdot F, \quad \frac{\partial^2 F}{\partial r^2} = B^2(t,T)\cdot F$$

Subbing in this information gives:

$$(A_t - B_t\cdot r)\cdot F - (\theta(t)-ar)B(t,T)\cdot F + \frac{\sigma^2}{2}B^2(t,T)\cdot F - rF = 0$$

Dividing by F and grouping like terms leads to

$$\left[A_t - \theta(t)B(t,T) + \frac{\sigma^2}{2}B^2(t,T)\right] + r\left[\left(-B_t + aB(t,T) - 1\right)\right] = 0$$

And this leads to the following ODEs:

- $B_t(t,T) = aB(t,T) - 1 \quad , \quad (B(T,T) = 0)$
- $A_t(t,T) = \theta(t)B(t,T) - \frac{\sigma^2}{2}B^2(t,T) \quad , \quad A(T,T) = 0$

Which can be solved to give:

- $B(t,T) = \frac{1}{a}\left(1 - e^{-a(T-t)}\right)$
- $A(t,T) = \int_t^T \left(\frac{\sigma^2}{2}B^2(s,T) - \theta(s)B(s,T)\right)ds$

Now, to fit theoretical prices to the empirical ones, we use forward rates because there is a one-to-one correspondence between forward rates and bond prices. So we want to fit the theoretical forward rate curve $\{f(0,T);T>0\}$ to the empirical curve $\{f^*(0,T);T>0\}$ where $f^*(t,T) = -\frac{\partial \log(p^*(t,T))}{\partial T}$.

In an affine model, forward rates are given by

$$f(0,T) = B_T(0,T)r(0) - A_T(0,T)$$

and we can fill in our values from above to get

$$f(0,T) = e^{-aT}r(0) + \int_0^T e^{-a(T-s)}\theta(s)\,ds - \frac{\sigma^2}{2a^2}(1-e^{-aT})^2$$

We want to find θ s.t. $f^*(0,T) = f(0,T)$, $\forall\, T>0$,
and we solve this by writing $f^*(0,T) = x(T) - g(T)$
where x and g are defined by

- $\dot{x} = -ax(t) + \theta(t)$
  with $x(0) = r(0)$.

$$\cdot\ g(t) = \frac{\sigma^2}{2a^2}(1-e^{-at})^2 = \frac{\sigma^2}{2}B^2(0,t)$$

This leads to

$$\theta(T) = \dot{x}(T) + ax(T) = f_T^*(0,T) + \dot{g}(T) + ax(T)$$
$$= f^*(0,T) + \dot{g}(T) + a\left(f^*(0,T) + g(T)\right)$$

From the book, we have

$$p(t,T) = \frac{p^*(0,T)}{p^*(0,t)} \cdot \exp\left(B(t,T)\cdot f^*(0,t) - \frac{\sigma^2}{4a}B^2(t,T)(1-e^{-2at}) - B(t,T)r(t)\right)$$

(we know the expression for B from above)

And we have that the price of a European call option with maturity, T, strike K, on an S-bond $(T<S)$ is given by

$$c(t,T,K,S) = p(t,S)\cdot N(d) - p(t,T)\cdot K\cdot N(d-\sigma_p),$$

where $d = \frac{1}{\sigma_p}\log\left(\frac{p(t,S)}{p(t,T)\cdot K}\right) + \frac{\sigma_p}{2}$

and $\sigma_p = \frac{1}{a}\left(1-e^{-a(S-T)}\right)\cdot\sqrt{\frac{\sigma^2}{2a}\left(1-e^{-2a(T-t)}\right)}$

Chapter 27 "LIBOR and Swap Market Models"

Firstly, from chapter 26:

Caps - An interest rate cap is a financial insurance contract which protects you from having to pay more than a prespecified rate, known as the cap rate, even though you have a loan with floating rate interest. Caps are made up of a series of caplets.

Caplets - Cover one specific reset period.

So a cap is like a series of call options on the interest rate over multiple periods and a caplet is a call option on the interest rate for one time period.

Now, we can start by defining the LIBOR forward rate as

$$L_i(t) = \frac{1}{\alpha_i} \cdot \frac{P_{i-1}(t) - P_i(t)}{P_i(t)}$$

where α_i is the tenor $(\alpha_i = T_i - T_{i-1})$

$p_i(t)$ is the price of a zero-coupon bond $p(t, T_i)$

T_i - Forward Measure $(\mathbb{Q}^{T_i})$ - A risk-neutral probability measure in which the numeraire is the zero-coupon bond maturing at T_i $(p(t, T_i))$. Under $\mathbb{Q}^{T_i}$ the discounted price of any T_i-payable payoff is a martingale.

We can easily prove that the LIBOR process L_i is a martingale under the corresponding forward measure $\mathbb{Q}^{T_i}$ on $[0, T_{i-1}]$.

We know $L_i(t) = \frac{1}{\alpha_i} \cdot \frac{P_{i-1}(t) - p_i(t)}{P_i(t)}$

so $\alpha_i \cdot L_i(t) = \frac{P_{i-1}(t)}{p_i(t)} - 1$.

$\frac{P_{i-1}(t)}{p_i(t)}$ is the price of the T_{i-1} bond normalised by the numeraire (the T_i bond). Since $p_i(t)$ is the numeraire, $\frac{P_{i-1}(t)}{p_i(t)}$ is a martingale on $[0, T_{i-1}]$. Thus $\alpha_i L_i$ is a martingale and L_i too.

If the LIBOR forward rates have the dynamics

$$dL_i(t) = L_i(t)\sigma_i(t)\, dW^i(t), \quad i = 1, \dots, N,$$

then we say we have a discrete tenor LIBOR market model with volatilities $\sigma_1, \dots, \sigma_N$.

Pricing an exotic option like a Bermudan Swaption using a LIBOR model typically has two steps.

- Use implied Black volatilities in order to calibrate the model parameters to market data.
- Use MC simulation to price the exotic instrument.

For the calibration aspect, we may be given an empirical term structure for implied forward volatilities $(\bar{\sigma}_1, \dots, \bar{\sigma}_N)$ for resettlement dates $T_1, \dots, T_N$.

In order to calibrate the model, we choose deterministic LIBOR volatilities $\sigma_1(\cdot), \dots, \sigma_N(\cdot)$ such that

$$\bar{\sigma}_i = \frac{1}{T_i} \int_0^{T_{i-1}} \|\sigma_i(s)\|^2 \, ds , \quad i = 1, \dots, N$$

Alternatively, if using a scalar Wiener process for each LIBOR rate, we must choose the scalar function $\sigma_i(\cdot)$ such that

$$\bar{\sigma}_i = \frac{1}{T_i} \int_0^{T_{i-1}} \sigma_i^2(s) \, ds , \quad i = 1, \dots, N$$

It is common to make some sort of structural assumption about the shape of volatility functions and Björk outlines 5 of the most popular.