On the Modelling of the Quanser Magnetic Levitation System

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***Abstract*—In this paper, we explore techniques for modelling the magnetic levitation system designed and produced by Quanser Consulting. The system can be modelled using a series of non-linear equations effectively, however, this is not practical for the design of control systems. To progress in designing a sophisticated control system for the specified device, we develop two models, splitting the device into two different systems. The first model relates the voltage provided to the system with the current provided through the coil. This is done with a first-order transfer function and discrete sampling with noise-injection for an accurate model of current flowing through the coil of the apparatus. The second model relates the current running through the coil to the force provided to the ball, which is then related to the position of the ball. As this is typically modelled by a second order differential equation, the relationship must be linearized in order to design efficient control systems. In addition to discussing the process, we explore the accuracy of the created models through validation techniques. Lastly, we discuss potential improvements to the system, effects of the perturbation variables, and justification of design choices.**

***Index Terms*—Magnetic Levitation, Linearization, Control Theory, Model Validation**

I. Introduction

Magnetic suspension allows an object to levitate utilizing an electromagnet. This is considered the simplest method to levitate an object. To allow levitation, downward force on the object due to gravity should be equal to the upward force (i.e. zero net force). To balance the forces and result in a net force of zero, an electromagnet can be used to create a magnetic field. The strength of the magnetic field applies an upward force equal to gravitational force to create a net force of zero. Position sensors are utilized to keep track of the measured position. The sensor's output is fed back into the system to create a basic feedback system.

For the ball to successfully levitate, the upward magnetic force needs to match the weight of the metal ball exactly. The levitating ball is prone to disturbances, such as surrounding air motion, and slight fluctuations in coil current. Furthermore, it is extremely sensitive in its transient phase, in which there is a high risk of overshoot, resulting in the ball making contact with the magnet. Once the attractive force of the magnet overcomes the weight of the ball, the ball quickly attaches itself to the magnet. Due to system's sensitivity to disturbances, it undoubtedly requires a controller. However, to implement a feedback control system, a transfer function must model the system. To model the system as a transfer function, it must first be linearized. The system is inherently non-linear and unstable. Through careful selection of an operating point, and a thorough understanding of the system's perturbation variables, as well as its behaviour as defined by poles and zeros, one can begin the iterative control design process. This report outlines the modelling and linearization of the magnetic levitation system, which is necessary before attempting to control the system.

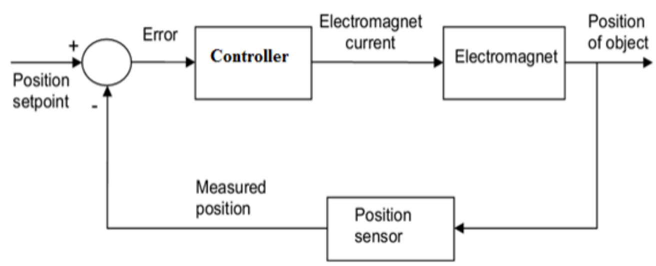

Fig. 1. Basic Magnetic Levitation System

Figure 1 presents a basic control arrangement of a magnetic levitation system. The system in Figure 1 takes only one force in consideration which is the object's weight. The controller increases or decreases the force of attraction between the electromagnet and the object depending on the measured position outputted from the position sensor. If the levitating object gets close to the electromagnet, the controller reduces the input current. If the levitating objects gets far from the electromagnet, the controller increases the input current.

II. System Model

A. Current Model

1) Design Process: To model the current portion of the system, the circuit equation,

$$v(t) = (R_c + R_s) + L\frac{di(t)}{dt} \tag{1}$$

is used to find the initial model of the system. The current model of the system can then be converted to a first order transfer function. Taking the Laplace transform of Equation 1 is equivalent to a transfer function:

$$H_I(s) = \frac{I(s)}{V(s)} = \frac{1}{R + Ls} \tag{2}$$

Using this transfer function, initial trials are run to determine the fitness of the model. Equation 2 uses the resistance $R = R_c + R_s$, where R_c is the resistance of the coil and R_s is the resistance of the current sensor. It is important to note that although the resistance values $R_c = 10\Omega$ and $R_s = 1\Omega$, these values are approximations assuming a constant resistance. These resistances, however, change based on the temperature of the coil apparatus. To account for this, the device is only operated for a short period of time, followed by a period of inactivity to allow for adequate cooling. The affects of temperature are later compared in the validation process, however, it was empirically found that a $R = 10\Omega$ value for the transfer function was most accurate when operated with proper cooling. This leads to a transfer function, Equation 3:

$$H_I(s) = \frac{1}{10 + 0.35s} \tag{3}$$

This transfer function converts the voltage signal into a current signal, however, does not incorporate the physical time-domain to discrete-domain conversion. To create a more accurate model, other components must be added to the system in order to properly simulate the signals being used by the system.

2) Simulink Analog-to-Digital Conversion: Simulink is used to implement the transfer function model previously derived in Equation 3. This transfer function, however, models the system in the Laplace domain. To accurately match the model to the system in the laboratory, we must replicate the digital conversion by sampling the continuous system in a discrete interval.

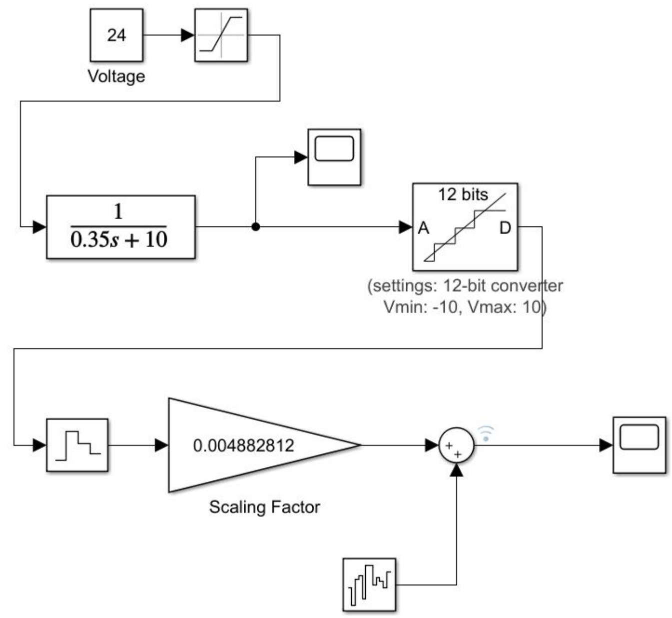

Fig. 2. Open-Loop Model of Voltage Input to Current Output

The voltage level is fed into the transfer function, creating the continuous-time model for the current. This is similar to in reality, where the model is a continuous-system. In the real-world magnetic levitation system, the current is sampled by an analog-to-digital converter to measure the current flowing through the device. To replicate this, we sample the transfer function output, the modelled current, with a 12-bit -10V to 10V ADC with a sampling time of T = 0.002s. The configuration used for digitally sampling the system can be found in Figure II-A2.

The output of the ADC must then be multiplied by a gain, where a gain G can be modelled as

$$G = V_{max} * 2^{-n} \tag{4}$$

where

- V_{max} is the maximum voltage in the coil
- n is the n-bit resolution

The maximum voltage is experimentally found, where the current saturates regardless of if a higher voltage is found. It was experimentally determined that this happened at approximately 20V, therefore, V_{max} is determined to be approximately 20, and n is determined to be 12. Applying these values to Equation 4 leads to

$$G = 20 * 2^{-12} = 0.00488 \tag{5}$$

After finding an appropriate gain for the output of the ADC block, a summer block is added with a white-noise source is added to the ADC output for a more realistic model of genuine sampled data.

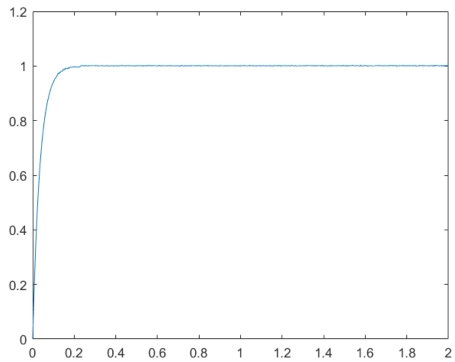

Fig. 3. Simulated Current Given 10V Input

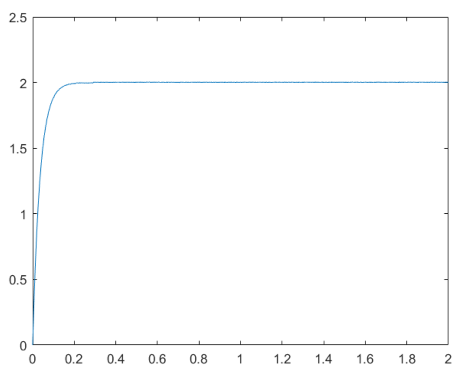

Fig. 4. Simulated Current Given 20V Input

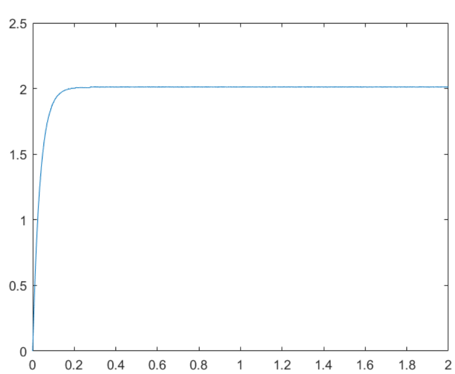

Fig. 5. Simulated Current Given 24V Input

The simulated currents in Figures 3, 4, 5 are comparable to the experimental currents, shows in Figures 6, 7, 8.

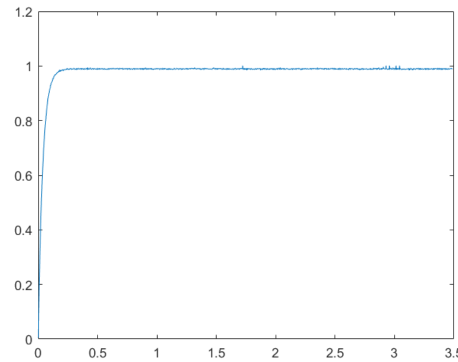

Fig. 6. Real Current Given 10V Input

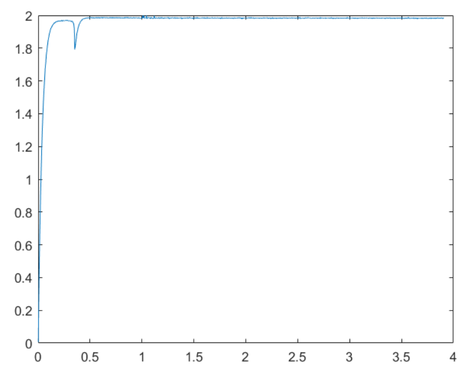

Fig. 7. Real Current Given 20V Input

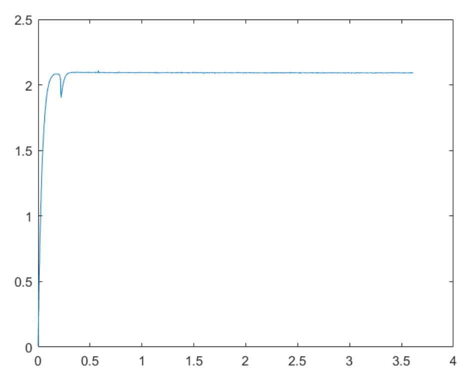

Fig. 8. Real Current Given 24V Input

Note: x-axis is time in seconds while the y-axis is current in amperes. The rise time may appear more rapid in the real data compared to that of the simulated data, however, the real data was simply extended to show the eventual steady-state of the model.

As can be seen above, an approximate comparison of the data shows the similarities of the generated data. Next, we will discuss the comparisons of the model and why certain design choices were selected over others.

3) Benefits and Limitations of Current Model: There are several noteworthy limitations of the current model discussed above. One important design decision made was prioritizing simplicity over extreme accuracy. The model is designed to follow the same general trend as the real-world data, which requires similar response times and steady-state values. Being able to easily modify these values is the reason why the first-order transfer function approach was selected.

The initial coefficients selected for the transfer function were already sufficient for the modelling purposes, as data verification can be found later in the report. As such, the inductance value was left at the value of $L = 350mH$, as shown in [2]. It was found, however, that the resistance value of 11 was causing a steady-state error where the model was producing less than that of the actual system. By replacing the current factor from $R = 11\Omega$ to $R = 10\Omega$, it was found that the model was dramatically more accurate. This may be due to one of two different possibilities:

- $R_s = 1\Omega$ being non-linear and negligible at low temperatures
- $R_{coil} = 10\Omega$ is closer to 9ω at the lower end of operating temperatures

As the model prioritizes both simplicity and accuracy, the transfer function was adjusted but kept at simple coefficients with clear significance of the permutation variables. The coefficients could have, however, been further tuned for the maximum fitness to experimental data recovered.

In addition, the system is modelled to have a saturation of 20.1V. This was selected as it was found that trials with over 20V of input would have a sharp drop after a rapid rise, eventually stabilizing roughly to the same point as 20V. A more accurate saturation voltage could have been found, however, at the expense of simplicity of the model ad having to account for the sharp drops in current at high voltages.

Lastly, a non-linear improvement to the model could be made by factoring in the temperature dependence of the resistance values. By conducting experiments at a variety of resistances and calculating the true resistance to temperature relationship, an improvement could be made to the transfer function accuracy at the expense of a simplified equation.

Considering all factors, the model of current is an accurate approximation of the current generated through the system given an initial input voltage. The rise time, steady-state value, and noise fluctuations of the generated data matches well with the actual data found experimentally, thus, the model satisfies its requirements. This is confirmed in the validation section.

B. Linearization

1) Non-Linear Model: The relationship between input voltage, and the position of the ball is inherently non-linear. Three equations are used to model the non-linear system, incorporating input voltage, coil current, and the resulting ball position:

$$v(t) = (R_c + R_s) + L\frac{di(t)}{dt} \tag{6}$$

$$f(t) = k_m \frac{i^2(t)}{2x^2(t)} \tag{7}$$

$$m_b \frac{d_2x(t)}{dt^2} = -f(t) + m_b g \tag{8}$$

Table I below presents the values relevant to the magnetic levitation system, extracted from the Quanser Consulting Magnetic Levitation User Manual [2].

TABLE I
MAGNETIC LEVITATION MODEL VALUES

Variable	Description	Value	Unit
$v(t)$	Input Voltage	$Control Input$	V
$i(t)$	Coil Current	$Control Variable$	mA
$f(t)$	Lift Force	$Control Variable$	kgm^2
$x(t)$	Ball Position	$Control Output$	m
R_c	Coil Resistance	10	Ω
R_s	Sensor Resistance	1	Ω
L	Coil Inductance	412.5	mH
k_m	Force Conversion const.	$6.5308x10^{-5}$	N-m$^2/A^2$
m_b	Ball Mass	0.068	kg
g	Gravitational const.	9.81	m/s^2

Combining Equation 7 and Equation 8 the following equation is produced relating ball acceleration to current and ball position.

$$\frac{d_2x(t)}{dt^2} = g - \frac{k_m i^2(t)}{2m_b x^2(t)} \tag{9}$$

Furthermore, the circuit equation which relates voltage to coil current can be simplified as follows:

$$\frac{di(t)}{dt} = -\frac{R}{L}i(t) + \frac{1}{L}v(t) \tag{10}$$

To linearize the system, an operating point must be carefully selected. In the case of a magnetic levitation system, this operating point, which the system will be modelled around, is the equilibrium point where the ball stays relatively stationary. The operating point and relative perturbation variables will be discussed in further detail in section *3) State Space Model*. At the equilibrium point, the ball will ideally not move, thus the acceleration term can be set to 0. Additionally, current should not change and the $\dot{i}$ term can also be set to 0, yielding the following equations:

$$0 = g - \frac{k_m i^2(t)k}{2mx^2(t)} \tag{11}$$

$$0 = -\frac{R}{L}i(t) + \frac{1}{L}v(t) \tag{12}$$

Equation 11 can then be rearranged to represent a non-linear relationship between current and ball position as follows:

$$i(t) = \sqrt{\frac{2mg}{k_m}}x(t) \tag{13}$$

In contrast, the relationship between voltage and current at the equilibrium point is linear.

$$v(t) = i(t)R \tag{14}$$

Utilizing equations 13 and 14, design parameters from Table I, and a selected equilibrium point $x(t)$, one can solve for current

(i) and voltage (v) values. However, the system remains non-linear an cannot be represented by a transfer function, nor can it be controlled unless it is linearized about the equilibrium point.

2) Taylor Series Expansion: To obtain a linear model of the system that represents the ball's motion, Taylor Series Expansion is used.

Defining variables:

- x_e = Equilibrium Position
- $\dot{x}_e$ = Ball Velocity at Equilibrium
- i_e = Equilibrium Current
- v_e = equilibrium input voltage

The following equation represents the first two terms of the system's equation.

$$\frac{d^2x}{dt^2} = a_1(x - x_e) + a_2(\dot{x} - \dot{x_e}) + a_3(i - i_e) + b_1(v - v_e) \quad (15)$$

a_1 represents magnitude of position at equilibrium point:

$$a_1 = \frac{\partial f_1}{\partial x}|_{equil} = \frac{\partial}{\partial_x}(g - \frac{K_m i^2}{2mx^2})|_{equil} = \frac{K_m i_e^2}{2mx^3} \quad (16)$$

a_2 represents velocity at equilibrium point which should have a value of 0 as the ball ideally should be moving:

$$a_2 = \frac{\partial f_1}{\partial \dot{x}}|_{equil} = \frac{\partial}{\partial x}(g - \frac{K_m i^2}{2mx^2})|_{equil} = 0 \quad (17)$$

a_3 represents current at equilibrium point:

$$a_3 = \frac{\partial f_1}{\partial i}|_{equil} = \frac{\partial}{\partial i}(g - \frac{K_m i^2}{2mx^2})|_{equil} = -\frac{K_m i}{2mx^2}|_{equil} \quad (18)$$

$$a_3 = -\frac{K_m i}{2mx^2}|_{equil} = -\frac{K_m i_e}{2mx_e^2} \quad (19)$$

a_2 represents voltage at equilibrium point:

$$b_1 = \frac{\partial f_1}{\partial v}|equil = \frac{\partial}{\partial v}(g - \frac{K_m i}{2mx^2})|equil = 0 \quad (20)$$

Using the following equations, a state-space model of the system can be defined.

3) State Space Model: Equations 9 and 10, which represent the non-linear system are functions of: $\ddot{x}(t)$, x, $\dot{i(t)}$, $i(t)$, and $v(t)$. Thus, the state variables are selected as follows:

- x_1: Ball Position
- x_2: Ball Velocity
- x_3: Coil Current
- u: Input voltage

These can be found with this set of equations.

$$x_1 = x - X_e \quad (21)$$

$$x_2 = \dot{x} - \dot{X}_e \quad (22)$$

$$x_3 = i - i_e \quad (23)$$

$$u = v - v_e \quad (24)$$

These variables present the difference between the equilibrium values in which the current position is known, which results in the difference between them. Thus the values needed to achieve to correct for error. This acts as a perturbation factor in which we can solve for the equilibrium values comparing them to a set operating points values for equilibrium states.

$$\dot{x_1} = 0x_1 + x_2 + 0x_3 + 0u(t) \quad (25)$$

$$\dot{x_2} = a_1x_1 + 0x_2 + a_3x_3 + 0u(t) \quad (26)$$

$$\dot{x_3} = 0x_1 + 0x_2 + \frac{-R}{L}x3 + \frac{1}{L}u(t) \quad (27)$$

This translates to a state space model:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ a_1 & 0 & a_3 \\ 0 & 0 & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} u(t)$$

This is when the x_1 ball position is equal to the x_e ball position at equilibrium. Substituting the variables in results in.

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{k_m i^2(t)}{m_b x^3(t)} & 0 & -\frac{k_m i(t)}{m_b x_e^2(t)} \\ 0 & 0 & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} u(t)$$

To test whether this model gives us a stable transfer function, we give a reasonable ball equilibrium position and check for stability.

Assume $x(t) = 0.005m$ then plugging that value into equation 13 results in, This will be where the operating point is set for the associated equilibrium values.

$$i(t) = \sqrt{\frac{2 * 0.068 * 9.81}{6.5308 * 10^{-5}}} * 0.005$$

$$i(t) = 0.715A$$

Using this value as the current for equation 14.

$$v(t) = 0.715 * (10 + 1)$$

$$v(t) = 7.865$$

These values are in expectation as the ball is fairly close to the magnet and would require less current than when the ball is initially lifted off from its resting point in which required a current of about 2A. With these values we can plug everything into our state space and use the MATLAB function ss2tf to produce a transfer function.

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 3927.9 & 0 & -27.467 \\ 0 & 0 & -26.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2.42 \end{bmatrix} u(t)$$

The resulting transfer function is

$$G(s) = \frac{-66.4701}{s^3 + 26.7s^2 - 3928s - 104870} \quad (28)$$

The transfer function is based off of the operating point assuming the equilibrium ball position of 0.005m. This transfer function therefore is set up to levitate at the x position so the equilibrium values are then put into the state space model and create this transfer function. This transfer function has poles at -26.7, -62.67 and 62.67. This means two stable poles and one unstable pole.

III. Model Validation

A. Current Model

Next, the fitness of the model is evaluated, specifically, the model of the current in the system as we have empirical data to compare against.

The Root Mean Square Error approach is extremely useful in control systems to compare the fitness of a model for a system [1]. In this experiment, we will use this metric to compare the average sample difference between the modelled current and the simulated current.

Three different sets of data were used in validating the model on the real system. The first set of trials showed slightly different results than that of the expected outputs, as well as having values shifted in comparison to those of sets 2 and 3. In each data set, a trial was conducted of the current response given voltages of 5V, 10V, 15V, 20V, 22V, 24V.

The equation for the RMSE error can be found below:

$$RMSE = \sqrt{(\frac{1}{n})\sum_{i=1}^{n}(x_{real,i} - x_{sim,i})^2} \qquad (29)$$

The following RMSE were found for the different sets of data and trials:

TABLE II
RMSE By Trial compared to Simulation Results

Voltage Supplied	Trial 1 RMSE	Trial 2 RMSE	Trial 3 RMSE
5V	0.0143	0.0036	0.0059
10V	0.0392	0.0136	0.0060
15V	0.0590	0.0234	0.0079
20V	0.0704	0.0312	0.0224
22V	0.0292	0.0815	0.0869
24V	0.0306	0.0811	0.0871

The RMSE data alone does not show much - to be able to truly extract meaningful information about it, the steady-state current is used to find an approximate error percentage:

TABLE III
Simulation Steady-State Current

Voltage Input	Simulation Steady-State Current
5V	0.498A
10V	1.001A
15V	1.499A
20V	2.002A
22V	2.011A
24V	2.011A

The accuracy per trial can then be calculated by:

$$[H]\%_{Error} = 100\% - \frac{i_{ss} - i_{RMSE}}{i_{ss}} \times 100 \qquad (30)$$

By applying this to all different trials, we can get a quantitative idea on the accuracy of the model:

TABLE IV
Percentage Error Per Trial

Voltage	Trial 1	Trial 2	Trial 3
5V	2.87	0.72	1.18
10V	3.91	1.36	0.60
15V	3.94	1.56	0.53
20V	3.52	4.05	4.32
22V	1.45	4.05	4.32
24V	1.52	4.03	4.33

It may look like the trials of Trial 1 are more accurate than the others, however, it is worth noting that Trials 2 and 3 decline in performance simply due to overshoot encountered by the system. Below is the current output on the real-world system using a step voltage of 24V:

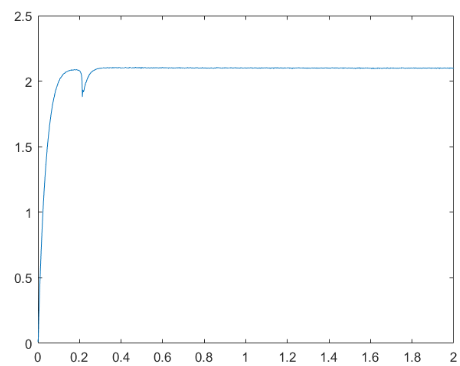

Fig. 9. Real Current for V = 24V Input

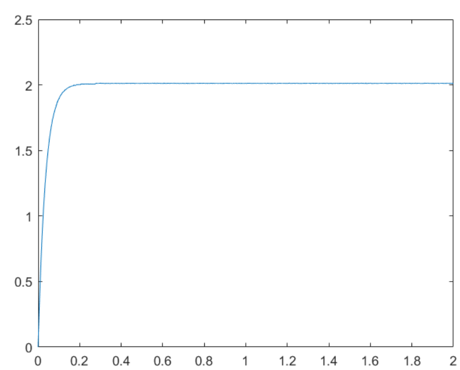

Fig. 10. Simulated Current for V = 24V Input

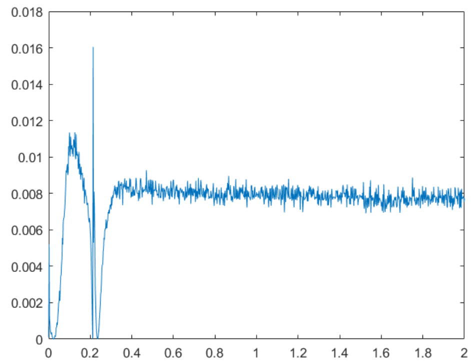

Fig. 11. Squared Error for V = 24V Comparing Trial 3 to the Simulation

It is clear that the real system has a rapid fluctuation that occurs when going past the saturation point set by our model. This was specifically not included in our design of the circuit model, as this abnormal behaviour will be avoided by not surpassing this threshold when attempting to control the coil current. This is the reason for the approximated error increasing at higher voltages - the real current has a flicker in the rise of the signal, this causing deviations between the real and simulated models.

B. Linearization

Through the use of perturbation theory, the solvable equation of what the voltage and current should be at a specific operating point assuming equilibrium, (zero acceleration and velocity), allows the creation of a transfer function in which is usable to simulate the motion of the ball in the magnetic levitation system. Using this transfer function we will create a controller which can properly ensure steady state levitation using the magnetic levitation system. For this transfer function the assumed operating point was x = 0.005m from the magnet, this operating point will tried and tested on the magnetic levitation system in the next couple of weeks.

IV. Discussion

One important factor to note for future design is the non-linearities introduced based on the temperature of the coil. It has been discovered that after prolonged experimentation, the coil will increase in temperature a non-negligible amount. This temperature dependence is significant when looking at the fluctuations of results based on the time between trials. Set 1 of the trials were conducted with the minimum time between experiments, causing disagreement with Sets 2 and 3 that differ with voltage supplied. Sets 2 and 3 are more accurate at lower voltages (5V, 10V, 15V) and are comparable at 20V, however, suffer in accuracy at higher voltages of 22V and 24V.

This is particularly intriguing, as it seems waiting approximately 15 minutes between trials leads to a more accurate model for lower voltages, while shorter periods of cooling between trials lead to a more accurate model at higher voltages. This is important to note as the high voltages are not the ideal area of operation for this system. By saturating the voltage provided at approximately 20V, the model will be an accurate prediction of how the system works while reducing the possibility of rapid current fluctuations.

Verified using the RMSE approach, it was demonstrated that the current model derived in this paper will adequately model the current provided by a voltage step function within the assumed bounds of operation. This will be used with the linearized system model to create an efficient control system demonstrating the control of ball position in the magnetic levitation system.

V. Conclusions

In this paper, a state space equation and two transfer functions for the two different sub-systems of the magnetic levitation system were developed. The first model relates voltage with the current provided through the coil. The transfer function used was a first order transfer function. The second model relates current to the force created by the electromagnet. The transfer function that describes this subsystem was a second order transfer function. A state space equation was developed for this subsystem. To test whether this model gives us a stable transfer function, a reasonable ball equilibrium position is provided and stability is monitored.

Using a controller and a position sensor, a negative feedback control system was developed. The negative feedback model (based on the approximate linear model of the two dimensional magnetic levitation system) was designed to stabilize the system and keep the steel ball levitating at a specific steady state value. Validation techniques such as using the root mean squared error between the coil current and the simulated coil current was used to explore the accuracy of the system.

References

[1] Prentice Hall, Digital Process Control: Analysis and Design [Textbook], 2014.

[2] Quanser, User Manual: Magnetic Levitation Experiment [Manual], 2012.