Solving Blasius Equation

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December 6, 2020

(a) Blasius Equation

Starting off with the continuity equation

$$\frac{D\rho}{Dt} + div(\rho V) = 0,$$

which due to considering both assumptions of the fluid being incompressible and steady, will be simplified to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \quad (1)$$

This equation may not seem that important now, but lucky for us, the continuity equation reduced down nicely to equation 1, which can be satisfied identically and substituted by what's known as a stream function ψ, which will come later in the analysis.

Moving on to the Navier-Stokes equation

$$\rho \frac{DV}{Dt} = \rho g - \nabla p + \mu \nabla^2 V$$

which like the continuity equation, will only be defined in the x and y directions. In order to simplify this equation, we need to have some idea of the physics of the problem first. First of all, we know that there is a no slip condition at the plate, meaning there is a veloctiy gradient in the y direction. By knowing that, we realize that the viscosity term μ must stay. Also, because the flow is driven by a streamline flow of constant velocity U in the x direction, we can conclude that within the boundary layer, velocity component in the y direction v is much smaller than that in the x direction

u. One thing to note is that outside the boundary layer, although the fluid has a viscosity, it is treated as an inviscid flow, and that's because it has a constant velocity U. Gravity effects can be neglected as the boundary layer is thin and the reynold's number is much greater than one. Same can be said about the pressure gradient in the flow.

With that out of the way, the Navier-Stokes equation in the x direction reduces down to

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

where ν is the kinematic viscosity term which equates to the ratio of the dynamic viscosity over density, $\frac{\mu}{\rho}$. The reason for the absense of the double derivative of velocity u with respect to x is because the change in the velocity in x is much smaller than that in y; and so the double derivative $\frac{\partial^2 u}{\partial x^2}$ will yield an even smaller quantity, which can be neglected. Because the flow is mainly in the x direction, the Navier-Stokes equation in the y direction will not be of interest or use to us in the analysis, as it will simplify down to

$$\frac{\partial p}{\partial y} = 0.$$

As far as the boundary conditions go, as I mentioned earlier, we do have a no slip condition for the flow at the plate, which yields two conditions for u and v such that,

$$u = v = 0; \quad y = 0.$$

Another boundary condition is that the velocity u approaches the streamline velocity U as we get closer to the boundary layer edge and beyond, and so

$$u = U; \quad y = \delta.$$

Here δ is some function of x which describes the height of the boundary layer edge moving further down the plate in the x direction. Recalling the derivation of $\delta(x)$ in White, 2006, we find it to be

$$\delta = \left(\frac{\nu x}{U}\right)^{\frac{1}{2}}. \tag{3}$$

Now in order for us to move forward and find the celebrated solution for this problem derived by Ludwig Prandtl, which was H. Blasius's first student, we need to propose some sort of implicit equation for u which has to satisfy all boundary conditions mentioned above. One can assumme that to be

$$u = U\Big(2\frac{y}{\delta} - \big(\frac{y}{\delta}\big)^2\Big) \tag{4}$$

We can also introduce a new variable η such that

$$\eta = \frac{y}{\delta}, \tag{5}$$

and so equation 4 can be rewritten as

$$u = U\phi(\eta), \tag{6}$$

where

$$\phi(\eta) = 2\eta - \eta^2$$

In the beginning of the analysis, I mentioned that I can use the stream-function here because I ended up with equation 1 for the continuity equation. Hence

$$u = \frac{\partial\psi}{\partial y}$$
$$v = -\frac{\partial\psi}{\partial x}.$$

Substituting both velocities in equation 2, we get an ungly equation looking like this

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = \nu\frac{\partial^3\psi}{\partial y^3} \tag{7}$$

Despite equation 7 still being an ugly PDE, we reduced down the dependent variables to only one, which is obviously ψ. But we still need to go a little further and transform this into a nice looking ODE. We can rewrite ψ in an integral form such that

$$\psi = \int_0^y u\,dy.$$

Applying the same change of variable as I did in equation 5, we can rewrite the integral, hence

$$\psi = \delta \int_0^\eta u \, d\eta = \delta \int_0^\eta U g(\eta) \, d\eta = \delta U f(\eta), \tag{8}$$

where

$$f(\eta) = \int_0^\eta g(\eta) \, d\eta.$$

Now that we know the value for ψ, we need to plug it back into equation 7, which won't be very easy, as it got messy as I was trying to find some of the values. So we can take this step by step and start with velocities u and v, such that

$$u = \frac{\partial \psi}{\partial y} = \delta U \frac{\partial f(\eta)}{\partial y},$$

but because we are aiming to obtain an ODE, we need to get rid of anything with the term y in it, and so, we can rewrite u to be

$$u = \delta U \frac{\partial f(\eta)}{\partial \eta} \cdot \frac{\partial \eta}{\partial y}.$$

Luckily for us,

$$\frac{\partial \eta}{\partial y} = \frac{1}{\delta}.$$

Hence, u becomes

$$u = U \frac{\partial f(\eta)}{\partial \eta} \tag{9}$$

And now finding v is a bit more work in terms of deriving it, as we would need to apply the product rule when finding the derivative of ψ with respect to x. However, I do not want to bore you with the math, and will give you the final result, which is

$$v = U \frac{\partial \delta}{\partial x} \Big(\eta \frac{f(\eta)}{\partial \eta} - f(\eta) \Big) \tag{10}$$

And just like we found u and v in terms of $f(\eta)$, we can find the terms in equation 7 in the same manner. It is a lot of messy math, but after some calculations, equation 7 reduces down to the celebrated solution, which is

$$\frac{1}{2} f \frac{d^2 f}{d\eta^2} + \frac{d^3 f}{d\eta^3} = 0 \tag{11}$$

The previous boundary condition still hold, and mathematically translate to $f(\eta)$ to become

$$f'(0) = f(0) = 0$$
$$f'(\infty) = 1$$

(b) Runge-Kutta Method

It is a lot more computationally efficient to use the Runge-Kutta method for solving the previously mentioned blasius equation (eq. 11). At it's core, this numerical method uses the values of multiple slopes at different points, gets their average and finds the solution for the next time step. Here, I am using the Runge-Kutta of the fourth order, meaning I will be using four different slopes. But before we start solving equation 11, we need to rewrite is as first order, hence

$$f' = G$$
$$f'' = G' = H$$
$$f''' = G'' = H' = -\frac{1}{2}fH.$$

According to the 4th order Runge-Kutta scheme, solution f at next time step $n+1$ would be

$$f_{n+1} = f_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \tag{12}$$

where values of k are slope values. Similarly, G and H can be defined as

$$G_{n+1} = G_n + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) \tag{13}$$

$$H_{n+1} = H_n + \frac{1}{6}(m_1 + 2m_2 + 2m_3 + m_4). \tag{14}$$

Similar to k, values of l and m are also slope values. These slope values can be defined as

$$k_1 = hF_1(f_n, G_n, H_n, \eta_n)$$

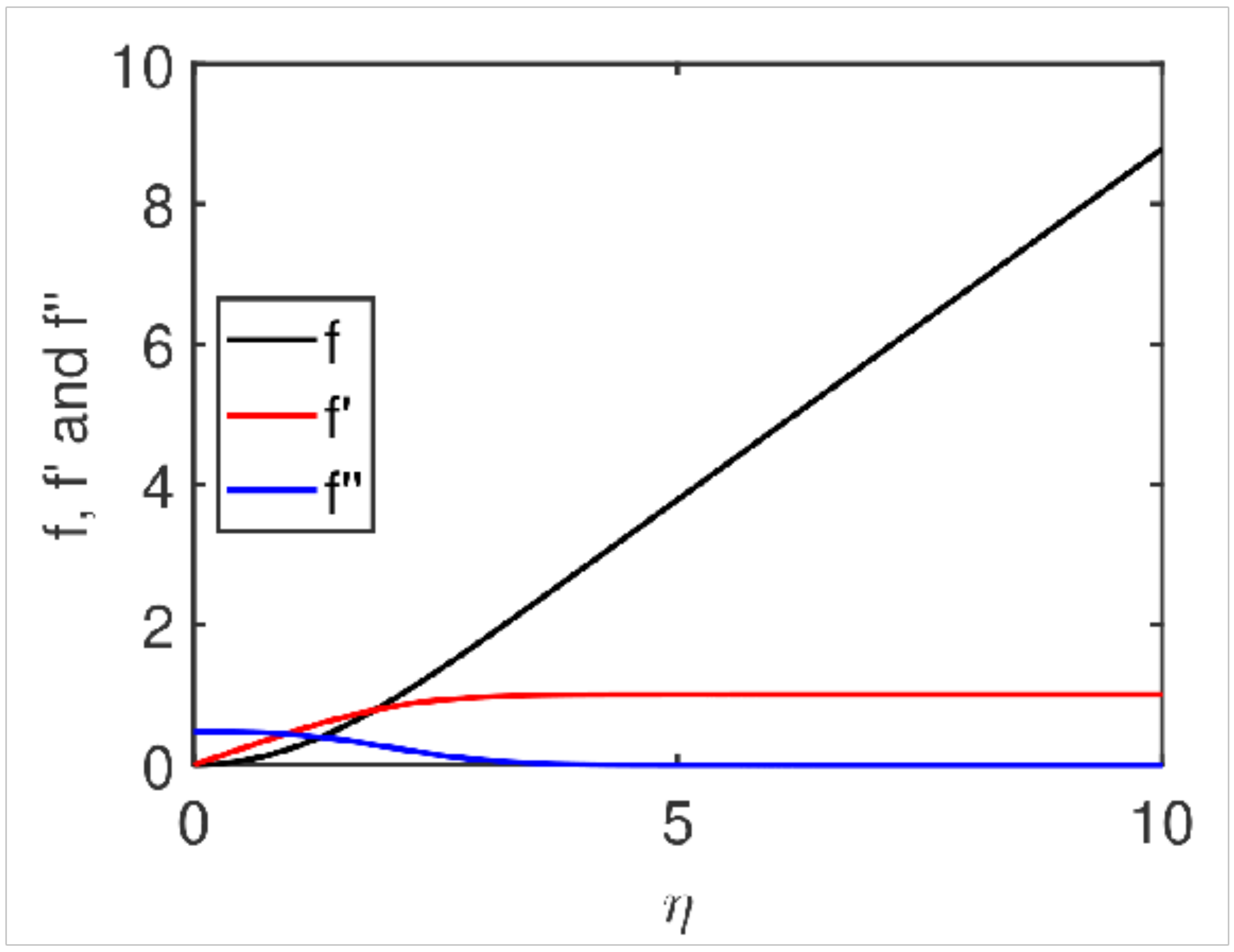

Figure 1: Solution of f, f' and f''

$$
\begin{aligned}
l_1 &= hF_2(f_n, G_n, H_n, \eta_n) \\
m_1 &= hF_3(f_n, G_n, H_n, \eta_n)
\end{aligned}
$$

$$
\begin{aligned}
k_2 &= hF_1\Big[(f_n + \frac{1}{2}k_1), (G_n + \frac{1}{2}l_1), (H_n + \frac{1}{2}m_1, (\eta_n + \frac{1}{2}h)\Big] \\
m_2 &= hF_2\Big[(f_n + \frac{1}{2}k_1), (G_n + \frac{1}{2}l_1), (H_n + \frac{1}{2}m_1, (\eta_n + \frac{1}{2}h)\Big] \\
l_2 &= hF_3\Big[(f_n + \frac{1}{2}k_1), (G_n + \frac{1}{2}l_1), (H_n + \frac{1}{2}m_1, (\eta_n + \frac{1}{2}h)\Big],
\end{aligned}
$$

where

$$
\begin{aligned}
f' &= F_1(f, G, H, \eta) \\
G' &= F_2(f, G, H, \eta) \\
H' &= F_3(f, G, H, \eta).
\end{aligned}
$$

The rest of the slopes can be found in a similar fasion to k_2, m_2 and l_2. Looping though the the slope values and substituting them for the next time step, while incrementing η values, we can solve equation 11 and get the solution seen in figure 1. According to White, 2006, f' approaches the value of 1 as η is aoround 3.5, and my solution agrees with that. f' is the ratio of u to that of the streamline velocity U, and so u approaches U when η is around 3.5. On the other hand, f'' is proportional the the wall shear stress τ_w which dies out as we move further down the plate. In my solution, f'' also dies out. The reason for this is because the shear stress at the wall is

$$\tau_w = \mu \frac{\partial u}{\partial y}\bigg|_w,$$

and the change in u with respect to y decreases because it u will need to cover more distance in y to get to the streamline velocity, whereas at the beginning of the plate, the term $\frac{\partial u}{\partial y}$ is so high, making the shear stress maximum there. f is proportional to the stream function ψ.

(c) Example

For this example, I will be working with water as my fluid. The parameters used are:

- Density $\rho = 997\ kg/m^3$
- Dynamic Viscosity $\mu = 8.9 * 10^{-4}$ Pa.s
- Streamline Velocity $U = 1\ m/s$
- Plate Length $L = 10$ m
- Step Size $h = 0.01$

(c1) Velocity Vector Field

Unfortunately, I was not able to plot the velocity vector field. I tried adding a meshgrid which included both velocities u and v, but I ended up with a one dimensional line, and I figured that was false. I then tried to plot the vectors in such a way that $|u| = \left(\frac{y}{\eta}\right)^2 \frac{U}{2\nu}$, but that did not work and I was getting weird dimensional errors.

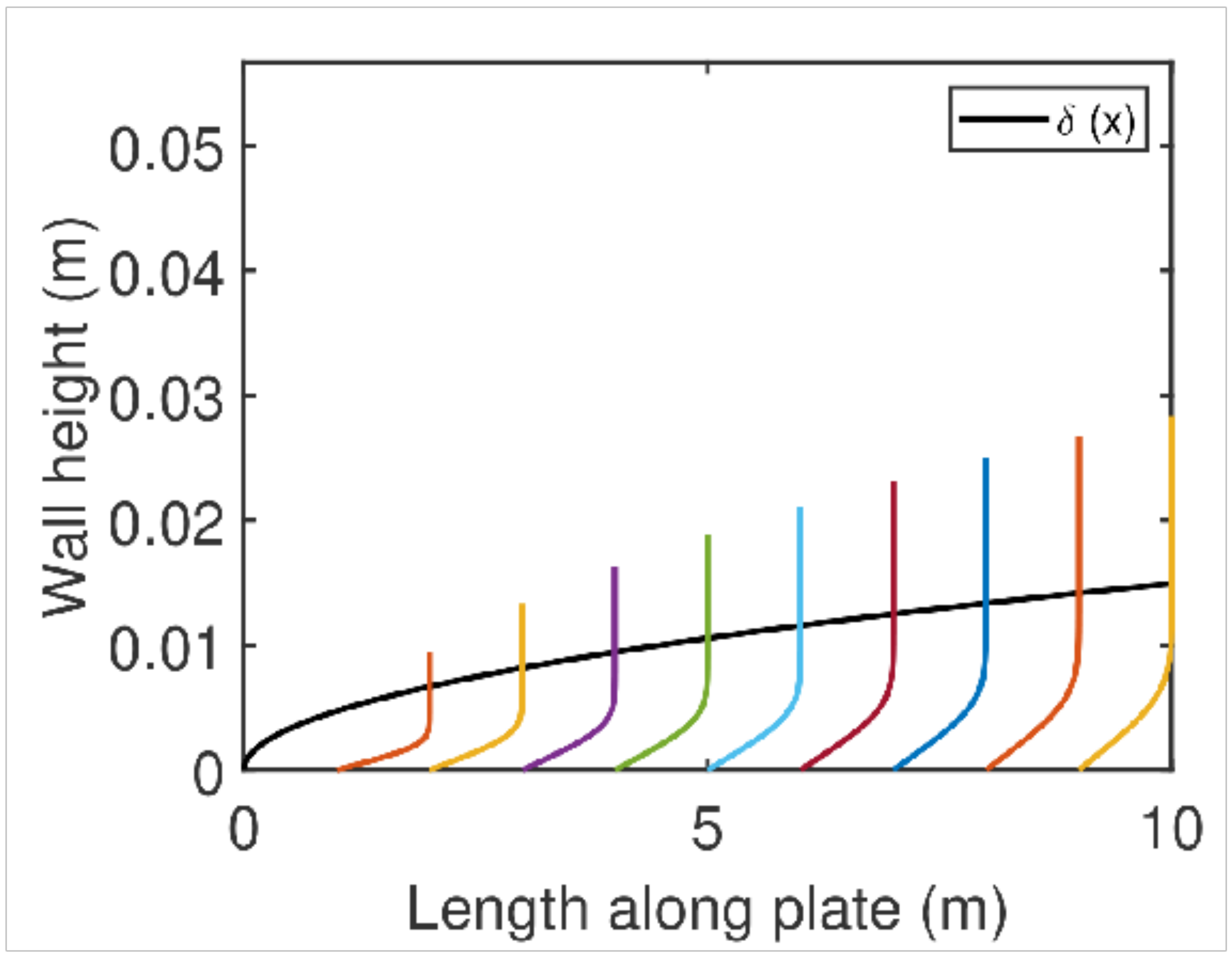

Figure 2: $\delta(x)$ and $u(x, y)$

(c2) Streamlines with Boundary Layer

Figure 2 shows the velocity streamlines $u(y)$ at different locations along the flat plat. In addition to the streamlines, there is also the boundary layer thickness as a function of x. Both the streamlines and the BL thickness plots make sense because $\delta(x)$ is supposed to increase in magnitude further down the plate, and the velocity streamlines $u(y)$ satisdy both the no-slip condition as well as reaching a constant velocity U at the BL and beyond.

(c3) Velocity Profiles

I did not understand this question.

(c4) Wall Shear Stress

Figure 3 shows the behavior of the shear stress as we move further down the plate. This is to be expected as I explained before. The shear stress will die out as x increases.

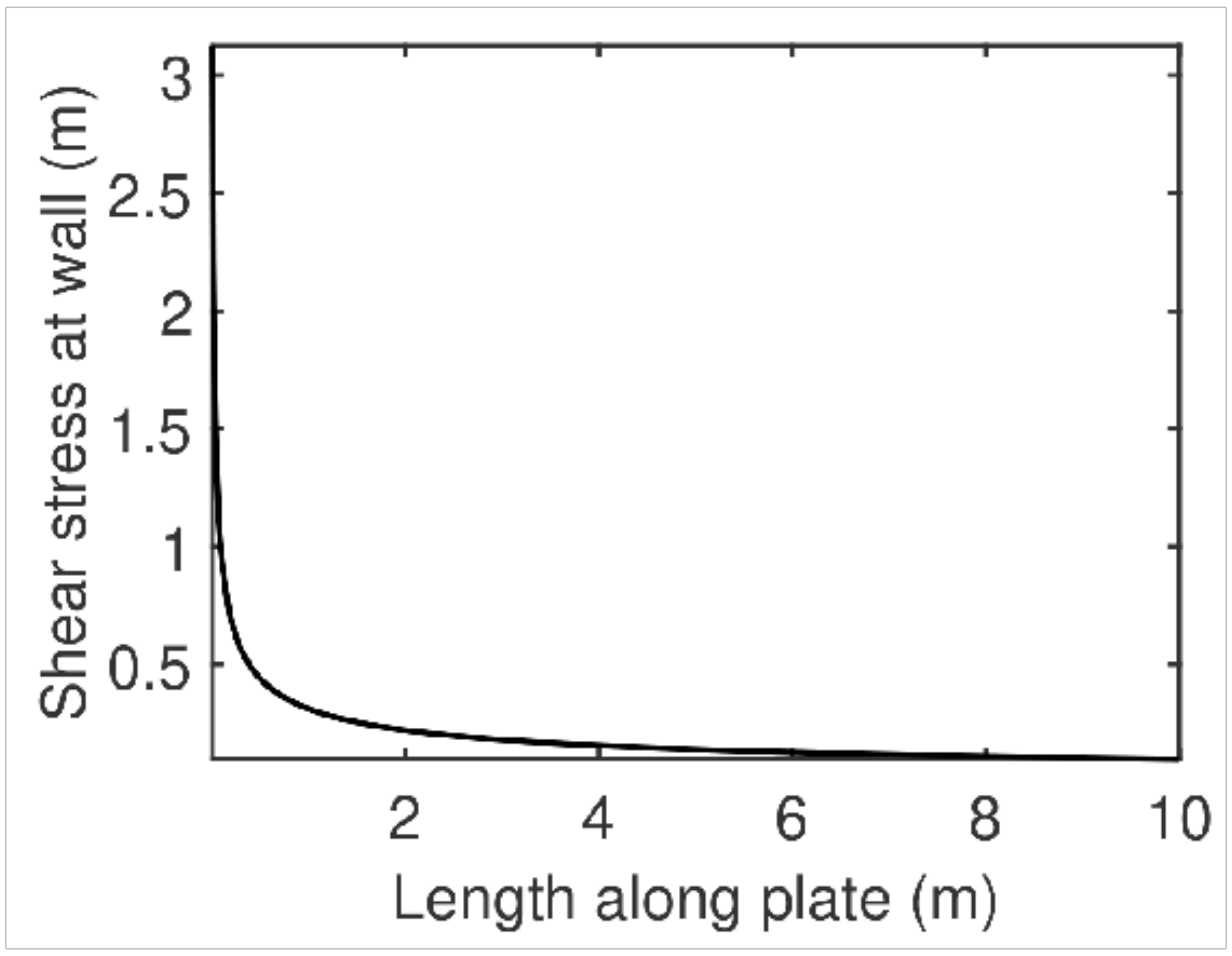

Figure 3: Wall shear stress along the plate length

(c5) Drag Force

The drag force is defined as the integral of the shear stress along the length of the plate, hence

$$D = \int_0^L \tau_w(x)\, dx, \tag{15}$$

and we know the formula for shear stress, which is

$$\tau_w = \mu \frac{\partial u}{\partial y}\Big|_{y=0} \tag{16}$$

The way I found the drag force was summation of drag force at every x. We know that coefficient of friction C_f and coefficient of drag C_D are

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2}$$

$$C_D = \frac{D}{\frac{1}{2}\rho U^2 L},$$

respectively. And so, we can find a relationship between both coefficients and solve for D, the drag force. Hence

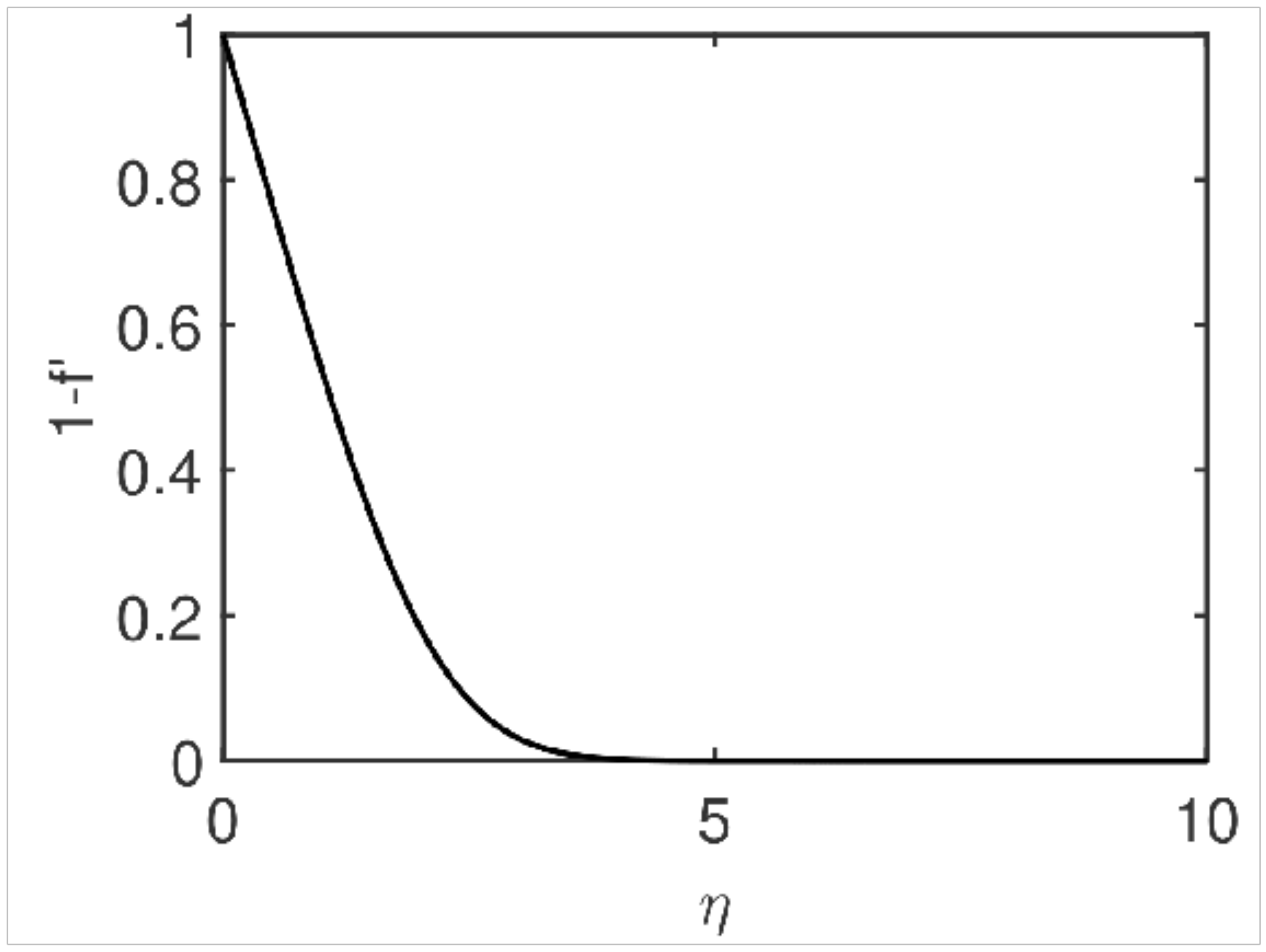

Figure 4: $(1 - f')$ as a function of η

$$D = \frac{C_D L \tau_w}{C_f}. \tag{17}$$

We have the values of τ_w along the plate, so all we need to do is loop through those values and substitute every instance of τ_w in equation 17 then add the values up, which would sum up to be $2550.216N$.

(c6) Numerical Integration & Comparison

For the displacement thickness δ^*, according to White, 2006, the integral

$$\int_0^\infty (1 - f')\, d\eta = 1.21678.$$

To find the integral numerically, I need to plot $1 - f'$ as a function of η, which I did in figure 4.

Now we can numerically integrate the area under the function $1 - f'$ and find it to be 1.221785, which is close to the previously mentioned area. Same thing can be said for the momentum thickness, where according to White, 2006, the integral

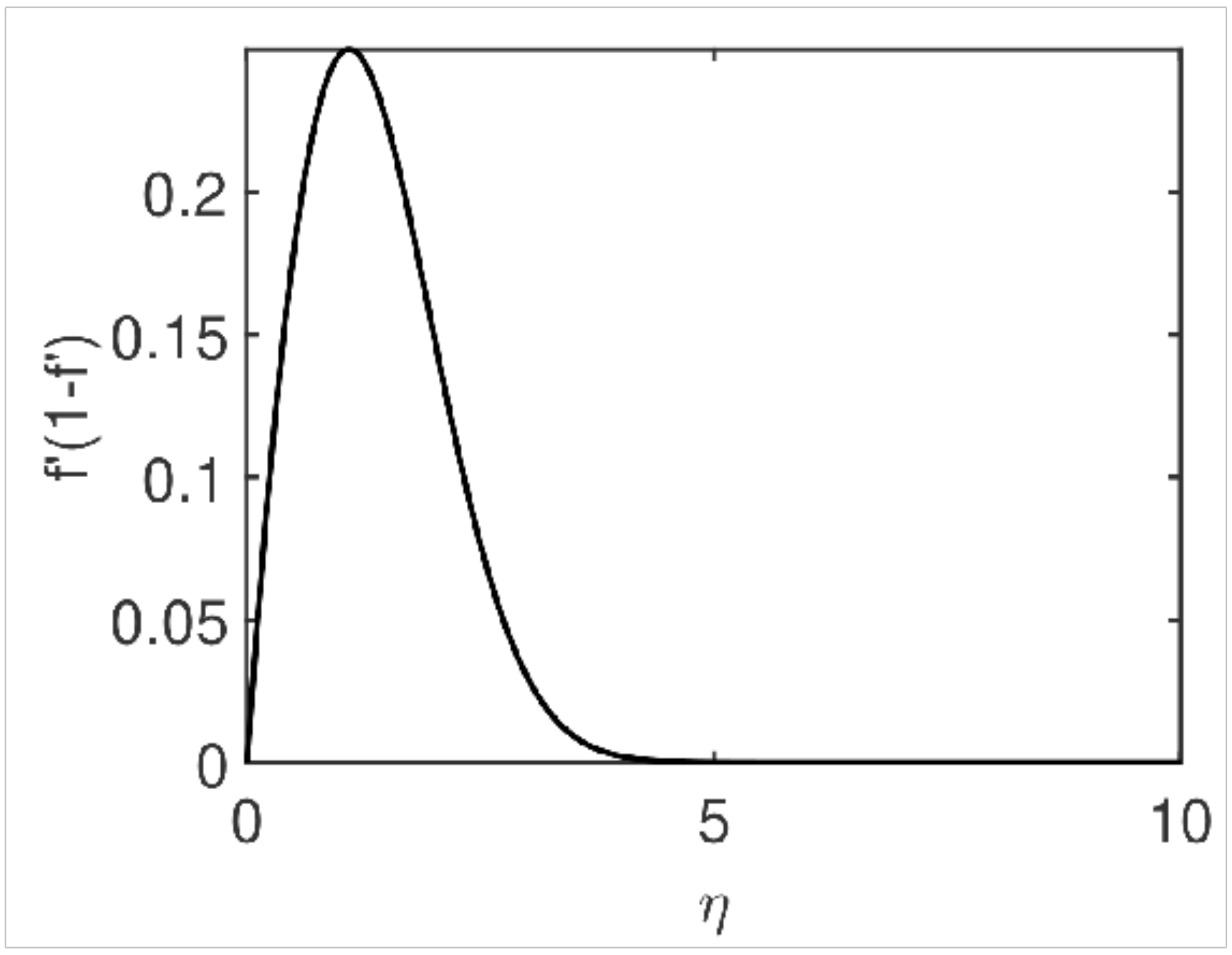

Figure 5: $f'(1-f')$ as a function of η

$$\int_0^\infty f'(1-f')\,d\eta = 0.4696.$$

So again, I plotted $f'(1-f')$ as a function of η, which can be seen in figure 5.

Integrating the function in figure 5 numerically, we find the area to be 0.469596107. I have proof of both numbers in a spreadsheet. I am not just making those numers up.

As for the MATLAB code [ElTahan, n.d.] I used, it was only for plotting the blasius solution in figure 1. The rest I ended up changing the code and generating the rest of the plots. I worked really hard on this report, and I tried my best to do part c1 and c3, but I couldn't.

References

ElTahan, A. (n.d.). Blaisus equation solution.
White, F. M. (2006). *Viscous fluid flow*. McGraw-Hill Higher Education.