Rayleigh Taylor Stability Analysis

Bilal Abdul Halim

November 24, 2020

Abstract

Studying the stability of a system can be very crucial to predicting its behaviour in the long run. But to do so analytically can be very complicated, as there starts to appear more and more variables and parameters, and the problems are almost never linear in nature. Obviously, we cannot discretize the problem and try to approximate it by hand, because that will take ages, and there are cpus that do it instantly. Because of that, we need to take as many assumptions as permitted, without over simplifying the problem, and still getting a somewhat valid solution. Here, I attempt to do the same thing and develop a model based on the Rayleigh Taylor Instability, by looking at the melting of ice on room-temperature water. This yields the colder fluid being on top and the lighter one on the bottom, creating an instability.

Introduction

The Rayleigh-Taylor instability was first studied by Lord Rayleigh in 1883, and Sir G.I. Taylor in 1950 [1]. The instability occurs at the interface of two fluids with different densities; and so, assuming the placement of the lighter fluid at the bottom and the heavier fluid at the top, the lighter fluid works it's way upward. And as the instability grows, spikes, curtains and bubbles start developing, and the flow quickly becomes turbulent and chaotic.

In this report, I apply the Rayleigh-Taylor instability to a slab of ice floating on room-temperature water. This causes an exchange of heat transfer, and a local displacement of cold water and warm water. This displacement is driven by the different densities of water (at different temperatures), which is driven by gravity. And so for my model, I will consider two completely plane-parallel layers of a miscible fluid (water), with the more dense on top and the less dense on the bottom. To solve less partial differential equations, I will completely ignore any effects of heat transfer happening. I know this is ironic considering the fact that my entire problem is based on heat transfer, but I am strictly looking at the fluid instability growth, and not what caused it. Other important assumptions will be mentioned later on in my analysis.

Stability Analysis

Governing Equations

For my problem, I will start with the base state of the two fluids without the addition of any kind of disturbance. This will yield the continuity equation of the form

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + div(\rho V) = 0$$

$$div(\rho V) = \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z}$$

Because I want to solve this model by hand and not a computer, I will assume the two fluids to be incompressible. After all, this is not a statistical mechanics class. And so the continuity equation reduces down to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

.

As for the Navier-Stokes equation,

$$\rho \frac{DV}{Dt} = \rho g - \nabla p + \mu \nabla^2 V$$

,

the viscosity term μ is zero as the flow is assumed to be inviscid. And the Navier-Stokes equation can be defined in all three directions x, y and z. It is also important to note that the term ρg only exists in the y direction as that is where gravity is defined.

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x}$$
$$\rho \frac{Dv}{Dt} = \rho g - \frac{\partial p}{\partial y}$$
$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z}$$

. Because we are looking at the base state of the two fluids, the term $\frac{DV}{Dt}$ is zero. And so all three equations reduce down to

$$0 = -\frac{\partial p}{\partial x}$$
$$0 = \rho g - \frac{\partial p}{\partial y}$$
$$0 = -\frac{\partial p}{\partial z}$$

.

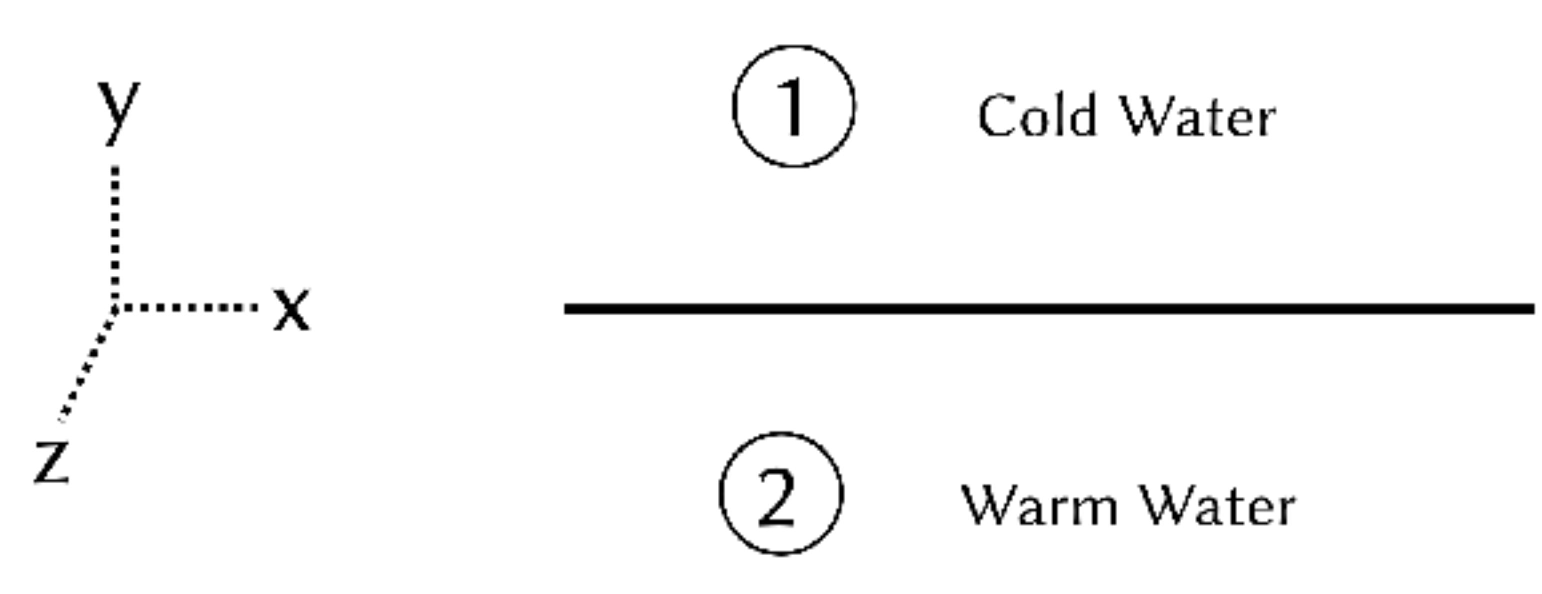

Figure 1: Problem schematic

From this, it can be concluded that the pressure is only a function of y, and can be solved to get the hydrostatic pressure equation

$$p = -\rho g y + C$$

,

where C is some constant. In order to find C, we can impose a boundary condition at the interface because we know that the pressure of both fluids is equal, which means $p_1 = p_2$. For the sake of simplicity, I will take the pressure at the interface to be 0. That value won't matter because flow is driven by pressure differences, and not actual values. And so, having taken the pressure to be zero, constant C becomes 0, simplifying the pressure to be

$$p = -\rho g y$$

.

Adding Disturbance

Let $u_1 = u_{b1} + \hat{u_1}$ be the sum of the velocity at base state u_{b1} and that of disturbance $\hat{u_1}$. The subscript 1 denotes fluid 1, but the analysis is the same for fluid 2. Similarly,

$$v_1 = v_{b1} + \hat{v_1}$$
$$w_1 = w_{b1} + \hat{w_1}$$
$$p_1 = p_{b1} + \hat{p_1}$$

.

One important thing to note is that the disturbance added is going to be very small. For example, $\hat{u_1} = \epsilon\phi$ where ϕ is some velocity. And so, I am going to neglect any term of order ϵ^2 or higher, as it is very small and only adds more complexity to the model.

Continuity Equation

Now the continuity equation, including the disturbance is going to be

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0$$

$$\frac{\partial(u_{b1}+\hat{u}_1)}{\partial x}+\frac{\partial(v_{b1}+\hat{v}_1)}{\partial y}+\frac{\partial(w_{b1}+\hat{w}_1)}{\partial z}=0$$

,

which simplifies to

$$\frac{\partial \hat{u}_1}{\partial x}+\frac{\partial \hat{v}_1}{\partial y}+\frac{\partial \hat{w}_1}{\partial z}=0 \tag{1}$$

because

$$\frac{\partial u_{b1}}{\partial x}=\frac{\partial v_{b1}}{\partial y}=\frac{\partial w_{b1}}{\partial z}=0$$

.

Navier-Stokes Equations

Navier-Stokes in x direction

$$\rho_1\Bigg(\frac{\partial u_{b1}}{\partial t}+\frac{\partial \hat{u}_1}{\partial t}+(u_{b1}+\hat{u}_1)\Big(\frac{\partial u_{b1}}{\partial x}+\frac{\partial \hat{u}_1}{\partial x}\Big)+(v_{b1}+\hat{v}_1)\Big(\frac{\partial u_{b1}}{\partial x}+\frac{\partial \hat{u}_1}{\partial x}\Big) \\ +(w_{b1}+\hat{w}_1)\Big(\frac{\partial u_{b1}}{\partial x}+\frac{\partial \hat{u}_1}{\partial x}\Big)\Bigg)=0$$

Now, we can easily simplify the above equation since $\hat{u}_1=\epsilon\phi$. This yields

$$\hat{u}_1\frac{\partial \hat{u}_1}{\partial x}=\epsilon^2\phi\frac{\partial \phi}{\partial x}=0$$

.

I also previously mentioned that

$$\frac{\partial u_{b1}}{\partial x}=\frac{\partial v_{b1}}{\partial y}=\frac{\partial w_{b1}}{\partial z}=0$$

.

Simplified Navier-Stokes equation in x direction becomes

$$\rho_1\frac{\partial \hat{u}_1}{\partial t}=-\frac{\partial \hat{p}_1}{\partial x}. \tag{2}$$

Similarly, Navier-Stokes equations in y and z directions are

$$\rho_1 \frac{\partial \hat{v}_1}{\partial t} = -\frac{\partial \hat{p}_1}{\partial y} \tag{3}$$

$$\rho_1 \frac{\partial \hat{w}_1}{\partial t} = -\frac{\partial \hat{p}_1}{\partial z}. \tag{4}$$

So now I need to solve these 4 PDES

$$\frac{\partial \hat{u}_1}{\partial x} + \frac{\partial \hat{v}_1}{\partial y} + \frac{\partial \hat{w}_1}{\partial z} = 0$$

$$\rho_1 \frac{\partial \hat{u}_1}{\partial t} = -\frac{\partial \hat{p}_1}{\partial x}$$

$$\rho_1 \frac{\partial \hat{v}_1}{\partial t} = -\frac{\partial \hat{p}_1}{\partial y}$$

$$\rho_1 \frac{\partial \hat{w}_1}{\partial t} = -\frac{\partial \hat{p}_1}{\partial z}$$

.

For fluid 2, we get the same 4 PDES to solve as well

$$\frac{\partial \hat{u}_2}{\partial x} + \frac{\partial \hat{v}_2}{\partial y} + \frac{\partial \hat{w}_2}{\partial z} = 0$$

$$\rho_2 \frac{\partial \hat{u}_2}{\partial t} = -\frac{\partial \hat{p}_2}{\partial x}$$

$$\rho_2 \frac{\partial \hat{v}_2}{\partial t} = -\frac{\partial \hat{p}_2}{\partial y}$$

$$\rho_2 \frac{\partial \hat{w}_2}{\partial t} = -\frac{\partial \hat{p}_2}{\partial z}$$

.

By Fourier's theorem and the general theory of disturbed equilibrium, Lord Rayleigh [2] proposes a solution of the form

$$\hat{u_1}(x, y, z, t) = u_1^*(y) e^{i(\omega_x x + \omega_z z) + \sigma t}$$

where $u_1^*(y)$ is some velocity, as we need to impose a boundary condition at the interface, in the y direction. ω_x and ω_z are spacial frequencies in x and z, respectively, assuming the two planes extend to infinity. This means we have a periodic solution in both these directions. σ is the temporal growth rate. We can propose similar solutions for the rest of the disturbances, hence

$$\hat{v_1}(x,y,z,t) = v_1^*(y)e^{i(\omega_x x+\omega_z z)+\sigma t}$$
$$\hat{w_1}(x,y,z,t) = w_1^*(y)e^{i(\omega_x x+\omega_z z)+\sigma t}$$
$$\hat{p_1}(x,y,z,t) = p_1^*(y)e^{i(\omega_x x+\omega_z z)+\sigma t}$$

. Substituting the proposed solutions into equation 1, we get

$$i\omega_x u_1^*(y)e^{i(\omega_x x+\omega_z z)+\sigma t} + \frac{dv_1^*(y)}{dy}e^{i(\omega_x x+\omega_z z)+\sigma t} + i\omega_z w_1^*(y)e^{i(\omega_x x+\omega_z z)+\sigma t} = 0$$

, but this can be simplified, since the term $e^{i(\omega_x x+\omega_z z)+\sigma t}$ can never be zero. This yields

$$i\omega_x u_1^*(y) + \frac{dv_1^*(y)}{dy} + i\omega_z w_1^*(y) = 0. \tag{5}$$

We also substitute the proposed solutions into equation 2, 3, and 4. After simplification, we get

$$u_1^*(y) = -\frac{i\omega_x p_1^*(y)}{\rho_1\sigma}$$
$$v_1^*(y) = -\frac{1}{\rho_1\sigma}\frac{dp_1^*(y)}{dy}$$
$$w_1^*(y) = -\frac{i\omega_z p_1^*(y)}{\rho_1\sigma}$$

.

These 3 equations can then be substituted into equation 5, and simplified nicely to become

$$\frac{d^2\big(v_1^*(y)\big)}{dy^2} - v_1^*(y)\big(\omega_x^2+\omega_z^2\big) = 0$$

. Similarly, the same equation can be obtained for fluid 2,

$$\frac{d^2\big(v_2^*(y)\big)}{dy^2} - v_2^*(y)\big(\omega_x^2+\omega_z^2\big) = 0$$

.

These two second-order linear differential equations can easily be solved. Hence

$$v_1^* = C_1 e^{\omega y} + C_2 e^{-\omega y}, \quad 0 < z < \infty$$
$$v_2^* = C_3 e^{\omega y} + C_4 e^{-\omega y}, \quad -\infty < z < 0,$$

where ω is the composite wave number such that

$$\omega^2 = (\omega_x^2 + \omega_z^2).$$

If we move really far from the interface in the y direction, there won't be any disturbance and the velocities v_1^* and v_2^* would be zero. Using that boundary condition yields C_1 and C_4 to be zero, and so we have

$$v_1^* = C_2 e^{-\omega y}, \quad 0 < z < \infty \tag{6}$$
$$v_2^* = C_3 e^{\omega y}, \quad -\infty < z < 0 \tag{7}$$

In order to find the other two constants C_2 and C_3, we have to use the kinematic boundary condition and the normal stress boundary condition. We cannot use the shear stress boundary condition because the fluids were assumed to be inviscid. I do not have a proper reason as to why I considered this assumption, other than the fact that I really want to be able to simplify the model enough for me to solve it and get some results out of it. Now Let

$$y = \hat{h}(x, z, t)$$

be a function of the interface as it deforms with time. It's important to keep in mind that the disturbance function is very small, and also, the two fluids share the same function. We write the function implicitly and apply the kinematic boundary condition, hence

$$F = y - \hat{h}(x, z, t) = 0$$
$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + V \cdot \nabla F = 0$$

Simplifying the substantial derivative, we get

$$-\frac{\partial \hat{h}}{\partial t} - u\frac{\partial \hat{h}}{\partial x} + v - w\frac{\partial \hat{h}}{\partial z} = 0.$$

Here, the velocities u, v and w are the sum of both the velocity at base stat $\left(u_b, v_b, w_b\right)$ and the disturbance velocities $\left(\hat{u}, \hat{v}, \hat{w}\right)$. Because of that, the above equation becomes

$$\hat{v} = \frac{\partial h}{\partial t}$$

As I mentioned, the deformed interface is the same for both fluids, so

$$\hat{v_1} = \hat{v_2} = \frac{\partial h}{\partial t} \tag{8}$$

We can propose a solution similar to those proposed earlier for the velocities and pressure, and so

$$h = C_5 e^{i(\omega_x x + \omega_z z) + \sigma t}$$

Substituting that along with equation 6 and 7 into equation 8, as well as setting $y = 0$ we get

$$C_2 = C_3 = \sigma C_5$$

We also need to impose the normal stress boundary condition. The complete stress balance is

$$\left(p_1 - n \cdot T_1 \cdot n\right) - \left(p_2 - n \cdot T_2 \cdot n\right) = \gamma(\nabla \cdot n)$$

where n is the normal vector of the interface, T_1 and T_2 are the hydrodynamic forces corresponding to fluid 1 and 2, respectively, and γ is the surface tension. What this equation implies is that the pressure differences are balanced by the surface tension and the curvature. Luckily for me, I do not need to deal with the surface tension, as I assumed it to be zero in the beginning of my analysis. I also do not need to deal with the forces corresponding to the viscosity as I am working with inviscid fluids, meaning $T_1 = T_2 = 0$. And so, the normal stress boundary condition reduces down to

$$p_1 - p_2 = 0 \tag{9}$$

But p_1 and p_2 are the sum of both the base state and disturbance, which means equation 9 becomes

$$gy(\rho_2 - \rho_1)\big|_{y=\hat{h}} = \hat{p_2} - \hat{p_1}$$

It is needless to say that this boundary condition be evaluated at the interface, hence $y = \hat{h}$. After substituting and doing some simplification, we arrive at an equation which combines both the composite wave number ω and the temporal growth rate σ, which is

$$\sigma^2 = \omega g \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} \tag{10}$$

From equation 10, we can conclude that the temporal growth rate squared σ^2 varies linearly with the wave number ω.

Results & Discussion

Now we certainly did not have to go through the trouble of deriving this mathematically, but we can conclude that the system would be unstable if $\rho_1 - \rho_2 > 0$. Recalling that the heavier fluid (fluid 1) is on the top and the lighter fluid (fluid 2) is on the bottom, this makes sense. In the case of this instability, the temporal growth rate squared σ^2 is always positive. It is also worth mentioning that the system is stable if the temporal growth rate squared was negative. This yields two strictly imaginary σ's of opposite signs. Physically, this means that the lighter fluid is on top, and the heavier one is on the bottom; and so there is no way for the system to possess an instability and have that develop with time. But in reality, the two densities are very close to each other, because after all, the two fluids are just water at different temperatures. If we try to find the limit of ρ_1 as it approaches ρ_2, we get

$$\lim_{\rho_1 \to \rho_2} \sigma^2 = \omega g \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} = 0.$$

This in fact yields the most boring case of the instability, which is that there really won't be any instability, or even mixing. We are dealing with the same fluid, and there isn't really that big of a difference in the densities to allow for the development of the instability. The growth rate $e^{\sigma t}$ becomes one. I do not think the result I got is realistic, because this means the instability will never grow or die. There will always be a periodic perturbation in the x, z and y direction. Another thing to point out is the fact that the flow is inviscid, meaning there is no loss in energy.

Now, for $\rho_1 > \rho_2$, meaning the density of the colder water is higher than that of warmer water, there will always be a slight instability. So this means

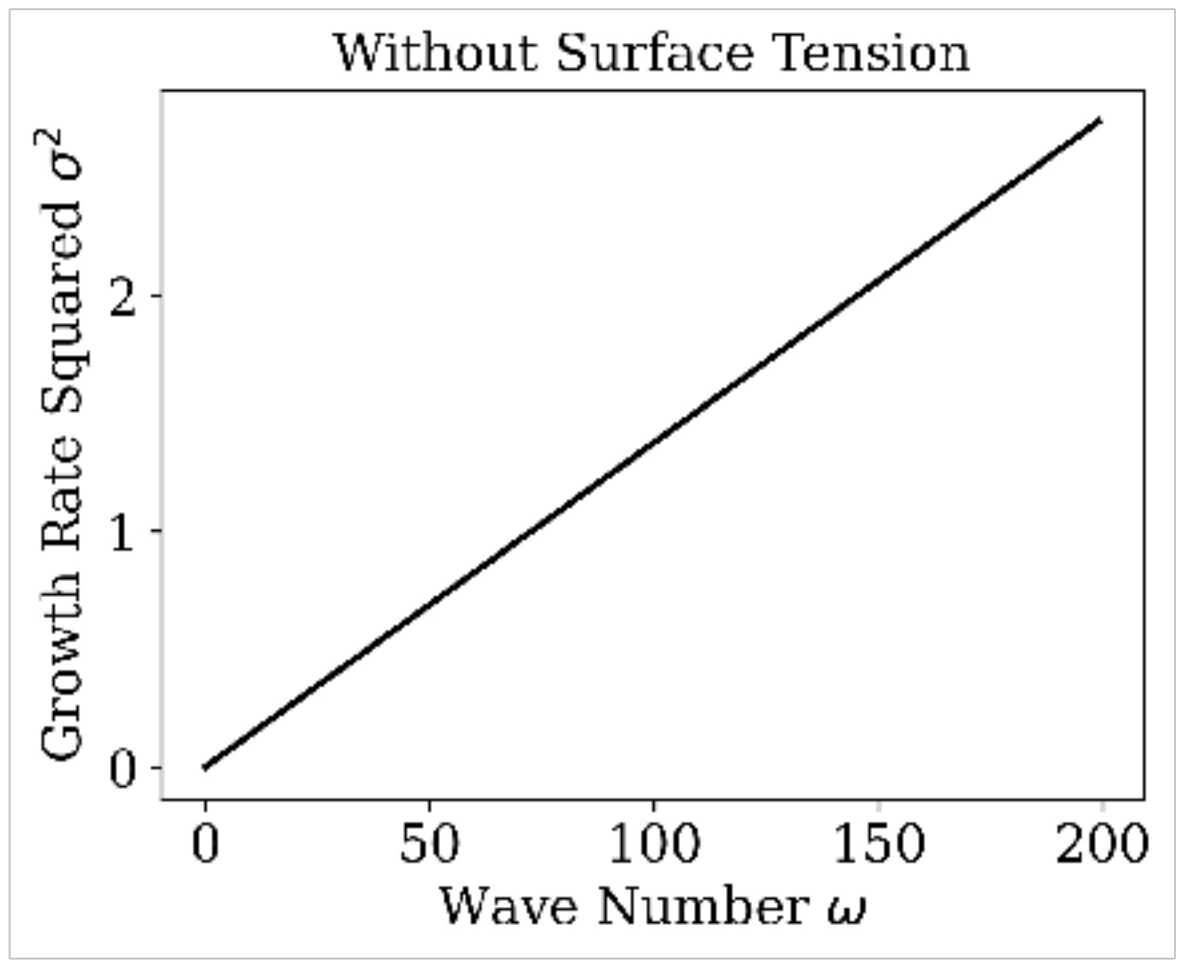

Figure 2: Dispersion relationship between σ^2 and ω

that for all wave numbers ω, there will always be an instability. This can be shown in both figures, 2 and 3 where σ^2 and σ keep growing.

But here is where things get a little interesting. If I were to consider the surface tension, the dispersion relationship I would have gotten would be

$$\sigma^2 = \omega \frac{(\rho_1 - \rho_2)g - \gamma\omega^2}{(\rho_1 + \rho_2)g} \tag{11}$$

and unlike the previous dispersion relationship, where having the cold water on top yielded only positive values for σ^2 and σ, this relationship will have σ^2 switch signs, meaning the model won't always be unstable. I plotted the relationship in figure 4, and by observing the figure, we can see that the solution eventually becomes stable for large values of ω, meaning large wave numbers. This does make sense as the surface tension does have a stabilizing effect on the disturbance and causes the instability to die out.

Conclusion

For my project, I considered a big slab of ice floating on top of warm water. This will cause the ice to melt and seep through. But I considered having cold water on top of warm water, an ideal case to make the analysis a bit easier. For the sake of time and deliverable, I also decided to exclude any

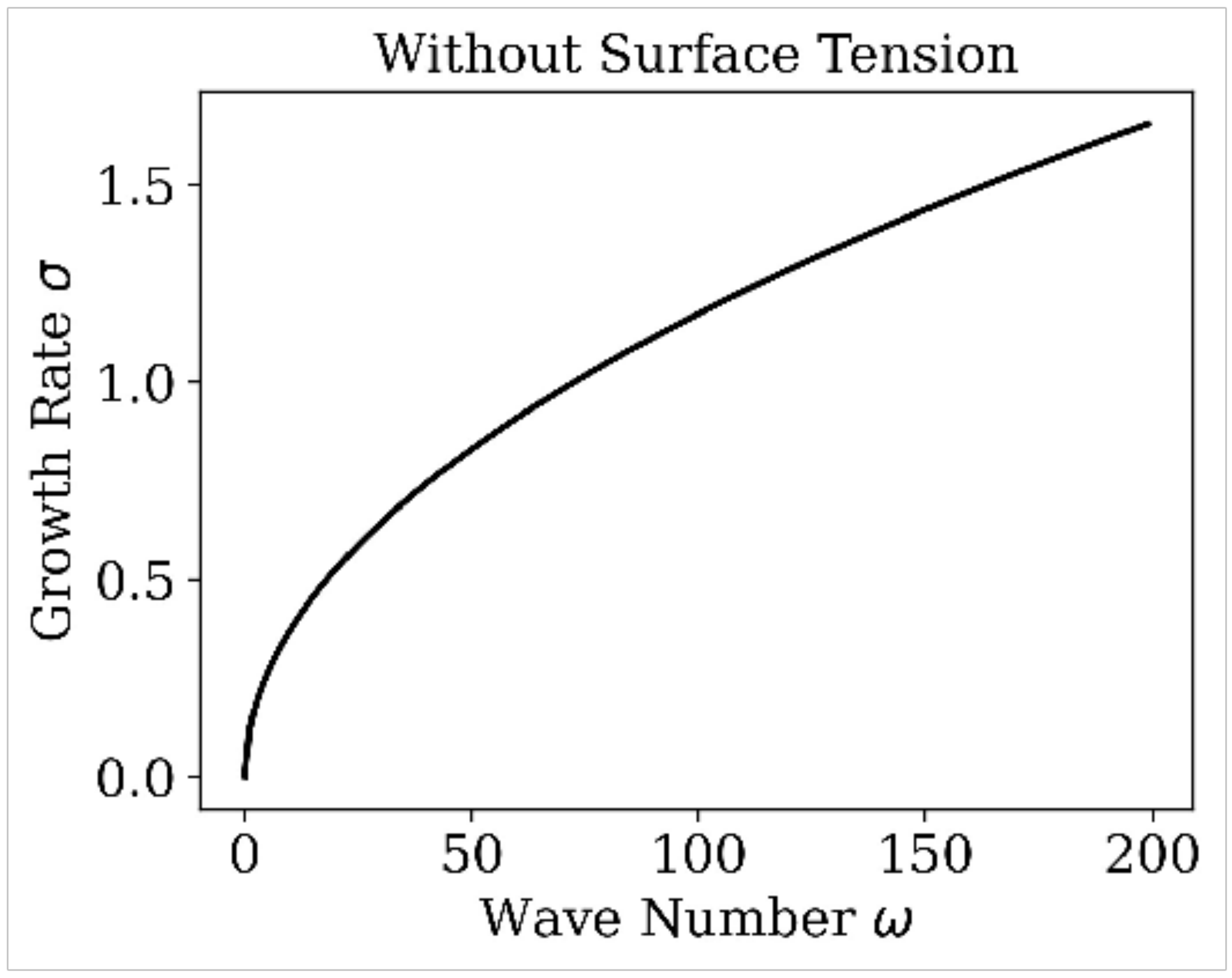

Figure 3: Dispersion relationship between σ and ω

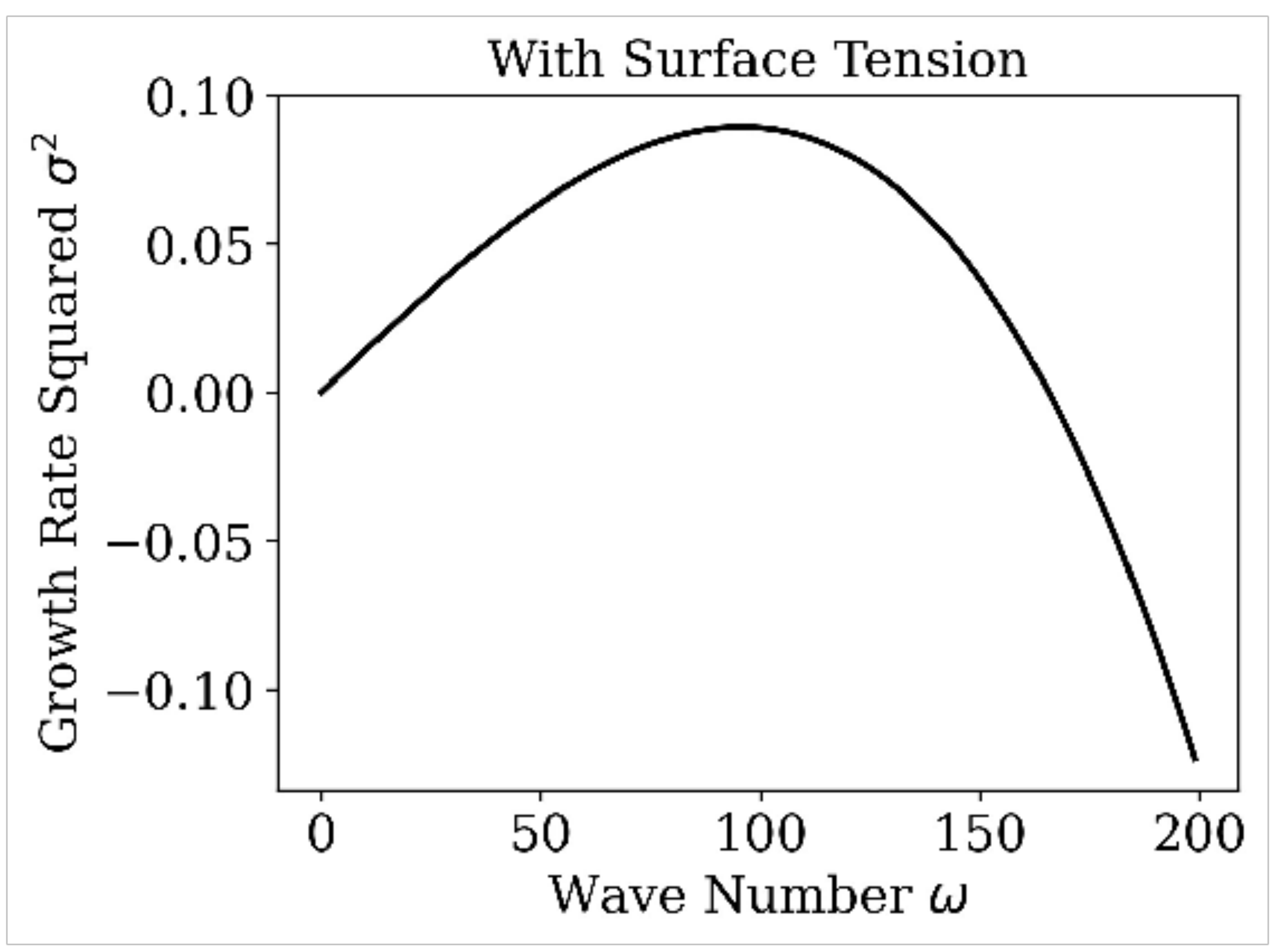

Figure 4: Dispersion relationship between σ and ω

heat transfer effects, and so I did not have to deal with the energy equation. I assumed an incompressible, inviscid flow with no surface tension. Considering these assumptions yielded an realistic result, as the plots I generated concluded that there will always be a growing instability. But, after considering the surface tension, I got somewhat of a realistic plot. The new result suggest that there will be a switch in the sign of the temporal growth rate σ^2 suggesting that the system will eventually reach stability. Now this was only due to surface tension causing a stabilizing effect, but I think if I considered the heat transfer between ice and water, I would have gotten a completely stable system without any instability.

Code

```
# Matplotlib is a python package for plotting
# Numpy is a python package for handling matrices

import matplotlib.pyplot as plt
import numpy as np
from matplotlib import rc
rc('font', family='serif')
rc('lines', linewidth=1.5)
rc('font', size=16)
plt.rc('legend',**{'fontsize':12})

# Defining Constants
rho_1 = 999.85 # Density of denser fluid
rho_2 = 997.05 # Density of lighter fluid
g = 9.81 # Gravitational acceleration
sigma = [] # Temporal growth rate
sigma_sq = [] # Temporal growth rate squared
omega = [] # Composite wave number
sur = [] # Dispersion relationship with surface tension
#----------------------------------------------------------------------#

# This for-loop generates dispersion relationship
# with and without surface tension
for i in np.arange(0,200,1):
    sigma_sq.append(( i * g * ( rho_1 - rho_2 ) / ( rho_1 + rho_2 ) ))
    sigma.append(( i * g * ( rho_1 - rho_2 ) / ( rho_1 + rho_2 ) )**0.5)
    sur.append(((rho_1 - rho_2)*g - 0.001*i**2)*i/((rho_1 + rho_2)*g))
    omega.append(i)

#----------------------------------------------------------------------#
# Plotting the generated values of sigma squared
fig, ax = plt.subplots(figsize=(5.5,4))
plt.xlabel("Wave Number " +  r'$ \omega $',fontsize=16)
plt.ylabel("Growth Rate Squared " +  r'$ \sigma^2 $',fontsize=16)
plt.title("Without Surface Tension",fontsize=16)
```

```
plt.plot(omega,sigma_sq,linewidth = 2,c = 'k')
plt.savefig("plot.jpg",dpi=300,bbox_inches='tight')

#----------------------------------------------------------------------#
# Dispersion relationship with surface tension
fig, ax = plt.subplots(figsize=(5.5,4))
plt.xlabel("Wave Number " +  r'$ \omega $',fontsize=16)
plt.ylabel("Growth Rate Squared " +  r'$ \sigma^2 $',fontsize=16)
plt.title("With Surface Tension",fontsize=16)
plt.plot(omega,sur,linewidth = 2,c = 'k')
plt.savefig("stability_sur.jpg",dpi=300,bbox_inches='tight')

#----------------------------------------------------------------------#
# Dispersion relationship without surface tension
fig, ax = plt.subplots(figsize=(5.5,4))
plt.xlabel("Wave Number " +  r'$ \omega $',fontsize=16)
plt.ylabel("Growth Rate " +  r'$ \sigma $',fontsize=16)
plt.title("Without Surface Tension",fontsize=16)
plt.plot(omega,sigma,linewidth = 2,c = 'k')
plt.savefig("stability.jpg",dpi=300,bbox_inches='tight')
```

References

[1] Lord Rayleigh. “On The Instability Of Jets”. In: *Proceedings of the London Mathematical Society* s1-10.1 (Nov. 1878), pp. 4–13. DOI: 10.1112/plms/s1-10.1.4. URL: https://doi.org/10.1112%2Fplms%2Fs1-10.1.4.

[2] Geoffrey Ingram Taylor. “The instability of liquid surfaces when accelerated in a direction perpendicular to their planes. I”. In: *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences* 201.1065 (1950), pp. 192–196.