

Solutions to Exercises from Le Gall's *Brownian Motion, Martingales, and Stochastic Calculus*

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1 Gaussian Variables and Gaussian Processes

1.1 Exercice 1.15

1. \Rightarrow Let us suppose that K is continuous. Let $(t, s) \in [0, 1]^2$. We have :

$$\begin{aligned}\|X_t - X_s\|_{L^2}^2 &= E(X_t^2) + E(X_s^2) - 2E(X_t X_s) \\ &= K(t, t) + K(s, s) - 2K(t, s) \\ &= K(t, t) - K(t, s) + K(s, s) - K(t, s)\end{aligned}$$

By continuity of K in $(t, t) : K(t, s) \xrightarrow{s \rightarrow t} K(t, t)$ and $K(s, s) \xrightarrow{s \rightarrow t} K(t, t)$

Thus : $\|X_t - X_s\|_{L^2}^2 \xrightarrow{s \rightarrow t} 0$ and X is continuous in t

\Leftarrow We suppose now that $t \mapsto X_t$ is continuous on $[0, 1]$. Let $(t, s), (t', s') \in [0, 1]^2$.

$$\begin{aligned}|K(t, s) - K(t', s')| &= |E(X_t X_s) - E(X_{t'} X_{s'})| \\ &\leq E(|X_t X_s - X_{t'} X_{s'}|) \\ &= \|X_t X_s - X_{t'} X_s + X_{t'} X_s - X_{t'} X_{s'}\|_{L^1} \\ &\leq \|X_t X_s - X_{t'} X_s\|_{L^1} + \|X_{t'} X_s - X_{t'} X_{s'}\|_{L^1} \\ &= \|(X_t - X_{t'}) X_s\|_{L^1} + \|X_{t'} (X_s - X_{s'})\|_{L^1} \\ &\leq \|X_t - X_{t'}\|_{L^2} \|X_s\|_{L^2} + \|X_{t'}\|_{L^2} \|X_s - X_{s'}\|_{L^2} \text{ using Cauchy-Schwartz inequality}\end{aligned}$$

By continuity of $t \mapsto X_t$ in t , $\|X_t - X_{t'}\|_{L^2} \xrightarrow{t' \rightarrow t} 0$ and $X_{t'}$ is bounded. Similarly, since $t \mapsto X_t$ is continuous on s , $\|X_s - X_{s'}\|_{L^2} \xrightarrow{s' \rightarrow s} 0$

Thus, we have $|K(t, s) - K(t', s')| \xrightarrow{(t', s') \rightarrow (t, s)} 0$ which proves the continuity of K on $[0, 1]^2$

2. Let us show that $\int_0^1 |h(t)| \|X_t\| dt \in L^1(\Omega)$ which will end the proof.

$$E \left(\int_0^1 |h(t)| \|X_t\| dt \right) = \int_0^1 |h(t)| E(\|X_t\|) dt \leq \int_0^1 |h(t)| \sqrt{E(X_t^2)} dt = \int_0^1 |h(t)| \sqrt{K(t, t)} dt < \infty$$

Moreover, $Z \in L^1$

3. First, let us show that $Z \in L^2$ and that for every $n \in \mathbb{N}, Z_n \in L^2$

Since Z_n is a linear combination of variables of the Gaussian process $(X_t)_{t \in \mathbb{R}}$, it is Gaussian thus in L^2

On the other hand :

$$E(Z^2) = E \left(\left(\int_0^1 h(t) X_t dt \right)^2 \right) = E \left(\int_0^1 \int_0^1 h(t) h(s) X_t X_s ds dt \right) = \int_0^1 \int_0^1 h(t) h(s) K(t, s) ds dt$$

Since K is continuous on the compact set $[0, 1]^2$, it is bounded let's say by a constant $C > 0$. Thus:

$$E(Z^2) \leq \int_0^1 \int_0^1 |h(t)| |h(s)| K(t, s) ds dt \leq C \left(\int_0^1 |h(t)| dt \right)^2 < \infty$$

Now let us show that $\|Z - Z_n\|_{L^2} \xrightarrow{n \rightarrow \infty} 0$

Let $\varepsilon > 0$. Since $t \mapsto X_t$ is continuous on the compact set $[0, 1]$, it is uniformly continuous. Thus, there exist $\eta > 0$ such that :

$$\forall t, s \in [0, 1], |t - s| < \eta \implies \|X_t - X_s\|_{L^2} < \varepsilon$$

Let $N \in \mathbb{N}$ such that $\frac{1}{N} < \eta$, and let $n \geq N$:

$$\begin{aligned}
\|Z - Z_n\|_{L^2} &= \left\| \int_0^1 h(t) X_t dt - \sum_{i=1}^n X_{\frac{i}{n}} \int_{\frac{i-1}{n}}^{\frac{i}{n}} h(t) dt \right\|_{L^2} \\
&= \left\| \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} h(t) X_t dt - \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} h(t) X_{\frac{i}{n}} dt \right\|_{L^2} \\
&= \left\| \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} h(t) (X_{\frac{i}{n}} - X_t) dt \right\|_{L^2} \\
&\leq \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} h(t) \|X_{\frac{i}{n}} - X_t\|_{L^2} dt
\end{aligned}$$

Since $\forall t \in [\frac{i-1}{n}, \frac{i}{n}]$, $|t - \frac{i}{n}| < \frac{1}{N} \leq \eta$ we have that $\|X_{\frac{i}{n}} - X_t\|_{L^2} < \varepsilon$
Thus :

$$\|Z - Z_n\|_{L^2} \leq \varepsilon \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} |h(t)| dt = \varepsilon \int_0^1 |h(t)| dt \leq \varepsilon M$$

since $\int_0^1 |h(t)| dt < \infty$

We showed that for all $n \in \mathbb{N}$, Z_n is Gaussian and $Z_n \xrightarrow{n \rightarrow \infty} Z$ in L^2 . It follows from Proposition 1.1 that Z is Gaussian.

4. Since $(L^2(\Omega), \|\cdot\|_2)$ is a complete metric space, to prove that \dot{X}_t exists, it suffices to show that :

$$\left\| \frac{X_t - X_s}{t - s} - \frac{X_t - X_r}{t - r} \right\|_{L^2} \xrightarrow{(s,r) \rightarrow (0,0)} 0$$

For $t \in \mathbb{R}$:

$$\begin{aligned}
\left\| \frac{X_t - X_s}{t - s} - \frac{X_t - X_r}{t - r} \right\|_{L^2}^2 &= E \left[\left(\frac{X_t - X_s}{t - s} \right)^2 + \left(\frac{X_t - X_r}{t - r} \right)^2 - 2 \cdot \frac{(X_t - X_s)(X_t - X_r)}{(t - s)(t - r)} \right] \\
&= \underbrace{\frac{1}{(t - s)^2} [K(t, t) + K(s, s) - 2K(t, s)]}_A \\
&\quad + \underbrace{\frac{1}{(t - r)^2} [K(t, t) + K(r, r) - 2K(t, r)]}_B \\
&\quad - \underbrace{\frac{2}{(t - s)(t - r)} [K(t, t) - K(t, r) - K(t, s) + K(s, r)]}_C
\end{aligned}$$

Now let us make a fundamental remark for the following. Since K is twice continuously differentiable, we have the following properties on the partial derivatives of K using symetry of K and Schwarz Theorem:

$$\forall t, s \in \mathbb{R}, \quad \frac{\partial K}{\partial t}(t, s) = \frac{\partial K}{\partial s}(s, t) \quad \text{and} \quad \frac{\partial^2 K}{\partial t^2}(t, s) = \frac{\partial^2 K}{\partial s^2}(s, t) \quad \text{and} \quad \frac{\partial^2 K}{\partial t \partial s}(t, s) = \frac{\partial^2 K}{\partial s \partial t}(s, t) = \frac{\partial^2 K}{\partial s \partial t}(t, s)$$

Let us write the general second-order Taylor expansion of K in (t, s) which all terms in A, B and C will derive from:

$$\begin{aligned}
K(u, r) &= K(t, s) + (u - t) \frac{\partial K}{\partial t}(t, s) + (r - s) \frac{\partial K}{\partial s}(t, s) + \frac{1}{2} (u - t)^2 \frac{\partial^2 K}{\partial t^2}(t, s) + \frac{1}{2} (r - s)^2 \frac{\partial^2 K}{\partial s^2}(t, s) \\
&\quad + (u - t)(r - s) \frac{\partial^2 K}{\partial t \partial s}(t, s) + o(\|(u, r) - (t, s)\|^2)
\end{aligned}$$

This gives us for the term A :

$$\begin{aligned} K(t, s) &\underset{s \rightarrow t}{=} K(t, t) + \frac{\partial K}{\partial s}(t, t)(s - t) + \frac{1}{2} \frac{\partial^2 K}{\partial s^2}(t, t)(s - t)^2 + o((s - t)^2) \\ &\underset{s \rightarrow t}{=} K(t, t) + \frac{\partial K}{\partial t}(t, t)(s - t) + \frac{1}{2} \frac{\partial^2 K}{\partial t^2}(t, t)(s - t)^2 + o((s - t)^2) \end{aligned}$$

and

$$\begin{aligned} K(s, s) &\underset{s \rightarrow t}{=} K(t, t) + \left(\frac{\partial K}{\partial t}(t, t) + \frac{\partial K}{\partial s}(t, t) \right) (s - t) \\ &\quad + \frac{1}{2} \left(\frac{\partial^2 K}{\partial t^2}(t, t) + 2 \frac{\partial^2 K}{\partial t \partial s}(t, t) + \frac{\partial^2 K}{\partial s^2}(t, t) \right) (s - t)^2 + o((s - t)^2) \\ &\underset{s \rightarrow t}{=} K(t, t) + 2 \frac{\partial K}{\partial t}(t, t)(s - t) + \left(\frac{\partial^2 K}{\partial t^2}(t, t) + \frac{\partial^2 K}{\partial t \partial s}(t, t) \right) (s - t)^2 + o((s - t)^2) \end{aligned}$$

After simplification, this leads to:

$$A \underset{s \rightarrow t}{=} \frac{\partial^2 K}{\partial t \partial s}(t, s) + o(1)$$

By symetry of the roles of s and r we also have:

$$B \underset{s \rightarrow t}{=} \frac{\partial^2 K}{\partial t \partial s}(t, t) + o(1)$$

Now let's take a look at a new term in C

$$\begin{aligned} K(s, r) &\underset{(s, r) \rightarrow (t, t)}{=} K(t, t) + \frac{\partial K}{\partial t}(t, t)(s - t) + \frac{\partial K}{\partial s}(t, t)(r - t) + \frac{1}{2} \frac{\partial^2 K}{\partial t^2}(t, t)(s - t)^2 \\ &\quad + \frac{1}{2} \frac{\partial^2 K}{\partial s^2}(t, t)(r - t)^2 + \frac{\partial^2 K}{\partial t \partial s}(t, t)(s - t)(r - t) + o(\|(s, r) - (t, t)\|^2) \\ &\underset{(s, r) \rightarrow (t, t)}{=} K(t, t) + \frac{\partial K}{\partial t}(t, t)(s + r - 2t) + \frac{1}{2} \frac{\partial^2 K}{\partial t^2}(t, t)((s - t)^2 + (r - t)^2) \\ &\quad + \frac{\partial^2 K}{\partial t \partial s}(t, t)(s - t)(r - t) + o(\|(s, r) - (t, t)\|^2) \end{aligned}$$

After simplification we have:

$$C \underset{(s, r) \rightarrow (t, t)}{=} 2 \frac{\partial^2 K}{\partial t \partial s}(t, t) + o(1)$$

Thus, we have:

$$\left\| \frac{X_t - X_s}{t - s} - \frac{X_t - X_r}{t - r} \right\|_{L^2} \xrightarrow{(s, r) \rightarrow (t, t)} 0$$

which leads to the existence of \dot{X}_t .

Let us now note $Y_{t,s} = \frac{X_t - X_s}{t - s}$. Since $(X_t)_{t \in \mathbb{R}}$ is a Gaussian process, $Y_{t,s}$ is Gaussian as linear combination of variables of $(X_t)_{t \in \mathbb{R}}$ and any finite linear combination of the variables $(Y_{t,s})_{t,s \in \mathbb{R}}$ is Gaussian.

Let (t_1, \dots, t_n) and $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$. Then, since all the limits exist in L^2 :

$$\sum_{i=1}^n \lambda_i \dot{X}_{t_i} = \sum_{i=1}^n \lambda_i \lim_{s_i \rightarrow t_i} Y_{t_i, s_i} = \lim_{(s_1, \dots, s_n) \rightarrow (t_1, \dots, t_n)} \sum_{i=1}^n \lambda_i Y_{t_i, s_i}$$

The last term is the L^2 limit of centered Gaussian variables, so it is centered Gaussian by Proposition 1.1, which proves that $(\dot{X}_t)_{t \in \mathbb{R}}$ is a centered Gaussian process.

For the covariance function, we can write for $(u, v) \in [0, 1]^2$:

$$\begin{aligned} E(\dot{X}_t \dot{X}_s) &= E \left(\lim_{u \rightarrow t} \frac{X_t - X_u}{t - u} \cdot \lim_{r \rightarrow s} \frac{X_s - X_r}{s - r} \right) = E \left(\lim_{(u, r) \rightarrow (t, s)} \frac{(X_t - X_u)(X_s - X_r)}{(t - u)(s - r)} \right) \text{ since the limits exist in } L^2 \\ &= \lim_{(s, r) \rightarrow (t, u)} E \left(\frac{(X_t - X_u)(X_s - X_r)}{(t - u)(s - r)} \right) \text{ (dominated convergence theorem)} \\ &= \lim_{(u, r) \rightarrow (t, s)} \frac{1}{(t - u)(s - r)} (K(t, s) - K(t, r) - K(u, s) + K(u, r)) \end{aligned}$$

Just as before we can use the Taylor expansions of K in (t, s) and simplifying using the previous properties. After calculus, we find :

$$E(\dot{X}_t \dot{X}_s) = \frac{\partial^2 K}{\partial t \partial s}(t, s)$$

With the properties stated above, we verify that this is indeed a symetric function.

1.2 Exercice 1.16 (Kalman filtering)

1. Let $n \geq 0$. Using linearity of the conditional expectation we can write:

$$\hat{X}_{n+1/n} = E(X_{n+1}|Y_0, \dots, Y_n) = a_n E(X_n|Y_0, \dots, Y_n) + E(\epsilon_{n+1}|Y_0, \dots, Y_n)$$

The recursive formula for $(X_n)_n$ shows that for all $k \in \mathbb{N}$, $X_i \in \text{Vect}(\epsilon_1, \dots, \epsilon_k)$ so $X_1, \dots, X_n \in \sigma(\epsilon_1, \dots, \epsilon_n)$ and since the $(\epsilon_i)_i$ are mutually independant, ϵ_{n+1} is independant from $\sigma(\epsilon_1, \dots, \epsilon_n)$, thus from X_1, \dots, X_n . Furthermore, the recursive formula for $(Y_n)_n$ gives us that for all $k \in \mathbb{N}$, $Y_k \in \sigma(\epsilon_1, \dots, \epsilon_k, \eta_k)$ and Since the $(\eta_i)_i$ and the $(\epsilon_i)_i$ are independant, ϵ_{n+1} is independant from Y_0, \dots, Y_n and so we can write:

$$E(\epsilon_{n+1}|Y_0, \dots, Y_n) = E(\epsilon_{n+1}) = 0 \text{ cause } \epsilon_{n+1} \text{ is centered}$$

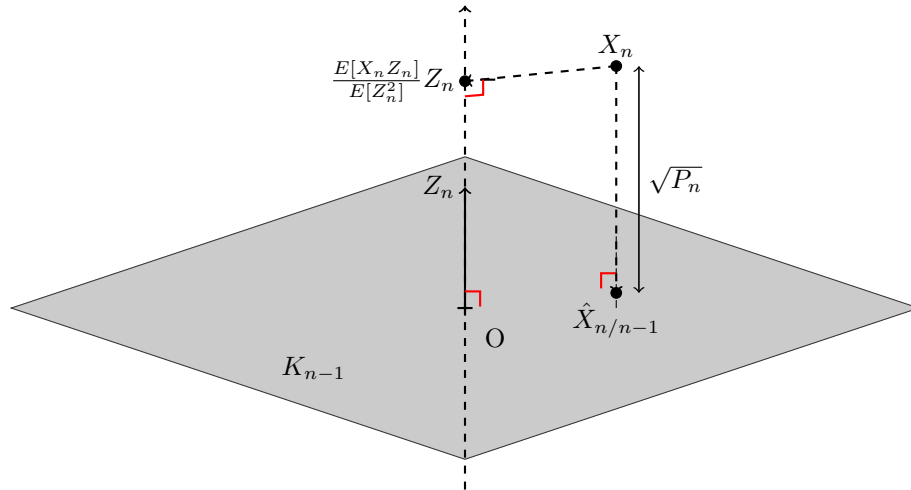
Thus,

$$\hat{X}_{n+1/n} = a_n E(X_n|Y_0, \dots, Y_n) = a_n \hat{X}_{n/n}$$

2. Let $(\cdot|\cdot)$ be the usual scalar product on L^2 . Consider $H_n = \text{Vect}(\epsilon_1, \dots, \epsilon_n, \eta_1, \dots, \eta_n)$. $\epsilon_1, \dots, \epsilon_n, \eta_1, \dots, \eta_n$ are mutually independant Gaussian random variables so $(\epsilon_1, \dots, \epsilon_n, \eta_1, \dots, \eta_n)$ is a Gaussian vector, thus H_n is a Gaussian Space. Let $K_n = \text{Vect}(Y_0, \dots, Y_n)$ and p_{K_n} be the orthogonal projection on K_n in L^2 . Such a projection exists since K_n is finite-dimensional.

Using these notations we can rewrite the equality we want to prove as:

$$E[X_n|\sigma(K_n)] = E[X_n|\sigma(K_{n-1})] + \frac{(X_n|Z_n)}{\|Z_n\|_{L^2}^2} \cdot Z_n$$



Since K_n and K_{n-1} are closed because finite-dimensional in the Gaussian Space H_n , we can use Corollary 1.10 to write this as a geometric equation using projections:

$$p_{K_n}(X_n) = p_{K_{n-1}}(X_n) + p_{\text{Vect}(Z_n)}(X_n)$$

Then it suffices to prove that $Z_n \in K_n \cap K_{n-1}^\perp$. In fact we will have $K_n = K_{n-1} \oplus^\perp \text{Vect}(Z_n)$ which gives us $p_{K_n} = p_{K_{n-1}} + p_{\text{Vect}(Z_n)}$

First of all: $Z_n = Y_n - cE[X_n|\sigma(K_{n-1})] = \underbrace{Y_n}_{\in K_n} - c \cdot \underbrace{p_{K_{n-1}}(X_n)}_{\in K_{n-1} \subset K_n}$, hence $Z_n \in K_n$.

To show that $Z_n \in K_{n-1}^\perp$ it suffices to prove that for all $k \leq n-1$, $Z_n \perp Y_k$. Let $k \leq n-1$

$$\begin{aligned} E[Y_k Z_n] &= E[Y_k Y_n - cY_k E[X_n|\sigma(K_{n-1})]] \\ &= cE[Y_k X_n] + E[Y_k \eta_n] - cE[Y_k X_n] \quad \text{because } Y_k \in L^2 \text{ is } \sigma(K_{n-1})\text{-mesurable} \\ &= E[Y_k \eta_n] \\ &= E[Y_k]E[\eta_n] = 0 \quad \text{since } Y_k \text{ and } \eta_n \text{ are independant and } \eta_n \text{ is centered} \end{aligned}$$

With that being proved, we have the desired equality for all $n \in \mathbb{N}$:

$$\hat{X}_{n/n} = \hat{X}_{n/n-1} + \frac{E[X_n Z_n]}{E[Z_n^2]} Z_n$$

3. First of all : $Z_n = c(X_n - \hat{X}_{n/n-1}) + \eta_n$. Since η_n is centered and independant of X_n and $\hat{X}_{n/n-1}$, we have:

$$E[Z_n^2] = c^2 E[(X_n - \hat{X}_{n/n-1})^2] + E[\eta_n^2] = c^2 P_n + \delta^2$$

In the other hand we can write :

$$\begin{aligned} E[X_n Z_n] &= cE[X_n(X_n - \hat{X}_{n/n-1})] + E[X_n \eta_n] \\ &= E[(X_n - \hat{X}_{n/n-1})^2] + cE[\underbrace{\hat{X}_{n/n-1}}_{\in K_{n-1}} \underbrace{(X_n - \hat{X}_{n/n-1})}_{\in K_{n-1}^\perp}] \quad \text{because } \eta_n \text{ is independent of } X_n \text{ and centered} \\ &= cP_n \end{aligned}$$

Let $n \geq 1$. Using question 1 and 2, and our formulas we have;

$$\begin{aligned} \hat{X}_{n+1/n} &= a_n \hat{X}_{n/n} = a_n \left(\hat{X}_{n/n-1} + \frac{E[X_n Z_n]}{E[Z_n^2]} Z_n \right) \\ &= a_n \left(\hat{X}_{n/n-1} + \frac{cP_n}{cP_n + \delta^2} Z_n \right) \end{aligned}$$

4. In the first place we have $X_1 = \epsilon_1$ and $Y_0 = \eta_0$. Hence;

$$X_{1/0} = E[X_1|Y_0] = E[\epsilon_1|\eta_0] = E[\epsilon_1] = 0 \quad \text{almost surely}$$

Then:

$$P_1 = E[(X_1 - X_{1/0})^2] = E[\epsilon_1^2] = \sigma^2$$

Now let's prove the recursive formula for P_n . First we remark that, using question 3:

$$\begin{aligned} \hat{X}_{n+1/n} - X_{n+1} &= a_n \left(\hat{X}_{n/n-1} + \frac{cP_n}{cP_n + \delta^2} Z_n \right) - a_n X_n - \epsilon_{n+1} \\ &= a_n (\hat{X}_{n/n-1} - X_n) + a_n \frac{cP_n}{cP_n + \delta^2} Z_n - \epsilon_{n+1} \end{aligned}$$

Let us note

$$b_n = a_n \frac{cP_n}{cP_n + \delta^2}$$

hence:

$$\begin{aligned}
P_{n+1} &= E[(\hat{X}_{n+1/n} - X_{n+1})^2] \\
&= E[(a_n(\hat{X}_{n/n-1} - X_n)^2] + 2E[a_n(\hat{X}_{n/n-1} - X_n)(b_n Z_n - \epsilon_{n+1})] + E[(b_n Z_n - \epsilon_{n+1})^2] \\
&= a_n^2 P_n + 2a_n b_n E[(\hat{X}_{n/n-1} - X_n)Z_n] + b_n^2 E[Z_n^2] + E[\epsilon_{n+1}^2] \quad \text{as } \epsilon_{n+1} \text{ centered and independent of } \hat{X}_{n/n-1}, X_n \text{ and } Z_n \\
&= a_n^2 P_n - 2a_n b_n c P_n + b_n^2 (c^2 P_n + \delta^2) + \sigma^1 \quad \text{as } \hat{X}_{n/n-1} \perp Z_n \text{ and with previous formulas} \\
&= a_n^2 P_n - 2a_n^2 \frac{c^2 P_n^2}{c P_n + \delta^2} + a_n^2 \frac{c^2 P_n^2}{c P_n + \delta^2} + \sigma^2 \\
&= \sigma^2 + a_n \frac{\delta^2 P_n}{c P_n + \delta^2}
\end{aligned}$$

which is the desired formula.

1.3 Exercice 1.17

A. Why the hint allows us to conclude

The hint suggests to consider the case where X_1 , resp. X_2 , is the indicator function of an event depending only on finitely many variables in H_1 , resp. in H_2 . Let us show why we can do this.

Let $\mathcal{A} = \{A \in \sigma(H_1), \exists X_1, \dots, X_m \in H_1, A \in \sigma(X_1, \dots, X_m)\}$ be the set of events depending only on finitely many variables of H_1 and $\mathcal{B} = \{B \in \sigma(H_2), \exists Y_1, \dots, Y_n \in H_2, B \in \sigma(Y_1, \dots, Y_n)\}$ be the set of events depending only on finitely many variables of H_2 .

We **suppose** that for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$:

$$E[\mathbb{1}_A \mathbb{1}_B | \sigma(K)] = E[\mathbb{1}_A | \sigma(K)] E[\mathbb{1}_B | \sigma(K)]$$

and we want to **show** that for all non-negative $\sigma(H_1)$ -mesurable random variable X_1 and for all non-negative $\sigma(H_2)$ -mesurable random variable X_2 :

$$E[X_1 X_2 | \sigma(K)] = E[X_1 | \sigma(K)] E[X_2 | \sigma(K)]$$

Frist, it is immediate that \mathcal{A} and \mathcal{B} are stable under finite intersection. Moreover;

$$\forall X_1 \in H_1, \forall A_1 \in \mathcal{B}(\mathbb{R}), X_1^{-1}(A_1) \in \mathcal{A} \quad \text{and} \quad \forall X_2 \in H_2, \forall A_2 \in \mathcal{B}(\mathbb{R}), X_2^{-1}(A_2) \in \mathcal{B}$$

Thus $\sigma(\mathcal{A}) = \sigma(H_1)$ and $\sigma(\mathcal{B}) = \sigma(H_2)$

Let us now consider

$$\mathcal{M}_1 = \{A \in \sigma(H_1), \forall B \in \mathcal{B}, E[\mathbb{1}_A \mathbb{1}_B | \sigma(K)] = E[\mathbb{1}_A | \sigma(K)] E[\mathbb{1}_B | \sigma(K)]\}$$

By hypothesis, $\mathcal{A} \subset \mathcal{M}_1$. Besides, \mathcal{M}_1 is a monotone class. In fact :

- $\mathbb{1}_\Omega = 1$ a.s thus, for all $B \in \mathcal{B}$ $E[\mathbb{1}_\Omega \mathbb{1}_B | \sigma(K)] = E[\mathbb{1}_B | \sigma(K)] = E[\mathbb{1}_B | \sigma(K)] E[\mathbb{1}_\Omega | \sigma(K)]$ a.s, hence $\Omega \in \mathcal{M}_1$
- If $A_1, A_2 \in \mathcal{M}_1, A_1 \subset A_2$ we know that $\mathbb{1}_{A_2 \setminus A_1} = \mathbb{1}_{A_2} - \mathbb{1}_{A_1}$. Then for all $B \in \mathcal{B}$, by linearity of conditional expectation:

$$\begin{aligned}
E[\mathbb{1}_{A_2 \setminus A_1} \mathbb{1}_B | \sigma(K)] &= E[\mathbb{1}_{A_2} \mathbb{1}_B | \sigma(K)] - E[\mathbb{1}_{A_1} \mathbb{1}_B | \sigma(K)] \\
&= E[\mathbb{1}_{A_2} | \sigma(K)] \cdot E[\mathbb{1}_B | \sigma(K)] - E[\mathbb{1}_{A_1} | \sigma(K)] \cdot E[\mathbb{1}_B | \sigma(K)] \quad \text{since } A_1, A_2 \in \mathcal{M}_1 \\
&= E[\mathbb{1}_{A_2} - \mathbb{1}_{A_1} | \sigma(K)] \cdot E[\mathbb{1}_B | \sigma(K)] \\
&= E[\mathbb{1}_{A_2 \setminus A_1} | \sigma(K)] \cdot E[\mathbb{1}_B | \sigma(K)]
\end{aligned}$$

Hence $A_2 \setminus A_1 \in \mathcal{M}_1$

- If $(A_n)_n \in (\mathcal{M}_1)^\mathbb{N}$ is an increasing sequence then for all $B \in \mathcal{B}$ and for all $n \geq 0$:

$$E[\mathbb{1}_{\bigcup_{k=0}^n A_k} \mathbb{1}_B | \sigma(K)] = E[\mathbb{1}_{A_n} \mathbb{1}_B | \sigma(K)] = E[\mathbb{1}_{A_n} | \sigma(K)] E[\mathbb{1}_B | \sigma(K)] \quad \text{since } A_n \in \mathcal{M}_1$$

Since the $(A_n)_n$ are increasing (in the sense of \subset) then, $(\mathbb{1}_{A_n})_n$ is an increasing sequence of non-negative functions. Moreover $\mathbb{1}_{A_n} \xrightarrow{n \rightarrow \infty} \mathbb{1}_{\bigcup_{n=0}^\infty A_n}$ a.s., and $\mathbb{1}_B$ is non-negative, thus by conditional monotone convergence theorem :

$$E[\mathbb{1}_{A_n} | \sigma(K)] \xrightarrow{n \rightarrow \infty} E[\mathbb{1}_{\bigcup_{n=0}^\infty A_n} | \sigma(K)] \quad \text{and} \quad E[\mathbb{1}_{A_n} \mathbb{1}_B | \sigma(K)] \xrightarrow{n \rightarrow \infty} E[\mathbb{1}_{\bigcup_{n=0}^\infty A_n} \mathbb{1}_B | \sigma(K)]$$

Hence $\bigcup_{n=0}^\infty A_n \in \mathcal{M}_1$

Under these conditions, according to the Monotone class lemma, we have that the monotone class generated by \mathcal{A} , denoted by $\mathcal{M}(\mathcal{A})$ is the sigma field $\sigma(\mathcal{A})$ generated by \mathcal{A}

$$\sigma(H_1) = \sigma(\mathcal{A}_1) = \mathcal{M}(\mathcal{A}_1) \subset \mathcal{M}_1 \quad \text{hence} \quad \mathcal{M}_1 = \sigma(H_1)$$

Then

$$\forall A \in \sigma(H_1), \forall B \in \mathcal{B} \quad E[\mathbb{1}_A \mathbb{1}_B | \sigma(K)] = E[\mathbb{1}_A | \sigma(K)] E[\mathbb{1}_B | \sigma(K)]$$

This equality being true for all $A \in \sigma(H_1)$, it is also true for all non-negative $\sigma(H_1)$ -mesurable step functions. Since every non-negative $\sigma(H_1)$ -mesurable function is the limit of an non-decreasing sequence of non-negative $\sigma(H_1)$ -mesurable step functions, the conditional monotone convergence theorem gives us for all non-negative $\sigma(H_1)$ -mesurable random variable X_1

$$\forall B \in \mathcal{B} \quad E[X_1 \mathbb{1}_B | \sigma(K)] = E[X_1 | \sigma(K)] E[\mathbb{1}_B | \sigma(K)]$$

Now if we consider

$$\mathcal{M}_2 = \{B \in \sigma(H_2), \forall X_1 \text{ } \sigma(H_1)\text{-mesurable } X_1 \geq 0, E[X_1 \mathbb{1}_B | \sigma(K)] = E[X_1 | \sigma(K)] E[\mathbb{1}_B | \sigma(K)]\}$$

We prove analogously that \mathcal{M}_2 is a monotone class.

Since $B \subset \mathcal{M}_2$ and $\sigma(\mathcal{B}) = \sigma(H_2)$ the monotone class lemma holds and

$$\forall B \in \sigma(H_2), \forall X_1 \text{ } \sigma(H_1)\text{-mesurable } X_1 \geq 0, \quad E[X_1 \mathbb{1}_B | \sigma(K)] = E[X_1 | \sigma(K)] E[\mathbb{1}_B | \sigma(K)]$$

With the same arguments as earlier, this equality holds for every non-negative $\sigma(H_2)$ -mesurable random variable X_2 and we have that for all non-negative $\sigma(H_2)$ -mesurable random variable X_2 and for all non-negative $\sigma(H_1)$ -mesurable random variable X_1

$$E[X_1 X_2 B | \sigma(K)] = E[X_1 | \sigma(K)] \cdot E[X_2 B | \sigma(K)]$$

which implies that the σ -fields $\sigma(H_1)$ and $\sigma(H_2)$ are conditionally independent given $\sigma(K)$

B. Proving the result for indicator functions

Let us show how we can prove the result when X_1 , resp. X_2 , is the indicator function of an event depending only on finitely many variables in H_1 , resp. in H_2 , as suggested in the hint. First, we recall a fundamental lemma useful for the following.

Lemma: Let X and Y be two random variables taking values in the measurable spaces E and F , respectively. Suppose that X is independent of \mathcal{B} , and that Y is \mathcal{B} -measurable. Then, for any measurable function $g : E \times F \rightarrow \mathbb{R}_+$, we have:

$$E[g(X, Y) | \mathcal{B}] = \int_E g(x, Y) \mathbb{P}_X(dx)$$

Let $A \in \sigma(H_1)$ and $B \in \sigma(H_2)$ depend, respectively, only on X_1, \dots, X_m and Y_1, \dots, Y_n . Thus we can write $A = \bigcap_{i=1}^m (X_i \in A_i)$ and $B = \bigcap_{i=1}^n (Y_i \in B_i)$ with $A_1, \dots, A_m, B_1, \dots, B_n \in \mathcal{B}(\mathbb{R})$.

Thus

$$\mathbb{1}_A = \mathbb{1}_{X_1 \in A_1} \cdots \mathbb{1}_{X_m \in A_m} \text{ and } \mathbb{1}_B = \mathbb{1}_{Y_1 \in B_1} \cdots \mathbb{1}_{Y_n \in B_n}$$

Then, what we would like is :

$$E[\mathbb{1}_{X_1 \in A_1} \cdots \mathbb{1}_{X_m \in A_m} \cdot \mathbb{1}_{Y_1 \in B_1} \cdots \mathbb{1}_{Y_n \in B_n} | \sigma(K)] = E[\mathbb{1}_{X_1 \in A_1} \cdots \mathbb{1}_{X_m \in A_m} | \sigma(K)] \cdot E[\mathbb{1}_{Y_1 \in B_1} \cdots \mathbb{1}_{Y_n \in B_n} | \sigma(K)]$$

Let us define

$$U = (X_1 - p_K(X_1), \dots, X_m - p_K(X_m), Y_1 - p_K(Y_1), \dots, Y_n - p_K(Y_n)) \text{ and } V = (p_K(X_1), \dots, p_K(X_m), p_K(Y_1), \dots, p_K(Y_n))$$

We will use the notation U_i (resp. V_i) for the i -th coordinate of U (resp. V). We will also use $U_X := (U_1, \dots, U_m)$, $V_X := (V_1, \dots, V_m)$, $U_Y := (U_{m+1}, \dots, U_{m+n})$, $V_Y := (V_{m+1}, \dots, V_{m+n})$.

Let us also define

$$g : \mathbb{R}^{m+n} \times \mathbb{R}^{m+n} \longrightarrow \mathbb{R}^+ \\ (u_1, \dots, u_{m+n}), (v_1, \dots, v_{m+n}) \longmapsto \prod_{i=1}^m \mathbb{1}_{u_i + v_i \in A_i} \cdot \prod_{i=m+1}^n \mathbb{1}_{u_i + v_i \in B_{m-j}}$$

We will use the notation U_i for the i -th coordinate of U (i.e. $X_i - p_K(X_i)$, resp. $Y_i - p_K(Y_i)$ for $i \leq m$, resp. for $m+1 \leq i \leq m+n$). We will also use $U_X := (U_1, \dots, U_m)$, $V_X := (V_1, \dots, V_m)$, $U_Y := (U_{m+1}, \dots, U_{m+n})$, $V_Y := (V_{m+1}, \dots, V_{m+n})$.

Then we can write :

$$E[\mathbb{1}_A \mathbb{1}_B | \sigma(K)] = E[\mathbb{1}_{X_1 \in A_1} \cdots \mathbb{1}_{X_m \in A_m} \cdot \mathbb{1}_{Y_1 \in B_1} \cdots \mathbb{1}_{Y_n \in B_n} | \sigma(K)] \\ = E[g(U, V) | \sigma(K)]$$

Or, for all $i \leq m$ and for all $j \leq n$ $X_i - p_K(X_i)$ and $Y_j - p_K(Y_j)$ are Gaussian and orthogonal to K hence independent of $\sigma(K)$.

In the other hand, for all $i \leq m$ and for all $j \leq n$ $p_K(X_i)$ and $p_K(Y_j)$ are in K hence $\sigma(K)$ -mesurable. Under those conditions we can apply the lemma and :

$$E[\mathbb{1}_A \mathbb{1}_B | \sigma(K)] = E[g(U, V) | \sigma(K)] \\ = \int \mathbb{1}_{u_1 + p_K(X_1) \in A_1} \cdots \mathbb{1}_{u_{m+n} + p_K(Y_n) \in B_n} \cdot P_U(du_1 \dots du_{m+n})$$

Now in order to split the integral into two parts we need to prove that the $(X_i - p_K(X_i))_{1 \leq i \leq m}$ and the $(Y_j - p_K(Y_j))_{1 \leq j \leq n}$ are orthogonal. Let $i \leq m$ and $j \leq n$ $p_K(X_i)$:

$$E[(X_i - p_K(X_i))(Y_j - p_K(Y_j))] = E[X_i(Y_j - p_K(Y_j))] - \underbrace{E[p_K(X_i)(Y_j - p_K(Y_j))]}_{=0} \\ = E[X_i Y_j] - E[X_i p_K(Y_j)] \quad \text{since } p_K(X_i) \in K \text{ and } Y_j - p_K(Y_j) \perp K \\ = E[X_i Y_j] - E[(p_K(X_i) + X_i - p_K(X_i))p_K(Y_j)] \\ = E[X_i Y_j] - E[p_K(X_i)p_K(Y_j)] - \underbrace{E[(X_i - p_K(X_i))p_K(Y_j)]}_{=0} \\ = 0 \quad \text{by hypothesis}$$

Since the $(U_i)_i$ are all Gaussian, what we proved is that $(U_i)_{1 \leq i \leq m}$ is independent of $(U_i)_{m+1 \leq i \leq m+n}$. From this we use the fact that when X and Y are independent, $P_{(X,Y)} = P_X \otimes P_Y$:

$$P_U = P_{(U_1, \dots, U_{m+n})} \\ = P_{(U_1, \dots, U_m)} \otimes P_{(U_{m+1}, \dots, U_{m+n})} \\ = P_{(X_1 - p_K(X_1), \dots, X_m - p_K(X_m))} \otimes P_{(Y_1 - p_K(Y_1), \dots, Y_n - p_K(Y_n))}$$

Thus,

$$\begin{aligned}
E[\mathbb{1}_A \mathbb{1}_B | \sigma(K)] &= \int \mathbb{1}_{u_1 + p_K(X_1) \in A_1} \cdots \mathbb{1}_{u_{m+n} + p_K(Y_n) \in B_n} \cdot P_U(du_1 \dots du_{m+n}) \\
&= \int \mathbb{1}_{u_1 + p_K(X_1) \in A_1} \cdots \mathbb{1}_{u_{m+n} + p_K(Y_n) \in B_n} \cdot P_{(U_1, \dots, U_m)}(du_1 \dots du_m) P_{(U_{m+1}, \dots, U_{m+n})}(du_{m+1} \dots du_{m+n}) \\
&= \left(\int \prod_{j=1}^m \mathbb{1}_{u_j + p_K(X_j) \in A_j} \cdot dP_{(U_1, \dots, U_m)} \right) \left(\int \prod_{j=1}^n \mathbb{1}_{u_{m+j} + p_K(Y_j) \in B_j} \cdot dP_{(U_{m+1}, \dots, U_{m+n})} \right) \\
&= E[h_X(U_X, V_X) | \sigma(K)] \cdot E[h_Y(U_Y, V_Y) | \sigma(K)] \quad \text{using the lemma} \\
&= E[\mathbb{1}_{X_1 \in A_1} \cdots \mathbb{1}_{X_m \in A_m} | \sigma(K)] \cdot E[\mathbb{1}_{Y_1 \in B_1} \cdots \mathbb{1}_{Y_n \in B_n} | \sigma(K)] \\
&= E[\mathbb{1}_A | \sigma(K)] E[\mathbb{1}_B | \sigma(K)]
\end{aligned}$$

where :

$$\begin{aligned}
h_X : \mathbb{R}^m \times \mathbb{R}^m &\longrightarrow \mathbb{R}^+ \\
(u_1, \dots, u_m), (v_1, \dots, v_m) &\longmapsto \prod_{i=1}^m \mathbb{1}_{u_i + v_i \in A_i} \quad \text{and} \quad h_Y : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^+ \\
(u_1, \dots, u_n), (v_1, \dots, v_n) &\longmapsto \prod_{i=1}^n \mathbb{1}_{u_i + v_i \in B_i}
\end{aligned}$$