Solutions to Exercises from Le Gall's *Brownian Motion, Martingales, and Stochastic Calculus*

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1 Gaussian Variables and Gaussian Processes

1.1 Exercice 1.15

1. $\boxed{\Rightarrow}$ Let us suppose that K is continuous. Let $(t,s) \in [0,1]^2$. We have :

$$\begin{aligned}\|X_t - X_s\|_{L^2}^2 &= E(X_t^2) + E(X_s^2) - 2E(X_tX_s)\\ &= K(t,t) + K(s,s) - 2K(t,s)\\ &= K(t,t) - K(t,s) + K(s,s) - K(t,s)\end{aligned}$$

By continuity of K in (t,t) : $K(t,s) \xrightarrow[s\to t]{} K(t,t)$ and $K(s,s) \xrightarrow[s\to t]{} K(t,t)$
Thus : $\|X_t - X_s\|_{L^2}^2 \xrightarrow[s\to t]{} 0$ and X is continuous in t

$\boxed{\Leftarrow}$ We suppose now that $t \mapsto X_t$ is continuous on $[0,1]$. Let $(t,s),(t',s') \in [0,1]^2$.

$$\begin{aligned}|K(t,s) - K(t',s')| &= |E(X_tX_s) - E(X_{t'}X_{s'})|\\ &\leq E(|X_tX_s - X_{t'}X_{s'}|)\\ &= \|X_tX_s - X_{t'}X_s + X_{t'}X_s - X_{t'}X_{s'}\|_{L^1}\\ &\leq \|X_tX_s - X_{t'}X_s\|_{L^1} + \|X_{t'}X_s - X_{t'}X_{s'}\|_{L^1}\\ &= \|(X_t - X_{t'})X_s\|_{L^1} + \|X_{t'}(X_s - X_{s'})\|_{L^1}\\ &\leq \|X_t - X_{t'}\|_{L^2}\|X_s\|_{L^2} + \|X_{t'}\|_{L^2}\|X_s - X_{s'}\|_{L^2} \text{ using Cauchy-Schwartz inequality}\end{aligned}$$

By continuity of $t \mapsto X_t$ in t, $\|X_t - X_{t'}\|_{L^2} \xrightarrow{t'\to t} 0$ and $X_{t'}$ is bounded. Similarly, since $t \mapsto X_t$ is continuous on s, $\|X_s - X_{s'}\|_{L^2} \xrightarrow{s'\to s} 0$
Thus, we have $|K(t,s) - K(t',s')| \xrightarrow{(t',s')\to(t,s)} 0$ which proves the continuity of K on $[0,1]^2$

2. Let us show that $\int_0^1 |h(t)||X_t|dt \in L^1(\Omega)$ which will end the proof.

$$E\left(\int_0^1 |h(t)||X_t|dt\right) = \int_0^1 |h(t)|E(|X_t|) \leq \int_0^1 |h(t)|\sqrt{E(X_t^2)}dt = \int_0^1 |h(t)|\sqrt{K(t,t)}dt < \infty$$

Moreover, $Z \in L^1$

3. First, let us show that $Z \in L^2$ and that for every $n \in \mathbb{N}$, $Z_n \in L^2$
Since Z_n is a linear combination of variables of the Gaussian process $(X_t)_{t\in\mathbb{R}}$, it is Gaussian thus in L^2

On the other hand :

$$E(Z^2) = E\left(\left(\int_0^1 h(t)X_tdt\right)^2\right) = E\left(\int_0^1\int_0^1 h(t)h(s)X_tX_sdsdt\right) = \int_0^1\int_0^1 h(t)h(s)K(t,s)dsdt$$

Since K is continuous on the compact set $[0,1]^2$, it is bounded let's say by a constant $C > 0$. Thus:

$$E(Z^2) \leq \int_0^1\int_0^1 |h(t)||h(s)||K(t,s)|dsdt \leq C\left(\int_0^1 |h(t)|dt\right)^2 < \infty$$

Now let us show that $||Z - Z_n||_{L^2} \xrightarrow{n\to\infty} 0$
Let $\varepsilon > 0$. Since $t \mapsto X_t$ is continuous on the compact set $[0,1]$, it is uniformly continuous. Thus, there exist $\eta > 0$ such that :

$$\forall t,s \in [0,1], |t-s| < \eta \implies ||X_t - X_s||_{L^2} < \varepsilon$$

Let $N \in \mathbb{N}$ such that $\frac{1}{N} < \eta$, and let $n \geq N$:

$$\begin{aligned}
||Z - Z_n||_{L^2} &= || \int_0^1 h(t) X_t dt - \sum_{i=1}^n X_{\frac{i}{n}} \int_{\frac{i-1}{n}}^{\frac{i}{n}} h(t) dt ||_{L^2} \\
&= || \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} h(t) X_t dt - \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} h(t) X_{\frac{i}{n}} dt ||_{L^2} \\
&= || \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} h(t) (X_{\frac{i}{n}} - X_t) dt ||_{L^2} \\
&\leq \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} h(t) ||X_{\frac{i}{n}} - X_t||_{L^2} dt
\end{aligned}$$

Since $\forall t \in [\frac{i-1}{n}, \frac{i}{n}], |t - \frac{i}{n}| < \frac{1}{N} \leq \eta$ we have that $||X_{\frac{i}{n}} - X_t||_{L^2} < \varepsilon$
Thus :

$$||Z - Z_n||_{L^2} \leq \varepsilon \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} |h(t)| dt = \varepsilon \int_0^1 |h(t)| dt \leq \varepsilon M$$

since $\int_0^1 |h(t)| dt < \infty$
We showed that for all $n \in \mathbb{N}, Z_n$ is Gaussian and $Z_n \xrightarrow{n \to \infty} Z$ in L^2. It follows from Proposition 1.1 that Z is Gaussian.

4. Since $(L^2(\Omega), ||\cdot||_2)$ is a complete metric space, to prove that $\dot{X}_t$ exists, it suffices to show that :

$$\left\| \frac{X_t - X_s}{t - s} - \frac{X_t - X_r}{t - r} \right\|_{L^2} \xrightarrow{(s,r) \to (0,0)} 0$$

For $t \in \mathbb{R}$:

$$\begin{aligned}
\left\| \frac{X_t - X_s}{t - s} - \frac{X_t - X_r}{t - r} \right\|_{L^2}^2 &= E\left[\left(\frac{X_t - X_s}{t - s} \right)^2 + \left(\frac{X_t - X_r}{t - r} \right)^2 - 2 \cdot \frac{(X_t - X_s)(X_t - X_r)}{(t - s)(t - r)} \right] \\
&= \underbrace{\frac{1}{(t - s)^2} [K(t,t) + K(s,s) - 2K(t,s)]}_{A} \\
&\quad + \underbrace{\frac{1}{(t - r)^2} [K(t,t) + K(r,r) - 2K(t,r)]}_{B} \\
&\quad - \underbrace{\frac{2}{(t - s)(t - r)} [K(t,t) - K(t,r) - K(t,s) + K(s,r)]}_{C}
\end{aligned}$$

Now let us make a fundamental remark for the following. Since K is twice continuously differentiable, we have the following properties on the partial derivatives of K using symetry of K and Schwarz Theorem:

$$\forall t, s \in \mathbb{R}, \quad \frac{\partial K}{\partial t}(t,s) = \frac{\partial K}{\partial s}(s,t) \quad \text{and} \quad \frac{\partial^2 K}{\partial t^2}(t,s) = \frac{\partial^2 K}{\partial s^2}(s,t) \quad \text{and} \quad \frac{\partial^2 K}{\partial t \, \partial s}(t,s) = \frac{\partial^2 K}{\partial s \, \partial t}(s,t) = \frac{\partial^2 K}{\partial s \, \partial t}(t,s)$$

Let us write the general second-order Taylor expansion of K in (t, s) which all terms in A, B and C will derive from:

$$\begin{aligned}
K(u,r) = K(t,s) &+ (u - t) \frac{\partial K}{\partial t}(t,s) + (r - s) \frac{\partial K}{\partial s}(t,s) + \frac{1}{2}(u - t)^2 \frac{\partial^2 K}{\partial t^2}(t,s) + \frac{1}{2}(r - s)^2 \frac{\partial^2 K}{\partial s^2}(t,s) \\
&+ (u - t)(r - s) \frac{\partial^2 K}{\partial t \partial s}(t,s) + o\left(\|(u,r) - (t,s)\|^2 \right)
\end{aligned}$$

This gives us for the term A:

$$\begin{aligned}K(t,s) &\underset{s\to t}{=} K(t,t)+\frac{\partial K}{\partial s}(t,t)(s-t)+\frac{1}{2}\frac{\partial^2 K}{\partial s^2}(t,t)(s-t)^2+o\big((s-t)^2\big)\\ &\underset{s\to t}{=} K(t,t)+\frac{\partial K}{\partial t}(t,t)(s-t)+\frac{1}{2}\frac{\partial^2 K}{\partial t^2}(t,t)(s-t)^2+o\big((s-t)^2\big)\end{aligned}$$

and

$$\begin{aligned}K(s,s) &\underset{s\to t}{=} K(t,t)+\left(\frac{\partial K}{\partial t}(t,t)+\frac{\partial K}{\partial s}(t,t)\right)(s-t)\\ &\quad+\frac{1}{2}\left(\frac{\partial^2 K}{\partial t^2}(t,t)+2\frac{\partial^2 K}{\partial t\partial s}(t,t)+\frac{\partial^2 K}{\partial s^2}(t,t)\right)(s-t)^2+o\big((s-t)^2\big)\\ &\underset{s\to t}{=} K(t,t)+2\frac{\partial K}{\partial t}(t,t)(s-t)+\left(\frac{\partial^2 K}{\partial t^2}(t,t)+\frac{\partial^2 K}{\partial t\partial s}(t,t)\right)(s-t)^2+o\big((s-t)^2\big)\end{aligned}$$

After simplification, this leads to:

$$A \underset{s\to t}{=} \frac{\partial^2 K}{\partial t\,\partial s}(t,s)+o(1)$$

By symetry of the roles of s and r we also have:

$$B \underset{s\to t}{=} \frac{\partial^2 K}{\partial t\,\partial s}(t,t)+o(1)$$

Now let's take a look at a new term in C

$$\begin{aligned}K(s,r) &\underset{(s,r)\to(t,t)}{=} K(t,t)+\frac{\partial K}{\partial t}(t,t)(s-t)+\frac{\partial K}{\partial s}(t,t)(r-t)+\frac{1}{2}\frac{\partial^2 K}{\partial t^2}(t,t)(s-t)^2\\ &\quad+\frac{1}{2}\frac{\partial^2 K}{\partial s^2}(t,t)(r-t)^2+\frac{\partial^2 K}{\partial t\partial s}(t,t)(s-t)(r-t)+o\big(\|(s,r)-(t,t)\|^2\big)\\ &\underset{(s,r)\to(t,t)}{=} K(t,t)+\frac{\partial K}{\partial t}(t,t)(s+r-2t)+\frac{1}{2}\frac{\partial^2 K}{\partial t^2}(t,t)\big((s-t)^2+(r-t)^2\big)\\ &\quad+\frac{\partial^2 K}{\partial t\partial s}(t,t)(s-t)(r-t)+o\big(\|(s,r)-(t,t)\|^2\big)\end{aligned}$$

After simplification we have:

$$C \underset{(s,r)\to(t,t)}{=} 2\frac{\partial^2 K}{\partial t\,\partial s}(t,t)+o(1)$$

Thus, we have:

$$\left\|\frac{X_t-X_s}{t-s}-\frac{X_t-X_r}{t-r}\right\|_{L^2}^2 \xrightarrow{(s,r)\to(t,t)} 0$$

which leads to the existence of $\dot{X}_t$.

Let us now note $Y_{t,s}=\frac{X_t-X_s}{t-s}$. Since $(X_t)_{t\in\mathbb{R}}$ is a Gaussian process, $Y_{t,s}$ is Gaussian as inear combination of variables of $(X_t)_{t\in\mathbb{R}}$ and any finite linear combination of the variables $(Y_{t,s})_{t,s\in\mathbb{R}}$ is Gaussian.
Let $(t_1,...,t_n)$ and $(\lambda_1,...,\lambda_n)\in\mathbb{R}^n$. Then, since all the limits exist in L^2:

$$\sum_{i=1}^n \lambda_i\dot{X}_{t_i}=\sum_{i=1}^n \lambda_i \lim_{s_i\to t_i} Y_{t_i,s_i}=\lim_{(s_1,...,s_n)\to(t_1,...,t_n)}\sum_{i=1}^n \lambda_i Y_{t_i,s_i}$$

The last term is the L^2 limit of centered Gaussian variables, so it is centered Gaussian by Proposition 1.1, which proves that $(\dot{X}_t)_{t\in\mathbb{R}}$ is a centered Gaussian process.

For the covariance function, we can write for $(u,v)\in[0,1]^2$:

$$\begin{aligned}E(\dot{X}_t\dot{X}_s)=E\left(\lim_{u\to t}\frac{X_t-X_u}{t-u}\cdot\lim_{r\to s}\frac{X_s-X_r}{s-r}\right) &= E\left(\lim_{(u,r)\to(t,s)}\frac{(X_t-X_u)(X_s-X_r)}{(t-u)(s-r)}\right) \text{ since the limits exist in } L^2\\ &= \lim_{(s,r)\to(t,u)} E\left(\frac{(X_t-X_u)(X_s-X_r)}{(t-u)(s-r)}\right) \text{ (dominated convergence theorem)}\\ &= \lim_{(u,r)\to(t,s)}\frac{1}{(t-u)(s-r)}\left(K(t,s)-K(t,r)-K(u,s)+K(u,r)\right)\end{aligned}$$

Just as before we can use the Taylor expansions of K in (t, s) and simplifying using the previous properties. After calculus, we find :

$$E(\dot{X}_t \dot{X}_s) = \frac{\partial^2 K}{\partial t \partial s}(t, s)$$

With the properties stated above, we verify that this is indeed a symetric function.

1.2 Exerccice 1.16 (Kalman filtering)

1. Let $n \geq 0$. Using linearity of the conditional expectation we can write:

$$\hat{X}_{n+1/n} = E(X_{n+1}|Y_0, ..., Y_n) = a_n E(X_n|Y_0, ..., Y_n) + E(\epsilon_{n+1}|Y_0, ..., Y_n)$$

The recusrive formula for $(X_n)_n$ shows that for all $k \in \mathbb{N}, X_i \in \mathrm{Vect}(\epsilon_1, \ldots, \epsilon_\mathrm{k})$ so $X_1, .., X_n \in \sigma(\epsilon_1, ..., \epsilon_n)$ and since the $(\epsilon_i)_i$ are mutually independant, ϵ_{n+1} is independant from $\sigma(\epsilon_1, ..., \epsilon_n)$, thus from $X_1, .., X_n$. Furthermore, the recursive formula for $(Y_n)_n$ gives us that for all $k \in \mathbb{N}, Y_k \in \sigma(\epsilon_1, ..., \epsilon_k, \eta_k)$ and Since the $(\eta_i)_i$ and the $(\epsilon_i)_i$ are independant, ϵ_{n+1} is independant from $Y_0, .., Y_n$ and so we can write:

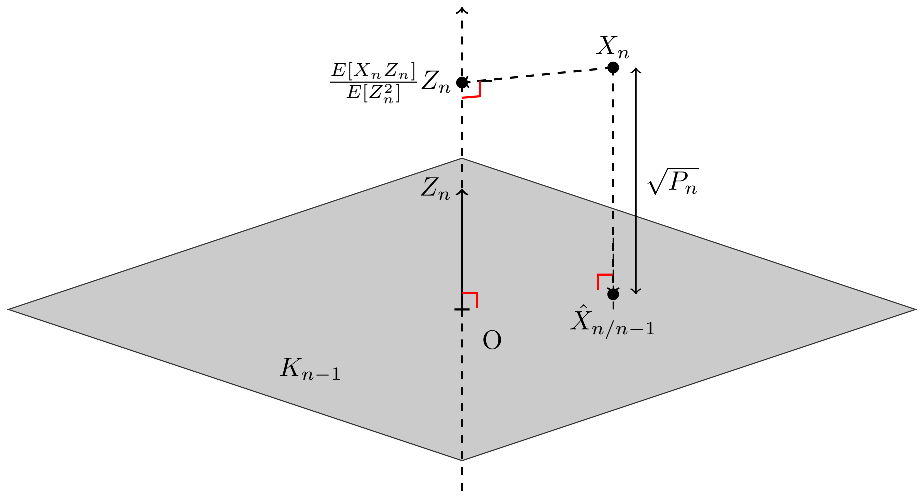
$$E(\epsilon_{n+1}|Y_0, ..., Y_n) = E(\epsilon_{n+1}) = 0 \text{ cause } \epsilon_{n+1} \text{ is centered}$$

Thus,

$$\hat{X}_{n+1/n} = a_n E(X_n|Y_0, ..., Y_n) = a_n \hat{X}_{n/n}$$

2. Let $(\cdot|\cdot)$ be the usual scalar product on L^2. Consider $H_n = Vect(\epsilon_1, ..., \epsilon_n, \eta_1, ..., \eta_n)$. $\epsilon_1, ..., \epsilon_n, \eta_1, ..., \eta_n$ are mutually independant Gaussian random variables so $(\epsilon_1, ..., \epsilon_n, \eta_1, ..., \eta_n)$ is a Gaussian vector, thus H_n is a Gaussian Space. Let $K_n = Vect(Y_0, ..., Y_n)$ and p_{K_n} be the orthogonal projection on K_n in L^2. Such a projection exists since K_n is finite-dimensional.
Using these notations we can rewrite the equality we want to prove as:

$$E[X_n|\sigma(K_n)] = E[X_n|\sigma(K_{n-1})] + \frac{(X_n|Z_n)}{\|Z_n\|^2_{L^2}} \cdot Z_n$$



Since K_n and K_{n-1} are closed because finite-dimensional in the Gaussian Space H_n, we can use Corollary 1.10 to write this as a geometric equation using projections:

$$p_{K_n}(X_n) = p_{K_{n-1}}(X_n) + p_{\mathrm{Vect(Z_n)}}(X_n)$$

Then it suffices to prove that $Z_n \in K_n \cap K_{n-1}^{\perp}$. In fact we will have $K_n = K_{n-1} \overset{\perp}{\oplus} \mathrm{Vect(Z_n)}$ which gives us $p_{K_n} = p_{K_{n-1}} + p_{\mathrm{Vect(Z_n)}}$

First of all: $Z_n = Y_n - cE[X_n|\sigma(K_{n-1})] = \underbrace{Y_n}_{\in K_n} - c \cdot \underbrace{p_{K_{n-1}}(X_n)}_{\in K_{n-1} \subset K_n}$, hence $Z_n \in K_n$.

To show that $Z_n \in K_{n-1}^{\perp}$ it suffices to prove that for all $k \leq n-1, Z_n \perp Y_k$. Let $k \leq n-1$

$$\begin{aligned}
E[Y_k Z_n] &= E[Y_k Y_n - cY_k E[X_n|\sigma(K_{n-1})]] \\
&= cE[Y_k X_n] + E[Y_k \eta_n] - cE[Y_k X_n] \quad \text{because } Y_k \in L^2 \text{ is } \sigma(K_{n-1})\text{−mesurable} \\
&= E[Y_k \eta_n] \\
&= E[Y_k]E[\eta_n] = 0 \quad \text{since } Y_k \text{ and } \eta_n \text{ are independant and } \eta_n \text{ is centered}
\end{aligned}$$

With that being proved, we have the desired equality for all $n \in \mathbb{N}$:

$$\hat{X}_{n/n} = \hat{X}_{n/n-1} + \frac{E[X_n Z_n]}{E[Z_n^2]} Z_n$$

3. First of all : $Z_n = c(X_n - \hat{X}_{n/n-1}) + \eta_n$. Since η_n is centered and independant of X_n and $\hat{X}_{n/n-1}$, we have:

$$E[Z_n^2] = c^2 E[(X_n - \hat{X}_{n/n-1})^2] + E[\eta_n^2] = c^2 P_n + \delta^2$$

In the other hand we can write :

$$\begin{aligned}
E[X_n Zn] &= cE[X_n(X_n - \hat{X}_{n/n-1})] + E[X_n \eta_n] \\
&= E[(X_n - \hat{X}_{n/n-1})^2] + cE[\underbrace{\hat{X}_{n/n-1}}_{\in K_{n-1}} \underbrace{(X_n - \hat{X}_{n/n-1})}_{\in K_{n-1}\perp}] \quad \text{because } \eta_n \text{ is independent of } X_n \text{ and centered} \\
&= cP_n
\end{aligned}$$

Let $n \geq 1$. Using question 1 and 2, and our formulas we have;

$$\begin{aligned}
\hat{X}_{n+1/n} = a_n \hat{X}_{n/n} &= a_n \left(\hat{X}_{n/n-1} + \frac{E[X_n Z_n]}{E[Z_n^2]} Z_n \right) \\
&= a_n \left(\hat{X}_{n/n-1} + \frac{cP_n}{cP_n + \delta^2} Z_n \right)
\end{aligned}$$

4. In the first place we have $X_1 = \epsilon_1$ and $Y_0 = \eta_0$. Hence;

$$X_{1/0} = E[X_1|Y_0] = E[\epsilon_1|\eta_0] = E[\epsilon_1] = 0 \text{ almost surely}$$

Then:

$$P_1 = E[(X_1 - X_{1/0})^2] = E[\epsilon_1^2] = \sigma^2$$

Now let's prove the recusrive formula for P_n. First we remark that, using question 3:

$$\begin{aligned}
\hat{X}_{n+1/n} - X_{n+1} &= a_n \left(\hat{X}_{n/n-1} + \frac{cP_n}{cP_n + \delta^2} Z_n \right) - a_n X_n - \epsilon_{n+1} \\
&= a_n(\hat{X}_{n/n-1} - X_n) + a_n \frac{cP_n}{cP_n + \delta^2} Z_n - \epsilon_{n+1}
\end{aligned}$$

Let us note

$$b_n = a_n \frac{cP_n}{cP_n + \delta^2}$$

hence:

$$
\begin{aligned}
P_{n+1} &= E[(\hat{X}_{n+1/n} - X_{n+1})^2] \\
&= E[(a_n(\hat{X}_{n/n-1} - X_n)^2] + 2E[a_n(\hat{X}_{n/n-1} - X_n)(b_n Z_n - \epsilon_{n+1})] + E[(b_n Z_n - \epsilon_{n+1}^2)] \\
&= a_n^2 P_n + 2a_n b_n E[(\hat{X}_{n/n-1} - X_n)Z_n] + b_n^2 E[Z_n^2] + E[\epsilon_{n+1}^2] \quad \text{as } \epsilon_{n+1} \text{ centered and independent of } \hat{X}_{n/n-1}, X_n \text{ and } Z_n \\
&= a_n^2 P_n - 2a_n b_n c P_n + b_n^2(c^2 P_n + \delta^2) + \sigma^1 \quad \text{as } \hat{X}_{n/n-1} \perp Z_n \text{ and with previous formulas} \\
&= a_n^2 P_n - 2a_n^2 \frac{c^2 P_n^2}{cP_n + \delta^2} + a_n^2 \frac{c^2 P_n^2}{cP_n + \delta^2} + \sigma^2 \\
&= \sigma^2 + a_n \frac{\delta^2 P_n}{cP_n + \delta^2}
\end{aligned}
$$

which is the desired formula.

1.3 Exerccice 1.17

A. Why the hint allows us to conclude

The hint suggests to consider the case where X_1, resp. X_2, is the indicator function of an event depending only on finitely many variables in H_1, resp. in H_2. Let us show why we can do this.

Let $\mathcal{A} = \{A \in \sigma(H_1), \exists X_1, ..., X_m \in H_1, A \in \sigma(X_1, ..., X_m)\}$ be the set of events depending only on finetly many variables of H_1 and $\mathcal{B} = \{B \in \sigma(H_2), \exists Y_1, ..., Y_n \in H_2, B \in \sigma(Y_1, ..., Y_n)\}$ be the set of events depending only on finetly many variables of H_2.

We **suppose** that for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$:

$$E[\mathbb{1}_A \mathbb{1}_B | \sigma(K)] = E[\mathbb{1}_A | \sigma(K)] E[\mathbb{1}_B | \sigma(K)]$$

and we want to **show** that for all non-negative $\sigma(H_1)$-mesurable random variable X_1 and for all non-negative $\sigma(H_2)$-mesurable random variable X_2:

$$E[X_1 X_2 | \sigma(K)] = E[X_1 | \sigma(K)] E[X_2 | \sigma(K)]$$

Frist, it is immediate that $\mathcal{A}$ and $\mathcal{B}$ are stable under finite intersection. Moreover;

$$\forall X_1 \in H_1, \forall A_1 \in \mathcal{B}(\mathbb{R}), X_1^{-1}(A_1) \in \mathcal{A}. \quad \text{and} \quad \forall X_2 \in H_2, \forall A_2 \in \mathcal{B}(\mathbb{R}), X_2^{-1}(A_2) \in \mathcal{B}$$

Thus $\sigma(\mathcal{A}) = \sigma(H_1)$ and $\sigma(\mathcal{B}) = \sigma(H_2)$

Let us now consider

$$\mathcal{M}_1 = \{A \in \sigma(H_1), \forall B \in \mathcal{B}, E[\mathbb{1}_A \mathbb{1}_B | \sigma(K)] = E[\mathbb{1}_A | \sigma(K)] E[\mathbb{1}_B | \sigma(K)]\}$$

.
By hypothesis, $\mathcal{A} \subset \mathcal{M}_1$. Besides, $\mathcal{M}_1$ is a monotone class. In fact :

- $\mathbb{1}_\Omega = 1$ a.s thus, for all $B \in \mathcal{B}$ $E[\mathbb{1}_\Omega \mathbb{1}_B | \sigma(K)] = E[\mathbb{1}_B | \sigma(K)] = E[\mathbb{1}_B | \sigma(K)] E[\mathbb{1}_\Omega | \sigma(K)]$ a.s, hence $\Omega \subset \mathcal{M}_1$
- If $A_1, A_2 \in \mathcal{M}_1, A_1 \subset A_2$ we know that $\mathbb{1}_{A_2 \backslash A_1} = \mathbb{1}_{A_2} - \mathbb{1}_{A_1}$. Then for all $B \in \mathcal{B}$, by linearity of conditional expectation:

$$
\begin{aligned}
E[\mathbb{1}_{A_2 \backslash A_1} \mathbb{1}_B | \sigma(K)] &= E[\mathbb{1}_{A_2} \mathbb{1}_B | \sigma(K)] - \cdot E[\mathbb{1}_{A_1} \mathbb{1}_B | \sigma(K)] \\
&= E[\mathbb{1}_{A_2} | \sigma(K)] \cdot E[\mathbb{1}_B | \sigma(K)] - E[\mathbb{1}_{A_1} | \sigma(K)] \cdot E[\mathbb{1}_B | \sigma(K)] \quad \text{since } A_1, A_2 \in \mathcal{M}_1 \\
&= E[\mathbb{1}_{A_2} - \mathbb{1}_{A_1} | \sigma(K)] \cdot E[\mathbb{1}_B | \sigma(K)] \\
&= E[\mathbb{1}_{A_2 \backslash A_1} | \sigma(K)] \cdot E[\mathbb{1}_B | \sigma(K)]
\end{aligned}
$$

  Hence $A_2 \backslash A_1 \in \mathcal{M}_1$

- If $(A_n)_n \in (\mathcal{M}_1)^{\mathbb{N}}$ is an increasing sequence then for all $B \in \mathcal{B}$ and for all $n \geq 0$:

$$E[\mathbb{1}_{\bigcup_{k=0}^n A_k}\mathbb{1}_B|\sigma(K)] = E[\mathbb{1}_{A_n}\mathbb{1}_B|\sigma(K)] = E[\mathbb{1}_{A_n}|\sigma(K)]E[\mathbb{1}_B|\sigma(K)] \quad \text{since } A_n \in \mathcal{M}_1$$

  Since the $(A_n)_n$ are increasing (in the sense of $\subset$) then, $(\mathbb{1}_{A_n})_n$ is an increasing sequence of non-negative functions. Moreover $\mathbb{1}_{A_n} \xrightarrow{n\to\infty} \mathbb{1}_{\bigcup_{n=0}^\infty A_k}$ a.s., and $\mathbb{1}_B$ is non-negative, thus by conditional monotone convergence theorem :

$$E[\mathbb{1}_{A_n}|\sigma(K)] \xrightarrow{n\to\infty} E[\mathbb{1}_{\bigcup_{n=0}^\infty A_n}|\sigma(K)] \quad \text{and} \quad E[\mathbb{1}_{A_n}\mathbb{1}_B|\sigma(K)] \xrightarrow{n\to\infty} E[\mathbb{1}_{\bigcup_{n=0}^\infty A_n}\mathbb{1}_B|\sigma(K)]$$

  Hence $\bigcup_{n=0}^\infty A_n \in \mathcal{M}_1$

Under theses conditions, according to the Monotone class lemma, we have that the monotone class generated by $\mathcal{A}$, denoted by $\mathcal{M}(\mathcal{A})$ is the sigma field $\sigma(\mathcal{A})$ generated by $\mathcal{A}$

$$\sigma(H_1) = \sigma(\mathcal{A}_1) = \mathcal{M}(\mathcal{A}_1) \subset \mathcal{M}_1 \quad \text{hence} \quad \mathcal{M}_1 = \sigma(H_1)$$

Then

$$\forall A \in \sigma(H_1), \forall B \in \mathcal{B} \quad E[\mathbb{1}_A\mathbb{1}_B|\sigma(K)] = E[\mathbb{1}_A|\sigma(K)]E[\mathbb{1}_B|\sigma(K)]$$

This equality being true for all $A \in \sigma(H_1)$, it is also true for all non-negative $\sigma(H_1)$-mesurable step functions. Since every non-negative $\sigma(H_1)$-mesurable function is the limit of an non-deacreasing sequence of non-negative $\sigma(H_1)$-mesurable step functions, the conditional monotone convergence theorem gives us for all non-negative $\sigma(H_1)$-mesurable random variable X_1

$$\forall B \in \mathcal{B} \quad E[X_1\mathbb{1}_B|\sigma(K)] = E[X_1|\sigma(K)]E[\mathbb{1}_B|\sigma(K)]$$

Now if we consider

$$\mathcal{M}_2 = \{B \in \sigma(H_2), \forall X_1 \ \sigma(H_1)\text{-mesurable } X_1 \geq 0, E[X_1\mathbb{1}_B|\sigma(K)] = E[X_1|\sigma(K)]E[\mathbb{1}_B|\sigma(K)]\}$$

.
We prove analogously that $\mathcal{M}_2$ is a monotone class.
Since $B \subset \mathcal{M}_2$ and $\sigma(\mathcal{B}) = \sigma(H_2)$ the monotone class lemma holds and

$$\forall B \in \sigma(H_2), \forall X_1 \ \sigma(H_1)\text{-mesurable } X_1 \geq 0, \quad E[X_1\mathbb{1}_B|\sigma(K)] = E[X_1|\sigma(K)]E[\mathbb{1}_B|\sigma(K)]$$

.
With the same arguments as earlier, this equality holds for evry non-negative $\sigma(H_2)$-mesurable random variable X_2 and we have that for all non-negative $\sigma(H_2)$-mesurable random variable X_2 and for all non-negative $\sigma(H_1)$-mesurable random variable X_1

$$E[X_1X_2B|\sigma(K)] = E[X_1|\sigma(K)] \cdot E[X_2|\sigma(K)]$$

which implies that the σ-fields $\sigma(H_1)$ and $\sigma(H_2)$ are conditionally independent give $\sigma(K)$

B. Proving the result for indicator functions

Let us show how we can prove the result when X_1, resp. X_2, is the indicator function of an event depending only on finitely many variables in H_1, resp. in H_2, as suggested in the hint. First, we recall a fundamental lemma useful for the following.

***Lemma**: Let X and Y be two random variables taking values in the measurable spaces E and F, respectively. Suppose that X is independent of $\mathcal{B}$, and that Y is $\mathcal{B}$-measurable. Then, for any measurable function $g : E \times F \to \mathbb{R}_+$, we have:*

$$E[g(X,Y) \mid \mathcal{B}] = \int_E g(x,Y)\,\mathbb{P}_X(dx)$$

Let $A \in \sigma(H_1)$ and $B \in \sigma(H_2)$ depend, respectively, only on $X_1, ..., X_m$ and $Y_1, ..., Y_n$. Thus we can write $A = \cap_{i=1}^m (X_i \in A_i)$ and $B = \cap_{i=1}^n (X_i \in B_i)$ with $A_1, .., A_m, B_1, .., B_n \in \mathcal{B}(\mathbb{R})$.

Thus

$$\mathbb{1}_A = \mathbb{1}_{X_1 \in A_1} \cdots \mathbb{1}_{X_m \in A_m} \text{ and } \mathbb{1}_B = \mathbb{1}_{Y_1 \in B_1} \cdots \mathbb{1}_{Y_n \in B_n}$$

Then, what we would like is :

$$E[\mathbb{1}_{X_1 \in A_1} \cdots \mathbb{1}_{X_m \in A_m} \cdot \mathbb{1}_{Y_1 \in B_1} \cdots \mathbb{1}_{Y_n \in B_n} | \sigma(K)] = E[\mathbb{1}_{X_1 \in A_1} \cdots \mathbb{1}_{X_m \in A_m} | \sigma(K)] \cdot E[\mathbb{1}_{Y_1 \in B_1} \cdots \mathbb{1}_{Y_n \in B_n} | \sigma(K)]$$

Let us define

$U = (X_1 - p_K(X_1), ..., X_m - p_K(X_m), Y_1 - p_K(Y_1), ..., Y_n - p_K(Y_n))$ and $V = (p_K(X_1), ..., p_K(X_m), p_K(Y_1), ..., p_K(Y_n))$

We will use the notation U_i (reps. V_i) for the i-th coordinate of U (resp. V). We will also use $U_X := (U_1, ..., U_m)$, $V_X := (V_1, ..., V_m)$, $U_Y := (U_{m+1}, ..., U_{m+n})$ $V_Y := (V_{m+1}, ..., V_{m+n})$.
Let us also define

$$\begin{aligned} g : \mathbb{R}^{m+n} \times \mathbb{R}^{m+n} &\longrightarrow \mathbb{R}^+ \\ (u_1, \dots, u_{m+n}), (v_1, \dots, v_{m+n}) &\longmapsto \prod_{i=1}^{m} \mathbb{1}_{u_i + v_i \in A_i} \cdot \prod_{i=m+1}^{n} \mathbb{1}_{u_i + v_i \in B_{m-j}} \end{aligned}$$

We will use the notation U_i for the i-th coordinate of U (i.e. $X_i - p_K(X_i)$, resp. $Y_i - p_K(Y_i)$ for $i \leq m$, resp. for $m+1 \leq i \leq m+n$). We will also use $U_X := (U_1, ..., U_m)$, $V_X := (V_1, ..., V_m)$, $U_Y := (U_{m+1}, ..., U_{m+n})$ $V_Y := (V_{m+1}, ..., V_{m+n})$.
Then we can write :

$$\begin{aligned} E[\mathbb{1}_A \mathbb{1}_B | \sigma(K)] &= E[\mathbb{1}_{X_1 \in A_1} \cdots \mathbb{1}_{X_m \in A_m} \cdot \mathbb{1}_{Y_1 \in B_1} \cdots \mathbb{1}_{Y_n \in B_n} | \sigma(K)] \\ &= E[g(U, V) | \sigma(K)] \end{aligned}$$

Or, for all $i \leq m$ and for all $j \leq n$ $X_i - p_K(X_i)$ and $Y_j - p_K(Y_j)$ are Gaussian and orthogonal to K hence independent of $\sigma(K)$.
In the other hand, for all $i \leq m$ and for all $j \leq n$ $p_K(X_i)$ and $p_K(Y_j)$ are in K hence $\sigma(K)$-mesurable. Under those conditions we can apply the lemma and :

$$\begin{aligned} E[\mathbb{1}_A \mathbb{1}_B | \sigma(K)] &= E[g(U, V) | \sigma(K)] \\ &= \int \mathbb{1}_{u_1 + p_K(X_1) \in A_1} \cdots \mathbb{1}_{u_{m+n} + p_K(Y_n) \in B_n} \cdot P_U(du_1 ... du_{m+n}) \end{aligned}$$

Now in order to split the integral into two parts we need to prove that the $(X_i - p_K(X_i))_{1 \leq i \leq m}$ and the $(Y_j - p_K(Y_j))_{1 \leq j \leq n}$ are orthogonal. Let $i \leq m$ and $j \leq n$ $p_K(X_i)$:

$$\begin{aligned} E[(X_i - p_k(X_i))(Y_j - p_K(Y_j))] &= E[X_i(Y_j - p_K(Y_j))] - \underbrace{E[p_k(X_i)(Y_j - p_K(Y_j))]}_{=0} \\ &= E[X_i Y_j] - E[X_i p_K(Y_j)] \quad \text{ since } p_k(X_i) \in K \text{ and } Y_j - p_K(Y_j) \perp K \\ &= E[X_i Y_j] - E[(p_K(X_i) + X_i - p_K(X_i)) p_K(Y_j)] \\ &= E[X_i Y_j] - E[p_K(X_i) p_K(Y_j)] - \underbrace{E[(X_i - p_K(X_i)) p_K(Y_j)]}_{=0} \\ &= 0 \quad \text{by hypothesis} \end{aligned}$$

Since the $(U_i)_i$ are all Gaussian, what we proved is that $(U_i)_{1 \leq i \leq m}$ is independent of $(U_i)_{m+1 \leq i \leq m+n}$. From this we use the fact that when X and Y are independent, $P_{(X,Y)} = P_X \otimes P_Y$:

$$\begin{aligned} P_U &= P_{(U_1, \dots, U_{m+n})} \\ &= P_{(U_1, \dots, U_m)} \otimes P_{(U_{m+1}, \dots, U_{m+n})} \\ &= P_{(X_1 - p_K(X_1), \dots, X_m - p_K(X_m))} \otimes P_{(Y_1 - p_K(Y_1), \dots, Y_n - p_K(Y_n))} \end{aligned}$$

Thus,

$$
\begin{aligned}
E[\mathbb{1}_A\mathbb{1}_B|\sigma(K)] &= \int \mathbb{1}_{u_1+p_K(X_1)\in A_1}\cdots\mathbb{1}_{u_{m+n}+p_K(Y_n)\in B_n}\cdot P_U(du_1...du_{m+n}) \\
&= \int \mathbb{1}_{u_1+p_K(X_1)\in A_1}\cdots\mathbb{1}_{u_{m+n}+p_K(Y_n)\in B_n}\cdot P_{(U_1,\dots,U_m)}(du_1...du_m)P_{(U_{m+1},\dots,U_{m+n})}(du_{m+1}...du_{m+n}) \\
&= \left(\int\prod_{j=1}^{m}\mathbb{1}_{u_j+p_K(X_j)\in A_j}\cdot dP_{(U_1,\dots,U_m)}\right)\left(\int\prod_{j=1}^{n}\mathbb{1}_{u_{m+j}+p_K(Y_j)\in B_j}\cdot dP_{(U_{m+1},\dots,U_{m+n})}\right) \\
&= E[h_X(U_X,V_X)|\sigma(K)]\cdot E[h_Y(U_Y,V_Y)|\sigma(K)] \quad \text{using the lemma} \\
&= E[\mathbb{1}_{X_1\in A_1}\cdots\mathbb{1}_{X_m\in A_m}|\sigma(K)]\cdot E[\mathbb{1}_{Y_1\in B_1}\cdots\mathbb{1}_{Y_n\in B_n}|\sigma(K)] \\
&= E[\mathbb{1}_A|\sigma(K)]E[\mathbb{1}_B|\sigma(K)]
\end{aligned}
$$

where :

$$
\begin{array}{rcl}
h_X:\mathbb{R}^m\times\mathbb{R}^m & \longrightarrow & \mathbb{R}^+ \\
(u_1,\dots,u_m),(v_1,\dots,v_m) & \longmapsto & \displaystyle\prod_{i=1}^{m}\mathbb{1}_{u_i+v_i\in A_i}
\end{array}
\quad \text{and} \quad
\begin{array}{rcl}
h_Y:\mathbb{R}^n\times\mathbb{R}^n & \longrightarrow & \mathbb{R}^+ \\
(u_1,\dots,u_n),(v_1,\dots,v_n) & \longmapsto & \displaystyle\prod_{i=1}^{n}\mathbb{1}_{u_i+v_i\in B_i}
\end{array}
$$

.

2 Brownian Motion

2.1 Exerccice 2.25 (Time inversion)

1. We begin by showing that $(W_t)_{t\geq 0}$ is a pre-Brownian motion.
It is immediate that $(W_t)_{t\geq 0}$ is a centered Gaussian process, since $(B_t)_{t\geq 0}$ is centered Gaussian.
Let K denote the covariance function of $(W_t)_{t\geq 0}$. If $t=0$, then for all $s\geq 0$ we have

$$K(0,s)=E[W_0W_s]=0=0\wedge s,$$

and for all $s,t>0$,

$$
\begin{aligned}
K(t,s)=E[W_tW_s] &= t\cdot s\cdot E[B_{1/t}B_{1/s}] \\
&= t\cdot s\cdot\left(\frac{1}{t}\wedge\frac{1}{s}\right) \\
&= t\wedge s.
\end{aligned}
$$

Therefore, $(W_t)_{t\geq 0}$ is a pre-Brownian motion.

Since $(B_t)_{t\geq 0}$ is a Brownian motion, for every $\omega\in\Omega$, the map $t\mapsto B_t(\omega)$ is continuous. Moreover, the functions $t\mapsto t$ and $t\mapsto\frac{1}{t}$ are continuous on $(0,+\infty)$. It follows that, for every $\omega\in\Omega$, the map

$$t\mapsto B_{1/t}(\omega)=W_t(\omega)$$

is continuous on $(0,\infty)$.
The issue now is whether $t\mapsto W_t(\omega)$ is continuous at 0; that is, whether for every $\omega\in\Omega$,

$$W_t\xrightarrow[t\to 0]{}W_0=0.$$

To address this, we use the observation following **Definition 2.8**.
In particular, $t\mapsto W_t(\omega)$ is right-continuous on $[0,+\infty)$.
Furthermore, by **Corollary 2.11**, $(W_t)_{t\geq 0}$ admits a modification $(\tilde{W}_t)_{t\geq 0}$ whose sample paths are continuous, and thus in particular right-continuous.

Since both $(W_t)_{t\geq 0}$ and $(\tilde{W}_t)_{t\geq 0}$ have right-continuous sample paths and $(\tilde{W}_t)_{t\geq 0}$ is a modification of $(W_t)_{t\geq 0}$, the two processes are indistinguishable.
But $(\tilde{W}_t)_{t\geq 0}$ is a pre-Brownian motion with continuous sample paths, hence it is a Brownian motion started from 0.

2. Now that we know $(W_t)_{t\geq 0}$ is indistinguishable from a Brownian motion started from 0, its sample paths are almost surely continuous. In particular, its sample paths are almost surely continuous at 0, and we have

$$W_t \xrightarrow[t\to 0]{} W_0 = 0,$$

which implies

$$\frac{B_t}{t} = W_{1/t} \xrightarrow[t\to+\infty]{} 0 \quad \text{almost surely.}$$

2.2 Exerccice 2.26

First, let us prove that $T_b - T_a$ and T_{b-a} have the same distribution. Intuitively, knowing the strong Markov property, it makes sense that B_t has the same probability getting to b starting from a than getting to $b-a$ starting from 0.
We're gonna write that rigourously. Let $t \geq 0$. B and $B^{(T_a)}$ are both Brownian motions, thus they have the same law under the measures P and $P(\cdot \mid T_a < \infty)$ respectively and

$$\begin{aligned}
P(B_t \geq b-a) &= P(B_t^{(T_a)} \geq b-a \mid T_a < \infty) \\
&= P(B_{t+T_a} - a \geq b-a) \quad \text{since a.s} \quad T_a < \infty \text{ and } B_{T_a} = a \\
&= P(B_{t+T_a} \geq b)
\end{aligned}$$

In addition, we know that for $c \in \mathbb{R}$ and $t \geq 0$

$$P(T_c \leq t) = P(S_t \geq c) = P(|B_t| \geq c) = 2P(B_t \geq c)$$

Now let $t \geq 0$.

$$P(T_{b-a} \leq t) = 2P(B_t \geq b-a) = 2P(B_{t+T_a} \geq b) = P(T_b \leq t + T_a) = P(T_b - T_a \leq t)$$

which proves that $T_b - T_a$ and T_{b-a} have the same distribution.

Now let us show why $T_b - T_a$ is independent of $\sigma(T_c, 0 \leq c \leq a)$. We know that $T_c \leq T_a$ a.s. and hence, $\mathcal{F}_{T_c} \subset \mathcal{F}_{T_a}$. Indeed if $A \in \mathcal{F}_{T_c}$:

$$\begin{aligned}
\forall t \geq 0, \quad A \cap (T_a \leq t) &= A \cap ((T_c \leq t) \cap (T_a \leq t)) \\
&= (A \cap (T_c \leq t)) \cap (T_a \leq t) \in \mathcal{F}_t
\end{aligned}$$

Thus, for all $0 \leq c \leq a$, T_c is $\mathcal{F}_{T_a}$-mesurable, and $\sigma(T_c, 0 \leq c \leq a) \subset \mathcal{F}_{T_a}$.
So now, it suffices to prove that $T_b - T_a$ is independent of $\mathcal{F}_{T_a}$.
Let us note $T'_{b-a} = \inf\{t \geq 0 \mid B_t^{(T_a)} = b-a\}$, then since the measures P and $P(\cdot \mid T_a < \infty)$ coincide:

$$\begin{aligned}
T'_{b-a} &= \inf\{t \geq 0 \mid B_t^{(T_a)} = b-a\} \\
&\overset{a.s.}{=} \inf\{t \geq 0 \mid B_{t+T_a} - a = b-a\} \quad \text{since } T_a < \infty \text{ a.s.} \\
&\overset{a.s.}{=} \inf\{t \geq 0 \mid B_{t+T_a} = b\} \\
&\overset{a.s.}{=} \inf\{s \geq T_a \mid B_s = b\} - T_a \\
&\overset{a.s.}{=} T_b - T_a \quad \text{because } (B_s = b) \subset (s \geq T_a)
\end{aligned}$$

But since T'_{b-a} only depends on $B^{(T_a)}$ independent of $\mathcal{F}_{T_a}$, $T_a - T_b$ is also independent of $\mathcal{F}_{T_a}$ thus, independent of $\sigma(T_c, 0 \leq c \leq a)$.