MAT 271E Probability and Statistics – 2015-2016 Spring - Midterm 2

Do all parts of all problems. Show your work for credit. Write your name on all submitted sheets. 100 minutes

1. Let the probability density function of random variable X be given as

$$f_X(x) = \begin{cases} \dfrac{1}{x+1} & 0 \le x \le a \\ 0 & \text{otherwise} \end{cases}$$

   a) Determine the constant a. (5 pts)
   b) Determine the probability distribution function $F_X(x)$. (10 pts)
   c) Determine the probability $\Pr\{X \ge 1\}$ and $\Pr\{X > 1\}$ (10 pts)
   d) Determine $f_X(x \mid X > 1)$ (10 pts)
   e) How many realizations does this random variable have? Describe all possible realizations. (5 pts)

2. Let random variables X and Y be related as $Y = \begin{cases} 1 - X & 0 < X \le 1 \\ 0 & \text{otherwise} \end{cases}$. X has density

$$f_X(x) = e^{-x} u(x)$$

   a) Determine the probability distribution function of random variable Y in terms of random variable X. (15 pts)
   b) Determine the probability density function of random variable Y. Note the jump discontinuity in $F_Y(y)$ whose derivative is a delta function. (5 pts)
   c) What kind of random variable is random variable Y (discrete/continuous/hybrid(mixed))? Explain. (5 pts)

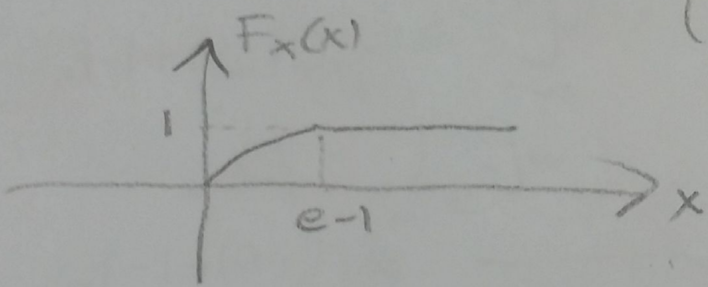
3. A Bernoulli random variable X assumes realizations -1 and 1 with equal probability.
   a) Determine the probability mass function and the probability density function of random variable $Y = 2X^2 + X + 1$. (10 pts)
   b) Determine the characteristic function of random variable X. (10 pts)
   c) Determine $E\left[(X-1)^3\right]$ by evaluating the moments directly. Check the moments by using the characteristic function. (15 pts)

MAT 271E Probability & Statistics Midterm 2

1. a) $$1=\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^a \frac{1}{x+1}\,dx = \ln(x+1)\Big|_0^a$$
$$= \ln(1+a)$$

$a = e-1 \cong 1.7$

b) $$F_X(x) = Pr\{X \le x\} = \begin{cases} 0 & x \le \emptyset \\ \int_0^x \frac{1}{\alpha+1}\,d\alpha & 0 \le x \le a \\ \int_0^a \frac{1}{\alpha+1}\,d\alpha = 1 & x > a \end{cases}$$
$$= \begin{cases} 0 & x \le \emptyset \\ \ln(1+x) & 0 \le x \le e-1 \\ 1 & x > e-1 \end{cases}$$



c) $$Pr\{X > 1\} = 1 - F_X(1) = 1 - \ln(2)$$
$$Pr\{X \ge 1\} = Pr\{X > 1\} + \underbrace{Pr\{X = 1\}}_{\emptyset} = 1 - \ln(2)$$

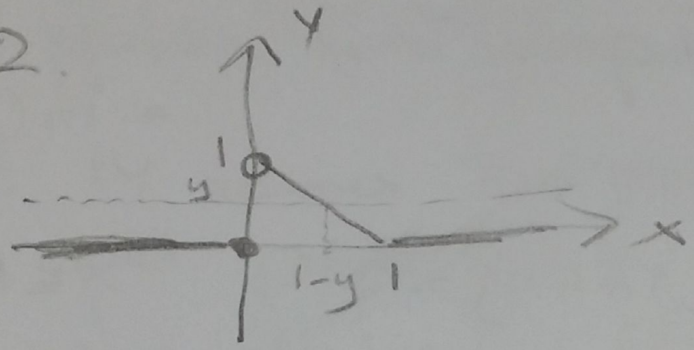
cont. r.v. $\Rightarrow \emptyset$

d) $$f_X(x \mid X > 1) = \begin{cases} \dfrac{f_X(x)}{Pr\{X > 1\}} & x > 1 \\ \emptyset & x \le 1 \end{cases}$$
$$= \begin{cases} \dfrac{1/x+1}{1-\ln(2)} & 1 \le x \le e-1 \\ \emptyset & \text{otherwise} \end{cases}$$

e) Since r.v. is continuous it has ∞'ly many realizations. They are.

$$\{x : 0 \le x \le e-1\}$$

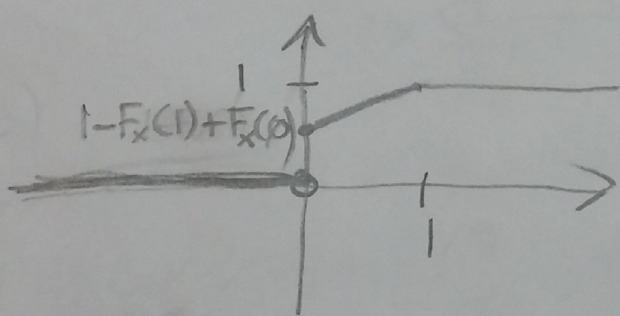
2.



a)

$$F_Y(y) = Pr\{Y \le y\} = \begin{cases} Pr\{-\infty < X < \infty\} & y \ge 1 \\ Pr\{X < 0\} + Pr\{X > 1-y\} & 0 \le y < 1 \\ Pr\{\emptyset\} & y < 0 \end{cases}$$

$$= \begin{cases} 1 & y \ge 1 \\ F_X(0) + 1 - F_X(1-y) & 0 \le y < 1 \\ 0 & y < 0 \end{cases}$$



$$= (F_X(0) + 1 - F_X(1-y))(u(y) - u(y-1)) + u(y-1)$$

b)

$$f_Y(y) = -f_X(1-y)\frac{d}{dy}(1-y)[u(y) - u(y-1)] + (F_X(0) + 1 - F_X(1-y))(\delta(y) - \delta(y-1)) + \delta(y-1)$$

$$= f_X(1-y)[u(y-1) - u(y)] + (1 - F_X(1) + F_X(0))\delta(y)$$

$$= e^{-(1-y)} u(1-y)[u(y-1) - u(y)] + e^{-1}\delta(y)$$

c) R.v. Y is hybrid since distribution function increases with a jump as well as increases gradually. Density function has both an impulsive and nonimpulsive component

3. Y assumes realizations 4 and 2 with

a) equal probability.

$$f_Y(y) = \frac{1}{2}\delta(y-4) + \frac{1}{2}\delta(y-2)$$

$$p_i = \Pr\{X = x_i\} = \frac{1}{2} \quad i = 1, 2$$

b)

$$\phi_X(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\,dx$$

$$= \frac{1}{2} e^{-j\omega} \int_{-\infty}^{\infty} \delta(x+1)\,dx + \frac{1}{2} e^{j\omega} \int_{-\infty}^{\infty} \delta(x-1)\,dx$$

$$= \frac{1}{2} e^{-j\omega} + \frac{1}{2} e^{j\omega}$$

$$= \cos \omega$$

c)

$$E[(X-1)^3] = E[X^3 - 3X^2 + 3X - 1]$$

$$= E[X^3] - 3E[X^2] + 3E[X] - 1$$

i) direct

$$E[X^3] = \sum_{i=1}^{2} x_i^3 \Pr\{X = x_i\} = \frac{(-1)^3}{2} + \frac{1^3}{2} = \emptyset$$

$$E[X^2] = \sum_{i=1}^{2} x_i^2 \Pr\{X = x_i\} = \frac{(-1)^2}{2} + \frac{1^2}{2} = 1$$

$$E[X] = \frac{-1}{2} + \frac{1}{2} = \emptyset$$

$$E[(X-1)^3] = (-3)(1) - 1 = -4.$$

$$m_1 = \frac{d}{j\,d\omega}\phi_X(\omega)\Big|_{\omega=\emptyset} = -\frac{\sin\omega}{j}\Big|_{\omega=\emptyset} = \emptyset$$

$$m_2 = \frac{d^2}{(-j)^2 d\omega^2}\phi_X(\omega)\Big|_{\omega=\phi} = -\frac{\cos\omega}{(-1)}\Big|_{\omega=\phi} = 1$$

$$m_3 = \frac{d^3}{(-j)^3 d\omega^3}\phi_X(\omega)\Big|_{\omega=\emptyset} = \frac{\sin\omega}{-j}\Big|_{\omega=\emptyset} = \emptyset$$