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Justify all your claims.

Exercise 1 (Recursive Sequences)

a) Let $(a_n)n \in N$ be a sequence in $\mathbb{R}$. Prove that if a_n is monotone increasing and has an upper bound, then it converges to its supremum.

b) Prove that the recursive sequence $a_{n+1} = \sqrt{a_n + 2}$ with $a_n = 0$ converges and determine its limit.

Solution

a) Let U be some upper bound such that $a_n \leq a_{n+1} \leq U$. It follows that that $|U - a_n| \geq |U - a_{n+1}|$ and therefore $\exists U' \leq U$ such that for every $\epsilon > 0$ $|S - a_n| < \epsilon$. If it weren't the case, then there would be a lesser bound which satisfies this, but by construction this cannot happen, as we would've chosen this as U' instead.

b) The sequence can easily be shown to be monotonically increasing: The base case is satisfied as $a_0 = 0 < a_1 = sqrt2$ and the derivative of $\sqrt{n+2}$ is positive $(\frac{1}{2\sqrt{x+2}})$. Since we know from a) this implies a limit exists, for sufficiently large n, it holds that $\underset{n\to\infty}{lim} a_{n+1} = \underset{n\to\infty}{lim} a_{n+2}$.

$$\begin{aligned}
\underset{n\to\infty}{lim} a_n &= \underset{n\to\infty}{lim} \sqrt{a_n + 2} \\
\underset{n\to\infty}{lim} a_n^2 &= \underset{n\to\infty}{lim} a_n + 2 \\
\underset{n\to\infty}{lim} a_n^2 - a_n - 2 = 0 &
\end{aligned}$$

This has Roots $\frac{1}{2} + -\frac{3}{2} = 2, -1$. We discard the negative root and obtain $L = 2$.

Exercise 2 (Continuity)

a) Prove that every Lipschitz continous function is uniformly continous.

b) Prove that $g : x \mapsto x^2$ is not uniformly continous.

c) Prove that $h : x \mapsto \sqrt{x}$ is uniformly continous but not Lipschitz continous.

Solution

a) Since $d(f(x), f(y)) \leq L \cdot d(x, y)$ from Lipschitz continuity and we require $that \exists \delta = d(x, y)$ such that $d(f(x), f(y)) < \epsilon$, we can choose δ as $\frac{\epsilon}{L}$. Then:

$$d(f(x), f(x + \delta)) < L \cdot \delta = L\frac{\epsilon}{L} = \epsilon$$

b) For any δ,

$$\begin{aligned} g(x) - g(x+\delta) &= x^2 - x^2 + 2x\delta + \delta^2 \\ &= 2x\delta + \delta^2 \end{aligned}$$

Since this is dependent on x on a term that dominates in the limit, for large x we cannot choose δ appropriately to get this distance arbitrarily small.

c) h is Lipschitz continous as:

$$h(x) - h(x+\delta) = \sqrt{x} - \sqrt{x+\delta}$$

$$\lim_{x\to\infty} \sqrt{x} - \sqrt{x+\delta} = 0$$

However as $h'(x) = \frac{1}{2\sqrt{x}}$, $\lim_{x\to 0} h'(x) = \infty$, thus $\nexists L$.

Exercise 3 (Uniform Convergence)

a) Analyze whether $f_n : x \mapsto \frac{1}{n} sin(nx)$ and $g : x \mapsto x + \frac{x}{n} cos(x)$ converge. If so, state limit and prove whether convergence is uniform.

b) Consider a sequence of functions $f_n : \mathcal{D} \to \mathbb{R}$ on a finite set $\mathcal{D}$ that converges pointwise. Prove that f_n converges uniformly. Consider a sequence of functions $f_n : [a, b]$ which are Lipschitz continous with the same $L > 0$. Assume that this sequence converges pointwise.

c) Prove that f is also Lipschitz continous with same L.

d) Prove that f_n converges uniformly to f.

Solution

a)

b)

c)

d)

Exercise 4 (Power and Taylor Series)

a) Determine the radius of convergence of:

$$\sum_{j=1}^{\infty} \frac{j^2}{2^j} x^j \text{ and } \sum_{j=1}^{\infty} 3^j x^{j^2}$$

b) Compute the Taylor polynomial of $f : x \mapsto e^{\pi - x} sin(x)$ with $a = 0$ of degree 3 and the corresponding Lagrange remainder.

c) Prove that f from b) is equal to its Taylor series.

Solution

a)

$$\lim_{j\to\infty} \frac{\frac{j^2}{2^j}}{\frac{(j+1)^2}{2^{j+1}}}$$

$$= \lim_{j\to\infty} \frac{2^j 2 \frac{j^2}{2^j}}{j^2+2j+1}$$

$$= \lim_{j\to\infty} \frac{2j^2}{j^2+2j+1}$$

$$\approx \lim_{j\to\infty} \frac{2j^2}{j^2} = 2 = r$$

Second series:

$$\sum_{j=1}^{\infty} 3^j x^{2j}$$

$$= \sum_{i=1}^{\infty} 3^{i/2} x^i$$

$$\lim_{i\to\infty} \frac{3^{i/2}}{3^{(i+1)/2}}$$

$$= \lim_{i\to\infty} \frac{\sqrt{3^i}}{\sqrt{3}\sqrt{3^i}}$$

$$= \frac{1}{\sqrt{3}} = r$$

b)

c)

d)