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Convolutions

Solution

a) If X and Y are two independent random variables and $Z = X + Y$, then the probability $P(Z = z)$ can be constructed by collecting all outcomes $x \in im(X)$ and inspecting the corresponding outcomes y, which cover the difference between z and x with $P(Y = y) = 0$ if $y \notin im(Y)$. By independence the $P(X = x) \cap P(Y = y) = P(X = x)P(Y = y)$. Thus the probability distribution of Z can be expressed as:

$$P(Z = z) = \sum_{x \in im(X)} P(X = x) \cdot P(Y = z - x)$$

b) Let $X, Y \sim \text{Poisson}(\lambda_{1,2}), Z = X + Y$.

$$P(Z = z) = \sum_{x \in im(X)} P(X = x) \cdot P(Y = z - x)$$

$$= \sum_{x \in im(X)} \frac{\lambda_1^x \cdot e^{-\lambda_1}}{x!} \cdot \frac{\lambda_2^{x-z} \cdot e^{-\lambda_2}}{(z - x)!}$$

$$= \sum_{x \in im(X)} \frac{\lambda_1^x \lambda_2^{x-z} \cdot e^{-\lambda_1 - \lambda_2}}{x!(z - x)!}$$

$$= e^{-\lambda_1 - \lambda_2} \sum_{x \in im(X)} \frac{\lambda_1^x \lambda_2^{x-z}}{x!(z - x)!}$$

Expand with $1/z!$ to complete binom coefficient

$$= e^{-\lambda_1 - \lambda_2} \frac{1}{z!} \sum_{x \in im(X)} \frac{z!}{x!(z - x)!} \lambda_1^x \lambda_2^{x-z}$$

$$= e^{-\lambda_1 - \lambda_2} \frac{1}{z!} \sum_{x \in im(X)} \binom{z}{k} \lambda_1^x \lambda_2^{x-z}$$

The sum is simply the binomial expansion of $(\lambda_1 + \lambda_2)^z$

$$P(Z = z) = \frac{(\lambda_1 + \lambda_2)^z \cdot e^{-(\lambda_1 + \lambda_2)}}{z!}$$

Which is the expression for a poisson distributed variable with $\lambda = \lambda_1 + \lambda_2$

c If X is normally distributed then the density is given by $p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$. We know:

$$P_Z(z) = \int_{-\infty}^{\infty} p_X(x) \cdot p_Y(z - x) dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-x)^2}{2}}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \cdot e^{-\frac{(z-x)^2}{2}}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\left(\frac{x^2+z^2-2zx+x^2}{2}\right)}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\left(x^2+\frac{z^2}{2}-zx\right)}$$

By completing the square of $(x - \frac{z}{2})^2$:

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-(x-\frac{z}{2})^2 - \frac{z^2}{4}}$$

$$= \frac{1}{2\pi} e^{-\frac{z^2}{4}} \int_{-\infty}^{\infty} e^{-(x-\frac{z}{2})^2}$$

Since the integral is a gaussian integral shifted along the x axis (which does not influence volume), we know it evaluates to $\sqrt{\pi}$:

$$= \frac{1}{2\pi} e^{-\frac{z^2}{4}} \sqrt{\pi}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{4}}$$

This matches the expression for the density of a normally distributed variable.

Weak law of large numbers

Solution

Given a sequence $X_1, X_2, \ldots$ and its sum $S_n = \sum_{i=1}^{n} X_i$, we know $E(S_n) = \sum_{i=1}^{n} E(X)$ and $Var(X)/n$ by subsample variance. Thus $\underset{lim}{n \to \infty} Var(S_n) = 0$ and $E(S_n) = S_n = \sum_{i=1}^{n} E(x)$

$$\Leftrightarrow \frac{1}{n} S_n = E(X)$$

$$\Leftrightarrow \frac{1}{n} S_n \xrightarrow{P} E(X_1)$$

Poisson distribution

Solution

a)

$$E(X) = \sum_{x=0}^{\infty} x P(X = x)$$

$$= \sum_{x=0}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{\lambda^x e^{-\lambda}}{(x-1)!}$$

Since the first term of the sequence equals 0:

$$= \sum_{x=1}^{\infty} \frac{\lambda^x e^{-\lambda}}{(x-1)!}$$

$$= \sum_{x=1}^{\infty} \frac{\lambda^x e^{-\lambda}}{(x-1)!}$$

Introduce $\tilde{x} = x - 1$

$$= \sum_{\tilde{x}=0}^{\infty} \frac{\lambda^{\tilde{x}+1} e^{-\lambda}}{\tilde{x}!}$$

$$= \lambda e^{-\lambda} \sum_{\tilde{x}=0}^{\infty} \frac{\lambda^{\tilde{x}}}{\tilde{x}!}$$

Since the sum is the taylor expansion of e:

$$= \lambda e^{-\lambda} e^{\lambda}$$

$$E(X) = \lambda$$

For the Variance:

$$Var(X) = E(X^2) - E(X)^2$$

$$= E(X^2) - \lambda^2$$

$$= \sum_{x=0}^{\infty} x^2 P(X = x) - \lambda^2$$

$$= \sum_{x=0}^{\infty} (x(x-1) + x) P(X = x) - \lambda^2$$

$$= \sum_{x=0}^{\infty} x(x-1) P(X = x) + \lambda - \lambda^2$$

$$= \sum_{x=0}^{\infty} x(x-1) \frac{\lambda^x e^{-\lambda}}{x!} + \lambda - \lambda^2$$

$$= \sum_{x=0}^{\infty} x(x-1) \frac{\lambda^x e^{-\lambda}}{x!} + \lambda - \lambda^2$$

Analogous reasoning to above, first 2 terms are 0, then reindex with $\tilde{x} = x - 2$:

$$= \sum_{x=2}^{\infty} x(x-1) \frac{\lambda^x e^{-\lambda}}{x!} + \lambda - \lambda^2$$

$$= \sum_{\tilde{x}=0}^{\infty} \frac{\lambda^{\tilde{x}-2} e^{-\lambda}}{\tilde{x}!} + \lambda - \lambda^2$$

$$= \lambda^2 e^{-\lambda} \sum_{\tilde{x}=0}^{\infty} \frac{\lambda^{\tilde{x}}}{\tilde{x}!} + \lambda - \lambda^2$$

$$= \lambda^2 e^{-\lambda} e^{\lambda} + \lambda - \lambda^2$$

$$= \lambda^2 + \lambda - \lambda^2$$

$$= \lambda$$

b)

$$\lim_{n\to\infty} \text{Bin}(n, \lambda/n)(k) = \lim_{n\to\infty} \binom{n}{k} \frac{\lambda^k}{n} \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

$$= \lim_{n\to\infty} \frac{1}{k!} \frac{n!}{(n-k)!} \lambda^k n^{-k} \left(1 - \frac{\lambda}{n}\right)^{-k} \left(1 - \frac{\lambda}{n}\right)^{n}$$

We can use that

$$\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n,$$

when the separate limits exists.

$$\lim_{n\to\infty} \frac{n!}{(n-k)!n^k} = \lim_{n\to\infty} \prod_{i=n-k}^{n} \frac{i}{n} = 1,$$

in the case where k remains fixed and thus $k \ll n$.

$$\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{-k} = \left(\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)\right)^{-k} = 1^{-k} = 1$$

And lastly, we use the hint from the exercise sheet to show:

$$\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{n} = 1 - e^{-\lambda}$$

So all the limits of the product exist and the total limit is therefore given by:

$$\lim_{n\to\infty} \text{Bin}(n, \lambda/n)(k) = \frac{\lambda^k}{k!}(1 - e^{-\lambda}),$$

which is the Poisson distribution.

c) This shows that the Poisson distribution comes from the limit, where we increase the number of trials, but keep the total number of succeses. For this reason also the probabilty of each event scales with the number of trials. As with a time derivative this gives means that λ then represents a rate of success in a certain intervall.

Convergence of random variables

Solution

We know from convergence in probability that $\lim_{n\to\infty} P(|X_n - X| > \epsilon) = 0$. In other words, for an arbitrarily small δ we can find a point N for which all $n \geq N$ are smaller. This means, that for any $\omega \in \Omega$, if $P(\omega) > 0$ and $\delta = P(\omega)$ there exists an N:

$$P(|X_n - X| > \epsilon) < P(\omega) \quad \forall n \geq N$$

Assume ω is part of the set where $|X_n - X| > \epsilon$, that is $|X_n(\omega) - X(\omega)| > \epsilon$. Then the probability of this set must be $\geq P(\omega)$, since it contains ω. This contradicts our previous assertion, that $P(|X_n - X| > \epsilon) < P(\omega)$. Since this holds for all $\omega \in \Omega$, a deviation bigger than ϵ cannot exist, and we conclude Convergence almost surely.