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Multivariate distributions with densities

Solution

a) Marginal densities:

$$\begin{aligned} f_X(x) &= \frac{1}{c}\int exp(-2x^2 - y^2 - x^2y^2)dy \\ &= \frac{1}{c}exp(-2x^2)\int exp(-y^2 - x^2y^2)dy \\ &= \frac{1}{c}exp(-2x^2)\int exp(-y^2(1+x^2))dy \end{aligned}$$

Recognize gaussian integral and apply hint:

$$= \frac{1}{c}exp(-2x^2)\sqrt{\frac{\pi}{1+x^2}}$$

$$\begin{aligned} f_Y(y) &= \frac{1}{c}\int exp(-2x^2 - y^2 - x^2y^2)dx \\ &= \frac{1}{c}exp(-y^2)\int exp(-x^2(2+y^2))dy \\ &= \frac{1}{c}exp(-y^2)\sqrt{\frac{\pi}{2+y^2}} \end{aligned}$$

Conditional densities:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$f_{X|Y}(x|y) = \frac{1}{c}\frac{c}{1}\frac{exp(-2x^2 - y^2 - x^2y^2)}{exp(-y^2)\sqrt{\frac{\pi}{2+y^2}}}$$

$$f_{X|Y}(x|y) = \sqrt{\frac{2+y^2}{\pi}}\frac{exp(-2x^2 - y^2 - x^2y^2)}{exp(-y^2)}$$

$$f_{X|Y}(x|y) = \sqrt{\frac{2+y^2}{\pi}}exp(-2x^2 - x^2y^2)$$

Analogously for $f_{Y|X}$

$$f_{Y|X}(y|x) = \frac{1}{c}\frac{c}{1}\frac{exp(-2x^2 - y^2 - x^2y^2)}{exp(-2x^2)\sqrt{\frac{\pi}{1+x^2}}}$$

$$f_{Y|X}(y|x) = \sqrt{\frac{1+x^2}{\pi}}exp(-y^2 - x^2y^2)$$

Ignoring the factor $\sqrt{\frac{1+x^2}{\pi}}$, the conditional densities resemble an unscaled normal distribution. With the example of $F_{Y|X}(y|x)$:

$$exp(-y^2(1+x^2))$$

$$= exp(\frac{-y^2}{\frac{1}{1+x^2}})$$

$$= exp(\frac{-(y-0)^2}{2\frac{1}{2(1+x^2)}})$$

Now that we know the variance, it becomes obvious that the scaling factor:

$$\frac{1}{\sqrt{2\pi\frac{1}{2(1+x^2)}}}$$

$$= \frac{1}{\sqrt{\frac{\pi}{1+x^2}}}$$

$$= \sqrt{\frac{1+x^2}{\pi}}$$

Thus we are left with $\mathcal{N}(0, \frac{1}{2(1+x^2)})$.

b) Bayes Formula (ommitting function arguments for convenience):

$$f_{X,Y} = f_{Y|X} f_X = f_{X|Y} f_Y$$

Divide by f_X:

$$f_{Y|X} = \frac{f_{X|Y} f_Y}{f_X}$$

Law of total probability:

$$f_Y = \int f_{X,Y} dx$$

Again since $f_{X,Y} = f_{Y|X} f_X = f_{X|Y} f_Y$:

$$f_Y = \int f_{Y|X} f_X dx$$

Conditional Expectation

Solution

a) We know:

$$E(X|Y=y) = \sum_x xP(X=x|Y=y)$$

and by the law of total probability:

$$E(X) = \sum_y \sum_x xP(X=x, Y=y)$$

Which can be rewritten as conditional probability:

$$= \sum_y \sum_x xP(X=x|Y=y)P(Y=y)$$

Here we recognize the conditional expectation in the expression:

$$= \sum_y E(X|Y=y)P(Y=y)$$

Which is nothing else as:

$$= EE(X|Y)$$

b)

c)

Unbiased estimators are not always useful

Solution

a) Unbiased $\Leftrightarrow E(U(x)) = \theta$

$$P(X = k) = \frac{(-0.5log\theta)^k}{k!} exp(0.5log\theta)$$

$$E(U) = \sum_k U(k)P(X = k)$$

$$= exp(0.5log\theta) \sum_k (-1)^k \frac{(-0.5log\theta)^k}{k!}$$

The factor $(-1)^k$ removes the sign of the fraction, which simultaneously proves that there cannot be another unbiased estimator, as it would have to have the same effect. According to the hint that would imply equality.

$$= exp(0.5log\theta) \sum_k \frac{(0.5log\theta)^k}{k!}$$

$$= exp(0.5log\theta)exp(0.5log\theta)$$

$$= exp(log\theta)$$

$$= \theta$$

b)

$$MSE(U) = Var(U) + Bias(U)^2$$

Since we know $Bias = 0$ from a):

$$= Var(U)$$

$$= E(U^2) - E(U)^2$$

$$\sum_k (-1)^{2k} P(X = k) - \theta^2$$

$$1 - \theta^2$$

c)

$$E(V) = exp(0.5log\theta) \sum_k 1_{2\mathbb{N}_0}(k) \frac{(-0.5log\theta)^k}{k!}$$

$$E(V) = exp(0.5log\theta) \sum_k \frac{(-0.5log\theta)^{2k}}{2k!}$$

$$E(V) = exp(0.5log\theta)cosh(-0.5log\theta)$$

This simplifies to:

$$= 0.5(\theta + 1)$$

Since squaring an indicator is a no-op:

$$E(V^2) = E(V)$$

$$Var(V) = 0.5(\theta + 1) - 0.25(\theta + 1)^2$$
$$= 0.25(1 - \theta^2)$$

Finally:

$$MSE(V) = 0.25(1 - \theta^2) + (0.5(\theta + 1) - \theta)^2$$
$$MSE(V) = 0.25(1 - \theta^2) + (-0.5\theta + 0.5)^2$$
$$= 0.25 - 0.25\theta^2 + 0.25 - 0.5\theta + 0.25\theta$$
$$= -\frac{\theta^2}{4} - \frac{\theta}{4} + \frac{1}{2}$$

To show that this is less than $MSE(V, \theta)$ consider:

$$(1 - \theta^2) - (0.5 - 0.25\theta^2 - 0.25\theta)$$

$$= 0.5 - 0.75\theta^2 + 0.25\theta$$

which is positive in (0,1), concluding the proof.

Consistency

Assume a statistical model of $\mathcal{F}_t = \{\text{Unif}([0, t]) | t \in (0, \infty)\}$ and independent samples $X_1, ..., X_n$.. Let the estimator be $\hat{t}_n = \max_{i=1,...,n} X_n$.

As we can assume uniform distribution the probability $\mathbb{P}(|t_n - t| > \epsilon) = (\frac{t-\epsilon}{t})^n$ as we draw n times from $[0, t - \epsilon]$. We now use the Borel-Cantonelli theorem and $\frac{t-\epsilon}{t} < 1$ to find:

$$\sum_{n=0}^{\infty} \mathbb{P}(|t_n - t| > \epsilon) = \sum_{n=0}^{\infty} \left(\frac{t - \epsilon}{t}\right)^n < \infty,$$

which follows from the convergence of the geometric series for all $\epsilon > 0$. From Borel-Cantonelli it follows that

$$\mathbb{P}(\lim_{n \to \infty} \sup |t_n - t| > \epsilon) = 0.$$

Therefore, t_n almost surely converges to t and thus the estimator is strongly consistent.