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1 Exercise 1 (Dictionary Learning)

1.1

Let $v = (2, 1, 1)$

$$D = \{d_1 = (1,0,0),\ d_2 = (0,1,0),\ d_3 = (0,0,1),\ d_4 = (1,1,0),\ d_5 = (0,1,1)\}$$

Let $A = [d_1, \dots, d_n]$
Express v as a linear combination of exactly n vectors:

i) $n = 3$:
$$v = A[2, 1, 1, 0, 0]^T$$

i) $n = 4$:
$$v = A[-1, -2, 1, 3, 0]^T$$

i) $n = 5$:
$$v = A[1, -2, -1, 1, 2]^T$$

1.2

Can $v \in \mathbb{R}^3$ be expressed as a linear combination in multiple ways given a dictionary D with five vectors spanning $\mathbb{R}^3$? Yes.
Let $D = \{d_1, \dots, d_5\}$. Since $dim(\mathbb{R}^3) < |D|, \exists \lambda'_1, \dots, \lambda'_{n-1}$, such that $d_n = \sum_{i=1}^{n-1} \lambda d_i$.
Thus for any $v = \sum_{i=1}^{n} \lambda_{vi} d_i = \sum_{i=1}^{n} \lambda_{vi} + d_n - \sum_{i=1}^{n-1} \lambda d_i$.

1.3

Find a dictionary D in $\mathbb{R}^3$ with minimal number of vectors, such that the following vectors can be expressed as a linear combination of exactly two dictionary vectors.

i) $v_1 = (1,1,1),\ v_2 = (2,1,1),\ v_3 = (1,2,2)$ $D = v_2, v_3$, optimal, since lower bound for $|D| = 2$.

ii) $v_1 = (1,1,1),\ v_2 = (2,1,0),\ v_3 = (2,1,0)$ $D = v_1, v_2, v_3$, no better solution exists, since v_1, v_2, v_3 are pairwise independent.

1.4

Upper and lower bounds on $|D|$ to represent any v in $\mathbb{R}^3$ with at most n vectors.

i) $n = 1$ lower: 1 upper: n

ii) $n = 2$ lower: 2 upper: n

iii) $n = 3$ lower: 3 upper: 3

2 Exercise 2 (Basis and Dimension)

Let V be a finite-dimensional vector space with basis $\mathcal{B} = (v_1, \ldots, v_n)$.
Let $v = \sum_{i=1}^{n}$ with $\lambda_k \neq 0$ for $1 \leq k \leq n$.
Prove that $\tilde{\mathcal{B}} = (v_1, \ldots, v_{k-1}, v, v_{k+1}, \ldots, v_n)$ is also a basis of V.

$$\begin{aligned} \tilde{B} &= (B \cup \{v\}\ \{v_k\} \\ v &= \lambda_k v_k + (v - \lambda_k v_k) \\ &= \lambda_k v_k + \sum_{i=1, i \neq k} \lambda_i v_i \end{aligned}$$

Since v_k is linearly independent of $v_{i \neq k}$, the resulting vector is linearly independent as well.

3 Exercise 3 (Linear Mappings and Vector Spaces)

Let V and W be vector spaces over F. Consider $\mathbb{L}(V, W)$ with:

$$(S + T)(v) \coloneqq Sv + Tv \text{ and } (\lambda T)(v) \coloneqq \lambda(Tv)$$

3.1

Verify that $S + T$ and λT are again linear maps in $\mathbb{L}(V, W)$.
Closed under addition:

$$(S + T)(v + w) = S(v + w) + T(v + w) = S(v) + S(w) + T(v) + T(w) = (S + T)(v) + (S + T)(w)$$

$$(\lambda T)(v + w) = \lambda(T(v + w)) = \lambda T(v) + \lambda T(w)$$

Closed under scalar multiplication:

$$\lambda(S + T)(v) = \lambda S(v) + \lambda T(v) = \lambda(S(v) + T(v))$$

$$(\lambda T)(\lambda' v) = \lambda T(\lambda' v) = \lambda \lambda' T(v)$$

3.2

Prove that $\mathbb{L}(V, W)$ with the above operations is a vector space.

- Closed under addition: see above
- Closed under scalar multiplication: see above
- Associativity of Addition:

$$((S + T) + U)(v) = S(v) + T(v) + U(v) = (S + (T + U))(v)$$

- Commutativity of Addition:

$$(S + T)(v) = S(v) + T(v) = T(v) + S(v) = (T + S)(v)$$

- Additive Identity: $0(v) :=$ zero-mapping

$$(S + 0)(v) = S(v)$$

- Multiplication Identity:

$$1(T(v)) = T(v)$$

- Additive inverse:

$$(S + (-S))(v) = 0$$

- Scalar distributivity:

$$\lambda(S + T)(v) = \lambda S(v) = \lambda T(v)$$

- Vector distributivity:

$$(\lambda + \lambda')S(v) = \lambda S(v) + \lambda' S(v)$$

4 Exercise 4 (Linear Mappings)

Let $v_1, \dots, v_n$ be vectors in V, $T \in \mathcal{L}(\mathbb{R}^n, V)$ with $T(\lambda_1, \dots, \lambda_n) = \lambda_1 v_1 + \cdots + \lambda_n v_n$ Which property does T need to have such that $v_1, \dots, v_n$:

4.1 span

T needs to be surjective, which means that it spans the co-domain i.e. V.

4.2 are linearly independent

T needs to be injective, which means that each element in the co-domain is mapped to by at most one in the domain.