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Justify all your claims.

1 Exercise 1 (Change of Basis, $2+2+1$ points)

Consider the linear map $T \in \mathcal{L}\left(\mathbb{R}^3, \mathbb{R}^3\right)$ with $T(x) = (-x_1, x_2, 2x_3)^T$ for $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$.

Consider the standard basis $\mathcal{B} = \{e_1, e_2, e_3\}$ and the basis $\mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ of $\mathbb{R}^3$.

a) Find the matrix $M(T, \mathcal{B}, \mathcal{B})$ which corresponds to the linear map T.

b) Find the transformation matrices $M(\mathrm{Id}, \mathcal{B}, \mathcal{C})$ and $M(\mathrm{Id}, \mathcal{C}, \mathcal{B})$.

c) Find the matrix $M(T, \mathcal{C}, \mathcal{C})$.

1.1 Solution

a) $M(T, \mathcal{B}, \mathcal{B}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

b) $M(\mathrm{Id}, \mathcal{B}, \mathcal{C}) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 1 \\ -1 & 2 & -1 \end{pmatrix}$, $M(\mathrm{Id}, \mathcal{C}, \mathcal{B}) = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix}$

c) $M(T, \mathcal{C}, \mathcal{C}) = M(\mathrm{Id}, \mathcal{B}, \mathcal{C}) M(T, \mathcal{B}, \mathcal{B}) M(\mathrm{Id}, \mathcal{C}, \mathcal{B}) = M(\mathrm{Id}, \mathcal{B}, \mathcal{C}) \begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 1 \\ 6 & 4 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 5 & 3 & 1 \\ -1 & -2 & 0 \end{pmatrix}$

2 Exercise 2 (Matrices, $1+1+1+1+1$ points).

Consider the differentiation operator $D = \mathrm{d}/\mathrm{d}t : \mathbb{R}^{\mathbb{R}} \to \mathbb{R}^{\mathbb{R}}, f \mapsto f'$ on the vector space $\mathbb{R}^{\mathbb{R}}$ of all real functions. Below we give different choices of bases $\mathcal{W}$. For each of them, we consider the corresponding subspace $\mathcal{U} := \mathrm{span}(\mathcal{W})$ and the restricted linear map $D|_{\mathcal{U}} : \mathcal{U} \to \mathbb{R}^{\mathbb{R}}$, which is the differentiation operator just applied to vectors in $\mathcal{U}$. Decide whether range $(D|_{\mathcal{U}}) \subseteq \mathcal{U}$ and if so, state the matrix $\mathcal{M}\left(D|_{\mathcal{U}}, \mathcal{W}, \mathcal{W}\right)$.

a) $\mathcal{W} = \left\{e^t, e^{2t}\right\}$
b) $\mathcal{W} = \left\{1, t^2, t^4\right\}$
c) $\mathcal{W} = \{e^t, te^t\}$
d) $\mathcal{W} = \{\sin t, \cos t\}$
e) $\mathcal{W} = \left\{t, (\sin t)^2, (\cos t)^2, \sin t \cos t\right\}$

2.1 Solution

a) $\mathcal{M}\left(D|_{\mathcal{U}}, \mathcal{W}, \mathcal{W}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

b) $(D|_{\mathcal{U}}) \subsetneq \mathcal{U}$, because $\mathrm{d}t^2/\mathrm{d}t = t \notin \mathrm{span}(\mathcal{W})$.

c) $\mathcal{M}\left(D|_{\mathcal{U}}, \mathcal{W}, \mathcal{W}\right) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

d)$\mathcal{M}\left(D|_{\mathcal{U}}, \mathcal{W}, \mathcal{W}\right) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

e)$\mathcal{M}\left(D|_{\mathcal{U}}, \mathcal{W}, \mathcal{W}\right) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & -2 & 0 \end{pmatrix}$

3 Exercise 3 (Eigenvalues, $2 + 3$ points).

a) Let $A \in \mathbb{R}^{n \times n}$ with $A^k = 0$ for some $k \in \mathbb{N}$. Prove that, if λ is an eigenvalue of A, then $\lambda = 0$.
b) Let V be a finite-dimensional vector space and $T : V \to V$ a linear map such that every $v \in V$ with $v \neq 0$ is an eigenvector of T. Prove that $T = \lambda \mathrm{Id}$ for some $\lambda \in \mathbb{R}$.

3.1 Solution

a) Let λ be an eigenvalue of A with eigenvector v. Then $A^k v = l^k v = 0$, where the last equality follows from $A^k = 0$. As the eigenvector cannot be the zero vector it immediately follows that $\lambda = 0$.
b) Let V be a finite-dimensional vector space with dimension n and $T : V \to V$ a linear map such that every $v \in V$ with $v \neq 0$ is an eigenvector of T.
If we choose a basis $\{b_i\}_i$ then $T(1, \dots, 1) = (\lambda_1, \dots, \lambda_n) = \lambda(1, \dots, 1)$. Here both equalities follow by the definition that b_i is an eigenvector with eigenvalue λ_i and $(1, \dots, 1)$ is an eigenvector with eigenvalue λ. Therefore, it must be true that $\lambda_i = \lambda$ for all i. Secondly, as the basis vectors "select" columns of the matrix $M(T, V, V)$ we know that the matrix must look like $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$ and $T = \lambda Id$.

4 Exercise 4 (Power Method, $1 + 4$ points).

Let $A \in \mathbb{R}^{n \times n}$ be a diagonalizable matrix with one unique largest eigenvalue, that is, $|\lambda_1| > |\lambda_2| > \ldots > |\lambda_n|$, where λ_i are the eigenvalues. We furthermore assume $\lambda_1 > 0$. We consider the power method, a method to numerically estimate an eigenvector to the largest eigenvalue λ_1. For an arbitrary initial vector $x_0 \in \mathbb{R}^n$ we recursively define

$$x_{k+1} := \frac{Ax_k}{\|Ax_k\|}$$

a) Prove that $x_k = \frac{A^k x_0}{\|A^k x_0\|}$.
b) Consider a basis of eigenvectors $v_1, \dots, v_n$, where v_i belongs to λ_i, and the representation

$$x_0 = c_1 v_1 + \ldots + c_n v_n$$

Prove that, if $c_1 \neq 0$, the sequence x_k converges to an eigenvector of λ_1 for $k \to \infty$.

4.1 Solution

a) We can prove the assumption by induction.
IA: For $k = 1 : x_1 = \frac{Ax_0}{\|Ax_0\|} = \frac{A^1 x_0}{\|A^1 x_0\|}$.
IH: Assume that$x_k = \frac{A^k x_0}{\|A^k x_0\|}$
IS: $x_k + 1 = \frac{Ax_k}{\|Ax_k\|} \overset{IH}{=} \frac{AA^k x_0}{\|AA^k x_0\|} = \frac{A^{k+1} x_0}{\|A^{k+1} x_0\|}$
b) As $x_k := \frac{A^k x_0}{\|A^k x_0\|} = \frac{\lambda_1^k c_1 v_1 + \cdots + \lambda^k c_n v_n}{\left\|\lambda_1^k c_1 v_1 + \cdots + \lambda^k c_n v_n\right\|}$. We bound $\frac{\lambda_i^k c_i v_i}{\left\|\lambda_1^k c_1 v_1 + \cdots + \lambda^k c_n v_n\right\|} \leq \frac{\lambda_i^k c_i v_i}{\left\|\lambda_1^k c_1 v_1\right\|} := d_k^i v_i$. Now d_k^i converges to zero for $i \neq 1$. (Proof: Let $\epsilon > 0$, we can define $\alpha_i = \frac{|\lambda_i|}{|\lambda_1|} < 1$. Therefore, we can find a N¿=0, such that $\alpha_i^m < \epsilon$ for $m \geq n$. From this is directly follows that $|d_k^i| = \frac{|c_i|}{|d_1|} \alpha_i^k$ converges to zero for $k \to \infty$). And therefore, $x_k \to \frac{\lambda_i^k c_1 v_1}{\left\|\lambda_1^k c_1 v_1 + \cdots + \lambda^k c_n v_n\right\|} \to c v_1$, where $\lambda_i^k c_1 \left\|\lambda_1^k c_1 v_1 + \cdots + \lambda^k c_n v_n\right\|$.