Friederike Horn & Bileam Scheuvens

Convolutions

Solution

a) If X and Y are two independent random variables and $Z = X + Y$, then the probability $P(Z = z)$ can be constructed by collecting all outcomes $x \in im(X)$ and inspecting the corresponding outcomes y, which cover the difference between z and x with $P(Y = y) = 0$ if $y \notin im(Y)$. By independence the $P(X = x) \cap P(Y = y) = P(X = x)P(Y = y)$. Thus the probability distribution of Z can be expressed as:

$$P(Z = z) = \sum_{x \in im(X)} P(X = x) \cdot P(Y = z - x)$$

b) Let $X, Y \sim \text{Poisson}(\lambda_{1,2}), Z = X + Y$.

$$P(Z = z) = \sum_{x \in im(X)} P(X = x) \cdot P(Y = z - x)$$

$$= \sum_{x \in im(X)} \frac{\lambda_1^x \cdot e^{-\lambda_1}}{x!} \cdot \frac{\lambda_2^{x-z} \cdot e^{-\lambda_2}}{(z-x)!}$$

$$= \sum_{x \in im(X)} \frac{\lambda_1^x \lambda_2^{x-z} \cdot e^{-\lambda_1 - \lambda_2}}{x!(z-x)!}$$

$$= e^{-\lambda_1 - \lambda_2} \sum_{x \in im(X)} \frac{\lambda_1^x \lambda_2^{x-z}}{x!(z-x)!}$$

Expand with $1/z!$ to complete binom coefficient

$$= e^{-\lambda_1 - \lambda_2} \frac{1}{z!} \sum_{x \in im(X)} \frac{z!}{x!(z-x)!} \lambda_1^x \lambda_2^{x-z}$$

$$= e^{-\lambda_1 - \lambda_2} \frac{1}{z!} \sum_{x \in im(X)} \binom{z}{k} \lambda_1^x \lambda_2^{x-z}$$

The sum is simply the binomial expansion of $(\lambda_1 + \lambda_2)^z$

$$P(Z = z) = \frac{(\lambda_1 + \lambda_2)^z \cdot e^{-(\lambda_1 + \lambda_2)}}{z!}$$

Which is the expression for a poisson distributed variable with $\lambda = \lambda_1 + \lambda_2$

c If X is normally distributed then the density is given by $p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$. We know:

$$P_Z(z) = \int_{-\infty}^{\infty} p_X(x) \cdot p_Y(z - x) dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-x)^2}{2}}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \cdot e^{-\frac{(z-x)^2}{2}}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\left(\frac{x^2+z^2-2zx+x^2}{2}\right)}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\left(x^2+\frac{z^2}{2}-zx\right)}$$

By completing the square of $(x - \frac{z}{2})^2$:

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-(x-\frac{z}{2})^2-\frac{z^2}{4}}$$

$$= \frac{1}{2\pi} e^{-\frac{z^2}{4}} \int_{-\infty}^{\infty} e^{-(x-\frac{z}{2})^2}$$

Since the integral is a gaussian integral shifted along the x axis (which does not influence volume), we know it evaluates to $\sqrt{\pi}$:

$$= \frac{1}{2\pi} e^{-\frac{z^2}{4}} \sqrt{\pi}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{4}}$$

This matches the expression for the density of a normally distributed variable.

Weak law of large numbers

Solution

Given a sequence $X_1, X_2, \ldots$ and its sum $S_n = \sum_{i=1}^{n} X_i$, we know $E(S_n) = \sum_{i=1}^{n} E(X)$ and $Var(X)/n$ by subsample variance. Thus $n \underset{lim}{\rightarrow} \infty Var(S_n) = 0$ and $E(S_n) = S_n = \sum_{i=1}^{n} E(x)$

$$\Leftrightarrow \frac{1}{n} S_n = E(X)$$

$$\Leftrightarrow \frac{1}{n} S_n \xrightarrow{P} E(X_1)$$

Poisson distribution

Solution

a)

$$E(X) = \sum_{x=0}^{\infty} x P(X = x)$$

$$= \sum_{x=0}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{\lambda^x e^{-\lambda}}{(x-1)!}$$

Since the first term of the sequence equals 0:

$$= \sum_{x=1}^{\infty} \frac{\lambda^x e^{-\lambda}}{(x-1)!}$$

$$= \sum_{x=1}^{\infty} \frac{\lambda^x e^{-\lambda}}{(x-1)!}$$

Introduce $\tilde{x} = x - 1$

$$= \sum_{\tilde{x}=0}^{\infty} \frac{\lambda^{\tilde{x}+1} e^{-\lambda}}{\tilde{x}!}$$

$$= \lambda e^{-\lambda} \sum_{\tilde{x}=0}^{\infty} \frac{\lambda^{\tilde{x}}}{\tilde{x}!}$$

Since the sum is the taylor expansion of e:

$$= \lambda e^{-\lambda} e^{\lambda}$$

$$E(X) = \lambda$$

For the Variance:

$$Var(X) = E(X^2) - E(X)^2$$

$$= E(X^2) - \lambda^2$$

$$= \sum_{x=0}^{\infty} x^2 P(X = x) - \lambda^2$$

$$= \sum_{x=0}^{\infty} (x(x-1) + x) P(X = x) - \lambda^2$$

$$= \sum_{x=0}^{\infty} x(x-1) P(X = x) + \lambda - \lambda^2$$

$$= \sum_{x=0}^{\infty} x(x-1) \frac{\lambda^x e^{-\lambda}}{x!} + \lambda - \lambda^2$$

$$= \sum_{x=0}^{\infty} x(x-1) \frac{\lambda^x e^{-\lambda}}{x!} + \lambda - \lambda^2$$

Analogous reasoning to above, first 2 terms are 0, then reindex with $\tilde{x} = x - 2$:

$$= \sum_{x=2}^{\infty} x(x-1) \frac{\lambda^x e^{-\lambda}}{x!} + \lambda - \lambda^2$$

$$= \sum_{\tilde{x}=0}^{\infty} \frac{\lambda^{\tilde{x}-2} e^{-\lambda}}{\tilde{x}!} + \lambda - \lambda^2$$

$$= \lambda^2 e^{-\lambda} \sum_{\tilde{x}=0}^{\infty} \frac{\lambda^{\tilde{x}}}{\tilde{x}!} + \lambda - \lambda^2$$

$$= \lambda^2 e^{-\lambda} e^{\lambda} + \lambda - \lambda^2$$

$$= \lambda^2 + \lambda - \lambda^2$$

$$= \lambda$$

b)

c)

Convergence of random variables

Solution

We know from convergence in probability that $\underset{n\to\infty}{lim} P(|X_n - X| > \epsilon) = 0$. In other words, for an arbitrarily small δ we can find a point N for which all $n \geq N$ are smaller. This means, that for any $\omega \in \Omega$, if $P(\omega) > 0$ and $\delta = P(\omega)$ there exists an N:

$$P(|X_n - X| > \epsilon) < P(\omega) \quad \forall n \geq N$$

Assume ω is part of the set where $|X_n - X| > \epsilon$, that is $|X_n(\omega) - X(\omega)| > \epsilon$. Then the probability of this set must be $\geq P(\omega)$, since it contains ω. This contradicts our previous assertion, that $P(|X_n - X| > \epsilon) < P(\omega)$. Since this holds for all $\omega \in \Omega$, a deviation bigger than ϵ cannot exist, and we conclude Convergence almost surely.

Bonus Exercise