ML 4101: Mathematics for Machine Learning — Fall 24

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Justify all your claims.

Exercise 1 (Recursive Sequences)

a) Let $(a_n)n \in N$ be a sequence in $\mathbb{R}$. Prove that if a_n is monotone increasing and has an upper bound, then it converges to its supremum.

b) Prove that the recursive sequence $a_{n+1} = \sqrt{a_n + 2}$ with $a_n = 0$ converges and determine its limit.

Solution

a) Let U be some upper bound such that $a_n \leq a_{n+1} \leq U$. If an upper bound exists, then logically a lowest upper bound must exist, which we call supremum. It follows that $|U - a_n| \geq |U - a_{n+1}|$ and therefore $\exists U' \leq U$ such that for every $\epsilon > 0$ $|S - a_n| < \epsilon$. If it weren't the case, then there would be a lesser bound which satisfies this, but by construction this cannot happen, as this bound would then be supremum and we would've chosen it as U' instead.

b) The sequence can easily be shown to be monotonically increasing: The base case is satisfied as $a_0 = 0 < a_1 = \sqrt{2}$ and the derivative of $\sqrt{n+2}$ is positive ($\frac{1}{2\sqrt{x+2}}$). Additionally the sequence is bounded since for any $x > 2$, $\sqrt{x+2} < x$. This can be seen for the example $x = 4$, since $\sqrt{6} < 4$ violating the proven property of monotonic increase. Since we know from a) this implies a limit exists, for sufficiently large n, it holds that $\underset{n\to\infty}{lim}\, a_{n+1} = \underset{n\to\infty}{lim}\, a_{n+2}$.

$$\underset{n\to\infty}{lim}\, a_n = \underset{n\to\infty}{lim} \sqrt{a_n + 2}$$
$$\underset{n\to\infty}{lim}\, a_n^2 = \underset{n\to\infty}{lim}\, a_n + 2$$
$$\underset{n\to\infty}{lim}\, a_n^2 - a_n - 2 = 0$$

This has Roots $\frac{1}{2} + -\frac{3}{2} = 2, -1$. We discard the negative root and obtain $L = 2$.

Exercise 2 (Continuity)

a) Prove that every Lipschitz continous function is uniformly continous.

b) Prove that $g : x \mapsto x^2$ is not uniformly continous.

c) Prove that $h : x \mapsto \sqrt{x}$ is uniformly continous but not Lipschitz continous.

Solution

a) Since $d(f(x), f(y)) \leq L \cdot d(x, y)$ from Lipschitz continuity and we require $that \exists \delta = d(x, y)$ such that $d(f(x), f(y)) < \epsilon$, we can choose δ as $\frac{\epsilon}{L}$. Then:

$$d(f(x), f(x + \delta)) < L \cdot \delta = L\frac{\epsilon}{L} = \epsilon$$

b) For any δ,

$$\begin{aligned} d(g(x), g(x+\delta)) &= x^2 - x^2 + 2x\delta + \delta^2 \\ &= 2x\delta + \delta^2 \end{aligned}$$

Since this is dependent on x on a term that dominates in the limit, for large x we cannot choose δ appropriately to get this distance arbitrarily small.

c) We first show that h is uniformly continuous. For this we need to show that for every $\epsilon > 0$ there exists a $\delta > 0$, such that for all pairs of points x, x_0 with $d(x, x_0) < \delta \rightarrow d(h(x), h(x_0)) < \epsilon$.

Let us assume that we have a ϵ, then we want:

$$d(h(x), h(x_0)) < \epsilon \leftrightarrow |\sqrt{x} - \sqrt{x_0}| < \epsilon$$

$$\leftrightarrow x + x_0 - 2\sqrt{xx_0} < \epsilon^2.$$

Without loss of generality we assume $x_0 = x + \delta$ and therefore $\sqrt{x, x_0} \geq x$. It follows:

$$x + x + \delta - 2\sqrt{xx_0} \leq x + x + \delta - 2x = \delta.$$

Therefore, we can choose $\delta = \sqrt{\epsilon}$ for any points x, x_0 and fulfill the defining inequaity for uniform continuity.

However $h(x)$ is not Lipschitz Continuous. This can be seen from the derivative $h'(x) = \frac{1}{2\sqrt{x}}$, $\underset{x \to 0}{lim}\, h'(x) = \infty$, thus $\nexists L$ around $x = 0$.

Exercise 3 (Uniform Convergence)

a) Analyze whether $f_n : x \mapsto \frac{1}{n} sin(nx)$ and $g : x \mapsto x + \frac{x}{n} cos(x)$ converge. If so, state limit and prove whether convergence is uniform.

b) Consider a sequence of functions $f_n : \mathcal{D} \rightarrow \mathbb{R}$ on a finite set $\mathcal{D}$ that converges pointwise. Prove that f_n converges uniformly.

Consider a sequence of functions $f_n : [a, b]$ which are Lipschitz continous with the same $L > 0$. Assume that this sequence converges pointwise.

c) Prove that f is also Lipschitz continous with same L.

d) Prove that f_n converges uniformly to f.

Solution

a)

$$\underset{n \to \infty}{lim} \frac{sin(nx)}{n} = 0$$

Thus the function sequences converges pointwise to $f(x) = 0$. For uniform convergence we require $|f_n - f| < \epsilon$:

$$\underset{n \to \infty}{lim} |\frac{sin(nx)}{n} - 0|$$

Since $sin(nx) \leq 1$:

$$\leq \lim_{n\to\infty} |\frac{1}{n}|$$

which is $\leq \epsilon$ for $n \geq \epsilon$, proving uniform convergence.

For the second sequence:

$$\lim_{n\to\infty} x + \frac{xcos(x)}{n} = x$$

Since the second term tends to 0, while x is unaffected. Investigating uniform convergence:

$$\lim_{n\to\infty} |x + \frac{xcos(x)}{n} - x| < \epsilon$$

$$\lim_{n\to\infty} |\frac{xcos(x)}{n}| < \epsilon$$

Since $cos(x) \leq 1$, $xcos(x) \leq x$ but otherwise unbounded. Therefore for any n we can choose $x > \frac{n\epsilon}{cos(x)}$ to achieve a function value not ϵ close and the sequence does not converge uniformly.

b) We show that a sequence of functions that converges pointwise on a finite set also covnerges uniformly. For this we have to find a N for any $\epsilon > 0$, such that for any f_m with $m > N$ $d(f_m(x) - f(x)) < \epsilon$. From the pointwise convergence we know that for each ϵ that for every point x_i there exists such an N_i such that it converges pointwise to $f(x)$.. We can now simply choose the largest out of the N_i. Then by construction the conditons for converges are fulfilled for every point x_i in the finite set with the same N and therefore it converges uniformly.

c) Since f_n is Lipschitz continous, for any deviation δ we know $d(f_n(x), f_n(x+\delta)) \leq L\delta$ for some distance metric d with $d(\delta) = \delta$. Since f_n converges pointwise, $\lim_{n\to\infty} |f_n - f| = 0$. It immediately follows that $d(f(x) - f(x+\delta)) \leq L\delta + 2 \cdot 0 = L\delta$, proving Lipschitz continuity.

d) We want to prove that a sequence of functions, that is defined over an interval a, b and is Lipschitz contious with the same Lipschitz constant and pointwise convergent also converges uniformly. This means that for any $\epsilon > 0$ there exists a N such that for any f_m with $m > N$ $d(f_m(x), f(x)) < \epsilon$ for all x.
To prove this we first introduce a finite set on the intervall [a, b] with distance ϵ as $D = \{a, a+\epsilon, a+2\epsilon, ...\}$ and define the functions:

$$[f](x) = f([x])$$

, where $[x]$ is the closest value in the constructed set D that is smaller or equal to x.

Thus when write out the distance between the f_n and f we can introduce zero in terms of these functions $[f](x)$ as:

$$d(f_n(x), f(x)) = |(f_n(x) - [f_n](x) + [f_n](x) - [f](x) + [f](x) - f(x)|$$

and bound this with the triangle inequality for norms:

$$\leq |f_n(x) - [f_n](x)| + |([f_n](x) - [f](x)| + |[f](x) - f(x)|$$

We can now use the Lipschitz continuity for $f_n(x)$ as we know that $d(x, [x]) < \epsilon$.

$$\leq L\epsilon + d([f_n](x), [f](x)) + L\epsilon = \epsilon(1 + 2L).$$

. As $[f_n](x)$ is defined on a finite set it also converges uniformly. Thus for every ϵ' we can find an ϵ defining the finite set and an N, such that:

$$d(f_m(x), f(x) \leq 2L\epsilon + d([f_m], [f]) \leq \epsilon 2L + \epsilon < \epsilon'.$$

Importantly, N and ϵ is independent of x and thererfore the sequence converges uniformly.

Exercise 4 (Power and Taylor Series)

a) Determine the radius of convergence of:

$$\sum_{j=1}^{\infty} \frac{j^2}{2^j} x^j \text{ and } \sum_{j=1}^{\infty} 3^j x^{j^2}$$

b) Compute the Taylor polynomial of $f : x \mapsto e^{\pi - x} sin(x)$ with $a = 0$ of degree 3 and the corresponding Lagrange remainder.

c) Prove that f from b) is equal to its Taylor series.

Solution

a)

$$\lim_{j\to\infty} \frac{\frac{j^2}{2^j}}{\frac{(j+1)^2}{2^{j+1}}}$$

$$= \lim_{j\to\infty} \frac{2^j 2 \frac{j^2}{2^j}}{j^2 + 2j + 1}$$

$$= \lim_{j\to\infty} \frac{2j^2}{j^2 + 2j + 1}$$

$$\approx \lim_{j\to\infty} \frac{2j^2}{j^2} = 2 = r$$

Second series:

$$\sum_{j=1}^{\infty} 3^j x^{2j}$$

$$= \sum_{i=1}^{\infty} 3^{i/2} x^i$$

$$\lim_{i\to\infty} \frac{3^{i/2}}{3^{(i+1)/2}}$$

$$= \lim_{i\to\infty} \frac{\sqrt{3^i}}{\sqrt{3}\sqrt{3^i}}$$

$$= \frac{1}{\sqrt{3}} = r$$

b)

$$\begin{aligned}
f(x) &= e^{\pi-x}sin(x)\\
f^{i}(x) &= e^{\pi-x}(cos(x)-sin(x))\\
f^{ii}(x) &= 2e^{\pi-x}cos(x)\\
f^{iii}(x) &= 2e^{\pi-x}(sin(x)+cos(x))\\
f^{iv}(x) &= -4e^{\pi-x}sin(x)
\end{aligned}$$

$$\begin{aligned}
T_3(x,\pi) &= \sum_{k=0}^{3}\frac{f^{(k)}(\pi)}{k!}(x-\pi)^k\\
&= \frac{e^0 sin(\pi)}{0!}(x-\pi)^0+\frac{e^0(cos(\pi)-sin(\pi))}{1!}(x-\pi)^1+\frac{2e^0 cos(\pi)}{2!}(x-\pi)^2+\frac{2e^0(cos(\pi)-sin(\pi))}{3!}(x-\pi)^3\\
&= \frac{0}{0!}1+\frac{(1-0)}{1}(x-\pi)^1+\frac{2\cdot 1}{2}(x-\pi)^2+\frac{2(1-0)}{6}(x-\pi)^3\\
&= (x-\pi)^1+(x-\pi)^2+\frac{1}{3}(x-\pi)^3
\end{aligned}$$

$$\begin{aligned}
R_3(x,\pi) &= \frac{f^{(4)}(\xi)}{4!}(x-pi)^4\\
&= \frac{8e^{\pi-\xi}sin(\xi)}{4!}(x-\pi)^4\\
&= \frac{e^{\pi-\xi}sin(\xi)}{3}(x-\pi)^4
\end{aligned}$$

Where $e^{\pi-\xi}$ is at most e^{π} for $\xi=0$ and $sin(\xi)$ is maximally 1 for $\xi=\frac{\pi}{2}$ yielding an upper bound on the error as $\frac{e^pi}{3}(x-\pi)^4$.

c) The taylor series is the taylor polynomial of degree ∞. Since the denominator of the Lagrange error terms is factorial, thus growing faster than the at most exponential remaining terms, the relative error approaches zero, converging to the original function and therefore the Taylor series.