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σ-Algebra

Solution

Since $\mathcal{A}$ is a σ-Algebra, it must be closed under complement and under countable unions. Using De Morgan's laws: $\overline{\bigcup_{i \in I} X_i} = \bigcap_{i \in I} \overline{X_i}$, we find that we can construct any intersection from unions and intersection, thus $\mathcal{A}$ is closed under intersection as well. Given that $\mathcal{A}$ separates points, it must contain every set of a single element, as this separates the remaining point from all others. If all singletons are in $\mathcal{A}$ then all possible subsets of X can trivially be constructed from the operations of countable intersection, union and complement without exiting $\mathcal{A}$. Thus $\mathcal{A} = \mathcal{P}(X)$.

Covariance and Correlation

Solution

a)

b)

$$\varrho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

Using the Cauchy-Schwarz Inequality:

$$|Cov(X,Y)|^2 \leq \sigma_X^2 \sigma_Y^2$$

$$\Rightarrow |Cov(X,Y)| \leq \sigma_X \sigma_Y$$

Assuming $\sigma_X, \sigma_Y \neq 0$, divide by the product:

$$\frac{|Cov(X,Y)|}{\sigma_X \sigma_Y} \leq 1$$

$$\Rightarrow -1 \leq \frac{Cov(X,Y)}{\sigma_X \sigma_Y} \leq 1$$

$$\Rightarrow -1 \leq \varrho_{X,Y} \leq 1$$

c)

Bernoulli Trials

Solution

a)

b) For a success on trial t we need the first $t-1$ trials to be failures, followed by a success. Thus $P(T_1 = t) = (1-p)^{t-1}p$.
For the second success to be on trial t, we need $t-2$ failures with a success in between at some point and at trial t. Since we have $t-1$ possible position for the first success, this leavus us with $P(T_2 = t) = (t-1)(1-p)^{t-1}p^2$.
As $X = T_2 - T_1$ describes the number of trials between the first and second success, this is identical distributed as T_1, namely: $(1-p)^{t-1}p$.

c)

Independance

Solution

a) Assume X is independant of itself. Then

$$\mathbb{P}(X \leq c) \cap \mathbb{P}(X \leq c) = \mathbb{P}(X \leq c) * \mathbb{P}(X \leq c)$$
$$= \mathbb{P}(X \leq c)^2$$

Additionally, regardless of whether $X \leq c$ it holds that:

$$\mathbb{P}(X \leq c) \cap \mathbb{P}(X \leq c) = \mathbb{P}(X \leq c)$$

Thus

$$\mathbb{P}(X \leq c) = \mathbb{P}(X \leq c)^2$$

Which can only be the case if the $\mathbb{P}(X \leq c) \in \{0, 1\}$. The same argument works $X < c$ to show that $\mathbb{P}(X < c) \in \{0, 1\}$ too. Since:

$$\mathbb{P}(X = c) = \mathbb{P}(X \leq c) - \mathbb{P}(X < c)$$

It follows that $\mathbb{P}(X = c) \in \{0, 1\}$ for any c. Thus X is constant if it is independant of itself.
It remains to show that X is independant if it is constant. Suppose $\mathbb{P}(X = c) = 1$, then:

$$\mathbb{P}(X = c) \cap \mathbb{P}(X = c) = 1 * 1 = \mathbb{P}(X = c)\,\mathbb{P}(X = c)$$

Alternatively suppose $\mathbb{P}(X = c) = 0$

$$\mathbb{P}(X = c) \cap \mathbb{P}(X = c) = 0 * 0 = \mathbb{P}(X = c)\,\mathbb{P}(X = c)$$

Thus X is independant of itself if it is constant, concluding the proof.