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Justify all your claims.

Exercise 1 (Extremal points)

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}; (x, y) \mapsto x^3 + 1/3y^3 - 12x - y$

a) (x, y) is an extremal point if $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$.

$$\frac{\partial f}{\partial x} = 3x^2 - 12$$
$$\frac{\partial f}{\partial y} = y^2 - 1$$

From this we can get four different extremal points:

$$(x_1, y_1) = (2, 1)$$
$$(x_2, y_2) = (-2, 1)$$
$$(x_3, y_3) = (2, -1)$$
$$(x_4, y_4) = (-2, -1)$$

In order to classify them we need to compute the Hessian:

$$H = \begin{pmatrix} 6x & 0 \\ 0 & 2y \end{pmatrix}.$$

We directly see that this is diagonal and thus the eigenvalues correspond to the diagonal entriess
We see that for (x_1, y_1) we have positive eigenvalues and therefore a strict local minimum. Likewise, for (x_4, y_4) we have negative eigenvalues and therefore a strict local maximum. For the other two points the Hessian matrix has a negative and a positive eigenvalue and thus it is indefinite and we have two saddle points.

b) (x, y) is a global maximum iff there exists no other (x', y') such that $f(x', y') > f(x, y))$ and likewise for global minimum.
In this case the function has no global maximum or minimum. E.g. $f(x_1, y_1) = 8+1/3-24-1 = -17.3$, but we find that $f(-3, 0) = -27 < -17.3$.
Similarly, $f(x_4, y_4) = -8 - 1/3 + 24 + 1 = 16.67$, but $f(3, 0) = 27 > 16.67 = f(x_4, y_4)$.

c) Consider $g : \mathbb{R}^3 \to \mathbb{R}, (x, y, z) \mapsto \alpha x^2 e^y + y^2 e^z + z^2 e^x$, with $\alpha \in \mathbb{R}$. The point $(0, 0, 0)$ is an extremal point (local, minimum, maximum or saddle point) iff the first derivative is zero i.e. $\nabla f = \mathbf{0}$.

$$\nabla f = \begin{pmatrix} \alpha 2xe^y + z^2 e^x \\ \alpha x^2 e^y + 2ye^z \\ y^2 e^z + 2ze^x \end{pmatrix}_{(0,0)}$$
$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

To differentiate between the different points we compute the Hessian:

$$H = \begin{pmatrix} \alpha 2e^y + z^2e^x & \alpha 2xe^y & 2ze^x \\ \alpha 2xe^y & x^2e^y + 2e^z & 2ye^z \\ 2ze^x & 2ye^z & y^2e^z + 2e^x \end{pmatrix}_{(0,0)}$$

$$\begin{pmatrix} \alpha 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

As this is again a diagonal matrix we can directly read out the eigenvalues from the diagonal entries and as two af them are greater than zero we can only have a local minimum or saddle point. If $\alpha < 0$ the matrix is indefinite and we therefore have a saddle point. If $a \geq 0$ we have a local minimum.

Exercise 2 (Derivatives)

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