Friederike Horn & Bileam Scheuvens

Justify all your claims.

Exercise 1 (Principal Component Analysis)

a) Suppose λ is an eigenvalue to the eigenvector v of Σ.

$$\lambda\langle x, x\rangle = \langle \lambda x, x\rangle$$

$$= \langle \Sigma x, x\rangle$$

Since $\Sigma = \bar{\Sigma}$

$$= \langle x, \Sigma x\rangle$$

$$= \langle x, \lambda x\rangle$$

$$= \bar{\lambda}\langle x, x\rangle$$

If $\lambda = \bar{\lambda}$ then λ must be in $\mathbb{R}$.

b) By the Rayleigh coefficient, vector that maximizes the variance is the normed eigenvector $\frac{v_1}{||v_1||}$ corresponding to the largest eigenvalue λ_1 of X^TX. The resulting variance is $\sqrt{\lambda_1}$ i.e. the first singular value σ_1 of X.

c) Analogously, the vector orthogonal to v_1 that maximizes the variance is the normed eigenvector $\frac{v_2}{||v_2||}$ corresponding to the largest eigenvalue λ_2 of X^TX. The resulting variance is the second singular value σ_2 of X.

d) The k-th principal component is obtained by eigendecomposition of X^TX and ordering the eigenvalues such that $\lambda_1 < \lambda_2 < \cdots < \lambda_d$. Then the k-th principal component is $X * v_k$ with variance $\sqrt{\lambda_k}$. To obtain the lower dimensional representation $\tilde{X}$ we multiply the matrix with the eigenvectors of XX^T as columns $\tilde{U}$ by the matrix with diagonal elements $\tilde{\sigma_i} = \sqrt{\lambda_i}$ $\tilde{\Sigma}$ and the matrix with eigenvectors of X^TX as rows $\tilde{V}^T$.

$$\tilde{X} = \tilde{U}\tilde{\Sigma}\tilde{V}^T$$

Exercise 2 (Singular Value Decomposition)

a) The singular values are equal to the ordered eigenvalues of MM^T and M^TM.

$$M^TM = \begin{bmatrix} -4 & 1 & 3 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} -4 & 2 \\ 1 & 2 \\ 3 & 2 \end{bmatrix} = \begin{pmatrix} 27 & 0 \\ 0 & 12 \end{pmatrix}$$

Since the result is diagonal, the eigenvalues are easily determined as $\lambda_1 = 27, \lambda_2 = 12$ which results in the singular values $\sigma_1 = \sqrt{27}, \sigma_2 = \sqrt{12}, \sigma_3 = 0$.

b) The singular values of a matrix A are the union of the roots of the eigenvalues of A^TA and AA^T. Since the union is commutative, $eigs(AA^T) \cup eigs(A^TA) = eigs(A^TA) \cup eigs(AA^T)$, so the singular values are identical.

c) Note that $\Sigma * \Sigma^{\#} = I$ only if $\sigma_i \neq 0 \forall i$, otherwise it acts as I on the subspace on nonzero singular values.

- $$\begin{aligned} AA^{\#}A &\stackrel{!}{=} A \\ U\Sigma V^T V\Sigma^{\#}U^T U\Sigma V^T &= A \\ U\Sigma\Sigma^{\#}\Sigma V^T &= A \\ U\Sigma V^T &= A \end{aligned}$$

- $$\begin{aligned} A^{\#}AA^{\#} &\stackrel{!}{=} A^{\#} \\ V\Sigma^{\#}U^T U\Sigma V^T V\Sigma^{\#}U^T &= A^{\#} \\ V\Sigma^{\#}\Sigma\Sigma^{\#}U^T &= A^{\#} \\ V\Sigma^{\#}U^T &= A^{\#} \end{aligned}$$

- $$\begin{aligned} (AA^{\#})^T &\stackrel{!}{=} AA^{\#} \\ (U\Sigma V^T V\Sigma^{\#}U^T)^T &= AA^{\#} \\ U\Sigma^{\#}V^T V\Sigma U^T &= AA^{\#} \\ U\Sigma^{\#}V^T V\Sigma U^T &= U\Sigma V^T V\Sigma^{\#}U^T \\ U\Sigma^{\#}\Sigma U^T &= U\Sigma\Sigma^{\#}U^T \end{aligned}$$

- $$\begin{aligned} (A^{\#}A)^T &\stackrel{!}{=} A^{\#}A \\ (V\Sigma^{\#}U^T U\Sigma V^T)^T &= A^{\#}A \\ V\Sigma U^T U\Sigma^{\#}V^T &= A^{\#}A \\ V\Sigma U^T U\Sigma^{\#}V^T &= V\Sigma^{\#}U^T U\Sigma V^T \\ V\Sigma\Sigma^{\#}V^T &= V\Sigma^{\#}\Sigma V^T \end{aligned}$$

Exercise 3 (Condition Number)

a) First prove that $||Av|| \stackrel{!}{\leq} ||A||_2 \cdot ||v||$

$$||Av|| \leq \max_{x \neq 0} \frac{||Ax||}{||x||} ||v||$$

If $||v|| = 0$ this holds since $0 = ||A||_2 \cdot 0$. Otherwise we divide by $||v||$ to obtain:

$$\frac{||Av||}{||v||} \leq \max_{x \neq 0} \frac{||Ax||}{||x||}$$

This clearly holds.

b)