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Random Graphs

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Abstract: A random graph is given by a pair $(\mathbb{G}, \mathbb{P})$, where $\mathbb{G}$ is a set of graphs and $\mathbb{P}$ is a probability distribution with support $\mathbb{G}$. Random graphs have been studied since the middle of the twentieth century and have witnessed a surge of interest since the turn of the twenty-first century, fueled by the rise of the Internet and social networks and the growing realization that today's world is a connected world.

The classic and most widely studied models of random graphs are:

1. *Bernoulli random graph model*[1, 2]. Let $\mathbb{G}_n$ be the set of all labelled graphs with n vertices. The Bernoulli(p) random graph model assumes that edges are independent and identically distributed Bernoulli(p) random variables, where $p \in [0, 1]$ is the probability of an edge. Thus, the model assumes that each graph $G_n \in \mathbb{G}_n$ with n vertices and $m \in \left\{0, \ldots, \binom{n}{2}\right\}$ edges occurs with probability

$$\mathbb{P}(G_n) = p^m \, (1-p)^{\binom{n}{2}-m}, \qquad G_n \in \mathbb{G}_n$$

2. *Uniform random graph model*[3]. Let $\mathbb{G}_{n,m}$ be the set of all labelled graphs with n vertices and $m \in \left\{0, \ldots, \binom{n}{2}\right\}$ edges. The uniform random graph model assumes that each graph $G_{n,m} \in \mathbb{G}_{n,m}$ occurs with equal probability:

$$\mathbb{P}(G_{n,m}) = \binom{\binom{n}{2}}{m}^{-1}, \qquad G_{n,m} \in \mathbb{G}_{n,m}$$

Both models are known as Erdős and Rényi random graphs. Indeed, the two models are closely related: the Bernoulli(p) random graph model with edge probability $p \in (0, 1)$ implies that the conditional probability of G_n, given that G_n has $m \in \left\{0, \ldots, \binom{n}{2}\right\}$ edges, is the discrete uniform distribution with support $\mathbb{G}_{n,m}$:

$$\mathbb{P}(G_n \mid G_n \in \mathbb{G}_{n,m}) = \frac{p^m \, (1-p)^{\binom{n}{2}-m}}{\binom{\binom{n}{2}}{m} \, p^m \, (1-p)^{\binom{n}{2}-m}} = \binom{\binom{n}{2}}{m}^{-1}, \qquad G_n \in \mathbb{G}_{n,m}$$

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Under conditions which ensure that p is not too small and n is large, the behaviour of these two models is similar: such conditions are reviewed in, for example, Section 1.1 of Frieze and Karoński[4].

Sections 1–3 focus on Erdős and Rényi random graphs, with an emphasis on the Bernoulli(p) random graph model. An introduction to Erdős and Rényi random graphs and related random graph models can be found in the monographs of Bollobás[5], Palmer[6], Janson *et al.*[7], Lovász[8] and Frieze and Karoński[4], while the probabilistic method used to study random graphs is reviewed in the books of Alon and Spencer[9], Molloy and Reed[10] and Chung and Lu[11]. We then discuss more complex random graph models that go beyond Erdős and Rényi random graphs (Section 4). We conclude with the important topic of statistical inference for random graphs (Section 5).

1 Phase Transitions of Erdős and Rényi Random Graphs

One of the most intriguing discoveries of classic random graph theory is that the structure of random graphs undergoes dramatic changes as the edge probability p of the Bernoulli(p) random graph model increases from 0 to 1, that is, as the random graph evolves from an empty graph without edges to a complete graph with all possible edges. Such drastic changes in the structure of random graphs are known as *phase transitions.*

To illustrate phase transitions, consider any monotone increasing graph property $\mathscr{P} \subseteq \mathbb{G}_n$, that is, any graph property that is preserved by adding edges (e.g. connectivity). Bollobás and Thomason[12] showed that, for every non-trivial, monotone increasing graph property $\mathscr{P} \subseteq \mathbb{G}_n$, there exists a threshold function $t_{\mathscr{P}}(n)$ such that

$$\lim_{n\to\infty} \mathbb{P}(G_n \in \mathscr{P}) = \begin{cases} 0 & \text{if } \dfrac{p}{t_{\mathscr{P}}(n)} \to 0 \text{ as } n \to \infty \\ 1 & \text{if } \dfrac{p}{t_{\mathscr{P}}(n)} \to \infty \text{ as } n \to \infty \end{cases}$$

In other words: with high probability, the random graph does not possess property $\mathscr{P}$ when $p \ll t_{\mathscr{P}}(n)$, but does possess property $\mathscr{P}$ when $p \gg t_{\mathscr{P}}(n)$; note that $p = p(n)$ is a function of n in the sparse-graph regime where some of the most interesting phase transitions occur.

We give two examples of phase transitions. Consider the evolution of a Bernoulli(p) random graph as $p = p(n)$ increases from 0 to 1. If the random graph is very sparse in the sense that $p\, n^{3/2} \to 0$ as $n \to \infty$, then the random graph is the union of isolated vertices and edges with high probability. If the random graph is less sparse in the sense that $p\, n \to 0$, then with high probability the random graph is a forest. The structure of the random graph undergoes a dramatic change when p reaches c/n, where $c = 1$: while, with high probability, the random graph consists of small components of order $O(\log n)$ when $c < 1$, it consists of a giant component with a constant proportion of n vertices along with small components of order $O(\log n)$ when $c > 1$. The abrupt emergence of a giant component is known as a phase transition. A second phase transition occurs when p reaches the order of magnitude $\log n/n$: while the random graph with high probability is not connected when $p \ll \log n/n$, it is connected with high probability when $p \gg \log n/n$.

These and other properties of Erdős and Rényi random graphs are reviewed in much more depth in the books of Bollobás[5], Janson *et al.*[7] and Frieze and Karoński[4].

2 Other Properties of Erdős and Rényi Random Graphs

The theory of random graphs is not limited to dichotomous properties of random graphs, such as connectivity.

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The distributions of non-dichotomous properties of functions of random graphs have been investigated as well. Examples include the number of vertices of a given degree and **Rank Statistics** associated with random graphs, such as the smallest and largest degree.

Often, the asymptotic distribution is concentrated at one or two discrete values: for example, with high probability, the size of the largest clique (maximal connected subset of vertices) and the size of the largest independent set (largest subset of vertices such that no pair of vertices is connected) assume one of at most two values in the neighbourhood of $2\log n/\log(1/p)$ and $2\log n/\log(1/(1-p))$, respectively[13, 14]. More interesting results concerning the chromatic number and other non-dichotomous properties of random graphs are discussed in the books of Bollobás[5], Janson *et al.*[7] and Frieze and Karoński[4].

3 Applications of Erdős and Rényi Random Graphs

We mention some applications of Erdős and Rényi random graphs in classic random graph theory: random trees, random tournaments and percolation on random lattices.

3.1 Random Trees

A random tree is a tree sampled at random from a given set of trees (*see* **Tree-Structured Methods**). Properties studied include the number of vertices of a given degree, the distance of a given vertex from the root (altitude), the number of vertices at a given altitude (width), the height of a rooted tree (the maximal altitude of its vertices) and the diameter of the random tree. Characteristics related to the length of paths between specified vertices in a random tree have been studied as well. These and more general structures, such as **Random Forests**, are reviewed in Moon[15] and Karoński[16].

3.2 Random Tournaments

A tournament with n players can be represented by a directed graph, where a directed edge (i, j) denotes that player i defeated player j. If the outcomes of the tournament are random, the resulting random graph is called a *random tournament*. Random tournaments can serve as a model of **Multiple Comparison Procedures**. A wide range of problems and results on random tournaments is presented in the monograph by Moon[17].

3.3 Percolation on Random Lattices

Consider a regular lattice and color edges (vertices) white or black with probabilities p and $1-p$, respectively. Assume that a fluid spreads through the lattice, with the white edges (vertices) interpreted as "open" and black ones as "closed." **Percolation Theory** is concerned with the structural properties of such random lattices. An overview on percolation theory can be found in monographs of Grimmet[18] and Bollobàs and Riordan[19].

4 Beyond Erdős and Rényi Random Graphs

Since the turn of the twenty-first century, an increasing number of more complex random graph models have been developed, fueled by the rise of the Internet and social networks and the growing realization

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that today's world is a connected world (*see* **Social Networks**). Those more complex random graph models are motivated by the fact that edges in real-world networks are not believed to be independent and identically distributed Bernoulli(p) random variables[20–24], necessitating the development of more complex random graph models[25]. In general, more complex random graph models attempt to capture heterogeneity in the propensities of vertices to form edges and other subgraphs of interest and dependence among edges[25].

4.1 Exchangeable Random Graph Models

A natural class of random graph models are models that possess invariance properties, such as **Exchangeability**[26]. Two broad classes of exchangeable random graph models can be distinguished, depending on whether random graphs are invariant to the labelling of vertices[27] or the labelling of edges[28, 29]. Exchangeable random graph models include both dense and sparse random graph models[30].

4.2 Exponential-Family Models of Random Graphs

A large statistical framework that extends the Bernoulli(p) random graph model is given by exponential-family models of random graphs, which are known as **Exponential Random Graph Models** (*see* **General Exponential Families**; **Special Exponential Families**)[31–35]. The class of exponential-family random graph models includes the Bernoulli(p) random graph model as a special case, because it is an exponential-family model with the number of edges as sufficient statistic and the log odds of the edge probability p as natural parameter. Well-specified exponential-family random graph models can model sparse and dense random graphs, short- and long-tailed degree distributions and dependencies among edges. Exponential-family models for dependent edges are closely related to other exponential-family models for dependent random variables, such as Ising models in physics[36], **Markov Random Fields** in spatial statistics[37] and **Markov Networks** in artificial intelligence, machine learning and statistics[38–40]. Several classes of models can be distinguished according to the dependence assumptions made. The simplest class of models assumes that edges independent, which includes the Bernoulli(p) random graph model[1, 2], β-models with vertex-dependent propensities to form edges[41] and p_1-models with vertex-dependent propensities to send and receive edges in directed random graphs[31]. A second class of exponential-family models assumes Markov dependence[32, 42]. While some of those models have been shown to be ill-behaved[43–47], well-behaved alternatives in the form of curved exponential-family models have been developed[46, 48, 49]. Other exponential-family models induce local dependence within subsets of vertices[50].

4.3 Random Graph Models with Latent Structure

A popular class of models are random graph models with latent structure, such as stochastic block models with an unobserved partition of a set of vertices into subsets, which can be used to detect communities in social networks[27, 51], and generalizations known as mixed membership models;[52] stochastic block models with dependent edges within subsets of vertices;[50] latent space models which assume that vertices have positions in an unobserved, metric space, which may be a Euclidean space or an ultrametric space[53, 54], and which sometimes can be interpreted as an unobserved, social space;[53] combinations of stochastic block models and latent space models;[55] random effects models and other latent variable models;[30, 52, 56–60] and graphons[8], which can be viewed as latent variable models.

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4.4 Temporal Random Graph Models

Many real-world networks evolve over time, and models of time-evolving random graphs are therefore of great interest. Two broad classes of models can be distinguished, depending on whether edges have a duration (e.g. links between websites) or edges do not have a duration (e.g. e-mails). The first class of models includes continuous-time Markov processes[61–63] and discrete-time Markov processes[64, 65] and extensions of stochastic block models and latent space models to time-dependent random graphs[66–68]. The second class of models is known as *relational event models*[69]. Most of these models assume that the set of vertices is time-invariant, but some models with time-evolving sets of vertices have been developed[70].

4.5 Small World and Scale-Free Random Graph Models

In physics and related fields, small world and scale-free random graph models with power-law degree distributions have attracted much attention[71, 72]. While the mathematical properties of scale-free random graphs are worth studying[73], the usefulness of scale-free random graphs in real-world applications has been debated[74, 75].

5 Statistical Inference for Random Graphs

Sections 1–4 are concerned with deduction – deducing properties of random graphs when the model is known. By contrast, Section 5 is concerned with induction – inferring properties of random graphs when the model is unknown. In other words, Section 5 is concerned with statistical inference for random graphs.

Consider a set of vertices of interest, called a population, and a population graph defined on the population. Statistical inference may be of interest in the following scenarios:

- The whole population graph is observed, but the probability model that generated the population graph is unknown and the goal of statistical inference is to estimate the probability model, based on the observed population graph.
- The population graph is unobserved, but subgraphs of the population graph are sampled, and statistical inference focuses on:
  - *Model-based inference given sampled subgraphs:* Estimating the probability model that generated the population graph, based on sampled subgraphs.
  - *Design-based inference given sampled subgraphs:* Estimating functions of the population graph, based on sampled subgraphs, without making assumptions about the probability model that generated the population graph.

We discuss sampling along with design-based and model-based inference below.

5.1 Sampling

There are many sampling designs for sampling subgraphs from population graphs: for example, one can sample vertices at random and observe all edges of sampled vertices, which is a simple form of ego-centric sampling[76, 77]. In addition, it may be possible to observe the edges of those vertices that are connected to sampled vertices, which is a simple form of link-tracing[76, 78]. Special cases of link-tracing are

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snowball sampling[79] and respondent-driven sampling[80–82]. An alternative is sampling edges instead of vertices[28].

5.2 Design-Based Versus Model-Based Inference

Two broad classes of statistical inference problems can be distinguished, design-based and model-based inference.

In the first class of inference problems, no assumption is made about the probability model that generated the population graph. A possible goal of statistical inference is to estimate functions of the population graph based on sampled subgraphs, for example, the number of edges in the population graph. A classic example of design-based estimators is Horvitz–Thompson estimators[83]. A disadvantage of Horvitz–Thompson estimators is that they require knowledge of the sampling inclusion probabilities, which are unknown in link-tracing sampling designs and other complex network sampling designs, limiting the usefulness of design-based estimators in applications to random graphs[84].

In the second class of inference problems, it is assumed that the population graph was generated by a probability model. The goal of statistical inference is to infer the probability model that generated the population graph based on either a complete observation of the population graph or an incomplete observation of the population graph in the form of subgraphs sampled from the population graph.

5.3 Statistical Theory: Design-Based Inference

The properties of design-based estimators depend on the sampling design used to sample subgraphs of the population graph.

One desirable property of statistical estimators is consistency. In design-based inference, there are at least two forms of consistency. The first form of consistency is Fisher-consistency[85]. An estimator of a population quantity is Fisher-consistent if the estimator is equal to the population quantity when the whole population graph is observed: for example, an estimator of the proportion of edges in the population graph based on the proportion of edges in sampled subgraphs is Fisher-consistent. The second form of consistency is consistency and asymptotic normality under sampling. Depending on the sampling design, design-based estimators may be consistent and asymptotically normal under sampling[77, 82].

5.4 Statistical Theory: Model-Based Inference

In general, the properties of model-based estimators depend on both the probability model that generates the population graph and the sampling design that generates samples of subgraphs from the population graph.

But, under some conditions, the sampling design is ignorable for the purpose of model-based inference. The classic work of Rubin[86, 87], applied to random graphs, implies that the sampling design is ignorable for the purpose of likelihood-based inference as long as the following two conditions are satisfied:[76, 84, 88]

- The probability of not observing whether two vertices are connected does not depend on whether the two vertices are connected.
- The parameters of the complete-data generating process that generates the population graph and the incomplete-data generating process that samples subgraphs are distinct.

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Two examples of ignorable sampling designs are ego-centric sampling and link-tracing sampling designs based on random samples of vertices[76, 84, 88]. In both examples, the likelihood function can be obtained by summing the probability mass function of the population graph with respect to the unobserved edges, and the sampling design used to generate the sampled subgraphs can be ignored for the purpose of likelihood-based inference[76, 84, 88].

One of the most studied statistical inference problems is the recovery of latent structure underlying random graphs. The best-known example is the recovery of an unobserved partition of a set of vertices,which arises in the study of stochastic block models and helps detect communities in social networks[27, 89–102]. A related example is the recovery of latent structure in latent space models[103, 104] and graphons[99, 105, 106]. In addition to the recovery of latent structure, statistical theory has studied the properties of estimators for the parameters of random graph models, for example, random graph models with vertex-dependent propensities to form edges, including β-models and p_1-models[41, 107–109], and canonical and curved exponential-family models of random graphs[110].

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Related Articles

Cluster Analysis, Graph-Theoretic; **Exchangeability**; **Exponential Random Graph Models**; **General Exponential Families**; **Graph Theory**; **Markov Networks**; **Markov Random Fields**; **Markov Random Field Models**; **Multivariate Directed Graphs**; **Regression Trees**; **Social Networks**; **Special Exponential Families**; **Trees, Probabilistic Functional**; **Tree-Structured Statistical Methods**; **Tree Models**.

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