Supplement: Disaster Response on September 11, 2001 Through the Lens of Statistical Network Analysis

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We are interested in maximising the lower bound on $\log P(\mathbf{Y} = \mathbf{y})$ given by (23). We note that, despite of the fact that the family of auxiliary distributions is fully factorised, the lower bound on $\log P(\mathbf{Y} = \mathbf{y})$ is intractable, because

$$\int \left[\log \frac{P(\mathbf{Y} = \mathbf{y} \mid \boldsymbol{\delta})\, p(\boldsymbol{\delta})}{q(\boldsymbol{\delta} \mid \boldsymbol{\vartheta})}\right] q(\boldsymbol{\delta} \mid \boldsymbol{\vartheta})\, \mathrm{d}\,\boldsymbol{\delta} \;=\; \sum_{i<j}^{n} E_{\boldsymbol{\vartheta}} \left[\log P(Y_{ij} = y_{ij} \mid \boldsymbol{\delta})\right] + \; E_{\boldsymbol{\vartheta}} \left[\log p(\boldsymbol{\delta})\right] \;-\; E_{\boldsymbol{\vartheta}} \left[\log q(\boldsymbol{\delta} \mid \boldsymbol{\vartheta})\right], \tag{27}$$

where

$$\begin{aligned} E_{\boldsymbol{\vartheta}}[\log P(Y_{ij} = y_{ij}|\boldsymbol{\delta})] \quad &= \quad \left[\sum_{k=1}^{K} \vartheta_{Z_{ik}}\, \vartheta_{\theta_k,1} + \sum_{l=1}^{K} \vartheta_{Z_{jl}}\, \vartheta_{\theta_l,1}\right] y_{ij} \\ &+ \quad \sum_{k=1}^{K}\sum_{l=1}^{K} \vartheta_{Z_{ik}} \vartheta_{Z_{jl}} E_{\boldsymbol{\vartheta}} \left\{-\log\left[1 + \exp(\theta_k + \theta_l)\right]\right\} \end{aligned} \tag{28}$$

includes the intractable expectations $E_{\boldsymbol{\vartheta}} \{-\log[1 + \exp(\theta_k + \theta_l)]\}$. To obtain a tractable lower bound on $\log P(\mathbf{Y} = \mathbf{y})$, we lower bound $E_{\boldsymbol{\vartheta}} \{-\log[1 + \exp(\theta_k + \theta_l)]\}$ by Jensen's inequality:

$$\begin{aligned} &E_{\boldsymbol{\vartheta}} \{-\log[1 + \exp(\theta_k + \theta_l)]\} \;\geq\; -\log E_{\boldsymbol{\vartheta}} \left[1 + \exp(\theta_k + \theta_l)\right] \\ &= \; -\log\left[1 + \exp(\vartheta_{\theta_k,1} + \tfrac{1}{2}\vartheta_{\theta_k,2} + \vartheta_{\theta_l,1} + \tfrac{1}{2}\vartheta_{\theta_l,2})\right]. \end{aligned} \tag{29}$$

The lower bound on $\log P(\mathbf{Y} = \mathbf{y})$ given by (27), (28), and (29) can be maximised with respect to $\boldsymbol{\vartheta}$ by coordinate ascent algorithms.

The lower bound on $\log P(\mathbf{Y} = \mathbf{y})$ can be maximised with respect to $\vartheta_{\alpha,1}$, $\vartheta_{\alpha,2}$, $\vartheta_{\mu,1}$, $\vartheta_{\mu,2}$, $\vartheta_{\sigma^{-2},1}$, $\vartheta_{\sigma^{-2},2}$, $\vartheta_{V_k,1}$, $\vartheta_{V_k,2}, \vartheta_{Z_{ik}}$ as follows:

$$\vartheta_{\alpha,1} \quad = \quad A_1 + K - 1, \tag{30}$$

$$\vartheta_{\alpha,2} \quad = \quad B_1 - \sum_{k=1}^{K-1} \left[\Psi(\vartheta_{V_k,2}) - \Psi(\vartheta_{V_k,1} + \vartheta_{V_k,2})\right], \tag{31}$$

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$$\vartheta_{\mu,1} = \frac{MS^{-2} + E_{\boldsymbol{\vartheta}}[\sigma^{-2}] \sum_{k=1}^{K} E_{\boldsymbol{\vartheta}}[\theta_k]}{S^{-2} + K\, E_{\boldsymbol{\vartheta}}[\sigma^{-2}]}, \tag{32}$$

$$\vartheta_{\mu,2} = \frac{1}{S^{-2} + K\, E_{\boldsymbol{\vartheta}}[\sigma^{-2}]}, \tag{33}$$

$$\vartheta_{\sigma^{-2},1} = A_2 + \frac{K}{2}, \tag{34}$$

$$\vartheta_{\sigma^{-2},2} = B_2 + \sum_{k=1}^{K} \frac{\vartheta_{\theta_k,1}^2 - 2\,\vartheta_{\theta_k,1}\vartheta_{\mu,1} + \vartheta_{\theta_k,2}}{2} + \frac{K(\vartheta_{\mu,1}^2 + \vartheta_{\mu,2})}{2}, \tag{35}$$

$$\vartheta_{V_k,1} = 1 + \sum_{i=1}^{n} \vartheta_{Z_{ik}},\ k = 1, \ldots, K-1, \tag{36}$$

$$\vartheta_{V_k,2} = E_{\boldsymbol{\vartheta}}[\alpha] + \sum_{i=1}^{n} \sum_{l=k+1}^{K} \vartheta_{Z_{il}},\ k = 1, \ldots, K-1, \tag{37}$$

$$\begin{aligned}\log \vartheta_{Z_{ik}} \quad &\propto \quad \Psi(\vartheta_{V_k,1}) - \Psi(\vartheta_{V_k,1} + \vartheta_{V_k,2}) + \sum_{l=1}^{k-1} [\Psi(\vartheta_{V_l,2}) - \Psi(\vartheta_{V_l,1} + \vartheta_{V_l,2})] \\ &+ \sum_{j \neq i}^{n} \vartheta_{\theta_k,1}\, y_{ij} - \sum_{j \neq i}^{n} \sum_{l=1}^{K} \vartheta_{Z_{jl}} \log\left[1 + \exp(\vartheta_{\theta_k,1} + \frac{1}{2}\vartheta_{\theta_k,2} + \vartheta_{\theta_l,1} + \frac{1}{2}\vartheta_{\theta_l,2})\right],\end{aligned} \tag{38}$$

where $\Psi(.)$ denotes the digamma function and the expectations $E_{\boldsymbol{\vartheta}}[\alpha]$, $E_{\boldsymbol{\vartheta}}[\theta_k]$, and $E_{\boldsymbol{\vartheta}}[\sigma^{-2}]$ under the auxiliary distribution indexed by $\boldsymbol{\vartheta}$ are tractable.

The lower bound on $\log P(\mathbf{Y} = \mathbf{y})$ with respect to $\vartheta_{\theta_k,1}$ and $\vartheta_{\theta_k,2}$ can be maximised by maximising

$$\begin{aligned} &- \sum_{k=1}^{K} \frac{\vartheta_{\sigma^{-2},1}}{2\vartheta_{\sigma^{-2},2}} \left[\vartheta_{\theta_k,1}^2 - 2\,\vartheta_{\theta_k,1}\vartheta_{\mu,1} + \vartheta_{\theta_k,2}\right] + \frac{1}{2}\sum_{k=1}^{K} \log \vartheta_{\theta_k,2} \\ &+ \sum_{i<j}^{n} \left[\sum_{k=1}^{K} \vartheta_{Z_{ik}}\vartheta_{\theta_k,1} + \sum_{l=1}^{K} \vartheta_{Z_{jl}}\vartheta_{\theta_l,1}\right] y_{ij} \\ &- \sum_{i<j}^{n} \sum_{k=1}^{K} \sum_{l=1}^{K} \vartheta_{Z_{ik}}\vartheta_{Z_{jl}} \log\left[1 + \exp(\vartheta_{\theta_k,1} + \frac{1}{2}\vartheta_{\theta_k,2} + \vartheta_{\theta_l,1} + \frac{1}{2}\vartheta_{\theta_l,2})\right].\end{aligned} \tag{39}$$

Since $\vartheta_{\theta_k,2}$ are the precision parameters of the marginal auxiliary distributions of θ_k, the constraints $\vartheta_{\theta_k,2} > 0$ must be respected ($k = 1, \ldots, K$). Two possible approaches are constrained maximisation subject to the constraints $\vartheta_{\theta_k,2} > 0$ and unconstrained maximisation by reparameterising $\rho_{\theta_k,2} : \vartheta_{\theta_k,2} \mapsto \log \vartheta_{\theta_k,2}$ and maximising with respect to $\rho_{\theta_k,2}$ ($k = 1, \ldots, K$).