Lecture 06

Application to Fluid Flow

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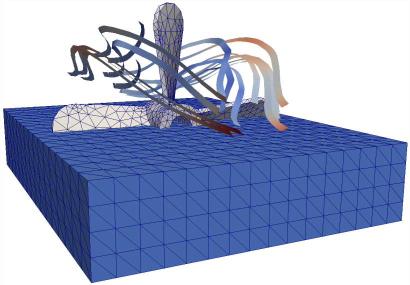
What will you learn?

- *How to solve the Stokes equations*
- *How to solve the Navier–Stokes equations*
- *How to formulate the boundary value problem*
- *How to derive the variational problem*
- *FEniCS programming*
  - Loading meshes from file
  - Applying boundary conditions to subspaces
- *Exercise*

Fluid mechanics

Compute the velocity u (vector) and pressure p (scalar)



A coupled system with unknowns u and p

(Or a system with unknown u and Lagrange multiplier p)

How to formulate the boundary value problem

Partial differential equations (“physical” formulation)

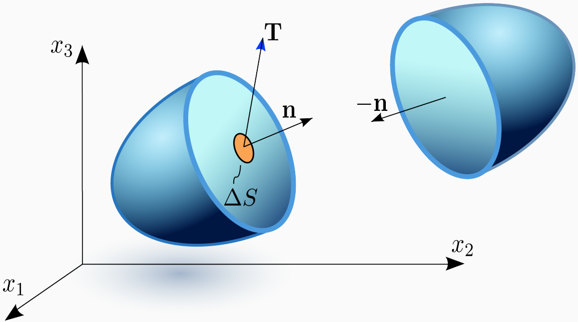
The Stokes equations

$$
\begin{aligned}
-\nabla \cdot \sigma(u, p) &= f \quad && \text{in } \Omega \\
\nabla \cdot u &= 0 \quad && \text{in } \Omega
\end{aligned}
$$

The (incompressible) Navier–Stokes equations

$$
\begin{aligned}
\rho \frac{\partial u}{\partial t} + u \cdot \nabla u - \nabla \cdot \sigma(u, p) &= f \quad && \text{in } \Omega \times (0, T) \\
\nabla \cdot u &= 0 \quad && \text{in } \Omega \times (0, T)
\end{aligned}
$$

The stress tensor



σ is the stress tensor [force per unit area]

n is the unit normal [dimensionless]

$T = \sigma \cdot n$ is the boundary traction [force per unit area]

$F \approx T\Delta S$ is the force acting on ΔS

Isotropic Newtonian fluids

The viscous stress tensor

$$\sigma(u,p) = 2\mu\varepsilon(u) - pI$$

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The symmetric gradient (of a vector v)

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Note that if $\nabla \cdot u = 0$ then

$$-\nabla \cdot \sigma(u, p) = -\mu(\nabla \cdot \nabla u + \underbrace{\nabla \cdot (\nabla u)^\top}_{=0}) + \underbrace{\nabla \cdot (pI)}_{=\nabla p} = -\mu\Delta u + \nabla p$$

μ is called the *dynamic viscosity*

Partial differential equations (“mathematical” formulation)

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Boundary conditions

Dirichlet boundary condition

$$u = u_{\mathrm{D}} \quad \text{on } \Gamma_{\mathrm{D}} \subseteq \partial\Omega$$

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$$\mu \partial_n u - pn = g \quad \text{on } \Gamma_{\mathrm{N}} \subseteq \partial\Omega$$

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Boundary conditions for p are a longer story...

How to derive the variational problem

From strong to weak form: (1) $\rightarrow$ (2) ("physical" formulation)

Partial differential equation (strong form)

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Multiply by test functions $(v, q) \in W = V \times Q$:

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Multiply by test functions $(v, q) \in W = V \times Q$:

$\Rightarrow$ Continuous variational problem (weak form)

Find $(u, p) \in W$ such that

$$
\int_\Omega \sigma(u, p) : \varepsilon(v) + \nabla \cdot u \, q \, \mathrm{d}x = \int_\Omega f \cdot v \, \mathrm{d}x + \int_{\Gamma_\mathrm{N}} g \cdot v \, \mathrm{d}s
$$

for all $(v, q) \in W$

From strong to weak form: (1) $\rightarrow$ (2) ("math" formulation)

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Multiply by test functions $(v, q) \in W = V \times Q$:

$\Rightarrow$ Continuous variational problem (weak form)

Find $(u, p) \in W$ such that

$$\int_\Omega \nabla u : \nabla v - (\nabla \cdot v) p \pm (\nabla \cdot u) q \,\mathrm{d}x = \int_\Omega f \cdot v \,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} g \cdot v \,\mathrm{d}s$$

for all $(v, q) \in W$

The saddle-point structure of the Stokes equations

Consider the “mathematical” formulation:

$$\underbrace{\int_\Omega \nabla u : \nabla v \,\mathrm{d}x}_{=a(u,v)} - \underbrace{\int_\Omega (\nabla \cdot v) p \,\mathrm{d}x}_{=b(v,p)} - \underbrace{\int_\Omega (\nabla \cdot u) q \,\mathrm{d}x}_{=b(u,q)} = L(v)$$

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Variational problem:

$$\begin{aligned} a(u,v) + b(v,p) &= L(v) \\ b(u,q) &= 0 \end{aligned}$$

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Linear system:

$$\begin{aligned} Au + B^\top p &= b \\ Bu &= 0 \end{aligned}$$

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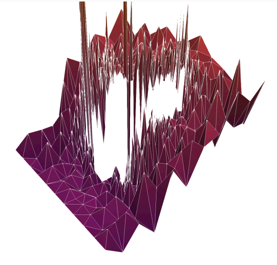
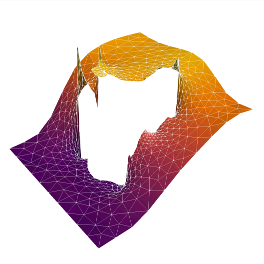
Requires special considerations for solvers and elements!

Stable finite elements for the Stokes equations

$P_r - P_r$ elements are unstable!





Stable elements:

- Taylor–Hood: $P_{r+1} - P_r$
- Mini element: $P_r^b - P_r$

How to solve the incompressible Navier–Stokes equations

Multiply by test functions v and q as for the Stokes problem

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Gives a saddle-point problem to be solved in each time step

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Apply an efficient iterative method …or

How to solve the incompressible Navier–Stokes equations

Multiply by test functions v and q as for the Stokes problem

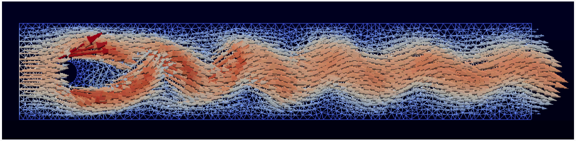
Gives a saddle-point problem to be solved in each time step

Apply an efficient iterative method …or

Use a splitting scheme (simple and efficient)

See FEniCS Book for a comparison of methods!

See FEniCS Tutorial for how to implement a simple and efficient splitting scheme!



FEniCS programming

Loading meshes from file

Meshes are loaded from file by specifying the filename to the `Mesh` constructor:

```
mesh = Mesh('mesh.xml.gz')
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Meshes must be stored in the FEniCS XML format

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Use `meshio` to convert between mesh formats:

```
meshio-convert mesh.msh mesh.xml
```

Supported formats: ANSYS msh, FEniCS XML, Exodus, H5M, Medit, MED/Salome, Gmsh, OFF, PERMAS, STL, VTK, VTU, XDMF

See `https://github.com/nschloe/meshio`

Applying boundary conditions to subspaces

Boundary conditions are applied to subsystems of a mixed function space `W` by accessing `W.sub(0)`, `W.sub(1)` and so on:

```
bc_u = DirichletBC(W.sub(0), u_D, boundary_u)
bc_p = DirichletBC(W.sub(1), p_D, boundary_p)
```

Exercise

Exercise 6: Application to Fluid Flow

In this exercise, we will solve the Stokes equations with FEniCS:

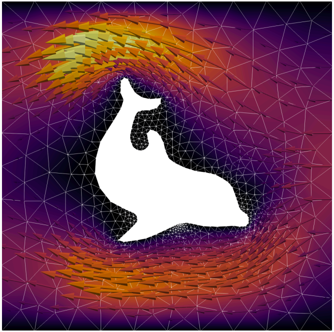
$$
\begin{aligned}
-\nabla \cdot \sigma(u, p) &= f && \text{in } \Omega \\
\nabla \cdot u &= 0 && \text{in } \Omega \\
u &= u_{\mathrm{R}} && \text{on } \Gamma_{\mathrm{R}} \\
\sigma \cdot n &= (0, 0) && \text{on } \Gamma_{\mathrm{L}} \\
p &= p_{\mathrm{L}} && \text{on } \Gamma_{\mathrm{L}} \\
u &= (0, 0) && \text{on } \Gamma_0,
\end{aligned}
$$

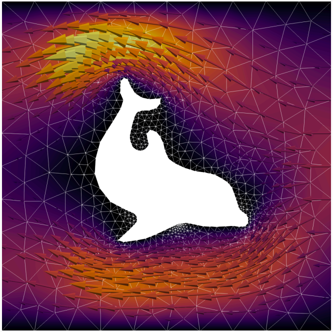
where $\sigma(u) = 2\mu\varepsilon(u) - pI$ is the viscous stress tensor, $\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^\top)$ is the strain tensor (symmetric gradient) and I is the identity matrix.

Exercise 6: Problem data

- Ω defined by the mesh `dolfin_coarse.xml.gz`
- $f = (0, 0)$
- $\Gamma_{\mathrm{L}} = \{(x, y, z) \in \partial\Omega \mid x = 0\}$ (the left boundary)
- $\Gamma_{\mathrm{R}} = \{(x, y, z) \in \partial\Omega \mid x = 1\}$ (the right boundary)
- $\Gamma_0 = \partial\Omega \setminus (\Gamma_{\mathrm{L}} \cup \Gamma_{\mathrm{R}})$
- $p_{\mathrm{L}} = 0$ (outflow)
- $u_{\mathrm{R}} = (-\sin(\pi y), 0)$ (inflow)
- $\mu = 1$ (dynamic viscosity)

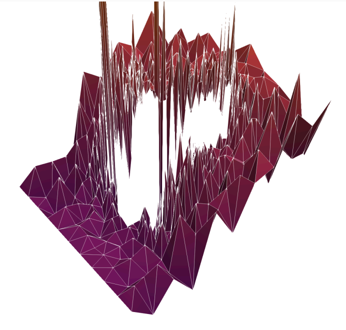
Exercise 6: Velocity solution

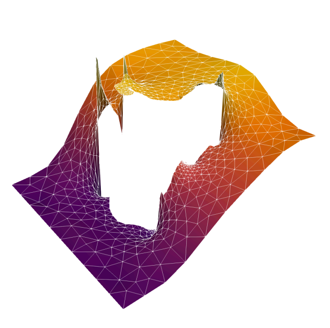




Solution to Exercise 6 plotted in Paraview using the "Glyph" filter. The left figure shows the $P_1 - P_1$ velocity solution and the right figure shows the $P_2 - P_1$ velocity solution.

Exercise 6: Pressure solution





Solution to Exercise 6 plotted in Paraview using the "Warp By Scalar" filter. The left figure shows the $P_1 - P_1$ pressure solution and the right figure shows the $P_2 - P_2$ pressure solution.