Lecture 03

Solving Nonlinear Problems

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What will you learn?

- ***How to solve nonlinear problems***
- *How to formulate the boundary value problem*
- *How to derive the variational problem*
- *How to solve the nonlinear problem*
- *FEniCS programming*
    - Defining nonlinear variational problems
    - Solving nonlinear variational problems
    - Computing the derivative (Jacobian)
    - Setting multiple boundary conditions
- *Exercise*

How to formulate the boundary value problem

Partial differential equation

Abstract nonlinear PDE

$$\mathcal{A}(u) = f \quad \text{in } \Omega$$

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u is the solution to be computed

f is a given source term

Ω is the computational domain

$\mathcal{A}$ is a nonlinear differential operator

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Examples:

$$\mathcal{A}(u) = -\nabla \cdot ((1 + \sin(u))\nabla u)$$
$$\mathcal{A}(u) = -\nabla \cdot ((1 + u^4)\nabla u) - u^2$$

Boundary conditions

Dirichlet boundary condition

$$u = u_{\mathrm{D}} \quad \text{on } \Gamma_{\mathrm{D}} \subseteq \partial\Omega$$

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Neumann boundary condition

$$-\partial_n u = g \quad \text{on } \Gamma_{\mathrm{N}} \subseteq \partial\Omega$$

The Dirichlet condition $u = u_{\mathrm{D}}$ is also called a *strong* boundary condition

The Neumann condition $-\partial_n u = g$ is also called a *natural* boundary condition

How to derive the variational problem

The FEM cookbook (for a nonlinear PDE)

Partial differential equation (strong form)

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Continuous variational problem (weak form)

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$$F(u; v) = 0 \quad \forall v \in V \tag{2}$$

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Discrete variational problem (finite element method)

Find $u_h \in V_h$ such that

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Discrete system of equations (nonlinear system)

$$R(U) = 0 \tag{4}$$

From strong to weak form: (1) $\rightarrow$ (2)

Partial differential equation (strong form)

$$\mathcal{A}(u) - f = 0$$

From strong to weak form: (1) $\to$ (2)

Partial differential equation (strong form)

$$\mathcal{A}(u) - f = 0$$

Multiply by a test function v and integrate over the domain Ω:

$$\underbrace{\int_\Omega (\mathcal{A}(u) - f)\, v \,\mathrm{d}x}_{=F(u;v)} = 0$$

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$\Rightarrow$ Continuous variational problem (weak form)

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From weak form to finite element method: (2) $\to$ (3)

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Let V_h be a discrete finite element subspace of V

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Let V_h be a discrete finite element subspace of V

$\Rightarrow$ Discrete variational problem (finite element method)

Find $u_h \in V_h$ such that

$$F(u_h; v) = L(v) \quad \forall v \in V_h$$

From finite element method to nonlinear system: (3) → (4)

Discrete variational problem (finite element method)

Find $u_h \in V_h$ such that

$$F(u_h; v) = 0 \quad \forall v \in V_h$$

From finite element method to nonlinear system: (3) $\rightarrow$ (4)

Discrete variational problem (finite element method)

Find $u_h \in V_h$ such that

$$F(u_h; v) = 0 \quad \forall v \in V_h$$

Let $\{\phi_j\}_{i=1}^{N}$ be a basis for V_h and make the ansatz

$$u_h(x) = \sum_{j=1}^{N} U_j \phi_j(x)$$

From finite element method to nonlinear system: (3) $\rightarrow$ (4)

Discrete variational problem (finite element method)

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$\Rightarrow$ Discrete system of equations (nonlinear system)

$$R(U) = 0$$

$U \in \mathbb{R}^N$ is the vector of *degrees of freedom*

From finite element method to nonlinear system: details

Insert the ansatz $u_h(x) = \sum_{j=1}^{N} U_j \phi_j(x)$ into the variational problem $F(u_h; v) = 0$ and take $v = \phi_i$ for $i = 1, 2, \ldots, N$:

$$\underbrace{F\Big(\sum_{j=1}^{N} U_j \phi_j; \phi_i\Big)}_{=R_i(U)} = 0, \quad i = 1, 2, \ldots, N$$

From finite element method to nonlinear system: details

Insert the ansatz $u_h(x) = \sum_{j=1}^{N} U_j \phi_j(x)$ into the variational problem $F(u_h; v) = 0$ and take $v = \phi_i$ for $i = 1, 2, \ldots, N$:

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Gives a system of algebraic equations:

$$R(U) = 0$$

where

$$R_i(U) = F\Big(\sum_{j=1}^{N} U_j \phi_j; \phi_i\Big), \quad i = 1, 2, \ldots, N$$

How to solve the nonlinear problem

Newton's method

System of algebraic equations

$$
\begin{aligned}
&R : \mathbb{R}^N \to \mathbb{R}^N \\
&R(U) = 0
\end{aligned}
$$

Newton's method

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$$R : \mathbb{R}^N \to \mathbb{R}^N$$
$$R(U) = 0$$

Newton's method

$$\begin{aligned} U^0 &= \text{initial guess} \\ U^{k+1} &= U^k - (R'(U^k))^{-1} R(U^k), \quad k = 1, 2, \dots \end{aligned}$$

Computing the derivative (Jacobian matrix) R'

$$
\begin{aligned}
R'_{ij} &= \frac{\partial R_i(U)}{\partial U_j} \\
&= \frac{\partial F(u_h;\phi_i)}{\partial U_j} = \frac{\partial F(\sum_{j=1}^{N} U_j\phi_j;\phi_i)}{\partial U_j} \\
&= \frac{\partial F(u_h;\phi_i)}{\partial u}\,\phi_j \\
&= F'(u_h;\phi_i)\,\phi_j = F'(u_h;\phi_j,\phi_i)
\end{aligned}
$$

Computing the derivative (Jacobian matrix) R'

$$\begin{aligned} R'_{ij} &= \frac{\partial R_i(U)}{\partial U_j} \\ &= \frac{\partial F(u_h;\phi_i)}{\partial U_j} = \frac{\partial F(\sum_{j=1}^{N} U_j\phi_j;\phi_i)}{\partial U_j} \\ &= \frac{\partial F(u_h;\phi_i)}{\partial u}\,\phi_j \\ &= F'(u_h;\phi_i)\,\phi_j = F'(u_h;\phi_j,\phi_i) \end{aligned}$$

F and F' are (possibly) nonlinear in u (or u_h)

$F = F(u;v)$ is a linear form in v

$F' = F'(u;\delta u, v)$ is a bilinear form in δu and v

F' is the functional (Fréchet) derivative

Example: A nonlinear Poisson problem

Consider the nonlinear variational problem

$$F(u;v) = \int_\Omega (1 + \sin(u^2))\, \nabla u \cdot \nabla v \,\mathrm{d}x - \int_\Omega f v \,\mathrm{d}x = 0$$

Example: A nonlinear Poisson problem

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$$F(u; v) = \int_\Omega (1 + \sin(u^2))\, \nabla u \cdot \nabla v \,\mathrm{d}x - \int_\Omega f v \,\mathrm{d}x = 0$$

Differentiate wrt u:

$$\begin{aligned} F'(u; v) = &\int_\Omega 2u \cos(u^2) \cdot (\,\cdot\,)\, \nabla u \cdot \nabla v \,\mathrm{d}x \\ &+ \int_\Omega (1 + \sin(u^2))\, \nabla(\,\cdot\,) \cdot \nabla v \,\mathrm{d}x = 0 \end{aligned}$$

Example / contd.

Apply to differential δu:

$$F'(u;\delta u, v) = F'(u;v)\,\delta u = \int_\Omega 2u\cos(u^2)\,\delta u\,\nabla u\cdot\nabla v\,\mathrm{d}x + \int_\Omega (1+\sin(u^2))\,\nabla\delta u\cdot\nabla v\,\mathrm{d}x$$

Example / contd.

Apply to differential δu:

$$F'(u;\delta u, v) = F'(u;v)\,\delta u = \int_\Omega 2u\cos(u^2)\,\delta u\,\nabla u\cdot\nabla v\,\mathrm{d}x + \int_\Omega (1+\sin(u^2))\,\nabla\delta u\cdot\nabla v\,\mathrm{d}x$$

Compute Jacobian matrix:

$$R'_{ij} = F'(u_h;\phi_j,\phi_i) = \int_\Omega 2u_h\cos(u_h^2)\,\phi_j\,\nabla u_h\cdot\nabla\phi_i\,\mathrm{d}x + \int_\Omega (1+\sin(u_h^2))\,\nabla\phi_j\cdot\nabla\phi_i\,\mathrm{d}x$$

FEniCS programming

Defining nonlinear variational problems

Recall the example nonlinear variational problem:

$$F(u; v) = \int_\Omega (1 + \sin(u^2))\, \nabla u \cdot \nabla v \,\mathrm{d}x - \int_\Omega f v \,\mathrm{d}x = 0$$

FEniCS implementation:

```
u = Function(V)
v = TestFunction(V)
f = Expression(...)
F = (1 + sin(u**2))*dot(grad(u), grad(v))*dx - f*v*dx
```

Solving nonlinear problems

Nonlinear problems are solved by calling the `solve()` function:

```
solve(F == 0, u, bc)
```

The `solve()` function automatically computes the derivative F' and solves the nonlinear variational problem using Newton's method

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The derivative $J = F'$ can be given as an optional argument:

```
solve(F == 0, u, bc, J=J)
```

Computing the derivative (Jacobian) $J = F'$

Manual implementation:

```
du = TrialFunction(V)
J = 2*u*cos(u**2)*du*dot(grad(u), grad(v))*dx \
  + (1 + sin(u**2))*dot(grad(du), grad(v))*dx
```

Computing the derivative (Jacobian) $J = F'$

Manual implementation:

```
du = TrialFunction(V)
J = 2*u*cos(u**2)*du*dot(grad(u), grad(v))*dx \
  + (1 + sin(u**2))*dot(grad(du), grad(v))*dx
```

Automatic differentation:

```
du = TrialFunction(V)
J = derivative(F, u, du)
```

Setting multiple boundary conditions

Both `solve(a == L)` and `solve(F == 0)` can set either a single boundary condition `bc` or a list of boundary conditions `bcs`:

```
bc0 = DirichletBC(V, u0, boundary0)
bc1 = DirichletBC(V, u1, boundary1)
bc2 = DirichletBC(V, u2, boundary2)
bcs = [bc0, bc1, bc2]

solve(a == L, u, bcs)
solve(F == 0, u, bcs)
```

Exercise

Exercise 3: Solving Nonlinear Problems

In this exercise, we will solve the following nonlinear PDE:

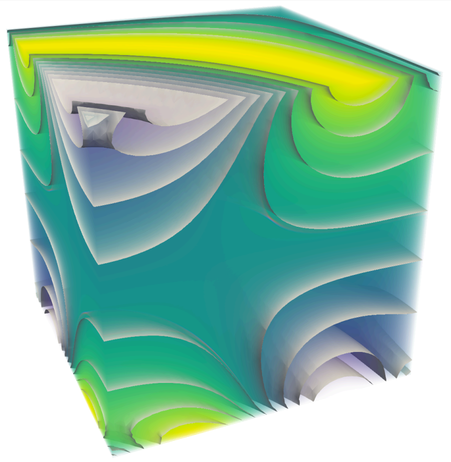
$$
\begin{aligned}
-\nabla \cdot (a(u)\nabla u) &= f(u) && \text{in } \Omega \\
u &= u_{\mathrm{L}} && \text{on } \Gamma_{\mathrm{L}} \\
u &= u_{\mathrm{R}} && \text{on } \Gamma_{\mathrm{R}} \\
-a(u)\partial_n u &= g && \text{on } \Gamma_{\mathrm{N}}
\end{aligned}
$$

Write a FEniCS program to compute and plot the solution, and save the solution to file for visualization in Paraview!

Exercise 3: Problem data

- $\Omega = (0,1)^3$ (the unit cube)
- $a(u) = 1 + u^2$
- $f(u) = u^3$
- $\Gamma_\mathrm{L} = \{(x,y,z) \in \partial\Omega \mid x = 0\}$
- $\Gamma_\mathrm{R} = \{(x,y,z) \in \partial\Omega \mid x = 1\}$
- $\Gamma_\mathrm{N} = \partial\Omega \setminus (\Gamma_\mathrm{L} \cup \Gamma_\mathrm{R})$
- $u_\mathrm{L} = \sin(2\pi yz)$
- $u_\mathrm{R} = \sin(2\pi(1 - yz))$
- $g = 0$

Exercise 3: Solution



Solution to Exercise 3 plotted in Paraview using a volume rendering with the "Viridis" colormap in combination with the "Contour" filter