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1. Introduction

2. Preliminaries

2.1. **Notation.** In the article all algebras, including Lie algebras, are complex. For Lie algebra $\mathfrak{g}$ we will use the notation $U\mathfrak{g}$ to denote its enveloping algebra. We will denote by $\mathfrak{g}$-**mod** and **mod**-$\mathfrak{g}$ the categories of left and right $\mathfrak{g}$-modules respectively. We write $\hat{\mathfrak{g}}$ for the set of set of isomorphism classes of simple finite-dimensional $\mathfrak{g}$-modules and $\mathbb{C}$ for trivial bimodule. For $\mathfrak{g}$-module V can be defined vector spaces

$$V^{\mathfrak{g}} = \{v \in V : g \cdot v = 0 \ \forall g \in \mathfrak{g}\}, \tag{1}$$

called invariants and

$$V_{\mathfrak{g}} = V/gV, \tag{2}$$

called coinvariants. It is known , that $\square^{\mathfrak{g}}$ and $\square_{\mathfrak{g}}$ are actually functors from $\mathfrak{g}$-**mod** ref (or **mod**-$\mathfrak{g}$) to the category of vector spaces over $\mathbb{C}$, isomorphic to $\mathrm{Hom}_{\mathfrak{g}}(\mathbb{C}, V)$ and $\mathbb{C} \otimes_{U\mathfrak{g}} V$ respectively.

2.2. **Functors between categories of modules.** For the rest of this section we will denote by $\mathfrak{g}$ an arbitary finite-dimensional Lie algebra. We define two functors $\square^* \colon \mathfrak{g}\textbf{-mod}^{op} \to \textbf{mod-}\mathfrak{g}$ and $\square^\circ \colon \mathfrak{g}\textbf{-mod} \to \textbf{mod-}\mathfrak{g}$ as follows. The first $\square^*$, called duality functor, sends $\mathfrak{g}$-module V to it's dual vector space, on which the right action of $\mathfrak{g}$ is defined as

$$(f \cdot g)(v) = f(g \cdot v), \quad \text{for all } f \in V^*,\ v \in V,\ g \in \mathfrak{g}.$$

The second $\square^\circ$, called antipode functor, sends V to itself as a vector space with right action

$$v \cdot g = -g \cdot v, \quad \text{for all } v \in V,\ g \in \mathfrak{g}.$$

These two functors define equivalence of categories $\mathfrak{g}$-**mod**, **mod**-$\mathfrak{g}$, $\mathfrak{g}\textbf{-mod}^{op}$ and $\textbf{mod-}\mathfrak{g}^{op}$. We will also denote by $\square^*$ and $\square^\circ$ functors from category of right $\mathfrak{g}$-modules to left $\mathfrak{g}$-modules, defined the same way. It is easy to see, that $(\square^*)^*$ and $(\square^\circ)^\circ$ are naturally isomorphic to the identity functor.

Another pair of very important functors are $\square \otimes_{\mathbb{C}} \square \colon \textbf{mod-}\mathfrak{g} \times \mathfrak{g}\textbf{-mod} \to \mathfrak{g}\textbf{-mod}$ and $\mathrm{Hom}_{\mathbb{C}}(\square, \square) \colon \mathfrak{g}\textbf{-mod}^{op} \times \mathfrak{g}\textbf{-mod} \to \mathfrak{g}\textbf{-mod}$. If $V \in \textbf{mod-}\mathfrak{g}$ and $W \in \mathfrak{g}\textbf{-mod}$, then $V \otimes_{\mathbb{C}} W$ is the tensor product of V and W as vector space with action of $\mathfrak{g}$, fully determined by the formula

$$g \cdot v \otimes w = v \otimes (g \cdot w) - (v \cdot g) \otimes w, \text{ for all } w \in W,\ v \in V,\ g \in \mathfrak{g}$$

. The Hom functor is defined as

$$\mathrm{Hom}_{\mathbb{C}}(V, W) = V^* \otimes_{\mathbb{C}} W.$$

For $V, W \in \mathfrak{g}$**-mod** (resp. **mod**-$\mathfrak{g}$), we will denote by $V \otimes W$ left $\mathfrak{g}$-module $V^\circ \otimes_{\mathbb{C}} W$ (resp. $V \otimes_{\mathbb{C}} W^\circ$).

For $V \in \mathfrak{g}$**-mod** and $S \in \hat{\mathfrak{g}}$, we will write V_S for the $\mathfrak{g}$-module $S \otimes_{\mathbb{C}} V$. If S is one-dimensional, it is fully determined by the character $\lambda \in (\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])$ and in this case we will simply write V_λ for it. For example, $\mathbb{C}_\lambda$ stands for one-dimensional module with action, given by $g \cdot s = \lambda(g)s$ for all $s \in \mathbb{C}_\lambda$ and $g \in \mathfrak{g}$. We will also use the notation V_{-S} for the module $S^* \otimes_{\mathbb{C}} V$, which is motivated by the fact, that if S is again one-dimensional with character λ, then V_{-S} is isomorphic to $V_{-\lambda}$.

2.3. **Homology and cohomology of Lie algebras.** In this paragraph we recall the definitions of Lie algebra cohomology, which can be found in any related textbook .

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Definition 1. *For $V \in \mathfrak{g}$**-mod** and for all $k \in \mathbb{Z}_{\geq 0}$ the homology functors are defined as*

$$H_k(\mathfrak{g}, V) = \mathrm{Tor}_k^{U\mathfrak{g}}(\mathbb{C}, V), \tag{3}$$

and, dually, the cohomology as

$$H^k(\mathfrak{g}, V) = \mathrm{Ext}^k_{U\mathfrak{g}}(\mathbb{C}, V). \tag{4}$$

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The homology can be computed using Chevalley-Eilenberg free resolution of the trivial $\mathfrak{g}$ module $\mathbb{C}$. It has $F_k = U\mathfrak{g} \otimes_{\mathbb{C}} \bigwedge^k \mathfrak{g}$ in degree k with the differential given by

$$d(u \otimes g_1 \wedge \dots \wedge g_p) = \sum_{i=1}^{p} (-1)^{i+1} u g_i \otimes g_1 \wedge \dots \wedge \hat{g}_i \wedge \dots \wedge g_p +$$

$$+ \sum_{i<j} (-1)^{i+j} u \otimes [g_i, g_j] \wedge \dots \wedge \hat{g}_i \dots \wedge \hat{g}_j \dots \wedge g_p, \text{ where } u \in U\mathfrak{g},\ g_i \in \mathfrak{g}. \tag{5}$$

The following fact about homology will be used in the text.

Lemma 1. *Let $V \in$ **mod**-$\mathfrak{g}$ and $W \in \mathfrak{g}$**-mod**. Then*

$$\mathrm{Tor}_k^{U\mathfrak{g}}(V, W) \cong \mathrm{Tor}_k^{U\mathfrak{g}}(\mathbb{C}, V \otimes_{\mathbb{C}} W) = H_k(\mathfrak{g}, V \otimes_{\mathbb{C}} W),$$

for all $k \in \mathbb{Z}_{\geq 0}$.

Proof. Since functors $\mathbb{C} \otimes_{U\mathfrak{g}} (\square \otimes_{\mathbb{C}} W)$ and $\square \otimes_{U\mathfrak{g}} W$ are naturally isomorphic, it suffices to show, that if $P_\bullet \to V$ is a flat resolution of V, then $P_\bullet \otimes_{\mathbb{C}} W$ is a flat resolution of $V \otimes_{\mathbb{C}} W$.

By definition, flatness of P_k means exactness of functor $P_k \otimes_{U\mathfrak{g}} \square$. Using properties of tensor product we obtain an isomorphism of functors $(P_k \otimes_{\mathbb{C}} W) \otimes_{U\mathfrak{g}} \square$ and $P_k \otimes_{U\mathfrak{g}} (W \otimes_{\mathbb{C}} \square)$. The last functor is the composition of exact functors, so it is also exact, hence $P_k \otimes_{\mathbb{C}} W$ is flat. ■

A useful variation of Poincare duality holds for finite dimensional Lie algebras. Let $n = \dim \mathfrak{g}$. We endow one-dimensional vector space $\bigwedge^n \mathfrak{g}$ with structure of left $\mathfrak{g}$-module, which extends adjoint action by Leibnitz rule.

Theorem 2 (Poincare duality)**.** *For $0 \leq k \leq n$ and any left $\mathfrak{g}$-module V, there are vector space isomorphism*

$$H^k(\mathfrak{g}, V^*) \cong H_k(\mathfrak{g}, V)^*,$$

and

$$H^k(\mathfrak{g}, V) \cong H_{n-k}(\mathfrak{g}, (\bigwedge^n \mathfrak{g})^* \otimes_{\mathbb{C}} V),$$

natural in V*. Consequently*

$$H^k(\mathfrak{g}, V^*) \cong H^{n-k}(\mathfrak{g}, \bigwedge^n \mathfrak{g} \otimes_{\mathbb{C}} V)^*.$$

Proof. Theorem 6.10 in ■ cite knapp

3. Taylor spectrum of $\mathfrak{g}$-module

Let $\mathfrak{g}$ be an arbitary Lie algebra and V be a left $\mathfrak{g}$-module. Recall, that $\hat{\mathfrak{g}}$ is the set of isomorphism classes of simple finite dimensional $\mathfrak{g}$-modules.

Definition 2. *The Taylor spectrum of* V *is the subset of* $\hat{\mathfrak{g}}$*, defined as*

$$\sigma_{\mathfrak{g}}(V) = \{S \in \hat{\mathfrak{g}} \mid \exists k \colon \operatorname{Tor}_k^{U\mathfrak{g}}(S^*, V) \neq 0\}.$$

We will simply write $\sigma(V)$ instead of $\sigma_{\mathfrak{g}}(V)$ if it is clear what Lie algebra is considered.

Let us provide several equivalent definitions, in order to use later.

Theorem 3. *For an arbitary* $S \in \mathfrak{g}$*, the following are equivalent:*

(1) $S \in \sigma(V)$*;*
(2) $H_k(\mathfrak{g}, V_{-S}) \neq 0$ *for some* k*;*
(3) $H^k(\mathfrak{g}, \bigwedge^n \mathfrak{g} \otimes_{\mathbb{C}} V_{-S}) \neq 0$ *for some* k*;*
(4) $S^* \otimes_{\mathbb{C}} \bigwedge^n \mathfrak{g} \in \sigma(V^*)$

Proof. $1 \Leftrightarrow 2$ This follows immediatly from Lemma 1.
$2 \Leftrightarrow 3$ Poincare duality (Theorem 2).
$3 \Leftrightarrow 4$ Again, by Poincare duality we have

$$H^k(\mathfrak{g}, \bigwedge^n \mathfrak{g} \otimes_{\mathbb{C}} V_{-S})^* = H^{n-k}(\mathfrak{g}, V^*_{-S}).$$

First observe that V^*_{-S} is isomorphic to $\bigwedge^n \mathfrak{g} \otimes_{\mathbb{C}} V^*_{-S^* \otimes_{\mathbb{C}} \bigwedge^n \mathfrak{g}}$. Using the equivalence of the 1'st and the 3'rd statements we obtain the desired result. ■

In case of abelian Lie algebra $\mathfrak{g}$ the set $\hat{\mathfrak{g}}$, due to the Lie's Theorem, can be identified with the space of characters $\mathfrak{g}^*$. prove that definitions coincide

4. Case of semisimple Lie algebra

In this section $\mathfrak{g}$ denotes finite-dimensional semisimple Lie algebra.

5. Spectrum of one-dimensional extensions

In this section we provide a tool for computing spectrum of some modules by induction.

5.1. **Extensions of Lie algebras.** Let $\mathfrak{h}$ be an arbitary Lie algebra. By one-dimensional extension of $\mathfrak{h}$ we call the exact sequence of Lie algebras

$$0 \to \mathbb{C}_\lambda \to \mathfrak{g} \xrightarrow{\pi} \mathfrak{h} \to 0,$$

where $\mathbb{C}_\lambda$ is one-dimensional $\mathfrak{h}$-module, such the commutator in $\mathfrak{g}$ is given by $[g, c] = \lambda(\pi(g))c$ for all $g \in \mathfrak{g}$ and $c \in \mathbb{C}_\lambda$. The fundamental result of Lie algebra cohomology theory states, that isomorphism classes of such extensions are in one-to-one correspondence with the set $H^2(\mathfrak{g}, \mathbb{C}_\lambda) = \mathrm{Ext}^2_{U\mathfrak{h}}(\mathbb{C}, \mathbb{C}_\lambda)$. Let us recall how to construct the bijection. Let $\xi \in Hom_{\mathbb{C}}(\bigwedge^2 \mathfrak{g}, \mathbb{C}_\lambda)$ be a cocycle. We define Lie bracket on the vector space $\mathfrak{g} = \mathbb{C}_\lambda \oplus \mathfrak{h}$ by

$$[(h_1, v_1), (h_2, v_2)]_\xi = ([h_1, h_2], \lambda(h_1)v_2 - \lambda(h_2)v_1 + \xi(h_1 \wedge h_2)),$$

for all $h_i \in \mathfrak{h}$, $c_i \in \mathbb{C}_\lambda$. Jacobi identity is obtained from definition of cocycle and it can be shown, that cohomologous cocycles induce isomorphic extensions (comp.).

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From now and till the end of the paragraph, we write $\mathfrak{g}$ for one-dimensional extension of $\mathfrak{h}$, represented by cocycle $\xi \in Hom_{\mathbb{C}}(\bigwedge^2 \mathfrak{g}, \mathbb{C}_\lambda)$. Note, that if $S \in \hat{\mathfrak{h}}$ – ireducible $\mathfrak{h}$-module, than S^π is also irreducible as $\mathfrak{g}$-module, thus we will identify $\hat{\mathfrak{h}}$ with subset of $\hat{\mathfrak{g}}$. Assume that we have a $\mathfrak{h}$-module V, and we want to compute it's spectrum $\sigma_{\mathfrak{g}}(V^\pi)$, here V^π is V considered as $\mathfrak{g}$-module via the homomorphism π. The following Theorem gives us some approximations for the desired result.

Theorem 4. *Let $\mathfrak{h}$, $\mathfrak{g}$ and V be as above. The*

(1) $\sigma_{\mathfrak{g}}(V^\pi) \subset \hat{\mathfrak{h}} \subset \hat{\mathfrak{g}}$
(2)

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6. Case of solvable Lie algebra

7. Case of nilpotent Lie algebra

8. Case of borel subalgebra of semisimple Lie algebra