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1. Introduction

2. Preliminaries

2.1. **Notation.** In the article all algebras, including Lie algebras, are complex. $\mathbb{C}_\lambda$
For Lie algebra $\mathfrak{g}$ we will use the notation $U\mathfrak{g}$ to denote its enveloping algebra. We will denote by $\mathfrak{g}$**-mod** and **mod-**$\mathfrak{g}$ the categories of left and right $\mathfrak{g}$-modules respectively. We write $\hat{\mathfrak{g}}$ for the set of set of isomorphism classes of simple finite-dimensional $\mathfrak{g}$-modules and $\mathbb{C}$ for trivial bimodule. For $\mathfrak{g}$-module V can be defined vector spaces

$$V^{\mathfrak{g}} = \{v \in V : g \cdot v = 0 \ \forall g \in \mathfrak{g}\}, \tag{1}$$

called invariants and

$$V_{\mathfrak{g}} = V/gV, \tag{2}$$

called coinvariants. It is known , that $\square^{\mathfrak{g}}$ and $\square_{\mathfrak{g}}$ are actually functors from $\mathfrak{g}$**-mod** ref
(or **mod-**$\mathfrak{g}$) to the category of vector spaces over $\mathbb{C}$, isomorphic to $\mathrm{Hom}_{\mathfrak{g}}(\mathbb{C}, V)$ and $\mathbb{C} \otimes_{U\mathfrak{g}} V$ respectively.

2.2. **Functors between categories of modules.** For the rest of this section we will denote by $\mathfrak{g}$ an arbitary Lie algebra. We define two functors $\square^* \colon$
$\mathfrak{g}$**-mod**^{op} $\to$ **mod-**$\mathfrak{g}$ and $\square^\circ \colon$ $\mathfrak{g}$**-mod** $\to$ **mod-**$\mathfrak{g}$ as follows. The first $\square^*$, called duality functor, sends $\mathfrak{g}$-module V to it's dual vector space, on which the right action of $\mathfrak{g}$ is defined as

$$(f \cdot g)(v) = f(g \cdot v), \quad \text{for all } f \in V^*,\ v \in V,\ g \in \mathfrak{g}.$$

The second $\square^\circ$, called antipode functor, sends V to itself as a vector space with right action

$$v \cdot g = -g \cdot v, \quad \text{for all } v \in V,\ g \in \mathfrak{g}.$$

These two functors define equivallence of categories $\mathfrak{g}$**-mod**, **mod-**$\mathfrak{g}$, $\mathfrak{g}$**-mod**^{op} and **mod-**$\mathfrak{g}^{op}$. We will also denote by $\square^*$ and $\square^\circ$ functors from category of right $\mathfrak{g}$-modules to left $\mathfrak{g}$-modules, defined the same way. It is easy to see, that $(\square^*)^*$ and $(\square^\circ)^\circ$ are naturally isomorphic to the identity functor. For some reasons, that will be clear later, for any $V \in \mathfrak{g}$**-mod** we denote by $-V$ the left $\mathfrak{g}$-module $(V^*)^\circ$.

Another pair of very important functors are $\square \otimes_{\mathbb{C}} \square \colon$ **mod-**$\mathfrak{g} \times \mathfrak{g}$**-mod** $\to \mathfrak{g}$**-mod** and $\mathrm{Hom}_{\mathbb{C}}(\square, \square) \colon$ $\mathfrak{g}$**-mod**$^{op} \times \mathfrak{g}$**-mod** $\to \mathfrak{g}$**-mod**. If $V \in$ **mod-**$\mathfrak{g}$ and $W \in \mathfrak{g}$**-mod**, then $V \otimes_{\mathbb{C}} W$ is the tensor product of V and W as vector space with action of $\mathfrak{g}$, fully determined by the formula

$$g \cdot v \otimes w = v \otimes (g \cdot w) - (v \cdot g) \otimes w, \ \text{for all } w \in W,\ v \in V,\ g \in \mathfrak{g}$$

. The Hom functor is defined as

$$\mathrm{Hom}_{\mathbb{C}}(V, W) = V^* \otimes_{\mathbb{C}} W.$$

For $V, W \in \mathfrak{g}\textbf{-mod}$ (resp. $\textbf{mod-}\mathfrak{g}$), we will denote by $V \otimes W$ left $\mathfrak{g}$-module $V^\circ \otimes_{\mathbb{C}} W$ (resp. $V \otimes_{\mathbb{C}} W^\circ$).

For $V \in \mathfrak{g}\textbf{-mod}$ and $S \in \hat{\mathfrak{g}}$, we will write V_S for the $\mathfrak{g}$-module $S \otimes_{\mathbb{C}} V$. If S is one-dimensional, it is fully determined by the character $\lambda \in (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])$ and in this case we will simply write V_λ for it. For example, $\mathbb{C}_\lambda$ stands for one-dimensional module with action, given by $g \cdot s = \lambda(g)s$ for all $s \in \mathbb{C}_\lambda$ and $g \in \mathfrak{g}$.

2.3. **Homology and cohomology of Lie algebras.** In this paragraph we recall the definitions of Lie algebra cohomology, which can be found in any related textbook .

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Definition 1. *For $V \in \mathfrak{g}\textbf{-mod}$ and for all $i \in \mathbb{Z}_{\geq 0}$ the homology functors are defined as*

$$H_i(\mathfrak{g}, V) = \mathrm{Tor}_i^{U\mathfrak{g}}(\mathbb{C}, V), \tag{3}$$

and, dually, the cohomology as

$$H^i(\mathfrak{g}, V) = \mathrm{Ext}^i_{U\mathfrak{g}}(\mathbb{C}, V). \tag{4}$$

The homology can be computed using Chevalley-Eilenberg projective resolution of

Chevalley-Eilenberg, Poincare duality, Tor(A, B)=Tor(C, AxB)

3. Taylor spectrum of $\mathfrak{g}$-module

Let $\mathfrak{g}$ be an arbitary Lie algebra and E be a left $\mathfrak{g}$-module. We will denote by $\hat{\mathfrak{g}}$ the set of isomorphism classes of simple finite dimensional $\mathfrak{g}$-modules.

Definition 2. *The Taylor spectrum of E is the set, defined as*

$$\sigma(E) = \{V \in \hat{\mathfrak{g}} \mid \exists k \colon \mathrm{Tor}_k^{U\mathfrak{g}}(V^*, E) \neq 0\}.$$

prove it

From it follows, that the definition above coincides with the original Taylor's definition in case of abelian $\mathfrak{g}$.

4. Case of semisimple Lie algebra

5. Spectrum of one-dimensional extensions

6. Case of solvable Lie algebra

7. Case of nilpotent Lie algebra

8. Case of borel subalgebra of semisimple Lie algebra