TAYLOR SPECTRUM FOR MODULES OVER LIE ALGEBRA

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Abstract. J. Taylor [Tay70a] introduced the notion of the joint spectrum of several commuting operators in Banach space and proved the existence of the holomorphic functional calculus in the neighbourhood of the spectrum. The natural question was, if it is possible to generalize the spectrum for tuples of not necceserily commuting operators. A.S. Fainshtein extended the notion of the Taylor spectrum to the tuples of operators, generating nilpotent Lie algebra. A bit later, E. Boasso and A. Larotonda defined it for solvable Lie algebras and proved the projection property. In 2010, A. Dosi proved a version of noncommutative holomorphic functional calculus for nilpotent Lie algebras.

In the paper we generalize the notion of Taylor spectrum to modules over an arbitary Lie algebra and study it for finite-dimensional modules. We show, that in case of nilpotent and semisimple Lie algebras, the spectrum can be described as the set of simple submodules. We also show, that this result does not hold for solvable Lie algebras and give the precise description of the spectrum in case of the Borel subalgebra of semisimple Lie algebra.

Contents

1. Introduction

In 1970, J. Taylor[Tay70a] introduced the notion of the joint spectrum for a finite tuple of commuting operators $T = (T_1, ..., T_n)$, acting on a Banach space V. It was defined as a subset of $\mathbb{C}^n$ (where n is number of operators), which consists of such elements $(\lambda_1, ..., \lambda_n)$, that the Koszul complex for algebra $\mathcal{B}(V)$ of bounded operators and elements $T_1-\lambda_1, ..., T_n-\lambda_n$ is not exact. For one operator, the Taylor spectrum coincides with the classical one. In the same year, Taylor estabilished the existence of the holomorphic functional calculus in the neighbourhood of the spectrum [Tay70b]. Two years later, he proposed a framework for noncommutative functional calculus[Tay72], but the notion of spectrum for non-commuting tuples of operators was not yet developed.

The step in this direction was made by A.S. Fainshtein in his work [Fai93], where he generalized the Taylor spectrum to tuples of operators, generating nilpotent Lie algebra $\mathfrak{g}$. This definiton was soon improved by E. Boasso and A. Larotonda [BL93] to fit also for solvable Lie algebras. Let us recall the definiton, given by them.

Definition 1. *Let $\mathfrak{g}$ be a solvable Lie algebra, and V be a right $\mathfrak{g}$-module. The Taylor spectrum of V is the subset of the space of characters $(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])^*$ of the form*

$$\sigma^{BL}_{\mathfrak{g}}(V) = \{\lambda \in (\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])^* \colon \operatorname{Tor}^{U\mathfrak{g}}_k(V, \mathbb{C}_\lambda) \neq 0 \text{ for some } k \in \mathbb{Z}_{\geq 0}\}, \tag{1}$$

where $\mathbb{C}_\lambda$ denotes a one-dimensional $\mathfrak{g}$-module, on which $\mathfrak{g}$ acts by multiplication on λ.

Further researches showed, that the spectrum has several nice properties, such as projection property and different variations of the spectral mapping theorem. The details can be found in the monograph [BŞ01] and in series of papers by A. Dosi (see for example [Dos01][Dos02][Dos03]). In 2010, A.Dosi proved, that for a special class of Banach modules over nilpotent Lie algebra, there exists some sort of noncommutative holomorphic functional calculus [Dos10].

In this paper, we introduce the notion of the Taylor joint spectrum for modules over an arbitary finite-dimensional Lie algebra $\mathfrak{g}$ and study it for, mostly, finite-dimensional modules. Our main results are Theorems 5, 10 and 13 in which we describe the spectrum for semisimple, nilpotent and Borel subalgebras respectively.

2. Preliminaries

2.1. **Notation.** In the article all algebras, including Lie algebras, are complex. For Lie algebra $\mathfrak{g}$ we will use the notation $U\mathfrak{g}$ to denote its enveloping algebra. We will denote by **$\mathfrak{g}$-mod** and **mod-$\mathfrak{g}$** the categories of left and right $\mathfrak{g}$-modules respectively. We write $\hat{\mathfrak{g}}$ for the set of isomorphism classes of simple finite-dimensional $\mathfrak{g}$-modules and $\mathbb{C}$ for trivial bimodule. For $\mathfrak{g}$-module V one can define vector spaces

$$V^{\mathfrak{g}} = \{v \in V \colon g \cdot v = 0 \ \forall g \in \mathfrak{g}\}, \tag{2}$$

called invariants and

$$V_{\mathfrak{g}} = V/gV, \tag{3}$$

called coinvariants. It is known [Wei94], that $\square^{\mathfrak{g}}$ and $\square_{\mathfrak{g}}$ are actually functors from **$\mathfrak{g}$-mod** (or **mod-$\mathfrak{g}$**) to the category of vector spaces over $\mathbb{C}$, isomorphic to $\operatorname{Hom}_{\mathfrak{g}}(\mathbb{C}, V)$ and $\mathbb{C} \otimes_{U\mathfrak{g}} V$ respectively.

2.2. **Functors between categories of modules.** For the rest of this section we will denote by $\mathfrak{g}$ an arbitary finite-dimensional Lie algebra. We define two functors $\square^* \colon$ **$\mathfrak{g}$-mod**$^{op} \to$ **mod-$\mathfrak{g}$** and $\square^\circ \colon$ **$\mathfrak{g}$-mod** $\to$ **mod-$\mathfrak{g}$** as follows. The first $\square^*$, called duality functor, sends a $\mathfrak{g}$-module V to it's dual vector space, on which the right action of $\mathfrak{g}$ is defined as

$$(f \cdot g)(v) = f(g \cdot v), \quad \text{for all } f \in V^*, \ v \in V, \ g \in \mathfrak{g}.$$

The second $\Box^\circ$, called antipode functor, sends V to itself as a vector space with right action

$$v \cdot g = -g \cdot v, \quad \text{for all } v \in V,\ g \in \mathfrak{g}.$$

These two functors define equivallence of categories $\mathfrak{g}$-**mod**, **mod**-$\mathfrak{g}$, $\mathfrak{g}$-**mod**^{op} and **mod**-$\mathfrak{g}^{op}$. We will also denote by $\Box^*$ and $\Box^\circ$ functors from category of right $\mathfrak{g}$-modules to left $\mathfrak{g}$-modules, defined in the same way. It is easy to see, that $(\Box^*)^*$ and $(\Box^\circ)^\circ$ are naturally isomorphic to the identity functor.

Another very important functor is $\Box \otimes_{\mathbb{C}} \Box$: **mod**-$\mathfrak{g} \times \mathfrak{g}$-**mod** $\to$ $\mathfrak{g}$-**mod**. If $V \in$ **mod**-$\mathfrak{g}$ and $W \in \mathfrak{g}$-**mod**, then $V \otimes_{\mathbb{C}} W$ is the tensor product of V and W as vector space with action of $\mathfrak{g}$, fully determined by the formula

$$g \cdot v \otimes w = v \otimes (g \cdot w) - (v \cdot g) \otimes w, \text{ for all } w \in W,\ v \in V,\ g \in \mathfrak{g}.$$

For $V, W \in \mathfrak{g}$-**mod** (resp. **mod**-$\mathfrak{g}$), we will denote by $V \otimes W$ left $\mathfrak{g}$-module $V^\circ \otimes_{\mathbb{C}} W$ (resp. $V \otimes_{\mathbb{C}} W^\circ$).

For $V \in \mathfrak{g}$-**mod** and $S \in \hat{\mathfrak{g}}$, we will write V_S for the $\mathfrak{g}$-module $S \otimes_{\mathbb{C}} V$. If S is one-dimensional, it is fully determined by the character $\lambda \in (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])$ and in this case we will simply write V_λ for V_S. For example, $\mathbb{C}_\lambda$ stands for one-dimensional module with action, given by $g \cdot s = \lambda(g)s$ for all $s \in \mathbb{C}_\lambda$ and $g \in \mathfrak{g}$. We will also use the notation V_{-S} for the module $S^* \otimes_{\mathbb{C}} V$, which is motivated by the fact, that if S is again one-dimensional with character λ, then V_{-S} is isomorphic to $V_{-\lambda}$.

2.3. **Homology and cohomology of Lie algebras.** In this paragraph we recall the definitions of Lie algebra cohomology, which can be found in any related textbook (for example [Wei94]).

Definition 2. *For $V \in \mathfrak{g}$-**mod** and for all $k \in \mathbb{Z}_{\geq 0}$ the homology functors are defined as*

$$H_k(\mathfrak{g}, V) = \operatorname{Tor}_k^{U\mathfrak{g}}(\mathbb{C}, V), \tag{4}$$

and, dually, the cohomology as

$$H^k(\mathfrak{g}, V) = \operatorname{Ext}^k_{U\mathfrak{g}}(\mathbb{C}, V). \tag{5}$$

The homology can be computed using Chevalley-Eilenberg [Gui80] free resolution of the trivial $\mathfrak{g}$ module $\mathbb{C}$. It has $F_k = U\mathfrak{g} \otimes_{\mathbb{C}} \bigwedge^k \mathfrak{g}$ in degree k with the differential given by

$$d(u \otimes g_1 \wedge \cdots \wedge g_p) = \sum_{i=1}^{p} (-1)^{i+1} u g_i \otimes g_1 \wedge \cdots \wedge \hat{g}_i \wedge \cdots \wedge g_p +$$

$$+ \sum_{i<j} (-1)^{i+j} u \otimes [g_i, g_j] \wedge \cdots \wedge \hat{g}_i \cdots \wedge \hat{g}_j \cdots \wedge g_p, \text{ where } u \in U\mathfrak{g},\ g_i \in \mathfrak{g}. \tag{6}$$

The following fact about homology will be used in the text.

Lemma 1. *Let $V \in$ **mod**-$\mathfrak{g}$ and $W \in \mathfrak{g}$-**mod**. Then*

$$\operatorname{Tor}_k^{U\mathfrak{g}}(V, W) \cong \operatorname{Tor}_k^{U\mathfrak{g}}(\mathbb{C}, V \otimes_{\mathbb{C}} W) = H_k(\mathfrak{g}, V \otimes_{\mathbb{C}} W),$$

for all $k \in \mathbb{Z}_{\geq 0}$.

Proof. Since the functors $\mathbb{C} \otimes_{U\mathfrak{g}} (\Box \otimes_{\mathbb{C}} W)$ and $\Box \otimes_{U\mathfrak{g}} W$ are naturally isomorphic, it suffices to show, that if $P_\bullet \to V$ is a flat resolution of V, then $P_\bullet \otimes_{\mathbb{C}} W$ is a flat resolution of $V \otimes_{\mathbb{C}} W$.

By definition, flatness of P_k means exactness of functor $P_k \otimes_{U\mathfrak{g}} \Box$. Using properties of tensor product we obtain an isomorphism of functors $(P_k \otimes_{\mathbb{C}} W) \otimes_{U\mathfrak{g}} \Box$ and $P_k \otimes_{U\mathfrak{g}} (W \otimes_{\mathbb{C}} \Box)$. The last functor is the composition of exact functors, so it is also exact, hence $P_k \otimes_{\mathbb{C}} W$ is flat. ■

A useful variation of Poincaré duality holds for finite dimensional Lie algebras. Let $n = \dim \mathfrak{g}$. We endow the one-dimensional vector space $\bigwedge^n \mathfrak{g}$ with structure of left $\mathfrak{g}$-module, which extends adjoint action by Leibnitz rule.

Theorem 2 (Poincaré duality)**.** *For $0 \leq k \leq n$ and any left module V over Lie algebra $\mathfrak{g}$ of dimension n, there are vector space isomorphism*

$$H^k(\mathfrak{g}, V^*) \cong H_k(\mathfrak{g}, V)^*,$$

and

$$H^k(\mathfrak{g}, V) \cong H_{n-k}(\mathfrak{g}, (\bigwedge^n \mathfrak{g})^* \otimes_{\mathbb{C}} V),$$

natural in V. Consequently

$$H^k(\mathfrak{g}, V^*) \cong H^{n-k}(\mathfrak{g}, \bigwedge^n \mathfrak{g} \otimes_{\mathbb{C}} V)^*.$$

Proof. Theorem 6.10 in [Kna88]. ■

3. Taylor spectrum of $\mathfrak{g}$-module

Let $\mathfrak{g}$ be an arbitary Lie algebra of dimension n and V be a left $\mathfrak{g}$-module. Recall, that $\hat{\mathfrak{g}}$ is the set of isomorphism classes of simple finite dimensional $\mathfrak{g}$-modules.

Definition 3. *The Taylor spectrum of V is the subset of $\hat{\mathfrak{g}}$, defined as*

$$\sigma_{\mathfrak{g}}(V) = \{S \in \hat{\mathfrak{g}} \mid \exists k \colon \operatorname{Tor}_k^{U\mathfrak{g}}(S^*, V) \neq 0\}.$$

We will simply write $\sigma(V)$ instead of $\sigma_{\mathfrak{g}}(V)$ if it is clear what Lie algebra is considered. Till the and of the article *the spectrum* stands for *the Taylor spectrum.*

Let us provide several equivalent definitions, in order to use later.

Lemma 3. *For an arbitary $S \in \mathfrak{g}$, the following are equivalent:*

1. $S \in \sigma(V)$;
2. $H_k(\mathfrak{g}, V_{-S}) \neq 0$ *for some* k;
3. $H^k(\mathfrak{g}, \bigwedge^n \mathfrak{g} \otimes_{\mathbb{C}} V_{-S}) \neq 0$ *for some* k;
4. $S^* \otimes_{\mathbb{C}} \bigwedge^n \mathfrak{g} \in \sigma(V^*)$

Proof. $1 \Leftrightarrow 2$ This follows immediatly from Lemma 1.

$2 \Leftrightarrow 3$ Poincaré duality (Theorem 2).

$3 \Leftrightarrow 4$ Again, by Poincaré duality we have

$$H^k(\mathfrak{g}, \bigwedge^n \mathfrak{g} \otimes_{\mathbb{C}} V_{-S})^* = H^{n-k}(\mathfrak{g}, V^*_{-S}).$$

First observe that V^*_{-S} is isomorphic to $\bigwedge^n \mathfrak{g} \otimes_{\mathbb{C}} V^*_{-S^* \otimes_{\mathbb{C}} \bigwedge^n \mathfrak{g}}$. Using the equivalence of the 1'st and the 3'rd statements we obtain the desired result. ■

In case of solvable Lie algebra $\mathfrak{g}$ the set $\hat{\mathfrak{g}}$, due to the Lie's Theorem, can be identified with the space of characters $(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])^*$. If V is left $\mathfrak{g}$-module, then the defined spectrum has the following relation, with the spectrum, given in (1):

$$\sigma_{\mathfrak{g}}(V) = -\sigma_{\mathfrak{g}}^{BL}(V^\circ),$$

so we may think about $\sigma_{\mathfrak{g}}$ as a generilization of $\sigma_{\mathfrak{g}}^{BL}$.

4. Case of semisimple Lie algebras

In this section $\mathfrak{g}$ denotes semisimple Lie algebra of dimension n. Note, that $\bigwedge^n \mathfrak{g}$ is isomorphic to trivial $\mathfrak{g}$-module, therefore, by Lemma 3, the Taylor spectrum of $\mathfrak{g}$-module V can be described as the set

$$\sigma_{\mathfrak{g}}(V) = \{S \in \hat{\mathfrak{g}} \colon H^k(\mathfrak{g}, V_{-S}) \neq 0 \text{ for some } k\}.$$

Let us recall the Theorem 7.8.9 from [Wei94], that we will use to describe the spectrum of $\mathfrak{g}$-modules.

Theorem 4. *If S is a simple module over the semisimple Lie algebra $\mathfrak{g}$, $S \neq \mathbb{C}$, then*

$$H^k(\mathfrak{g}, S) = 0 \textit{ for all } k.$$

If the module is trivial, then, obviously, $H^k(\mathfrak{g}, \mathbb{C}) \cong \mathbb{C}$. Now we are ready to proove the main result of the section.

Theorem 5. *For any finite-dimensional $\mathfrak{g}$ module V, its Taylor spectrum coincides with the set of simple components of V. Moreover the dimension of $H^0(\mathfrak{g}, V_{-S})$ is the number of copies of S in the decomposition of V.*

Proof. As the cohomology commute with finite sums it suffices to show, that the assertion of Theorem holds for simple V. By Theorem 4, we only need to show that $\mathbb{C}$ occurs in decomposition of V_{-S} if and only if $V \cong S$. But V_{-S} is $\mathrm{Hom}_{\mathbb{C}}(S, V)$ and $(V_{-S})^{\mathfrak{g}}$ is $\mathrm{Hom}_{\mathfrak{g}}(S, V)$, so it is one-dimensional if $S \cong V$ and zero otherwise. ■

In fact, the statement of the theorem above holds even for Banach $\mathfrak{g}$-modules. In order to prove this, we need the following fact, which can be found in the book [BŞ01].

Theorem 6. *Any Banach module V over semisimple Lie algebra $\mathfrak{g}$ is the union of its finite-dimensional submodules.*

Proof. Corollary 5 in §30 of [BŞ01]. ■

In other words, V is the colimit of the filtered diagram $\mathfrak{V}$ of its finite-dimensional submodules with inclusions as morphisms. Recall, that homology and the tensor product commute with filtered colimits. Thus, we can strengthen the Theorem 5.

Corollary 6.1. *For any Banach $\mathfrak{g}$ module V, its Taylor spectrum coincides with the set of simple submodules of V.*

Proof. As all the maps in $\mathfrak{V}$ are inclusions, so are maps in $H_*(\mathfrak{g}, \mathfrak{V}_{-S})$, hence $H_*(\mathfrak{g}, V_{-S})$ is nonzero if and only if $S \in \sigma(W)$ for some $W \in \mathfrak{V}$. ■

5. Spectrum of one-dimensional extensions

In this section we provide a tool for computing spectrum of some modules by induction.

5.1. **Extensions of Lie algebras.** Let $\mathfrak{h}$ be an arbitary Lie algebra. By one-dimensional extension of $\mathfrak{h}$ we mean the exact sequence of Lie algebras

$$0 \to \mathbb{C}_\lambda \to \mathfrak{g} \xrightarrow{\pi} \mathfrak{h} \to 0,$$

where $\mathbb{C}_\lambda$ is one-dimensional $\mathfrak{h}$-module, such the commutator in $\mathfrak{g}$ is given by $[g, c] = \lambda(\pi(g))c$ for all $g \in \mathfrak{g}$ and $c \in \mathbb{C}_\lambda$. The fundamental result of Lie algebra cohomology theory states that isomorphism classes of such extensions are in one-to-one correspondence with the set $H^2(\mathfrak{g}, \mathbb{C}_\lambda) = \mathrm{Ext}^2_{U\mathfrak{h}}(\mathbb{C}, \mathbb{C}_\lambda)$. Let us recall how to construct the bijection. Let $\xi \in Hom_{\mathbb{C}}(\bigwedge^2 \mathfrak{g}, \mathbb{C}_\lambda)$ be a cocycle. We define Lie bracket on the vector space $\mathfrak{g} = \mathbb{C}_\lambda \oplus \mathfrak{h}$ by

$$[(h_1, v_1), (h_2, v_2)]_\xi = ([h_1, h_2], \lambda(h_1)v_2 - \lambda(h_2)v_1 + \xi(h_1 \wedge h_2)),$$

for all $h_i \in \mathfrak{h},\ c_i \in \mathbb{C}_\lambda$. Jacobi identity is obtained from definition of a cocycle and it can be shown, that cohomologous cocycles induce isomorphic extensions (comp. [Wei94]).

From now on and untill the end of the paragraph, we write $\mathfrak{g}$ for a one-dimensional extension of $\mathfrak{h}$, represented by cocycle $\xi \in Hom_{\mathbb{C}}(\bigwedge^2 \mathfrak{g}, \mathbb{C}_\lambda)$. We also use the notation $\mathfrak{c}$ for an ideal $\mathbb{C}_\lambda \subset \mathfrak{g}$. Note, that if $S \in \hat{\mathfrak{h}}$ – ireducible $\mathfrak{h}$-module, than S^π is also irreducible as $\mathfrak{g}$-module, thus we will identify $\hat{\mathfrak{h}}$ with subset of $\hat{\mathfrak{g}}$. Assume that we have a $\mathfrak{h}$-module V, and we want to compute it's spectrum $\sigma_{\mathfrak{g}}(V^\pi)$, here V^π is V considered as $\mathfrak{g}$-module via the homomorphism π. The following Theorem gives us some approximations for the desired result.

Theorem 7. *Let $\mathfrak{h}$, $\mathfrak{g}$ and V be as above. Then:*

1. $\sigma_{\mathfrak{g}}(V^\pi) \subset \hat{\mathfrak{h}} \subset \hat{\mathfrak{g}}$;
2. *For any $S \in \hat{\mathfrak{h}}$, ξ induces the maps $\xi^*_k \colon H_k(\mathfrak{h}, V_{-S}) \to H_{k-2}(\mathfrak{h}, V_{-S} \otimes_{\mathbb{C}} \mathbb{C}_\lambda)$ for $2 \le k \le n$. Moreover, $S \in \sigma_{\mathfrak{g}}(V^\pi)$ if and only if ξ^*_k is not an isomoprhism for some k;*
3. *If $S \in \sigma_{\mathfrak{g}}(V^\pi)$, then either $S \in \sigma_{\mathfrak{h}}(V)$ or $S_{-\lambda} \in \sigma_{\mathfrak{h}}(V)$.*

Proof. Let $S \in \hat{\mathfrak{g}} \setminus \hat{\mathfrak{h}}$. It is easy to verify that $\mathfrak{c}S = \{c \cdot s \colon s \in S\, c \in \mathfrak{c}\}$ is a submodule of S. Since S is ireducible it is either 0 or S. But if it is 0, then S is actually an $\mathfrak{h}$ module and it contradicts our assumption, so $\mathfrak{c}S = S$ and $S_{\mathfrak{c}} = 0$. The same argument is used to show that the set $S^{\mathfrak{c}}$ equals to 0. The Hochsild-Serre spectral sequence $E^2_{p,q} = H_p(\mathfrak{h}, H_q(\mathfrak{c}, V_{-S}))$ convergers to $H_k(\mathfrak{g}, V_{-S})$, so it suffices to prove that $H_q(\mathfrak{c}, V_{-S}) = 0$. The only possible nonzero homology groups are $H_0(\mathfrak{c}, V_{-S}) = (V_{-S})_{\mathfrak{c}} \cong (S^*)_{\mathfrak{c}}.\otimes_{\mathbb{C}} V = 0$ and $H_1(\mathfrak{c}, V_{-S}) = (V_{-S})^{\mathfrak{c}} \cong (S^*)^{\mathfrak{c}} \otimes_{\mathbb{C}} V = 0$. Therefore $S \notin \sigma_{\mathfrak{g}}(V^\pi)$ by Lemma 3, which gives us the first assertion.

Now let $S \in \hat{\mathfrak{h}}$. Then, as $\mathfrak{h}$ modules, the homology groups $H_0(\mathfrak{c}, V_{-S})$ and $H_1(\mathfrak{c}, V_{-S})$ are isomorphic to V_{-S} and $V \otimes_{\mathbb{C}} \mathbb{C}_\lambda$ respectively. The Hochsild-Serre spectral sequence stabilizes on the third page and the nonzero differntials in E^2 are $d_k \colon H_k(\mathfrak{h}, V_{-S}) \to H_{k-2}(\mathfrak{h}, V_{-S} \otimes_{\mathbb{C}} \mathbb{C}_\lambda)$. These differentials can be described as the Yoneda product on the class $[\xi] \in \mathrm{Ext}_2(\mathbb{C}, \mathbb{C}_\lambda)$ and the spectral sequence collapses if and only if all the differentials are isomorphisms of vector spaces.

The last assertion follows from the fact that if d_k is not an isomoprhism, then either $H_{k-2}(\mathfrak{h}, V_{-S} \otimes_{\mathbb{C}} \mathbb{C}_\lambda)$ or $H_k(\mathfrak{h}, V_{-S})$ is nonzero. ■

In case of central extensions we can strengthen the theorem.

Corollary 7.1. *If $\mathfrak{g}$ is one-dimensional central extension of $\mathfrak{h}$ and V is $\mathfrak{h}$-module, then the Taylor spectrum of V, regarded as $\mathfrak{g}$-module, is equal to the spectrum of V, regarded as $\mathfrak{h}$-module:*

$$\sigma_{\mathfrak{g}}(V^\pi) = \sigma_{\mathfrak{h}}(V).$$

Proof. By Theorem 7 we already know, that $\sigma_{\mathfrak{g}}(V^\pi) \subset \sigma_{\mathfrak{h}}(V)$. On the other hand, for any $S \in \sigma_{\mathfrak{h}}(V)$ denote by k_0 the index of the first non-zero homology group $H_*(\mathfrak{h}, V_{-S})$. Then, the map $\xi^*_{k_0}$, defined in the Theorem, has $H_{k_0-2}(\mathfrak{h}, V_{-S}) = 0$ in its' image, and hence is not an isomorphism. Applying the second statement of the Theorem 7, we deduce the assertion. ∎

This statement will help us a lot in studying the spectrum of nilpotent Lie algebras, because they are always can be presented as a sequence of central extensions of an abelian Lie algebra.

6. Case of solvable Lie algebras

6.1. **The spectrum of the trivial module.** In this section $\mathfrak{g}$ will denote an arbitary solvable Lie algebra of dimension n. Due to Lie's theorem, every simple $\mathfrak{g}$-module is one-dimensional, so we will identify $\hat{\mathfrak{g}}$ with the space of characters $(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$. In this case, computing Taylor spectrum is a hard job even for the trivial module $\mathbb{C}$. For a finite-dimensional $\mathfrak{g}$-module V by the set of weights $\omega(V) \subset \hat{\mathfrak{g}}$ we mean the set of diagonal matrix entries in triangular basis for V. It is independant on the choice of upper triangular basis and can be also described as the set of one-dimensional subfactors of V. Consider the adjoint representation $\operatorname{ad}\mathfrak{g} \in \mathfrak{g}\text{-}\mathbf{mod}$. The set of weights $\omega(\operatorname{ad}\mathfrak{g})$ is called Jordan-Hölder values of $\mathfrak{g}$. We denote by 2ρ the sum of all Jordan-Hölder values with multiplicites. It is exactly the weight of the module $\bigwedge^n \mathfrak{g}$, defined in section 2. The following restriction on possible elements in the spectrum of the trivial module holds.

Theorem 8. *For any solvable Lie algebra $\mathfrak{g}$ of dimension n, if $\lambda \in \hat{\mathfrak{g}}$ is in the spectrum $\sigma_{\mathfrak{g}}(\mathbb{C})$, then it is the sum of at most n Jordan-Hölder values of $\mathfrak{g}$. Moreover, if $\lambda \in \sigma_{\mathfrak{g}}(\mathbb{C})$, then $2\rho - \lambda$ is also in $\sigma_{\mathfrak{g}}(\mathbb{C})$.*

Proof. The proof is by induction on n. Assertion of the Theorem, obviously, holds for one-dimensional Lie Algebra, so we have $n = 1$ as the base of induction. Assume, that it holds for all solvable Lie algebras of dimension $n-1$. Choose any one-dimensional ideal $\mathfrak{c}$ in $\mathfrak{g}$ and denote the corresponding character by μ. By the third statement of the Theorem 7 we know, that the spectrum $\sigma_{\mathfrak{g}}(\mathbb{C})$ lies in the set $\{0, \mu\} + \sigma_{\mathfrak{g}/\mathfrak{c}}(\mathbb{C})$. By induction, any $\nu \in \sigma_{\mathfrak{g}/\mathfrak{c}}(\mathbb{C})$ is the sum of at most $n-1$ Jordan-Hölder values of $\mathfrak{g}/\mathfrak{c}$, which are also Jordan-Hölder values of $\mathfrak{g}$. This is the desired conclusion.

For the second assertion we use Lemma 3. The trivial module is isomorphic to its dual, so if $\mathbb{C}_\lambda$ is in the spectrum of $\mathbb{C}$, then $\mathbb{C}^*_\lambda \otimes \bigwedge^n \mathfrak{g} \cong_{\mathbb{C}} \mathbb{C}_{2\rho-\lambda}$ is also in the spectrum. ∎

The first part of the theorem was originally obtained by S. Wadsley in the converstion on the mathoverflow website.

Obviously, 0 is always in the spectrum and, hence, so is 2ρ. So, generally, there is often more than one element in the spectrum of $\mathbb{C}$.

Example 1. *Let $\mathfrak{g}$ be a 3-dimensional solvable Lie algebra with basis e_1, e_2, e_3 and the commutator, given by $[e_1, e_2] = e_2$ and $[e_1, e_3] = \lambda e_3$ for some $\lambda \in \mathbb{C}$. The space of characters is one dimensional, so we identify it with $\mathbb{C}$ by evaluation on e_1. Then $\sigma_{\mathfrak{g}}(\mathbb{C}) = \{0, 1, \lambda, 1 + \lambda\}$. Indeed, 0 and $2\rho = 1 + \lambda$ is always in the spectrum, and for 1 and λ the first homology groups are non-vanishing.*

Nevertheless, for some nice classes of solvable Lie algebras the situation is more clear, than in general case. One of such classes is nilpotent Lie algebras.

6.2. **The spectrum of finite-dimensional modules over nilpotent Lie algebras.** The following well-known fact in representation theory for nilpotent Lie algebras plays major role in this paragraph.

Lemma 9. *For V – a finite-dimensional indecomposable module over nilpotent Lie algebra $\mathfrak{g}$, $\omega(V)$ consists of one element.*

Proof. Proposition 9 in chapter VII of [Bou05]. ■

We call modules with only one weight monoweighted. The last Lemma show us, that any finite-dimensional module over nilpotent Lie algebra can be decomposed in the sum of monoweighted submodules. Now we are ready to formulate the main result.

Theorem 10. *Let $\mathfrak{g}$ be a nilpotent Lie algebra and V – finite-dimensional $\mathfrak{g}$-module. Then the spectrum $\sigma(V)$ coincides with the set of weights $\omega(V)$.*

Proof. As in the proof of Theorem 5 we may assume V is monoweighted. The spectrum of module V_λ is $\sigma(V) + \lambda$, so we may additionally assume, that $\omega(V) = \{0\}$. Observe, that by Engel's theorem, all Jordan-Hölder values of $\mathfrak{g}$ are zero, so by Theorem 8, we conclude that the assertion holds for trivial V. We now proceed by induction on $m = \dim V$. Choose some one-dimensional submodule in V. We have a short exact sequence

$$0 \to \mathbb{C} \to V \to V/\mathbb{C} \to 0.$$

It follows easily from long exact sequence of cohomologies, that $\sigma(V) \subset \sigma(\mathbb{C}) \cup \sigma(V/\mathbb{C})$ and the latest is equal $\{0\}$ by induction. On the other hand 0 is in the spectrum of V, because $H_0(\mathfrak{g}, V) = V_{\mathfrak{g}} \neq 0$. This finishes the proof. ■

6.3. **Case of borel subalgebras of semisimple Lie algebras.** Let $\mathfrak{s}$ be a semisimple Lie algeba and $\mathfrak{g}$ its Borel subalgebra. It is known [Hum78], that $\mathfrak{g}$ is isomorphic to semidirect product $\mathfrak{h} \ltimes \mathfrak{n}$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{s}$ and $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$. By $\Delta \subset \mathfrak{h}^* = \hat{\mathfrak{g}}$ we will denote the root system of $\mathfrak{s}$ relative to $\mathfrak{h}$ and $\mathfrak{g}$. We write $\Delta^+ \subset \Delta$ for denote the subset of positive roots. In fact, elements of Δ^+ are nonzero Jordan-Hölder values of $\mathfrak{g}$. Let $W(\Delta)$ denote the Weil group. For any element $w \in W$ its length is denoted by $l(w)$.

The aim of the paragraph is to describe the spectrum of irreducible $\mathfrak{s}$ modules, regarded as $\mathfrak{g}$ modules. The plan is to compute relveant cohomologies of $\mathfrak{n}$ and then take advantage of the Hochshild-Serre spectral sequence. Fortunately, these two steps were alredy done by different authors and we only need to glue them together. The first one is Kostant Theorem.

Theorem 11 (Kostant)**.** *Let V be irreducible representation of semisiple Lie algebra $\mathfrak{s}$ with highest weight λ, relative to borel subalgebra $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{n}$. Then, as $\mathfrak{h}$-module, $H^k(\mathfrak{n}, V)$ is the sum of one-dimensional modules of weights*

$$w(\lambda + \rho) - \rho, \ \in W(\Delta), \ l(w) = k,$$

where ρ is half-sum of positive roots (or, equivalently, Jordan-Hölder values).

Proof. Theorem 6.12 in [Kna88]. ∎

We say, that an abelian Lie algebra $\mathfrak{h}$ acts torally on $\mathfrak{h}$-module W if W is direct sum of one-dimensional submodules. When $\mathfrak{h}$ is a Cartan subalgebra of semisiple Lie algebra $\mathfrak{s}$, $\mathfrak{h}$ acts torally on any finite dimensional $\mathfrak{s}$-module. If $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{n}$ – the Borel subalgebra of $\mathfrak{s}$, then $\mathfrak{h}$ also acts torally on $\mathfrak{n}$. The following Theorem guarantees convergence of the Hochsild-Serre spectral sequence in the special case of semidirect products.

Theorem 12. *Let $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{n}$ be a semidirect product where $\mathfrak{h}$ is abelian and acts torally both on $\mathfrak{g}$ and a $\mathfrak{g}$-module V. Then with respect to the weighting induced by the action of $\mathfrak{h}$ we have*

$$H^k(\mathfrak{g}, V) = \bigoplus_{p+q=k} \bigwedge^p \mathfrak{h}^* \otimes_{\mathbb{C}} H^q(\mathfrak{n}, V)^{\mathfrak{h}} \tag{7}$$

Proof. Theorem 4 in [CG16]. ∎

Let $\mu \in \mathfrak{h}^*$ a weight. For any $\mathfrak{g}$-module V, one may observe, that $H^k(\mathfrak{n}, V_{-\mu}) \cong H^k(\mathfrak{n}, V)_{-\mu}$ for all k. Hence, if V is irreducible $\mathfrak{s}$-module of highest weight λ, then $H^q(\mathfrak{n}, V_{-\mu})^{\mathfrak{h}}$ from (7) is nonzero for some q if and only if $\mu = w(\lambda + \rho) - \rho$. Combining it with Lemma 3 we obtain the main result.

Theorem 13. *For an irreducible module V of highest weight λ over the semisimple Lie algebra $\mathfrak{s}$, the Taylor spectrum of V, regarded as the module over Borel subalgebra $\mathfrak{g} \subset \mathfrak{s}$ has the form*

$$\sigma_{\mathfrak{g}}(V) = \{\rho + w(\lambda + \rho) \colon w \in W(\Delta)\}$$

Proof. Indeed, by Lemma 3, $\nu \in \sigma_{\mathfrak{g}}(V)$ if and only if $H^k(\mathfrak{g}, \bigwedge^n \mathfrak{g} \otimes_{\mathbb{C}} V_{-\nu})$ are nonzero for some k. As stated in the begining of the section, $\bigwedge^n \mathfrak{g}$ is isomorphic to $\mathbb{C}_{2\rho}$, so $\bigwedge^n \mathfrak{g} \otimes_{\mathbb{C}} V_{-\nu} \cong V_{2\rho-\nu}$. As shown above, the cohomology groups $H^k(\mathfrak{g}, V_{2\rho-\nu})$ are nonzero exactly when $2\rho - \nu = \rho - w(\lambda + \rho)$. ∎

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