

TAYLOR SPECTRUM

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1. INTRODUCTION

2. PRELIMINARIES

2.1. Notation. In the article all algebras, including Lie algebras, are complex. For Lie algebra \mathfrak{g} we will use the notation $U\mathfrak{g}$ to denote its enveloping algebra. We will denote by $\mathfrak{g}\text{-mod}$ and $\text{mod-}\mathfrak{g}$ the categories of left and right \mathfrak{g} -modules respectively. We write $\hat{\mathfrak{g}}$ for the set of set of isomorphism classes of simple finite-dimensional \mathfrak{g} -modules and \mathbb{C} for trivial bimodule. For \mathfrak{g} -module V can be defined vector spaces

$$(1) \quad V^{\mathfrak{g}} = \{v \in V : g \cdot v = 0 \ \forall g \in \mathfrak{g}\},$$

called invariants and

$$(2) \quad V_{\mathfrak{g}} = V/gV,$$

called coinvariants. It is known, that $\square^{\mathfrak{g}}$ and $\square_{\mathfrak{g}}$ are actually functors from $\mathfrak{g}\text{-mod}$ (or $\text{mod-}\mathfrak{g}$) to the category of vector spaces over \mathbb{C} , isomorphic to $\text{Hom}_{\mathfrak{g}}(\mathbb{C}, V)$ and $\mathbb{C} \otimes_{U\mathfrak{g}} V$ respectively.

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2.2. Functors between categories of modules. For the rest of this section we will denote by \mathfrak{g} an arbitrary finite-dimensional Lie algebra. We define two functors $\square^*: \mathfrak{g}\text{-mod}^{op} \rightarrow \text{mod-}\mathfrak{g}$ and $\square^{\circ}: \mathfrak{g}\text{-mod} \rightarrow \text{mod-}\mathfrak{g}$ as follows. The first \square^* , called duality functor, sends \mathfrak{g} -module V to it's dual vector space, on which the right action of \mathfrak{g} is defined as

$$(f \cdot g)(v) = f(g \cdot v), \text{ for all } f \in V^*, v \in V, g \in \mathfrak{g}.$$

The second \square° , called antipode functor, sends V to itself as a vector space with right action

$$v \cdot g = -g \cdot v, \text{ for all } v \in V, g \in \mathfrak{g}.$$

These two functors define equivalence of categories $\mathfrak{g}\text{-mod}$, $\text{mod-}\mathfrak{g}$, $\mathfrak{g}\text{-mod}^{op}$ and $\text{mod-}\mathfrak{g}^{op}$. We will also denote by \square^* and \square° functors from category of right \mathfrak{g} -modules to left \mathfrak{g} -modules, defined the same way. It is easy to see, that $(\square^*)^*$ and $(\square^{\circ})^{\circ}$ are naturally isomorphic to the identity functor.

Another pair of very important functors are $\square \otimes_{\mathbb{C}} \square: \text{mod-}\mathfrak{g} \times \mathfrak{g}\text{-mod} \rightarrow \mathfrak{g}\text{-mod}$ and $\text{Hom}_{\mathbb{C}}(\square, \square): \mathfrak{g}\text{-mod}^{op} \times \mathfrak{g}\text{-mod} \rightarrow \mathfrak{g}\text{-mod}$. If $V \in \text{mod-}\mathfrak{g}$ and $W \in \mathfrak{g}\text{-mod}$, then $V \otimes_{\mathbb{C}} W$ is the tensor product of V and W as vector space with action of \mathfrak{g} , fully determined by the formula

$$g \cdot v \otimes w = v \otimes (g \cdot w) - (v \cdot g) \otimes w, \text{ for all } w \in W, v \in V, g \in \mathfrak{g}$$

. The Hom functor is defined as

$$\text{Hom}_{\mathbb{C}}(V, W) = V^* \otimes_{\mathbb{C}} W.$$

For $V, W \in \mathfrak{g}\text{-mod}$ (resp. $\mathbf{mod}\text{-}\mathfrak{g}$), we will denote by $V \otimes W$ left \mathfrak{g} -module $V^\circ \otimes_{\mathbb{C}} W$ (resp. $V \otimes_{\mathbb{C}} W^\circ$).

For $V \in \mathfrak{g}\text{-mod}$ and $S \in \hat{\mathfrak{g}}$, we will write V_S for the \mathfrak{g} -module $S \otimes_{\mathbb{C}} V$. If S is one-dimensional, it is fully determined by the character $\lambda \in (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])$ and in this case we will simply write V_λ for it. For example, \mathbb{C}_λ stands for one-dimensional module with action, given by $g \cdot s = \lambda(g)s$ for all $s \in \mathbb{C}_\lambda$ and $g \in \mathfrak{g}$. We will also use the notation V_{-S} for the module $S^* \otimes_{\mathbb{C}} V$, which is motivated by the fact, that if S is again one-dimensional with character λ , then V_{-S} is isomorphic to $V_{-\lambda}$.

2.3. Homology and cohomology of Lie algebras. In this paragraph we recall the definitions of Lie algebra cohomology, which can be found in any related textbook.

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Definition 1. For $V \in \mathfrak{g}\text{-mod}$ and for all $k \in \mathbb{Z}_{\geq 0}$ the homology functors are defined as

$$(3) \quad H_k(\mathfrak{g}, V) = \mathrm{Tor}_k^{U\mathfrak{g}}(\mathbb{C}, V),$$

and, dually, the cohomology as

$$(4) \quad H^k(\mathfrak{g}, V) = \mathrm{Ext}_{U\mathfrak{g}}^k(\mathbb{C}, V).$$

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The homology can be computed using Chevalley-Eilenberg free resolution of the trivial \mathfrak{g} module \mathbb{C} . It has $F_k = U\mathfrak{g} \otimes_{\mathbb{C}} \bigwedge^k \mathfrak{g}$ in degree k with the differential given by

$$(5) \quad d(u \otimes g_1 \wedge \cdots \wedge g_p) = \sum_{i=1}^p (-1)^{i+1} u g_i \otimes g_1 \wedge \cdots \wedge \hat{g}_i \wedge \cdots \wedge g_p + \\ + \sum_{i < j} (-1)^{i+j} u \otimes [g_i, g_j] \wedge \cdots \wedge \hat{g}_i \cdots \wedge \hat{g}_j \cdots \wedge g_p, \text{ where } u \in U\mathfrak{g}, g_i \in \mathfrak{g}.$$

The following fact about homology will be used in the text.

Lemma 1. Let $V \in \mathbf{mod}\text{-}\mathfrak{g}$ and $W \in \mathfrak{g}\text{-mod}$. Then

$$\mathrm{Tor}_k^{U\mathfrak{g}}(V, W) \cong \mathrm{Tor}_k^{U\mathfrak{g}}(\mathbb{C}, V \otimes_{\mathbb{C}} W) = H_k(\mathfrak{g}, V \otimes_{\mathbb{C}} W),$$

for all $k \in \mathbb{Z}_{\geq 0}$.

Proof. Since functors $\mathbb{C} \otimes_{U\mathfrak{g}} (\square \otimes_{\mathbb{C}} W)$ and $\square \otimes_{U\mathfrak{g}} W$ are naturally isomorphic, it suffices to show, that if $P_\bullet \rightarrow V$ is a flat resolution of V , then $P_\bullet \otimes_{\mathbb{C}} W$ is a flat resolution of $V \otimes_{\mathbb{C}} W$.

By definition, flatness of P_k means exactness of functor $P_k \otimes_{U\mathfrak{g}} \square$. Using properties of tensor product we obtain an isomorphism of functors $(P_k \otimes_{\mathbb{C}} W) \otimes_{U\mathfrak{g}} \square$ and $P_k \otimes_{U\mathfrak{g}} (W \otimes_{\mathbb{C}} \square)$. The last functor is the composition of exact functors, so it is also exact, hence $P_k \otimes_{\mathbb{C}} W$ is flat. \blacksquare

A useful variation of Poincare duality holds for finite dimensional Lie algebras. Let $n = \dim \mathfrak{g}$. We endow one-dimensional vector space $\bigwedge^n \mathfrak{g}$ with structure of left \mathfrak{g} -module, which extends adjoint action by Leibnitz rule.

Theorem 2 (Poincare duality). For $0 \leq k \leq n$ and any left \mathfrak{g} -module V , there are vector space isomorphism

$$H^k(\mathfrak{g}, V^*) \cong H_k(\mathfrak{g}, V)^*,$$

and

$$H^k(\mathfrak{g}, V) \cong H_{n-k}(\mathfrak{g}, (\bigwedge^n \mathfrak{g})^* \otimes_{\mathbb{C}} V),$$

natural in V . Consequently

$$H^k(\mathfrak{g}, V^*) \cong H^{n-k}(\mathfrak{g}, \bigwedge^n \mathfrak{g} \otimes_{\mathbb{C}} V)^*.$$

Proof. Theorem 6.10 in

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3. TAYLOR SPECTRUM OF \mathfrak{g} -MODULE

Let \mathfrak{g} be an arbitrary Lie algebra and V be a left \mathfrak{g} -module. Recall, that $\hat{\mathfrak{g}}$ is the set of isomorphism classes of simple finite dimensional \mathfrak{g} -modules.

Definition 2. The Taylor spectrum of V is the subset of $\hat{\mathfrak{g}}$, defined as

$$\sigma_{\mathfrak{g}}(V) = \{S \in \hat{\mathfrak{g}} \mid \exists k: \text{Tor}_k^{U_{\mathfrak{g}}}(S^*, V) \neq 0\}.$$

We will simply write $\sigma(V)$ instead of $\sigma_{\mathfrak{g}}(V)$ if it is clear what Lie algebra is considered.

Let us provide several equivalent definitions, in order to use later.

Theorem 3. For an arbitrary $S \in \hat{\mathfrak{g}}$, the following are equivalent:

- (1) $S \in \sigma(V)$;
- (2) $H_k(\mathfrak{g}, V_{-S}) \neq 0$ for some k ;
- (3) $H^k(\mathfrak{g}, \bigwedge^n \mathfrak{g} \otimes_{\mathbb{C}} V_{-S}) \neq 0$ for some k ;
- (4) $S^* \otimes_{\mathbb{C}} \bigwedge^n \mathfrak{g} \in \sigma(V^*)$

Proof. $1 \Leftrightarrow 2$ This follows immediatly from Lemma 1.

$2 \Leftrightarrow 3$ Poincare duality (Theorem 2).

$3 \Leftrightarrow 4$ Again, by Poincare duality we have

$$H^k(\mathfrak{g}, \bigwedge^n \mathfrak{g} \otimes_{\mathbb{C}} V_{-S})^* = H^{n-k}(\mathfrak{g}, V_{-S}^*).$$

First observe that V_{-S}^* is isomorphic to $\bigwedge^n \mathfrak{g} \otimes_{\mathbb{C}} V_{-S^* \otimes_{\mathbb{C}} \bigwedge^n \mathfrak{g}}^*$. Using the equivalence of the 1'st and the 3'rd statements we obtain the desired result.

In case of abelian Lie algebra \mathfrak{g} the set $\hat{\mathfrak{g}}$, due to the Lie's Theorem, can be identified with the space of characters \mathfrak{g}^* .

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prove that definitions coincide

4. CASE OF SEMISIMPLE LIE ALGEBRA

5. SPECTRUM OF ONE-DIMENSIONAL EXTENSIONS

6. CASE OF SOLVABLE LIE ALGEBRA

7. CASE OF NILPOTENT LIE ALGEBRA

8. CASE OF BOREL SUBALGEBRA OF SEMISIMPLE LIE ALGEBRA