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1. Introduction

2. Preliminaries

2.1. **Notation.** In the article all algebras, including Lie algebras, are complex. For Lie algebra $\mathfrak{g}$ we will use the notation $U\mathfrak{g}$ to denote its enveloping algebra. We will denote by $\mathfrak{g}$**-mod** and **mod-**$\mathfrak{g}$ the categories of left and right $\mathfrak{g}$-modules respectively. We write $\hat{\mathfrak{g}}$ for the set of set of isomorphism classes of simple finite-dimensional $\mathfrak{g}$-modules and $\mathbb{C}$ for trivial bimodule. For $\mathfrak{g}$-module V can be defined vector spaces

$$V^{\mathfrak{g}} = \{v \in V \colon g \cdot v = 0 \ \forall g \in \mathfrak{g}\}, \tag{1}$$

called invariants and

$$V_{\mathfrak{g}} = V/gV, \tag{2}$$

called coinvariants. It is known , that $\square^{\mathfrak{g}}$ and $\square_{\mathfrak{g}}$ are actually functors from $\mathfrak{g}$**-mod** ref (or **mod-**$\mathfrak{g}$) to the category of vector spaces over $\mathbb{C}$, isomorphic to $\mathrm{Hom}_{\mathfrak{g}}(\mathbb{C}, V)$ and $\mathbb{C} \otimes_{U\mathfrak{g}} V$ respectively.

2.2. **Functors between categories of modules.** For the rest of this section we will denote by $\mathfrak{g}$ an arbitary finite-dimensional Lie algebra. We define two functors $\square^* \colon$
$\mathfrak{g}\textbf{-mod}^{op} \to \textbf{mod-}\mathfrak{g}$ and $\square^\circ \colon \mathfrak{g}\textbf{-mod} \to \textbf{mod-}\mathfrak{g}$ as follows. The first $\square^*$, called duality functor, sends $\mathfrak{g}$-module V to it's dual vector space, on which the right action of $\mathfrak{g}$ is defined as

$$(f \cdot g)(v) = f(g \cdot v), \quad \text{for all } f \in V^*, \ v \in V, \ g \in \mathfrak{g}.$$

The second $\square^\circ$, called antipode functor, sends V to itself as a vector space with right action

$$v \cdot g = -g \cdot v, \quad \text{for all } v \in V, \ g \in \mathfrak{g}.$$

These two functors define equivallence of categories $\mathfrak{g}$**-mod**, **mod-**$\mathfrak{g}$, $\mathfrak{g}\textbf{-mod}^{op}$ and $\textbf{mod-}\mathfrak{g}^{op}$. We will also denote by $\square^*$ and $\square^\circ$ functors from category of right $\mathfrak{g}$-modules to left $\mathfrak{g}$-modules, defined the same way. It is easy to see, that $(\square^*)^*$ and $(\square^\circ)^\circ$ are naturally isomorphic to the identity functor.

Another pair of very important functors are $\square \otimes_{\mathbb{C}} \square \colon \textbf{mod-}\mathfrak{g} \times \mathfrak{g}\textbf{-mod} \to \mathfrak{g}\textbf{-mod}$ and $\mathrm{Hom}_{\mathbb{C}}(\square, \square) \colon \mathfrak{g}\textbf{-mod}^{op} \times \mathfrak{g}\textbf{-mod} \to \mathfrak{g}\textbf{-mod}$. If $V \in \textbf{mod-}\mathfrak{g}$ and $W \in \mathfrak{g}\textbf{-mod}$, then $V \otimes_{\mathbb{C}} W$ is the tensor product of V and W as vector space with action of $\mathfrak{g}$, fully determined by the formula

$$g \cdot v \otimes w = v \otimes (g \cdot w) - (v \cdot g) \otimes w, \ \text{for all } w \in W, \ v \in V, \ g \in \mathfrak{g}$$

. The Hom functor is defined as

$$\mathrm{Hom}_{\mathbb{C}}(V, W) = V^* \otimes_{\mathbb{C}} W.$$

For $V, W \in \mathfrak{g}$**-mod** (resp. **mod-**$\mathfrak{g}$), we will denote by $V \otimes W$ left $\mathfrak{g}$-module $V^\circ \otimes_{\mathbb{C}} W$ (resp. $V \otimes_{\mathbb{C}} W^\circ$).

For $V \in \mathfrak{g}$**-mod** and $S \in \hat{\mathfrak{g}}$, we will write V_S for the $\mathfrak{g}$-module $S \otimes_{\mathbb{C}} V$. If S is one-dimensional, it is fully determined by the character $\lambda \in (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])$ and in this case we will simply write V_λ for it. For example, $\mathbb{C}_\lambda$ stands for one-dimensional module with action, given by $g \cdot s = \lambda(g)s$ for all $s \in \mathbb{C}_\lambda$ and $g \in \mathfrak{g}$. We will also use the notation V_{-S} for the module $S^* \otimes_{\mathbb{C}}$, which is motivated by the fact, that if S is again one-dimensional with character λ, then V_{-S} is isomorphic to $V_{-\lambda}$.

2.3. **Homology and cohomology of Lie algebras.** In this paragraph we recall the definitions of Lie algebra cohomology, which can be found in any related textbook .

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Definition 1. *For $V \in \mathfrak{g}$**-mod** and for all $k \in \mathbb{Z}_{\geq 0}$ the homology functors are defined as*

$$H_k(\mathfrak{g}, V) = \operatorname{Tor}_k^{U\mathfrak{g}}(\mathbb{C}, V), \tag{3}$$

and, dually, the cohomology as

$$H^k(\mathfrak{g}, V) = \operatorname{Ext}^k_{U\mathfrak{g}}(\mathbb{C}, V). \tag{4}$$

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The homology can be computed using Chevalley-Eilenberg free resolution of the trivial $\mathfrak{g}$ module $\mathbb{C}$. It has $F_k = U\mathfrak{g} \otimes_{\mathbb{C}} \bigwedge^k \mathfrak{g}$ in degree k with the differential given by

$$d(u \otimes g_1 \wedge \cdots \wedge g_p) = \sum_{i=1}^{p} (-1)^{i+1} u g_i \otimes g_1 \wedge \cdots \wedge \hat{g}_i \wedge \cdots \wedge g_p +$$

$$+ \sum_{i<j} (-1)^{i+j} u \otimes [g_i, g_j] \wedge \cdots \wedge \hat{g}_i \cdots \wedge \hat{g}_j \cdots \wedge g_p, \text{ where } u \in U\mathfrak{g},\ g_i \in \mathfrak{g}. \tag{5}$$

The following fact about homology will be used in the text.

Theorem 1. *Let $V \in$ **mod-**$\mathfrak{g}$ and $W \in \mathfrak{g}$**-mod**. Then*

$$\operatorname{Tor}_k^{U\mathfrak{g}}(V, W) \cong \operatorname{Tor}_k^{U\mathfrak{g}}(\mathbb{C}, V \otimes_{\mathbb{C}} W) = H_k(\mathfrak{g}, V \otimes_{\mathbb{C}} W),$$

for all $k \in \mathbb{Z}_{\geq 0}$.

Proof. Since functors $\mathbb{C} \otimes_{U\mathfrak{g}} (\square \otimes_{\mathbb{C}} W)$ and $\square \otimes_{U\mathfrak{g}} W$ are naturally isomorphic, it suffices to show, that if $P_\bullet \to V$ is a flat resolution of V, then $P_\bullet \otimes_{\mathbb{C}} W$ is a flat resolution of $V \otimes_{\mathbb{C}} W$.

By definition, flatness of P_k means exactness of functor $P_k \otimes_{U\mathfrak{g}} \square$. Using properties of tensor product we obtain an isomorphism of functors $(P_k \otimes_{\mathbb{C}} W) \otimes_{U\mathfrak{g}} \square$ and $P_k \otimes_{U\mathfrak{g}} (W \otimes_{\mathbb{C}} \square)$. The last fucntor is the composition of exact functors, so it is also exact, hence $P_k \otimes_{\mathbb{C}} W$ is flat. ■

A useful variation of Poincare duality holds for finite dimensional Lie algebras. Let $n = \dim \mathfrak{g}$. We endow one-dimensional vector space $\bigwedge^n \mathfrak{g}$ with structure of left $\mathfrak{g}$-module, which extends adjoint action by Leibnitz rule.

Theorem 2 (Poincare duality)**.** *For $0 \leq k \leq n$ and any left $\mathfrak{g}$-module V, there are vector space isomorphism*

$$H^k(\mathfrak{g}, V^*) \cong H_k(\mathfrak{g}, V)^*,$$

and

$$H^k(\mathfrak{g}, V) \cong H_{n-k}(\mathfrak{g}, (\bigwedge^n \mathfrak{g})^* \otimes_{\mathbb{C}} V),$$

natural in V. Consequently

$$H^k(\mathfrak{g}, V^*) \cong H^{n-k}(\mathfrak{g}, \bigwedge^n \mathfrak{g} \otimes_{\mathbb{C}} W)^*.$$

Proof. Theorem 6.10 in ■ cite knapp

3. Taylor spectrum of $\mathfrak{g}$-module

Let $\mathfrak{g}$ be an arbitary Lie algebra and E be a left $\mathfrak{g}$-module. We will denote by $\hat{\mathfrak{g}}$ the set of isomorphism classes of simple finite dimensional $\mathfrak{g}$-modules.

Definition 2. *The Taylor spectrum of E is the set, defined as*

$$\sigma(E) = \{V \in \hat{\mathfrak{g}} \mid \exists k \colon \operatorname{Tor}_k^{U\mathfrak{g}}(V^*, E) \neq 0\}.$$

From it follows, that the definition above coincides with the original Taylor's prove it
definition in case of abelian $\mathfrak{g}$.

4. Case of semisimple Lie algebra

5. Spectrum of one-dimensional extensions

6. Case of solvable Lie algebra

7. Case of nilpotent Lie algebra

8. Case of borel subalgebra of semisimple Lie algebra