TAYLOR SPECTRUM

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1. Introduction

2. Preliminaries

2.1. **Notation.** In the article all algebras, including Lie algebras, are complex. For the rest of this section we will denote by $\mathfrak{g}$ an arbitary Lie algebra. We will denote by $\mathfrak{g}$**-mod** and **mod-**$\mathfrak{g}$ the categories of left and right $\mathfrak{g}$-modules respectively. For $\mathfrak{g}$-module V defined the spaces

$\mathbb{C}_\lambda$

$$V^{\mathfrak{g}} = \{v \in V : g \cdot v = 0 \ \forall g \in \mathfrak{g}\}, \tag{1}$$

called invariants and

$$V_{\mathfrak{g}} = V/gV, \tag{2}$$

called coinvariants. It is known , that $\square^{\mathfrak{g}}$ and $\square_{\mathfrak{g}}$ are actually functors from $\mathfrak{g}$**-mod** (or **mod-**$\mathfrak{g}$) to the category of vector spaces over $\mathbb{C}$.

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2.2. **Functors between categories of modules.** We define two functors $\square^* \colon \mathfrak{g}\textbf{-mod}^{op} \to \textbf{mod-}\mathfrak{g}$ and $\square^\circ \colon \mathfrak{g}\textbf{-mod} \to \textbf{mod-}\mathfrak{g}$ as follows. The first $\square^*$, called duality functor, sends $\mathfrak{g}$-module V to it's dual vector space, on which the right action of $\mathfrak{g}$ is defined as

$$(f \cdot g)(v) = f(g \cdot v), \quad \text{for all } f \in V^*, \ v \in V, \ g \in \mathfrak{g}.$$

The second $\square^\circ$, called antipode functor, sends V to itself as a vector space with right action

$$v \cdot g = -g \cdot v, \quad \text{for all } v \in V, \ g \in \mathfrak{g}.$$

These two functors define equivallence of categories $\mathfrak{g}$**-mod**, **mod-**$\mathfrak{g}$, $\mathfrak{g}\textbf{-mod}^{op}$ and $\textbf{mod-}\mathfrak{g}^{op}$. We will also denote by $\square^*$ and $\square^\circ$ functors from category of right $\mathfrak{g}$-modules to left $\mathfrak{g}$-modules, defined the same way. It is easy to see, that $(\square^*)^*$ and $(\square^\circ)^\circ$ are naturally isomorphic to the identity functor.

Another pair of very important functors are $\square \otimes_{\mathbb{C}} \square \colon \textbf{mod-}\mathfrak{g} \times \mathfrak{g}\textbf{-mod} \to \mathfrak{g}\textbf{-mod}$ and $\mathrm{Hom}_{\mathbb{C}}(\square, \square) \colon \mathfrak{g}\textbf{-mod}^{op} \times \mathfrak{g}\textbf{-mod} \to \mathfrak{g}\textbf{-mod}$. If $V \in \textbf{mod-}\mathfrak{g}$ and $W \in \mathfrak{g}\textbf{-mod}$, then $V \otimes_{\mathbb{C}} W$ is the tensor product of V and W as vector space with action of $\mathfrak{g}$, fully determined by the formula

$$g \cdot v \otimes w = v \otimes (g \cdot w) - (v \cdot g) \otimes w, \ \text{for all } w \in W, \ v \in V, \ g \in \mathfrak{g}$$

. The Hom functor is defined as

$$\mathrm{Hom}_{\mathbb{C}}(V, W) = V^* \otimes_{\mathbb{C}} W.$$

2.3. **Homology and cohomology of Lie algebras.** In this paragraph we recall the definitions of Lie algebra cohomology, which can be found in any related textbook . ref weibel

Definition 1. *For $V \in \mathfrak{g}$-**mod** and for all $i \in \mathbb{Z}_{\geq 0}$ the homology functors are defined as*

$$H_i(\mathfrak{g}, V) = \mathrm{Tor}_i^{U\mathfrak{g}}(\mathbb{C}, V), \tag{3}$$

and, dually, the cohomology as

$$H^i(\mathfrak{g}, V) = \mathrm{Ext}^i_{U\mathfrak{g}}(\mathbb{C}, V). \tag{4}$$

3. Taylor spectrum of $\mathfrak{g}$-module

Let $\mathfrak{g}$ be an arbitary Lie algebra and E be a left $\mathfrak{g}$-module. We will denote by $\hat{\mathfrak{g}}$ the set of isomorphism classes of simple finite dimensional $\mathfrak{g}$-modules.

Definition 2. *The Taylor spectrum of E is the set, defined as*

$$\sigma(E) = \{V \in \hat{\mathfrak{g}} \mid \exists k \colon \mathrm{Tor}_k^{U\mathfrak{g}}(V^*, E) \neq 0\}.$$

prove it

From it follows, that the definition above coincides with the original Taylor's definition in case of abelian $\mathfrak{g}$.

4. Case of semisimple Lie algebra

5. Spectrum of one-dimensional extensions

6. Case of solvable Lie algebra

7. Case of nilpotent Lie algebra

8. Case of borel subalgebra of semisimple Lie algebra