GRASSMANNIAN CODES FROM STRATIFIED FRAMES

THESIS

William J. Brinkley, Second Lieutenant, USAF

AFIT-ENC-MS-25-M-240

DEPARTMENT OF THE AIR FORCE
AIR UNIVERSITY

AIR FORCE INSTITUTE OF TECHNOLOGY

Wright-Patterson Air Force Base, OH 45433

AFIT-ENC-MS-25-M-240

AFIT-ENC-MS-25-M-240

GRASSMANNIAN CODES FROM STRATIFIED FRAMES

THESIS

Presented to the Faculty

Department of Mathematics and Statistics
Graduate School of Engineering and Management
Air Force Institute of Technology
Air University
Air Education and Training Command

in Partial Fulfillment of the Requirements for the
Degree of Master of Science in Applied Mathematics

William J. Brinkley, B.S.
Second Lieutenant, USAF

March 2025

AFIT-ENC-MS-25-M-240

GRASSMANNIAN CODES FROM STRATIFIED FRAMES

William J. Brinkley, B.S.
Second Lieutenant, USAF

Committee Membership:

Matthew Fickus, Ph.D.
Chair

John Jasper, Ph.D.
Member

Maj Takayuki Iguchi, Ph.D.
Member

AFIT-ENC-MS-25-M-240

Abstract

An equichordal tight fusion frame (ECTFF) is a finite sequence of equi-dimensional subspaces of a Euclidean space that achieves equality in Conway, Hardin and Sloane's simplex bound. Every ECTFF is an optimal Grassmannian code with respect to the chordal distance. We introduce a method for constructing an ECTFF from any finite sequence of unit norm tight frames that happen to be "stratified" in a certain sense. We moreover show how to construct stratified unit norm tight frames from a difference family for a finite abelian group, as well as from a suitable combination of a resolvable balanced incomplete block design and an equiangular tight frame. These results streamline and unify several known constructions that were previously regarded as disparate, and moreover yield infinitely many apparently new ECTFFs.

AFIT-ENC-MS-25-M-240

Dedicated to my best friend, who kept me grounded, to my family, who supported me throughout, and to my wife, who without, I would have never come so far.

AFIT-ENC-MS-25-M-240

Acknowledgments

I would like to express my sincere gratitude to my committee for their invaluable support and expertise throughout this endeavor. Special thanks are extended to Dr. Jasper and Dr. Mixon, whose guidance and insights provided the tools and clarity that I greatly needed. My appreciation also goes out to Dr. Lunday, who went out of his way time and again to help me achieve my goals. Finally, I owe my deepest gratitude and appreciation to my advisor, Dr. Matt Fickus. Through his determination, patience, and passion for mathematics, I have seen further than I could ever have hoped.

William J. Brinkley

Table of Contents

GRASSMANNIAN CODES FROM STRATIFIED FRAMES

I. Introduction

Let C, R and N be integers such that $C \geq R \geq 1$ and $N \geq 2$. Further let $\mathcal{E}$ be a C-dimensional Euclidean space over $\mathbb{F}$, where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$, and let $\mathcal{R}$ and $\mathcal{N}$ be sets of cardinality R and N, respectively. For each $n \in \mathcal{N}$, let $\boldsymbol{\Pi}_n : \mathcal{E} \to \mathcal{E}$ be the (orthogonal) projection (operator) onto some R-dimensional subspace $\mathcal{V}_n$ of $\mathcal{E}$. Consider the problem of arranging these subspaces so that they are as far apart from each other as possible. This depends on one's choice of distance between the subspaces. One popular choice is the *chordal distance*, which Conway, Hardin and Sloane [10] define as

$$\operatorname{dist}(\mathcal{V}_{n_1}, \mathcal{V}_{n_2}) := \frac{1}{\sqrt{2}} \|\boldsymbol{\Pi}_{n_1} - \boldsymbol{\Pi}_{n_2}\|_{\mathrm{Fro}}, \tag{1.1}$$

where $\|\cdot\|_{\mathrm{Fro}}$ denotes the Frobenius norm. In that same paper, it is shown that any such finite sequence $(\mathcal{V}_n)_{n\in\mathcal{N}}$ of subspaces of $\mathcal{E}$ satisfies the *simplex bound*:

$$\min_{n_1 \neq n_2} \operatorname{dist}(\mathcal{V}_{n_1}, \mathcal{V}_{n_2}) \leq \sqrt{\frac{N}{N-1} \frac{R(C-R)}{C}}. \tag{1.2}$$

As shown in [10] and reviewed in the next chapter, $(\mathcal{V}_n)_{n\in\mathcal{N}}$ moreover achieves equality in this bound if and only if it is an *equichordal tight fusion frame* (ECTFF) for $\mathcal{E}$, namely if and only if $(\mathcal{V}_n)_{n\in\mathcal{N}}$ is both *equichordal*, meaning $\operatorname{dist}(\mathcal{V}_{n_1}, \mathcal{V}_{n_2})$ is constant over all distinct $n_1, n_2 \in \mathcal{N}$, and a *tight fusion frame* (TFF) for $\mathcal{E}$, meaning $\sum_{n\in\mathcal{N}} \boldsymbol{\Pi}_n = A\mathbf{I}$ for some $A > 0$. In particular, any ECTFF is a type of optimal *code* in the *Grassmannian* that consists of all R-dimensional subspaces of $\mathcal{E}$, being a sequence of N elements of this manifold for which the smallest chordal distance between any pair of its members is as large as possible

ECTFFs arise in quantum information theory [43], and *equi-isoclinic* instances of them are particularly well-suited for compressed sensing [14, 6]. The fundamental open problem regarding ECTFFs is that of existence. Since the defining properties of an ECTFF are preserved by unitary transformations, this depends solely on one's choice of $\mathbb{F}$, C, R and N. Accordingly, we refer to any such ECTFF as an $\mathrm{ECTFF}_{\mathbb{F}}(C,R,N)$. Only a few necessary conditions on their existence are known. *Gerzon's bound* [34, 38] states that if an $\mathrm{ECTFF}_{\mathbb{F}}(C,R,N)$ exists then N is at most $\frac{1}{2}C(C+1)$ or C^2 when $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$, respectively. It is also known [7, 4] that if an $\mathrm{ECTFF}_{\mathbb{F}}(C,R,N)$ exists then $R \leq C \leq NR$, that any $\mathrm{ECTFF}_{\mathbb{F}}(C,R,N)$ with $R < C$ has a *spatial complement*, which is an $\mathrm{ECTFF}_{\mathbb{F}}(C,C-R,N)$, and moreover that any $\mathrm{ECTFF}_{\mathbb{F}}(C,R,N)$ with $C < NR$ has a *Naimark complement*, which is an $\mathrm{ECTFF}_{\mathbb{F}}(NR-C,R,N)$. Sometimes, one can combine these facts to prove nonexistence. For example, an $\mathrm{ECTFF}_{\mathbb{F}}(3,1,5)$ does not exist since its Naimark complement would violate Gerzon's bound. See Examples 3.4 and 3.5 of [25] for additional instances of this type of argument.

Almost all sufficient conditions on the existence of ECTFFs are due to explicit construction. Each $\mathrm{ECTFF}_{\mathbb{F}}(C,R,N)$ for which $R=1$ is equivalent to an *equiangular tight frame* (ETF). ETFs have an extensive literature; see [26] for a survey of them, for example. Every *equi-isoclinic tight fusion frame* (EITFF) is also equichordal; see Theorem 5.2 of [24] for a survey of them, and see [18, 19, 20, 16, 21] for more recent developments. A number of works consider ECTFFs in generality. We now summarize some of the known constructions of them, drawing inspiration from the survey given in Theorem 5.3 of [24].

Directly summing an $\mathrm{ECTFF}_{\mathbb{F}}(C_1,R_1,N)$ and an $\mathrm{ECTFF}_{\mathbb{F}}(C_2,R_2,N)$ yields an $\mathrm{ECTFF}_{\mathbb{F}}(C_1+C_2,R_1+R_2,N)$ provided $C_1R_2 = C_2R_1$. Another way to construct ECTFFs follows from Hoggar's method of converting complex and quaternionic spaces into real and complex spaces of twice their dimension, respectively [30, 39]. Yet another construction, which notably achieves Gerzon's bound infinitely often in the real case, follows from Hadamard matrices of size $\frac{P+1}{2}$, where P is some prime [5]. ECTFFs also arise from ETFs that can be partitioned into regular simplices [22], as well as from finite abelian groups that have two difference sets that are *paired* in a particular way [17]. Any $\mathrm{ECTFF}_{\mathbb{F}}(C,R,N)$ can moreover be converted into a *harmonic* $\mathrm{ECTFF}_{\mathbb{C}}(RN,C,N)$ [19]. Remarkably, one can also formally prove the existence of certain ECTFFs in a nonconstructive way that relies on the Newton–Kantorovich method [9, 13].

Certain known constructions of ECTFFs that arise from algebro-combinatorial designs are particularly relevant to this thesis. One of these is Zauner's construction [43] of a real ECTFF from a *balanced incomplete block design* (BIBD). Another is King's construction [31] of an ECTFF from any *semiregular divisible difference set* or, more generally, from any *difference family* for a finite abelian group [25].

1.1 Contributions and Outline

In this thesis, we introduce a new method of constructing ECTFFs that apparently both unifies some of these previously known constructions and generalizes them in a way that yields infinitely many new ECTFFs. Our main idea is to construct them from a precursor object, namely a sequence of unit norm tight frames that happen to be *stratified* in a particular sense. We introduce our general theory in Chapter 3. In particular, we formally define a finite sequence of *stratified unit norm tight frames* (SUNTF) in Definition 3.1, and explain how to convert any such object into an ECTFF in Theorem 3.4. In Chapter 4, we then give several explicit constructions of SUNTFs. In Theorem 4.3 for example, we show how to convert any difference family for a finite abelian group into a SUNTF whose *layers* are harmonic frames. When combined with the construction of Theorem 3.4, this seemingly streamlines the "difference family to ECTFF" construction of [25]. Meanwhile, in Theorem 4.10, we show how to construct a SUNTF from any *resolvable* BIBD. When combined with Theorem 3.4, this seems to recover a special case of Zauner's construction [43]. Notably, while the construction of [43] appears to be disparate from that of [31, 25], our SUNTF theory reveals that they have much in common. In Theorem 4.14, we moreover generalize the approach of Theorem 4.10, showing how to combine a resolvable BIBD and a suitable ETF in a way that yields nontrivial SUNTFs. When combined with Theorem 3.4, this apparently yields infinitely many new ECTFFs. In Chapter 5, we conclude with a brief summary of our results and ideas for future work in this vein. Our formal discussion begins in the next chapter, in which we establish notation and review some previously known results.

II. Preliminaries

We begin with a brief discussion of notation and common definitions, primarily regarding topics within linear algebra. Let $\mathcal{N}$ be a finite nonempty set, and for any set $\mathcal{S}$, let $\mathcal{S}^{\mathcal{N}}$ be the set of functions from $\mathcal{N}$ into $\mathcal{S}$, namely $\mathcal{S}^{\mathcal{N}} := \{\boldsymbol{x} : \mathcal{N} \to \mathcal{S}\}$. We refer to an element $\boldsymbol{x}$ of $\mathcal{S}^{\mathcal{N}}$ as a *finite sequence of elements of $\mathcal{S}$ (indexed by $\mathcal{N}$)*, and often denote such a function *sequentially*, that is, as $(x_n)_{n\in\mathcal{N}}$, where $x_n = \boldsymbol{x}(n)$ for each $n \in \mathcal{N}$.

For any field $\mathbb{F}$, the set $\mathbb{F}^{\mathcal{N}} = \{\boldsymbol{x} : \mathcal{N} \to \mathbb{F}\}$ becomes a vector space over $\mathbb{F}$ under the usual (termwise) definitions of addition and scalar multiplication. We refer to any such vector space as a *standard space*, and denote its corresponding *standard basis* as $(\boldsymbol{\delta}_n)_{n\in\mathcal{N}}$ where

$$\boldsymbol{\delta}_n(n') := \begin{cases} 1, & n' = n, \\ 0, & n' \neq n. \end{cases}$$

Accordingly, the dimension of $\mathbb{F}^{\mathcal{N}}$ over $\mathbb{F}$ is the cardinality of $\mathcal{N}$, denoted $\#(\mathcal{N})$. For any $N \in \mathbb{N}$, we define $[N] := \{1, 2, \dots, N\}$ and $\mathbb{F}^N := \mathbb{F}^{[N]}$.

We often denote the *(Hadamard) termwise product* of $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{F}^{\mathcal{N}}$ as $\boldsymbol{x}_1 \odot \boldsymbol{x}_2 \in \mathbb{F}^{\mathcal{N}}$, meaning $(\boldsymbol{x}_1 \odot \boldsymbol{x}_2)(n) := \boldsymbol{x}_1(n)\boldsymbol{x}_2(n)$. For any finite nonempty sets $\mathcal{M}$ and $\mathcal{N}$, we refer to a member $\boldsymbol{A}$ of $\mathbb{F}^{\mathcal{M}\times\mathcal{N}}$ as an *$\mathcal{M} \times \mathcal{N}$ matrix (over $\mathbb{F}$)*. We define the *(Kronecker) tensor product* of $\boldsymbol{x} \in \mathbb{F}^{\mathcal{M}}$ and $\boldsymbol{y} \in \mathbb{F}^{\mathcal{N}}$ as $\boldsymbol{x} \otimes \boldsymbol{y} \in \mathbb{F}^{\mathcal{M}\times\mathcal{N}}$, $(\boldsymbol{x} \otimes \boldsymbol{y})(m, n) := \boldsymbol{x}(m)\boldsymbol{y}(n)$. We identify a linear map $\boldsymbol{L} : \mathbb{F}^{\mathcal{N}} \to \mathbb{F}^{\mathcal{M}}$ with a matrix $\boldsymbol{A} \in \mathbb{F}^{\mathcal{M}\times\mathcal{N}}$ in the standard way, namely so that

$$\boldsymbol{L}(\boldsymbol{\delta}_n) = \sum_{m\in\mathcal{M}} \boldsymbol{A}(m, n)\boldsymbol{\delta}_m, \quad \forall\, n \in \mathcal{N}. \tag{2.1}$$

Under this identification, it follows that $\boldsymbol{L}\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x}$ for any $\boldsymbol{x} \in \mathbb{F}^{\mathcal{N}}$. Accordingly, we often denote both such an $\boldsymbol{L}$ and $\boldsymbol{A}$ by a common symbol.

In the special case where $\mathbb{F}$ is either the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$, the vector space $\mathbb{F}^{\mathcal{N}}$ is a finite-dimensional Hilbert space under the inner product given by $\langle \boldsymbol{x}_1, \boldsymbol{x}_2 \rangle := \sum_{n\in\mathcal{N}} \overline{\boldsymbol{x}_1(n)}\boldsymbol{x}_2(n)$ for any $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{F}^{\mathcal{N}}$. Each inner product in this thesis is conjugate-linear in its first argument and linear in its second argument. We will refer to any finite-dimensional Hilbert space $\mathcal{E}$ (over $\mathbb{F}$) as a *Euclidean space (over* $\mathbb{F}$*)*. We denote the *adjoint* of any linear map $\boldsymbol{L}$ between such spaces by $\boldsymbol{L}^*$. As is typical, we say that such a map is *self-adjoint* when $\boldsymbol{L}^* = \boldsymbol{L}$, is an *isometry* when $\boldsymbol{L}^*\boldsymbol{L} = \mathbf{I}$, and is *unitary* if both $\boldsymbol{L}^*\boldsymbol{L} = \mathbf{I}$ and $\boldsymbol{L}\boldsymbol{L}^* = \mathbf{I}$, namely when $\boldsymbol{L}$ is invertible with $\boldsymbol{L}^{-1} = \boldsymbol{L}^*$. When identifying a linear map $\boldsymbol{L} : \mathbb{F}^{\mathcal{N}} \to \mathbb{F}^{\mathcal{M}}$ with a matrix $\boldsymbol{A} \in \mathbb{F}^{\mathcal{M}\times\mathcal{N}}$ as in (2.1), we have

$$\boldsymbol{A}(m,n) = \langle \boldsymbol{\delta}_m, \boldsymbol{L}\boldsymbol{\delta}_n \rangle \tag{2.2}$$

for all $m \in \mathcal{M}$, $n \in \mathcal{N}$, and identify $\boldsymbol{L}^*$ with the *conjugate-transpose* of $\boldsymbol{A}$, namely with $\boldsymbol{A}^* \in \mathbb{F}^{\mathcal{N}\times\mathcal{M}}$, $\boldsymbol{A}^*(n,m) := \overline{\boldsymbol{A}(m,n)}$.

In this thesis, a *linear operator* is a linear map whose codomain equals its domain. The *trace* of a linear operator $\boldsymbol{L}$ on some Euclidean space $\mathcal{E}$ is

$$\mathrm{Tr}(\boldsymbol{L}) := \sum_{n\in\mathcal{N}} \langle \boldsymbol{v}_n, \boldsymbol{L}\boldsymbol{v}_n \rangle,$$

where $(\boldsymbol{v}_n)_{n\in\mathcal{N}}$ is any orthonormal basis for $\mathcal{E}$. Properties of the trace are given in Appendix B. The *Frobenius (Hilbert-Schmidt) inner product* on the space of linear maps from one Euclidean space into another is

$$\langle \boldsymbol{L}_1, \boldsymbol{L}_2 \rangle_{\mathrm{Fro}} := \mathrm{Tr}(\boldsymbol{L}_1^*\boldsymbol{L}_2).$$

2.1 Finite Frame Theory

Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$, and let $(\boldsymbol{\varphi}_n)_{n\in\mathcal{N}}$ be a finite sequence of vectors in a Euclidean space $\mathcal{E}$ over $\mathbb{F}$. The *synthesis map* of $(\boldsymbol{\varphi}_n)_{n\in\mathcal{N}}$ is the map $\boldsymbol{\Phi} : \mathbb{F}^{\mathcal{N}} \to \mathcal{E}$ that is given by

$$\boldsymbol{\Phi}\boldsymbol{x} := \sum_{n\in\mathcal{N}} \boldsymbol{x}(n)\boldsymbol{\varphi}_n, \quad \forall\, \boldsymbol{x} \in \mathbb{F}^{N}.$$

Any such map is linear, and satisfies $\boldsymbol{\Phi}\boldsymbol{\delta}_n = \boldsymbol{\varphi}_n$ for all $n \in \mathcal{N}$. Conversely, any linear map $\boldsymbol{\Phi} : \mathbb{F}^{\mathcal{N}} \to \mathcal{E}$ is the synthesis map of the sequence $(\boldsymbol{\varphi}_n)_{n\in\mathcal{N}}$ of vectors in $\mathcal{E}$ that is defined by $\boldsymbol{\varphi}_n := \boldsymbol{\Phi}\boldsymbol{\delta}_n$ for each $n \in \mathcal{N}$. The image of any such map is the span of its associated vectors:

$$\operatorname{im}(\boldsymbol{\Phi}) = \{\boldsymbol{\Phi}\boldsymbol{x} : \boldsymbol{x} \in \mathbb{F}^{\mathcal{N}}\} = \left\{\sum_{n\in\mathcal{N}} \boldsymbol{x}(n)\boldsymbol{\varphi}_n : \boldsymbol{x} \in \mathbb{F}^{\mathcal{N}}\right\} = \operatorname{span}(\boldsymbol{\varphi}_n)_{n\in\mathcal{N}}.$$

Moreover, $(\boldsymbol{\varphi}_n)_{n\in\mathcal{N}}$ is linearly independent if and only if $\ker(\boldsymbol{\Phi}) = \{\boldsymbol{0}\}$. As such, $(\boldsymbol{\varphi}_n)_{n\in\mathcal{N}}$ is a basis for $\mathcal{E}$ if and only if $\boldsymbol{\Phi}$ is invertible. In the special case where $\mathcal{E} = \mathbb{F}^{\mathcal{M}}$ for some nonempty finite set $\mathcal{M}$, we identify any such $\boldsymbol{\Phi}$ with the matrix $\boldsymbol{\Phi} \in \mathbb{F}^{\mathcal{M}\times\mathcal{N}}$ whose nth column is $\boldsymbol{\varphi}_n$. In general, we refer to the adjoint $\boldsymbol{\Phi}^* : \mathcal{E} \to \mathbb{F}^{\mathcal{N}}$ of any synthesis map $\boldsymbol{\Phi} : \mathbb{F}^{\mathcal{N}} \to \mathcal{E}$ as the *analysis map* of $(\boldsymbol{\varphi}_n)_{n\in\mathcal{N}}$. For any $\boldsymbol{y} \in \mathcal{E}$ and $n \in \mathcal{N}$,

$$(\boldsymbol{\Phi}^*\boldsymbol{y})(n) = \langle \boldsymbol{\delta}_n, \boldsymbol{\Phi}^*\boldsymbol{y}\rangle = \langle \boldsymbol{\Phi}\boldsymbol{\delta}_n, \boldsymbol{y}\rangle = \langle \boldsymbol{\varphi}_n, \boldsymbol{y}\rangle.$$

Often, we identify a vector $\boldsymbol{\varphi} \in \mathcal{E}$ with the degenerate synthesis map $\boldsymbol{\varphi} : \mathbb{F} \to \mathcal{E}$ defined by $\boldsymbol{\varphi}(x) := x\boldsymbol{\varphi}$. Its adjoint is the linear functional $\boldsymbol{\varphi}^* : \mathcal{E} \to \mathbb{F}$ given by $\boldsymbol{\varphi}^*\boldsymbol{y} := \langle \boldsymbol{\varphi}, \boldsymbol{y}\rangle$.

The composition $\boldsymbol{\Phi}\boldsymbol{\Phi}^* : \mathcal{E} \to \mathcal{E}$ of these two maps is known as the *frame operator* of $(\boldsymbol{\varphi}_n)_{n\in\mathcal{N}}$. Note that for any $\boldsymbol{y} \in \mathcal{E}$,

$$\boldsymbol{\Phi}\boldsymbol{\Phi}^*\boldsymbol{y} = \sum_{n\in\mathcal{N}} (\boldsymbol{\Phi}^*\boldsymbol{y})(n)\boldsymbol{\varphi}_n = \sum_{n\in\mathcal{N}} \langle \boldsymbol{\varphi}_n, \boldsymbol{y}\rangle\boldsymbol{\varphi}_n = \sum_{n\in\mathcal{N}} \boldsymbol{\varphi}_n\boldsymbol{\varphi}_n^*\boldsymbol{y}. \tag{2.3}$$

As such, $\boldsymbol{\Phi}\boldsymbol{\Phi}^* = \sum_{n\in\mathcal{N}} \boldsymbol{\varphi}_n\boldsymbol{\varphi}_n^*$ is the sum of the outer products of each $\boldsymbol{\varphi}_n$ with itself. Meanwhile, the composition $\boldsymbol{\Phi}^*\boldsymbol{\Phi} : \mathbb{F}^{\mathcal{N}} \to \mathbb{F}^{\mathcal{N}}$ is known as the *Gram matrix* of $(\boldsymbol{\varphi}_n)_{n\in\mathcal{N}}$. Taking both $\boldsymbol{A}$ and $\boldsymbol{L}$ in (2.2) to be $\boldsymbol{\Phi}^*\boldsymbol{\Phi}$ gives that for any $n_1, n_2 \in \mathcal{N}$, the (n_1, n_2)th entry of $\boldsymbol{\Phi}^*\boldsymbol{\Phi} \in \mathbb{F}^{\mathcal{N}\times\mathcal{N}}$ is

$$(\boldsymbol{\Phi}^*\boldsymbol{\Phi})(n_1, n_2) = \langle \boldsymbol{\delta}_{n_1}, \boldsymbol{\Phi}^*\boldsymbol{\Phi}\boldsymbol{\delta}_{n_2}\rangle = \langle \boldsymbol{\Phi}\boldsymbol{\delta}_{n_1}, \boldsymbol{\Phi}\boldsymbol{\delta}_{n_2}\rangle = \langle \boldsymbol{\varphi}_{n_1}, \boldsymbol{\varphi}_{n_2}\rangle. \tag{2.4}$$

As such, $\boldsymbol{\Phi}^*\boldsymbol{\Phi}$ is a table of inner products of the $\boldsymbol{\varphi}_n$ vectors with each other. It follows that $(\boldsymbol{\varphi}_n)_{n\in\mathcal{N}}$ is orthonormal if and only if $\boldsymbol{\Phi}$ is an isometry, and moreover, that when this occurs, $\boldsymbol{\Phi}\boldsymbol{\Phi}^*$ is the projection onto the image of $\boldsymbol{\Phi}$. Accordingly, $(\boldsymbol{\varphi}_n)_{n\in\mathcal{N}}$ is an orthonormal basis for $\mathcal{E}$ if and only if $\boldsymbol{\Phi}$ is an invertible isometry, that is, a unitary map.

Every Gram matrix is positive semidefinite (PSD). Conversely, as explained in the appendix in Propositions A.7 and A.8, every PSD matrix $\boldsymbol{G}\in\mathbb{F}^{\mathcal{N}\times\mathcal{N}}$ is the Gram matrix of some spanning finite sequence $(\boldsymbol{\varphi}_n)_{n\in\mathcal{N}}$ of vectors in a Euclidean space over $\mathbb{F}$ of dimension $\operatorname{rank}(\boldsymbol{G})$, and moreover, this sequence and space are unique up to unitary transformations. In particular, one can always choose this space to be $\mathbb{F}^{\operatorname{rank}(\boldsymbol{G})}$ if so desired. One subtle consequence of these facts is that if a finite sequence of vectors in a complex Euclidean space has a real-valued Gram matrix then it can be regarded as lying in a real Euclidean space.

We say that $(\boldsymbol{\varphi}_n)_{n\in\mathcal{N}}$ is a *tight frame for* $\mathcal{E}$ if there exists a scalar $A>0$, referred to as the *frame constant*, such that $\boldsymbol{\Phi}\boldsymbol{\Phi}^*=A\mathbf{I}$. More generally, we say $(\boldsymbol{\varphi}_n)_{n\in\mathcal{N}}$ is a *tight frame* (without specifying the space) if it is a tight frame for its span, that is, if there exists a scalar $A>0$ such that $\boldsymbol{\Phi}\boldsymbol{\Phi}^*\boldsymbol{\Phi}=A\boldsymbol{\Phi}$. The following proposition details equivalent conditions under which this occurs.

Proposition 2.1 (Lemma 1 of [15])**.** For any scalar $A>0$ and any finite sequence $(\boldsymbol{\varphi}_n)_{n\in\mathcal{N}}$ of vectors in a Euclidean space with synthesis map $\boldsymbol{\Phi}$, the following are equivalent:

(i) $\boldsymbol{\Phi}\boldsymbol{\Phi}^*\boldsymbol{\Phi}=A\boldsymbol{\Phi}$,

(ii) $(\boldsymbol{\Phi}^*\boldsymbol{\Phi})^2=A\boldsymbol{\Phi}^*\boldsymbol{\Phi}$,

(iii) $(\boldsymbol{\Phi}\boldsymbol{\Phi}^*)^2=A\boldsymbol{\Phi}\boldsymbol{\Phi}^*$.

Proof. (i $\Rightarrow$ ii,iii) Multiplying (i) on the left or right by $\boldsymbol{\Phi}^*$ gives (ii) or (iii), respectively. (ii $\Rightarrow$ i) If $(\boldsymbol{\Phi}^*\boldsymbol{\Phi})^2=A\boldsymbol{\Phi}^*\boldsymbol{\Phi}$ then

$$\big\|\boldsymbol{\Phi}\boldsymbol{\Phi}^*\boldsymbol{\Phi}-A\boldsymbol{\Phi}\big\|_{\mathrm{Fro}}^2=\operatorname{Tr}[(\boldsymbol{\Phi}^*\boldsymbol{\Phi}-A\mathbf{I})\boldsymbol{\Phi}^*\boldsymbol{\Phi}(\boldsymbol{\Phi}^*\boldsymbol{\Phi}-A\mathbf{I})]=\operatorname{Tr}[(\boldsymbol{\Phi}^*\boldsymbol{\Phi}-A\mathbf{I})\mathbf{0}]=\operatorname{Tr}(\mathbf{0})=0,$$

implying $\boldsymbol{\Phi}\boldsymbol{\Phi}^*\boldsymbol{\Phi} = A\boldsymbol{\Phi}$. (iii $\Rightarrow$ i) Similarly, if $(\boldsymbol{\Phi}\boldsymbol{\Phi}^*)^2 = A\boldsymbol{\Phi}\boldsymbol{\Phi}^*$ then $\boldsymbol{\Phi}\boldsymbol{\Phi}^*\boldsymbol{\Phi} = A\boldsymbol{\Phi}$ since

$$\|\boldsymbol{\Phi}\boldsymbol{\Phi}\boldsymbol{\Phi}^* - A\boldsymbol{\Phi}\|_{\mathrm{Fro}}^2 = \mathrm{Tr}[(\boldsymbol{\Phi}\boldsymbol{\Phi}^* - A\mathbf{I})\boldsymbol{\Phi}\boldsymbol{\Phi}^*(\boldsymbol{\Phi}\boldsymbol{\Phi}^* - A\mathbf{I})] = \mathrm{Tr}[(\boldsymbol{\Phi}\boldsymbol{\Phi}^* - A\mathbf{I})\mathbf{0}] = \mathrm{Tr}(\mathbf{0}) = 0. \quad \square$$

Next, we note that if $\boldsymbol{\Phi}$ is the synthesis map of a tight frame, then dividing Proposition 2.1(iii) by A^2 reveals that $\frac{1}{A}\boldsymbol{\Phi}^*\boldsymbol{\Phi}$ is a projection. Since the trace of a projection equals its rank, this implies

$$\dim(\mathrm{span}(\boldsymbol{\varphi}_n)_{n\in\mathcal{N}}) = \mathrm{rank}(\boldsymbol{\Phi}) = \mathrm{rank}(\boldsymbol{\Phi}^*\boldsymbol{\Phi}) = \mathrm{rank}\Big(\frac{1}{A}\boldsymbol{\Phi}^*\boldsymbol{\Phi}\Big) = \mathrm{Tr}\Big(\frac{1}{A}\boldsymbol{\Phi}^*\boldsymbol{\Phi}\Big).$$

In light of (2.4), this in turn implies that

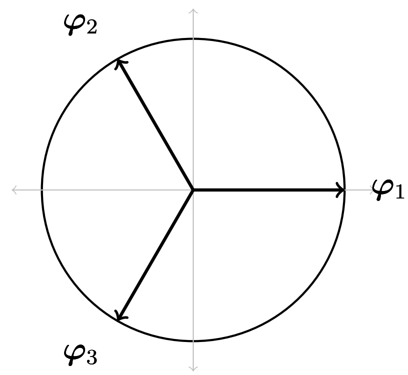
$$\dim(\mathrm{span}(\boldsymbol{\varphi}_n)_{n\in\mathcal{N}}) = \frac{1}{A}\sum_{n\in\mathcal{N}}\|\boldsymbol{\varphi}_n\|^2. \tag{2.5}$$

Meanwhile, the rank-nullity theorem gives

$$\dim(\mathrm{span}(\boldsymbol{\varphi}_n)_{n\in\mathcal{N}}) = \mathrm{rank}(\boldsymbol{\Phi}) \leq \mathrm{rank}(\boldsymbol{\Phi}) + \dim(\ker(\boldsymbol{\Phi})) = \dim(\mathbb{F}^{\mathcal{N}}) = \#(\mathcal{N}). \tag{2.6}$$

We say that $(\boldsymbol{\varphi}_n)_{n\in\mathcal{N}}$ is *unit norm* if each of its vectors is unit norm, that is, if $\|\boldsymbol{\varphi}_n\| = 1$ for each $n \in \mathcal{N}$. We call $(\boldsymbol{\varphi}_n)_{n\in\mathcal{N}}$ *equiangular* if it is unit norm and $|\langle\boldsymbol{\varphi}_{n_1}, \boldsymbol{\varphi}_{n_2}\rangle|$ is constant over all distinct $n_1, n_2 \in \mathcal{N}$. We say that $(\boldsymbol{\varphi}_n)_{n\in\mathcal{N}}$ is a *unit norm tight frame* (UNTF) (*for* $\mathcal{E}$) if it is both unit norm and a tight frame (for $\mathcal{E}$). When this occurs, (2.5) and (2.6) imply that its frame constant satisfies $A = \frac{N}{D} \geq 1$ where $D := \dim(\mathrm{span}(\boldsymbol{\varphi}_n)_{n\in\mathcal{N}})$ and $N := \#(\mathcal{N})$. We sometimes refer to such a finite sequence $(\boldsymbol{\varphi}_n)_{n\in\mathcal{N}}$ as a $\mathrm{UNTF}_{\mathbb{F}}(D, N)$. Similarly, we say that $(\boldsymbol{\varphi}_n)_{n\in\mathcal{N}}$ is an *equiangular tight frame* (ETF) (*for* $\mathcal{E}$) if it is both equiangular and a tight frame (for $\mathcal{E}$), and sometimes refer to it as an $\mathrm{ETF}_{\mathbb{F}}(D, N)$. The following example showcases an $\mathrm{ETF}_{\mathbb{R}}(2, 3)$.

Example 2.2. Consider the following arrangement of three unit vectors in $\mathbb{R}^2 \cong \mathbb{C}$, each pointing at a distinct cube root of unity:



Its synthesis map is the 2×3 real matrix

$$\boldsymbol{\Phi} := \begin{bmatrix} \varphi_1 & \varphi_2 & \varphi_3 \end{bmatrix} = \begin{bmatrix} \varphi_1(1) & \varphi_2(1) & \varphi_3(1) \\ \varphi_1(2) & \varphi_2(2) & \varphi_3(2) \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}.$$

Clearly, $(\varphi_1, \varphi_2, \varphi_3)$ is unit norm. It is moreover a tight frame for $\mathbb{R}^2$ since

$$\boldsymbol{\Phi\Phi}^* = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix} = \frac{3}{2}\mathbf{I}.$$

Thus, $(\varphi_1, \varphi_2, \varphi_3)$ is a $\mathrm{UNTF}_{\mathbb{R}}(2,3)$ for $\mathbb{R}^2$. Its Gram matrix is

$$\boldsymbol{\Phi}^*\boldsymbol{\Phi} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}. \tag{2.7}$$

Since the off-diagonal entries of $\boldsymbol{\Phi}^*\boldsymbol{\Phi}$ have constant modulus, $(\varphi_1, \varphi_2, \varphi_3)$ is moreover an $\mathrm{ETF}_{\mathbb{R}}(2,3)$ for $\mathbb{R}^2$. It is known as the *Mercedes-Benz frame.*

In general, we call any $\mathrm{ETF}_{\mathbb{F}}(D,N)$ with $N=D+1$ a *regular simplex*. It is well-known that such an ETF exists for any positive integer D. For an explicit iterative construction of them, see for example Lemma 2.4 of [16].

2.2 Equichordal Tight Fusion Frames

Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$, let $\mathcal{E}$ be a finite-dimensional Euclidean space over $\mathbb{F}$, and let N and R be integers such that $N\geq 2$ and $1\leq R\leq\dim(\mathcal{E})$. Further let $\mathcal{N}$ and $\mathcal{R}$ be sets of cardinality N and R, respectively. For each $n\in\mathcal{N}$, let $\mathcal{V}_n$ be an R-dimensional subspace of $\mathcal{E}$, and moreover let $\boldsymbol{\Phi}_n:\mathbb{F}^{\mathcal{R}}\rightarrow\mathcal{E}$ be the synthesis map of some orthonormal basis $(\boldsymbol{\varphi}_{n,r})_{r\in\mathcal{R}}$ for $\mathcal{V}_n$. It follows that $\boldsymbol{\Phi}_n$ is an isometry onto $\mathcal{V}_n$, and that $\boldsymbol{\Pi}_n:=\boldsymbol{\Phi}_n\boldsymbol{\Phi}_n^*$ is the projection onto $\mathcal{V}_n$. Further let $\boldsymbol{\Phi}$ be the synthesis map of $(\boldsymbol{\varphi}_{n,r})_{(n,r)\in\mathcal{N}\times\mathcal{R}}$, namely of the *concatenation* of these orthonormal bases for these subspaces. The corresponding Gram matrix $\boldsymbol{\Phi}^*\boldsymbol{\Phi}\in\mathbb{F}^{(\mathcal{N}\times\mathcal{R})\times(\mathcal{N}\times\mathcal{R})}$ is given by

$$\begin{aligned}(\boldsymbol{\Phi}^*\boldsymbol{\Phi})((n_1,r_1),(n_2,r_2))&=\langle\boldsymbol{\varphi}_{n_1,r_1},\boldsymbol{\varphi}_{n_2,r_2}\rangle\\&=\langle\boldsymbol{\Phi}_{n_1}\boldsymbol{\delta}_{r_1},\boldsymbol{\Phi}_{n_2}\boldsymbol{\delta}_{r_2}\rangle\\&=\langle\boldsymbol{\delta}_{r_1},\boldsymbol{\Phi}_{n_1}^*\boldsymbol{\Phi}_{n_2}\boldsymbol{\delta}_{r_2}\rangle\\&=(\boldsymbol{\Phi}_{n_1}^*\boldsymbol{\Phi}_{n_2})(r_1,r_2).\end{aligned}$$

Accordingly, we regard $\boldsymbol{\Phi}^*\boldsymbol{\Phi}$ as an $\mathcal{N}\times\mathcal{N}$ array of $\mathcal{R}\times\mathcal{R}$ blocks. Specifically, we refer to the (n_1,n_2)th block of $\boldsymbol{\Phi}^*\boldsymbol{\Phi}$ as the corresponding *cross-Gram matrix* $\boldsymbol{\Phi}_{n_1}^*\boldsymbol{\Phi}_{n_2}\in\mathbb{F}^{\mathcal{R}\times\mathcal{R}}$ whose (r_1,r_2)th entry is given by $(\boldsymbol{\Phi}_{n_1}^*\boldsymbol{\Phi}_{n_2})(r_1,r_2)=\langle\boldsymbol{\varphi}_{n_1,r_1},\boldsymbol{\varphi}_{n_2,r_2}\rangle$. We thus refer to $\boldsymbol{\Phi}^*\boldsymbol{\Phi}$ as the *fusion Gram matrix* of $(\boldsymbol{\Phi}_n)_{n\in\mathcal{N}}$.

Next, note that squaring the chordal distance (1.1) between $\mathcal{V}_{n_1}$ and $\mathcal{V}_{n_2}$ gives:

$$\begin{aligned}\big[\mathrm{dist}(\mathcal{V}_{n_1},\mathcal{V}_{n_2})\big]^2&=\frac{1}{2}\,\mathrm{Tr}\big((\boldsymbol{\Pi}_{n_1}-\boldsymbol{\Pi}_{n_2})^2\big)\\&=\frac{1}{2}\,\mathrm{Tr}\big(\boldsymbol{\Pi}_{n_1}+\boldsymbol{\Pi}_{n_2}-2\boldsymbol{\Pi}_{n_1}\boldsymbol{\Pi}_{n_2}\big)\\&=R-\mathrm{Tr}(\boldsymbol{\Pi}_{n_1}\boldsymbol{\Pi}_{n_2}).\end{aligned}\tag{2.8}$$

Here, cycling the trace gives

$$\operatorname{Tr}(\boldsymbol{\Pi}_{n_1}\boldsymbol{\Pi}_{n_2}) = \operatorname{Tr}(\boldsymbol{\Phi}_{n_1}\boldsymbol{\Phi}_{n_1}^*\boldsymbol{\Phi}_{n_2}\boldsymbol{\Phi}_{n_2}^*) = \operatorname{Tr}\big((\boldsymbol{\Phi}_{n_1}^*\boldsymbol{\Phi}_{n_2})^*\boldsymbol{\Phi}_{n_1}^*\boldsymbol{\Phi}_{n_2}\big) = \|\boldsymbol{\Phi}_{n_1}^*\boldsymbol{\Phi}_{n_2}\|_{\mathrm{Fro}}^2. \tag{2.9}$$

Together, (2.8) and (2.9) imply

$$\frac{1}{R}\|\boldsymbol{\Phi}_{n_1}^*\boldsymbol{\Phi}_{n_2}\|_{\mathrm{Fro}}^2 = \frac{1}{R}\operatorname{Tr}(\boldsymbol{\Pi}_{n_1}\boldsymbol{\Pi}_{n_2}) = 1 - \frac{1}{R}\big[\operatorname{dist}(\mathcal{V}_{n_1},\mathcal{V}_{n_2})\big]^2. \tag{2.10}$$

By the entrywise formulation of the Frobenius norm, this same quantity can be expressed as

$$\frac{1}{R}\|\boldsymbol{\Phi}_{n_1}^*\boldsymbol{\Phi}_{n_2}\|_{\mathrm{Fro}}^2 = \frac{1}{R}\sum_{r_1\in\mathcal{R}}\sum_{r_2\in\mathcal{R}}|(\boldsymbol{\Phi}_{n_1}^*\boldsymbol{\Phi}_{n_2})(r_1,r_2)|^2 = \frac{1}{R}\sum_{r_1\in\mathcal{R}}\sum_{r_2\in\mathcal{R}}|\langle\boldsymbol{\varphi}_{n_1,r_1},\boldsymbol{\varphi}_{n_2,r_2}\rangle|^2. \tag{2.11}$$

Now recall from the introduction that $(\mathcal{V}_n)_{n\in\mathcal{N}}$ is defined to be equichordal if $\operatorname{dist}(\mathcal{V}_{n_1},\mathcal{V}_{n_2})$ is constant over all distinct $n_1,n_2\in\mathcal{N}$. By (2.10), this is equivalent to the existence of some $\Lambda\geq 0$ such that $\frac{1}{R}\|\boldsymbol{\Phi}_{n_1}^*\boldsymbol{\Phi}_{n_2}\|_{\mathrm{Fro}}^2 = \Lambda$ for any distinct $n_1,n_2\in\mathcal{N}$.

In this context, the frame operator of $(\boldsymbol{\varphi}_{n,r})_{(n,r)\in\mathcal{N}\times\mathcal{R}}$ is the sum of the members of $(\boldsymbol{\Pi}_n)_{n\in\mathcal{N}}$, and it is called the *fusion frame operator* of $(\mathcal{V}_n)_{n\in\mathcal{N}}$:

$$\boldsymbol{\Phi}\boldsymbol{\Phi}^* = \sum_{(n,r)\in\mathcal{N}\times\mathcal{R}}\boldsymbol{\varphi}_{n,r}\boldsymbol{\varphi}_{n,r}^* = \sum_{n\in\mathcal{N}}\sum_{r\in\mathcal{R}}\boldsymbol{\varphi}_{n,r}\boldsymbol{\varphi}_{n,r}^* = \sum_{n\in\mathcal{N}}\boldsymbol{\Phi}_n\boldsymbol{\Phi}_n^* = \sum_{n\in\mathcal{N}}\boldsymbol{\Pi}_n. \tag{2.12}$$

We call $(\mathcal{V}_n)_{n\in\mathcal{N}}$ a *tight fusion frame* (TFF) (*for* $\mathcal{E}$) if $(\boldsymbol{\varphi}_{n,r})_{(n,r)\in\mathcal{N}\times\mathcal{R}}$ is a tight frame (for $\mathcal{E}$). In this event, $(\boldsymbol{\varphi}_{n,r})_{(n,r)\in\mathcal{N}\times\mathcal{R}}$ is a $\operatorname{UNTF}_{\mathbb{F}}(C,NR)$ where $C:=\dim(\operatorname{span}(\boldsymbol{\varphi}_{n,r})_{(n,r)\in\mathcal{N}\times\mathcal{R}})$, and so has frame constant $A=\frac{NR}{C}\geq 1$. When this occurs, we call $(\mathcal{V}_n)_{n\in\mathcal{N}}$ a $\operatorname{TFF}_{\mathbb{F}}(C,R,N)$, and furthermore call it an $\operatorname{ECTFF}_{\mathbb{F}}(C,R,N)$ if it is equichordal as well.

2.3 The Optimality of ECTFFs

Now let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$, and let C, R and N be integers such that $C\geq R\geq 1$ and $N\geq 2$. Further let $\mathcal{E}$ be a Euclidean space over $\mathbb{F}$ of dimension C. We define the corresponding *Grassmannian* to be the set of R-dimensional subspaces of $\mathcal{E}$, which is a metric space under the chordal distance (1.1). An N-member (optimal) *code* in this space is a sequence of N many R-dimensional subspaces of $\mathcal{E}$ with the property that the smallest distance between any pair of its members is as large as possible. The following well-known result implies that each ECTFF is a certifiably optimal such code.

Proposition 2.3 ([10]). Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$, and let C, R and N be integers such that $C \geq R \geq 1$ and $N \geq 2$. Further let $\mathcal{E}$ be a Euclidean space over $\mathbb{F}$ of dimension C, and let $\mathcal{R}$ and $\mathcal{N}$ be sets of cardinality R and N, respectively. For each $n \in \mathcal{N}$, let $\mathcal{V}_n$ be the image of some isometry $\boldsymbol{\Phi}_n : \mathbb{F}^{\mathcal{R}} \to \mathcal{E}$. Then:

$$\max_{n_1 \neq n_2} \frac{1}{R} \|\boldsymbol{\Phi}_{n_1}^* \boldsymbol{\Phi}_{n_2}\|_{\mathrm{Fro}}^2 \geq \frac{NR - C}{C(N-1)}. \tag{2.13}$$

Moreover, the following are equivalent:

(i) $\max_{n_1 \neq n_2} \frac{1}{R} \|\boldsymbol{\Phi}_{n_1}^* \boldsymbol{\Phi}_{n_2}\|_{\mathrm{Fro}}^2 = \frac{NR-C}{C(N-1)}$,

(ii) $(\mathcal{V}_n)_{n \in \mathcal{N}}$ is an ECTFF for $\mathcal{E}$,

(iii) $\frac{1}{R} \|\boldsymbol{\Phi}_{n_1}^* \boldsymbol{\Phi}_{n_2}\|_{\mathrm{Fro}}^2 = \frac{NR-C}{C(N-1)}$ for all distinct $n_1, n_2 \in \mathcal{N}$.

Proof. For each $n \in \mathcal{N}$, $\boldsymbol{\Pi}_n := \boldsymbol{\Phi}_n \boldsymbol{\Phi}_n^*$ is the projection onto $\mathcal{V}_n$. Note that

$$0 \leq \Big\| \sum_{n \in \mathcal{N}} \boldsymbol{\Pi}_n - \frac{NR}{C} \mathbf{I} \Big\|_{\mathrm{Fro}}^2 = \mathrm{Tr}\Big[\Big(\sum_{n \in \mathcal{N}} \boldsymbol{\Pi}_n - \frac{NR}{C} \mathbf{I} \Big)^2 \Big]. \tag{2.14}$$

Expanding the square and then using properties of the trace gives

$$\begin{aligned}
0 &\leq \Big\| \sum_{n \in \mathcal{N}} \boldsymbol{\Pi}_n - \frac{NR}{C} \mathbf{I} \Big\|_{\mathrm{Fro}}^2 \\
&= \sum_{n_1 \in \mathcal{N}} \sum_{n_2 \in \mathcal{N}} \mathrm{Tr}(\boldsymbol{\Pi}_{n_1} \boldsymbol{\Pi}_{n_2}) - 2\frac{NR}{C} \sum_{n \in \mathcal{N}} \mathrm{Tr}(\boldsymbol{\Pi}_n) + \Big(\frac{NR}{C}\Big)^2 \mathrm{Tr}(\mathbf{I}) \\
&= \sum_{n_1 \in \mathcal{N}} \sum_{\substack{n_2 \in \mathcal{N} \\ n_2 \neq n_1}} \mathrm{Tr}(\boldsymbol{\Pi}_{n_1} \boldsymbol{\Pi}_{n_2}) + \sum_{n \in \mathcal{N}} \mathrm{Tr}(\boldsymbol{\Pi}_n) - \frac{N^2 R^2}{C} \\
&= \sum_{n_1 \in \mathcal{N}} \sum_{\substack{n_2 \in \mathcal{N} \\ n_2 \neq n_1}} \mathrm{Tr}(\boldsymbol{\Pi}_{n_1} \boldsymbol{\Pi}_{n_2}) - \frac{NR(NR - C)}{C}.
\end{aligned}$$

Rearranging reveals that this is equivalent to

$$\frac{NR-C}{C(N-1)} \le \frac{1}{N(N-1)} \sum_{n_1\in\mathcal{N}} \sum_{\substack{n_2\in\mathcal{N}\\ n_2\neq n_1}} \frac{1}{R}\operatorname{Tr}(\boldsymbol{\Pi}_{n_1}\boldsymbol{\Pi}_{n_2}). \tag{2.15}$$

Moreover, equality holds above if and only if it holds in (2.14), namely if and only if $\sum_{n\in\mathcal{N}}\boldsymbol{\Pi}_n = \frac{NR}{C}\mathbf{I}$. That is, equality holds in (2.15) if and only if $(\mathcal{V}_n)_{n\in\mathcal{N}}$ is a TFF for $\mathcal{E}$. Next, since an average is no more than its largest term,

$$\frac{1}{N(N-1)} \sum_{n_1\in\mathcal{N}} \sum_{\substack{n_2\in\mathcal{N}\\ n_2\neq n_1}} \frac{1}{R}\operatorname{Tr}(\boldsymbol{\Pi}_{n_1}\boldsymbol{\Pi}_{n_2}) \le \max_{n_1\neq n_2}\frac{1}{R}\operatorname{Tr}(\boldsymbol{\Pi}_{n_1}\boldsymbol{\Pi}_{n_2}). \tag{2.16}$$

Moreover, equality holds above if and only if $\operatorname{Tr}(\boldsymbol{\Pi}_{n_1}\boldsymbol{\Pi}_{n_2})$ is constant over all distinct $n_1, n_2 \in \mathcal{N}$; by (2.10), this occurs if and only if $(\mathcal{V}_n)_{n\in\mathcal{N}}$ is equichordal. Combining (2.15) and (2.16) gives (2.13). Moreover, equality holds in (2.13) if and only if it holds in both (2.15) and (2.16), namely if and only if $(\mathcal{V}_n)_{n\in\mathcal{N}}$ is both a TFF for $\mathcal{E}$ and equichordal, that is, an ECTFF for $\mathcal{E}$. Thus, (i) and (ii) are equivalent. Moreover, (iii) clearly implies (i) and so also (ii). Conversely, if both (i) and (ii) hold, then (ii) gives that $(\mathcal{V}_n)_{n\in\mathcal{N}}$ is equichordal, implying there exists $\Lambda \ge 0$ such that $\frac{1}{R}\|\boldsymbol{\Phi}_{n_1}^*\boldsymbol{\Phi}_{n_2}\|_{\mathrm{Fro}}^2 = \Lambda$ for all distinct $n_1, n_2 \in \mathcal{N}$; combining this with (i) then implies (iii) since

$$\frac{NR-C}{C(N-1)} = \max_{n_1\neq n_2}\frac{1}{R}\|\boldsymbol{\Phi}_{n_1}^*\boldsymbol{\Phi}_{n_2}\|_{\mathrm{Fro}}^2 = \max_{n_1\neq n_2}\Lambda = \Lambda.$$

Therefore, (i),(ii), and (iii) are indeed equivalent. $\square$

In the special case of the previous result in which $R = 1$, each isometry $\boldsymbol{\Phi}_n$ is the synthesis map of an arbitrary unit vector $\boldsymbol{\varphi}_n$ in a one-dimensional subspace $\mathcal{V}_n$ of $\mathcal{E}$. In this case, (2.13) reduces to the *Welch bound* [40], namely that any finite sequence $(\boldsymbol{\varphi}_n)_{n\in\mathcal{N}}$ of unit vectors in a Euclidean space $\mathcal{E}$ of dimension C satisfies

$$\max_{n_1 \neq n_2} |\langle \boldsymbol{\varphi}_{n_1}, \boldsymbol{\varphi}_{n_2} \rangle|^2 \geq \frac{N-C}{C(N-1)}. \tag{2.17}$$

Moreover, in this case, the remaining conclusions of Proposition 2.3 recover the celebrated result that such a finite sequence $(\boldsymbol{\varphi}_n)_{n\in\mathcal{N}}$ achieves equality in this bound if and only if it is an ETF for $\mathcal{E}$ [37].

For this reason, we refer to (2.13) in general as the *chordal Welch bound*, and call $\frac{NR-C}{C(N-1)}$ the *chordal constant*. That said, one might alternatively refer to (2.13) as the simplex bound, as it immediately yields via (2.10) the inequality given that name by Conway, Hardin and Sloane [10], namely (1.2). Regardless, (2.10) moreover implies that $(\mathcal{V}_n)_{n\in\mathcal{N}}$ achieves equality in (1.2) if and only if it achieves equality in (2.13), namely if and only if it is an ECTFF for $\mathcal{E}$. Thus, every $\mathrm{ECTFF}_{\mathbb{F}}(C,R,N)$ indeed consists of N subspaces of a C-dimensional Euclidean space over $\mathbb{F}$, each of dimension R, with the property that the smallest chordal distance between any pair of them is as large as possible.

In later chapters, we give a new way to construct ECTFFs that yields infinitely many apparently new examples of them. Our proof of the validity of that construction technique relies on the following characterization of the fusion Gram matrix of an ECTFF in general.

Proposition 2.4 (Folklore)**.** Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$, and let $\mathcal{N}$ and $\mathcal{R}$ be finite sets of cardinality $N \geq 2$ and $R \geq 1$, respectively. Further suppose that $\boldsymbol{G} \in \mathbb{F}^{(\mathcal{N}\times\mathcal{R})\times(\mathcal{N}\times\mathcal{R})}$ satisfies both:

(i) $\boldsymbol{G}^* = \boldsymbol{G}$.

(ii) $\boldsymbol{G}^2 = A\boldsymbol{G}$ for some $A > 0$.

Then, $\boldsymbol{G}$ is the Gram matrix $\boldsymbol{\Phi}^*\boldsymbol{\Phi}$ of a tight frame $(\boldsymbol{\varphi}_{n,r})_{(n,r)\in\mathcal{N}\times\mathcal{R}}$ for a C-dimensional Euclidean space over $\mathbb{F}$, where $C := \frac{1}{A}\operatorname{Tr}(\boldsymbol{G})$. Now further suppose that:

(iii) $\boldsymbol{G}((n,r_1),(n,r_2)) = \mathbf{I}(r_1,r_2)$ for any $n \in \mathcal{N}$, $r_1, r_2 \in \mathcal{R}$.

Then, $(\boldsymbol{\varphi}_{n,r})_{r\in\mathcal{R}}$ is orthonormal for each $n \in \mathcal{N}$ and defining $(\mathcal{V}_n)_{n\in\mathcal{N}}$ by $\mathcal{V}_n := \operatorname{span}(\boldsymbol{\varphi}_{n,r})_{r\in\mathcal{R}}$ yields a $\mathrm{TFF}_{\mathbb{F}}(C,R,N)$ with $C = \frac{NR}{A}$, and it is an $\mathrm{ECTFF}_{\mathbb{F}}(C,R,N)$ if and only if:

(iv) There exists $\Lambda \geq 0$ such that for any distinct $n_1, n_2 \in \mathcal{N}$,

$$\frac{1}{R}\sum_{r_1\in\mathcal{R}}\sum_{r_2\in\mathcal{R}}\big|\boldsymbol{G}\big((n_1,r_1),(n_2,r_2)\big)\big|^2 = \Lambda. \tag{2.18}$$

Proof. Let $\boldsymbol{G} \in \mathbb{F}^{(\mathcal{N}\times\mathcal{R})\times(\mathcal{N}\times\mathcal{R})}$ satisfy both (i) and (ii), implying that $\frac{1}{A}\boldsymbol{G}$ is a projection and so also that $\boldsymbol{G}$ is PSD. Proposition A.8 thus implies that $\boldsymbol{G}$ is the Gram matrix of a spanning finite sequence $(\boldsymbol{\varphi}_{n,r})_{(n,r)\in\mathcal{N}\times\mathcal{R}}$ of vectors in a Euclidean space $\mathcal{E}$ over $\mathbb{F}$ of dimension $\operatorname{rank}(\boldsymbol{G}) = \operatorname{rank}(\frac{1}{A}\boldsymbol{G}) = \operatorname{Tr}(\frac{1}{A}\boldsymbol{G}) = \frac{1}{A}\operatorname{Tr}(\boldsymbol{G}) = C$. In particular, $\boldsymbol{G} = \boldsymbol{\Phi}^*\boldsymbol{\Phi}$ where $\boldsymbol{\Phi} : \mathbb{F}^{\mathcal{N}\times\mathcal{R}} \to \mathcal{E}$ is the synthesis map of $(\boldsymbol{\varphi}_{n,r})_{(n,r)\in\mathcal{N}\times\mathcal{R}}$. Since $(\boldsymbol{\Phi}^*\boldsymbol{\Phi})^2 = \boldsymbol{G}^2 = A\boldsymbol{G} = A\boldsymbol{\Phi}^*\boldsymbol{\Phi}$, Proposition 2.1 gives that $(\boldsymbol{\varphi}_{n,r})_{(n,r)\in\mathcal{N}\times\mathcal{R}}$ is a tight frame, namely a tight frame for $\operatorname{span}(\boldsymbol{\varphi}_{n,r})_{(n,r)\in\mathcal{N}\times\mathcal{R}} = \mathcal{E}$.

Now further assume that $\boldsymbol{G}$ satisfies (iii). For any $n \in \mathcal{N}$, $r_1, r_2 \in \mathcal{R}$, note that by (2.4),

$$\langle\boldsymbol{\varphi}_{n,r_1}, \boldsymbol{\varphi}_{n,r_2}\rangle = (\boldsymbol{\Phi}^*\boldsymbol{\Phi})\big((n,r_1),(n,r_2)\big) = \boldsymbol{G}\big((n,r_1),(n,r_2)\big) = \mathbf{I}(r_1,r_2) = \begin{cases} 1, & r_1 = r_2,\\ 0, & r_1 \neq r_2.\end{cases}$$

Thus, for each $n \in \mathcal{N}$, we have that $(\boldsymbol{\varphi}_{n,r})_{r\in\mathcal{R}}$ is orthonormal. Since $(\boldsymbol{\varphi}_{n,r})_{(n,r)\in\mathcal{N}\times\mathcal{R}}$ is a tight frame for $\mathcal{E}$, this implies that $(\mathcal{V}_n)_{n\in\mathcal{N}}$, $\mathcal{V}_n := \operatorname{span}(\boldsymbol{\varphi}_{n,r})_{r\in\mathcal{R}}$ defines a TFF for $\mathcal{E}$, and so an instance of a $\operatorname{TFF}_{\mathbb{F}}(C,R,N)$ where $C = \frac{1}{A}\operatorname{Tr}(\boldsymbol{G}) = \frac{NR}{A}$.

The final conclusion follows from the fact that for any $n_1, n_2 \in \mathcal{N}$,

$$\begin{aligned}\frac{1}{R}\sum_{r_1\in\mathcal{R}}\sum_{r_2\in\mathcal{R}}\big|\boldsymbol{G}\big((n_1,r_1),(n_2,r_2)\big)\big|^2 &= \frac{1}{R}\sum_{r_1\in\mathcal{R}}\sum_{r_2\in\mathcal{R}}\big|(\boldsymbol{\Phi}^*\boldsymbol{\Phi})\big((n_1,r_1),(n_2,r_2)\big)\big|^2\\ &= \frac{1}{R}\sum_{r_1\in\mathcal{R}}\sum_{r_2\in\mathcal{R}}\big|\langle\boldsymbol{\varphi}_{n_1,r_1},\boldsymbol{\varphi}_{n_2,r_2}\rangle\big|^2\\ &= \frac{1}{R}\|\boldsymbol{\Phi}_{n_1}^*\boldsymbol{\Phi}_{n_2}\|_{\mathrm{Fro}}^2.\end{aligned}$$

□

2.4 Harmonic Analysis over Finite Abelian Groups

Let $\mathcal{G}$ be a finite abelian group of order G, whose group operation we regard as addition. When potentially ambiguous, we indicate the binary operation of the group as a pair, e.g.

$(\mathcal{G},+)$. Regarding $\mathbb{T} := \{\boldsymbol{z} \in \mathbb{C} : |\boldsymbol{z}| = 1\}$ as a group under multiplication, we refer to a homomorphism $\gamma : \mathcal{G} \to \mathbb{T}$ as a *character* of $\mathcal{G}$. The set of all such characters is the *Pontryagin dual of* $\mathcal{G}$, which we denote as:

$$\widehat{\mathcal{G}} := \{\gamma : \mathcal{G} \to \mathbb{T} \mid \gamma(g_1 + g_2) = \gamma(g_1)\gamma(g_2),\ \forall\, g_1, g_2 \in \mathcal{G}\}.$$

As detailed in Appendix C, it is well-known that $\widehat{\mathcal{G}}$ is itself a group under the Hadamard (termwise) product, and moreover, that it is isomorphic to $\mathcal{G}$. For brevity, we often write the Hadamard product of two characters with typical multiplicative notation, i.e., $\gamma_1 \odot \gamma_2 = \gamma_1\gamma_2$.

Since $\mathbb{T}$ is a subset of $\mathbb{C}$, we often identify a character $\gamma : \mathcal{G} \to \mathbb{T}$ with the vector $\boldsymbol{\gamma} : \mathcal{G} \to \mathbb{C}$ in $\mathbb{C}^{\mathcal{G}}$ with $\boldsymbol{\gamma}(g) := \gamma(g)$ for all $g \in \mathcal{G}$. In particular, we denote the identity character and its associated vector as $1 \in \widehat{\mathcal{G}}$ and $\mathbf{1} \in \mathbb{C}^{\mathcal{G}}$, respectively, where $\mathbf{1}(g) := 1(g) := 1 \in \mathbb{T} \subseteq \mathbb{C}$ for all $g \in \mathcal{G}$. Now consider the finite sequence $(\boldsymbol{\gamma})_{\gamma\in\widehat{\mathcal{G}}}$ of all characters of $\mathcal{G}$, each serving as its own index. Its synthesis map $\boldsymbol{\Gamma} \in \mathbb{C}^{\mathcal{G}\times\widehat{\mathcal{G}}}$ is called the *character table* of $\mathcal{G}$, given by

$$\boldsymbol{\Gamma}(g,\gamma) := \boldsymbol{\gamma}(g).$$

As detailed in Appendix C, it is well-known that $(\frac{1}{\sqrt{G}}\boldsymbol{\gamma})_{\gamma\in\widehat{\mathcal{G}}}$ is an orthonormal basis for $\mathbb{C}^{\mathcal{G}}$; it is called the *discrete Fourier basis* over $\mathcal{G}$. Thus, $\frac{1}{\sqrt{G}}\boldsymbol{\Gamma}$ is unitary, yielding that

$$\boldsymbol{\Gamma}^*\boldsymbol{\Gamma} = G\mathbf{I}, \quad \boldsymbol{\Gamma}\boldsymbol{\Gamma}^* = G\mathbf{I}. \tag{2.19}$$

The *discrete Fourier transform* (DFT) *over* $\mathcal{G}$ is the corresponding analysis map

$$\boldsymbol{\Gamma}^* : \mathbb{C}^{\mathcal{G}} \to \mathbb{C}^{\widehat{\mathcal{G}}}, \quad (\boldsymbol{\Gamma}^*\boldsymbol{y})(\gamma) = \langle \boldsymbol{\gamma}, \boldsymbol{y}\rangle, \quad \forall\, \boldsymbol{y} \in \mathbb{C}^{\mathcal{G}}.$$

Since $\boldsymbol{\Gamma}$ is the synthesis map of $(\boldsymbol{\gamma})_{\gamma\in\widehat{\mathcal{G}}}$,

$$\boldsymbol{\Gamma}\boldsymbol{\delta}_1 = \mathbf{1}, \quad \text{i.e.,} \quad \boldsymbol{\Gamma}^*\mathbf{1} = G\boldsymbol{\delta}_1. \tag{2.20}$$

Analogously, since $\gamma(0) = 1$ for all $\gamma \in \widehat{\mathcal{G}}$,

$$\boldsymbol{\Gamma}^* \boldsymbol{\delta}_0 = \mathbf{1}, \quad \text{i.e.,} \quad \boldsymbol{\Gamma}\mathbf{1} = G\boldsymbol{\delta}_0. \tag{2.21}$$

We now illustrate these ideas with an example:

Example 2.5. Let N be a natural number, let $\mathcal{G}$ be the finite cyclic group $\mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z}$ of integers modulo N, and let $\omega := \mathrm{e}^{\frac{2\pi \mathrm{i}}{N}}$. For any $n \in \mathbb{Z}_N$, define

$$\mathbf{e}_n \in \mathbb{C}^{\mathbb{Z}_N}, \quad \mathbf{e}_n(m) := \mathrm{e}^{\frac{2\pi \mathrm{i} mn}{N}} = \omega^{mn}. \tag{2.22}$$

Since $\omega^N = 1$ and so $\omega^{m(n+N)} = \omega^{mn} = \omega^{(m+N)n}$ for all $m, n \in \mathbb{Z}$, this is well-defined. Moreover, each such $\mathbf{e}_n$ is a character of $\mathbb{Z}_N$, having $\mathbf{e}_n(m) \in \mathbb{T}$ for all $m \in \mathbb{Z}_N$ and

$$\mathbf{e}_n(m_1 + m_2) = \omega^{(m_1+m_2)n} = \omega^{m_1 n}\omega^{m_2 n} = \mathbf{e}_n(m_1)\mathbf{e}_n(m_2)$$

for all $m_1, m_2 \in \mathbb{Z}_N$. Since

$$\mathbf{e}_{n_1+n_2}(m) = \omega^{m(n_1+n_2)} = \omega^{mn_1}\omega^{mn_2} = \mathbf{e}_{n_1}(m)\mathbf{e}_{n_2}(m) = (\mathbf{e}_{n_1}\mathbf{e}_{n_2})(m)$$

for all $m \in \mathbb{Z}_N$, and moreover $\mathbf{e}_n(m) = 1$ for all $m \in \mathbb{Z}_N$ only if $n = 0$, the mapping $n \mapsto \mathbf{e}_n$ is an injective homomorphism from $\mathbb{Z}_N$ into its Pontryagin dual $\widehat{\mathbb{Z}}_N$. In Appendix C, we formally build upon these facts to show that this homomorphism is moreover surjective, and so is an isomorphism. There, we also explain how this implies that $(\frac{1}{\sqrt{N}}\mathbf{e}_n)_{n \in \mathbb{Z}_N}$ is an orthonormal basis for $\mathbb{C}^{\mathbb{Z}_N}$.

Below, we provide the character table for $\mathbb{Z}_5$, regarded as the synthesis map of $(\mathbf{e}_n)_{n \in \mathbb{Z}_5}$:

$$\boldsymbol{\Gamma} = \left[\ \mathbf{e}_0 \ \ \mathbf{e}_1 \ \ \mathbf{e}_2 \ \ \mathbf{e}_3 \ \ \mathbf{e}_4\ \right] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 \\ 1 & \omega^2 & \omega^4 & \omega & \omega^3 \\ 1 & \omega^3 & \omega & \omega^4 & \omega^2 \\ 1 & \omega^4 & \omega^3 & \omega^2 & \omega \end{bmatrix}, \quad \omega := \mathrm{e}^{\frac{2\pi \mathrm{i}}{5}}. \tag{2.23}$$

Since $(\frac{1}{\sqrt{5}}\mathbf{e}_n)_{n\in\mathbb{Z}_5}$ is an orthonormal basis for $\mathbb{C}^{\mathbb{Z}_5}$, this matrix satisfies $\boldsymbol{\Gamma}^*\boldsymbol{\Gamma} = 5\mathbf{I}$ and $\boldsymbol{\Gamma}\boldsymbol{\Gamma}^* = 5\mathbf{I}$. In particular, its rows are mutually orthogonal, and each has norm $\sqrt{5}$.

2.5 Filters and Convolution

As before, let $\mathcal{G}$ be a finite abelian group of order G whose operation is denoted as addition. For any $g \in \mathcal{G}$, the corresponding *translation operator* is

$$\boldsymbol{T}^g : \mathbb{C}^{\mathcal{G}} \to \mathbb{C}^{\mathcal{G}}, \quad \big(\boldsymbol{T}^g\boldsymbol{x}\big)(g') := \boldsymbol{x}(g'-g).$$

Properties of such operators are discussed in Appendix D. In particular, for any $g \in \mathcal{G}$, we have $(\boldsymbol{T}^g)^* = \boldsymbol{T}^{-g} = (\boldsymbol{T}^{-g})^{-1}$, so $\boldsymbol{T}^g$ is unitary. For any $\boldsymbol{f} \in \mathbb{C}^{\mathcal{G}}$, the corresponding *filter* $\boldsymbol{F} : \mathbb{C}^{\mathcal{G}} \to \mathbb{C}^{\mathcal{G}}$ is $\boldsymbol{F} := \sum_{g\in\mathcal{G}} \boldsymbol{f}(g)\boldsymbol{T}^g$. For any $\boldsymbol{x} \in \mathbb{C}^{\mathcal{G}}$ and $g \in \mathcal{G}$,

$$(\boldsymbol{F}\boldsymbol{x})(g) = \sum_{g'\in\mathcal{G}} \boldsymbol{f}(g')(\boldsymbol{T}^{g'}\boldsymbol{x})(g) = \sum_{g'\in\mathcal{G}} \boldsymbol{f}(g')\boldsymbol{x}(g-g') = (\boldsymbol{f} * \boldsymbol{x})(g), \tag{2.24}$$

where $* : \mathbb{F}^{\mathcal{G}} \times \mathbb{F}^{\mathcal{G}} \to \mathbb{F}^{\mathcal{G}}$ is the operation known as *convolution*. By Proposition D.3,

$$\boldsymbol{F}\boldsymbol{\delta}_0 = \sum_{g\in\mathcal{G}} \boldsymbol{f}(g)\boldsymbol{T}^g\boldsymbol{\delta}_0 = \sum_{g\in\mathcal{G}} \boldsymbol{f}(g)\boldsymbol{\delta}_g = \boldsymbol{f}.$$

Thus, $\boldsymbol{F}$ uniquely determines the vector $\boldsymbol{f}$, which is known as the *impulse response* of $\boldsymbol{F}$. The adjoint of $\boldsymbol{F}$ is itself a filter since

$$\boldsymbol{F}^* = \Big(\sum_{g\in\mathcal{G}} \boldsymbol{f}(g)\boldsymbol{T}^g\Big)^* = \sum_{g\in\mathcal{G}} \overline{\boldsymbol{f}(g)}(\boldsymbol{T}^g)^* = \sum_{g\in\mathcal{G}} \overline{\boldsymbol{f}(g)}\boldsymbol{T}^{-g} = \sum_{g'\in\mathcal{G}} \overline{\boldsymbol{f}(-g')}\boldsymbol{T}^{g'}.$$

In particular, $\boldsymbol{F}^*$ is the filter whose impulse response is the *involution* $\widetilde{\boldsymbol{f}}$ of $\boldsymbol{f}$, defined by $\widetilde{\boldsymbol{f}}(g) := \overline{\boldsymbol{f}(-g)}$. The *autocorrelation* of $\boldsymbol{x}$ is its convolution with its involution, as given by

$$(\boldsymbol{x} * \widetilde{\boldsymbol{x}})(g) = \sum_{g'\in\mathcal{G}} \boldsymbol{x}(g')\widetilde{\boldsymbol{x}}(g-g') = \sum_{g'\in\mathcal{G}} \overline{\boldsymbol{x}(g'-g)}\boldsymbol{x}(g') = \langle \boldsymbol{T}^g\boldsymbol{x}, \boldsymbol{x}\rangle. \tag{2.25}$$

More properties of filters and convolution are given in Appendix D.

We now discuss how the DFT distributes over convolution, involution, and autocorrelation; see Proposition D.6 for proofs of these well-known facts. To be precise, the DFT converts convolution into the termwise product, involution into termwise conjugation, and autocorrelation into the termwise modulus squared: for any $\boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2$ in $\mathbb{C}^{\mathcal{G}}$ and $\gamma \in \widehat{\mathcal{G}}$,

$$(\boldsymbol{\Gamma}^*(\boldsymbol{y}_1 * \boldsymbol{y}_2))(\gamma) = (\boldsymbol{\Gamma}^*\boldsymbol{y}_1)(\gamma)(\boldsymbol{\Gamma}^*\boldsymbol{y}_2)(\gamma), \tag{2.26}$$

$$(\boldsymbol{\Gamma}^*\widetilde{\boldsymbol{y}})(\gamma) = \overline{(\boldsymbol{\Gamma}^*\boldsymbol{y})(\gamma)}, \tag{2.27}$$

$$(\boldsymbol{\Gamma}^*(\boldsymbol{y} * \widetilde{\boldsymbol{y}}))(\gamma) = \left|(\boldsymbol{\Gamma}^*\boldsymbol{y})(\gamma)\right|^2. \tag{2.28}$$

2.6 Harmonic Frames

Again let $\mathcal{G}$ be a finite abelian group, and let $\widehat{\mathcal{G}}$ be its Pontryagin dual. Let $\mathcal{D}$ be a subset of $\mathcal{G}$ of order $D \geq 1$. The *characteristic function* of $\mathcal{D}$ is $\boldsymbol{\chi}_{\mathcal{D}} \in \mathbb{C}^{\mathcal{G}}$,

$$\boldsymbol{\chi}_{\mathcal{D}}(g) := \left\{\begin{array}{ll} 1, & g \in \mathcal{D}, \\ 0, & g \notin \mathcal{D}. \end{array}\right.$$

Equivalently, $\boldsymbol{\chi}_{\mathcal{D}} = \sum_{d\in\mathcal{D}} \boldsymbol{\delta}_d$. Translating $\boldsymbol{\chi}_{\mathcal{D}}$ by any given $g \in \mathcal{G}$ yields the characteristic function of $g + \mathcal{D} := \{g + d : d \in \mathcal{D}\}$ since

$$(\boldsymbol{T}^g\boldsymbol{\chi}_{\mathcal{D}})(g') = \boldsymbol{\chi}_{\mathcal{D}}(g' - g) = \left\{\begin{array}{ll} 1, & g' - g \in \mathcal{D} \\ 0, & g' - g \notin \mathcal{D} \end{array}\right\} = \left\{\begin{array}{ll} 1, & g' \in g + \mathcal{D} \\ 0, & g' \notin g + \mathcal{D} \end{array}\right\} = \boldsymbol{\chi}_{g+\mathcal{D}}(g'). \tag{2.29}$$

For any such $\mathcal{D}$, the corresponding *harmonic frame* is the normalized restrictions of the characters of $\mathcal{G}$ to $\mathcal{D}$, namely the finite sequence $(\boldsymbol{\varphi}_\gamma)_{\gamma\in\widehat{\mathcal{G}}}$ in $\mathbb{C}^{\mathcal{D}}$ defined $\boldsymbol{\varphi}_\gamma(d) := \frac{1}{\sqrt{D}}\boldsymbol{\gamma}(d)$. Any such frame is clearly unit norm, and is moreover tight since for any $d_1, d_2 \in \mathcal{D}$,

$$(\boldsymbol{\Phi}\boldsymbol{\Phi}^*)(d_1, d_2) = \sum_{\gamma\in\widehat{\mathcal{G}}} \boldsymbol{\varphi}_\gamma(d_1)\overline{\boldsymbol{\varphi}_\gamma(d_2)} = \frac{1}{D}\sum_{\gamma\in\widehat{\mathcal{G}}} \boldsymbol{\gamma}(d_1)\overline{\boldsymbol{\gamma}(d_2)} = \frac{1}{D}(\boldsymbol{\Gamma}\boldsymbol{\Gamma}^*)(d_1, d_2) = \frac{G}{D}\mathbf{I}(d_1, d_2).$$

In particular, $(\boldsymbol{\varphi}_\gamma)_{\gamma\in\widehat{\mathcal{G}}}$ is a $\mathrm{UNTF}_{\mathbb{C}}(D,G)$ for $\mathbb{C}^{\mathcal{D}}$. For any $\gamma_1,\gamma_2\in\widehat{\mathcal{G}}$,

$$\langle\boldsymbol{\varphi}_{\gamma_1},\boldsymbol{\varphi}_{\gamma_2}\rangle=\sum_{d\in\mathcal{D}}\overline{\boldsymbol{\varphi}_{\gamma_1}(d)}\boldsymbol{\varphi}_{\gamma_2}(d)=\frac{1}{D}\sum_{g\in\mathcal{G}}\overline{(\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2^{-1})(g)}\boldsymbol{\chi}_{\mathcal{D}}(g)=\frac{1}{D}\langle\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2^{-1},\boldsymbol{\chi}_{\mathcal{D}}\rangle.$$

Since $\widehat{\mathcal{G}}$ is a group, this yields the following expression for the corresponding entry of the Gram matrix of this frame in terms of the DFT of the characteristic function of $\mathcal{D}$:

$$(\boldsymbol{\Phi}^*\boldsymbol{\Phi})(\gamma_1,\gamma_2)=\langle\boldsymbol{\varphi}_{\gamma_1},\boldsymbol{\varphi}_{\gamma_2}\rangle=\frac{1}{D}(\boldsymbol{\Gamma}^*\boldsymbol{\chi}_{\mathcal{D}})(\gamma_1\gamma_2^{-1}). \tag{2.30}$$

As evidenced in (2.11), we are often interested in the modulus squared of such Gram matrix entries. Accordingly, we make note of the following relation, which follows by (2.28):

$$\big|\langle\boldsymbol{\varphi}_{\gamma_1},\boldsymbol{\varphi}_{\gamma_2}\rangle\big|^2=\Big|\frac{1}{D}(\boldsymbol{\Gamma}^*\boldsymbol{\chi}_{\mathcal{D}})(\gamma_1\gamma_2^{-1})\Big|^2=\frac{1}{D^2}\big[\boldsymbol{\Gamma}^*(\boldsymbol{\chi}_{\mathcal{D}}*\widetilde{\boldsymbol{\chi}}_{\mathcal{D}})\big](\gamma_1\gamma_2^{-1}). \tag{2.31}$$

Example 2.6. We now discuss an instance of a harmonic $\mathrm{UNTF}_{\mathbb{C}}(2,5)$. Let $\mathcal{G}$ be the finite abelian group $\mathbb{Z}_5$, whose characters are the corresponding discrete Fourier basis $(\mathbf{e}_n)_{n=0}^4$ discussed in Example 2.5. For our 2-element subset of $\mathbb{Z}_5$, we choose $\mathcal{D}:=\{1,4\}$. The synthesis map $\boldsymbol{\Phi}$ of the corresponding harmonic frame $(\boldsymbol{\varphi}_n)_{n\in\mathbb{Z}_5}$ in $\mathbb{C}^{\mathcal{D}}$ is given by

$$\boldsymbol{\Phi}=\frac{1}{\sqrt{2}}\begin{bmatrix}1&\omega&\omega^2&\omega^3&\omega^4\\1&\omega^4&\omega^3&\omega^2&\omega\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}1&\omega&\omega^2&\omega^3&\omega^4\\1&\overline{\omega}&\overline{\omega^2}&\overline{\omega^3}&\overline{\omega^4}\end{bmatrix}.$$

We note that $\boldsymbol{\Phi}$ is effectively constructed by extracting the rows indexed by $\mathcal{D}$ from the character table $\boldsymbol{\Gamma}$ given in (2.23) and scaling the result by $\frac{1}{\sqrt{2}}$. Since the rows of $\boldsymbol{\Gamma}$ are equal-norm orthogonal, the same is true for the rows of $\boldsymbol{\Phi}$, and so this frame is indeed a $\mathrm{UNTF}_{\mathbb{C}}(2,5)$ for $\mathbb{C}^{\mathcal{D}}$.

III. Stratified Unit Norm Tight Frames

As explained in the previous chapter, it is well known that every ECTFF is a type of optimal Grassmannian code, having the property that the smallest chordal distance between any pair of its members is as large as possible. In this chapter, we provide the first major contributions of this thesis. Specifically, in Definition 3.1, we introduce a new type of mathematical object which we call a *stratified* finite sequence of unit norm tight frames. In Theorem 3.4, we then show how to convert any such object into an ECTFF. In the next chapter, we will apply this general theory to nontrivial new constructions of such objects to obtain infinitely many apparently new ECTFFs.

Definition 3.1. Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$, let $D \geq 1$, $N \geq 2$ and $R \geq 1$ be integers, and let $\mathcal{N}$ and $\mathcal{R}$ be finite sets of cardinality N and R, respectively. For each $r \in \mathcal{R}$, let $(\boldsymbol{\psi}_{r,n})_{n\in\mathcal{N}}$ be a $\mathrm{UNTF}_{\mathbb{F}}(D,N)$. We say that $\bigl((\boldsymbol{\psi}_{r,n})_{n\in\mathcal{N}}\bigr)_{r\in\mathcal{R}}$ is *stratified* if there exists $\Lambda \geq 0$ such that

$$\frac{1}{R}\sum_{r\in\mathcal{R}}\bigl|\langle\boldsymbol{\psi}_{r,n_1},\boldsymbol{\psi}_{r,n_2}\rangle\bigr|^2 = \Lambda \tag{3.1}$$

for all distinct $n_1, n_2 \in \mathcal{N}$. When this occurs, we call $\bigl((\boldsymbol{\psi}_{r,n})_{n\in\mathcal{N}}\bigr)_{r\in\mathcal{R}}$ a *stratified finite sequence of unit norm tight frames* (SUNTF).

We now provide an example of a SUNTF constructed from two $\mathrm{UNTF}_{\mathbb{C}}(2,5)$:

Example 3.2. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two subsets of $\mathbb{Z}_5$ of cardinality 2, given by $\mathcal{D}_1 := \{1,4\}$ and $\mathcal{D}_2 := \{2,3\}$. It turns out these subsets form a difference family for $\mathbb{Z}_5$, a connection we further explore in Chapter 4. For each $r \in [2]$, we form the harmonic frame $(\boldsymbol{\psi}_{r,n})_{n\in\mathbb{Z}_5}$ corresponding to $\mathcal{D}_r$. As discussed in Example 2.6, this is done by extracting the rows indexed by $\mathcal{D}_r$ from the character table for $\mathbb{Z}_5$ and scaling by $\frac{1}{\sqrt{2}}$, as shown below:

$$\begin{aligned}
\boldsymbol{\Psi}_1 &= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \omega & \omega^2 & \omega^3 & \omega^4 \\ 1 & \omega^4 & \omega^3 & \omega^2 & \omega \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \omega & \omega^2 & \omega^3 & \omega^4 \\ 1 & \overline{\omega} & \overline{\omega^2} & \overline{\omega^3} & \overline{\omega^4} \end{bmatrix},\\
\boldsymbol{\Psi}_2 &= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \omega^2 & \omega^4 & \omega & \omega^3 \\ 1 & \omega^3 & \omega & \omega^4 & \omega^2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \omega^2 & \omega^4 & \omega & \omega^3 \\ 1 & \overline{\omega^2} & \overline{\omega^4} & \overline{\omega} & \overline{\omega^3} \end{bmatrix}.
\end{aligned}$$

For each $r \in [2]$, $\boldsymbol{\Psi}_r$ is the synthesis map of $(\boldsymbol{\psi}_{r,n})_{n\in\mathbb{Z}_5}$, which is a harmonic $\mathrm{UNTF}_{\mathbb{C}}(2,5)$. To then show these two harmonic frames are stratified, we note that for any $r \in [2]$ and $n_1, n_2 \in \mathbb{Z}_5$, it follows that

$$\langle \boldsymbol{\psi}_{r,n_1}, \boldsymbol{\psi}_{r,n_2}\rangle = \frac{1}{2}\big[\,\overline{\omega^{rn_1}}\ \ \omega^{rn_1}\,\big]\begin{bmatrix}\omega^{rn_2}\\ \overline{\omega^{rn_2}}\end{bmatrix} = \frac{1}{2}\big(\omega^{r(n_1-n_2)} + \overline{\omega^{r(n_1-n_2)}}\big) = \cos\left(\frac{2\pi r(n_1-n_2)}{5}\right). \tag{3.2}$$

We then define the scalars $c, d \in \mathbb{R}$ in the following manner:

$$\begin{aligned} c &:= \frac{-1+\sqrt{5}}{4} = \cos\left(\frac{2\pi}{5}\right) = \cos\left(\frac{8\pi}{5}\right) = \cos\left(\frac{12\pi}{5}\right),\\ d &:= \frac{-1-\sqrt{5}}{4} = \cos\left(\frac{4\pi}{5}\right) = \cos\left(\frac{6\pi}{5}\right) = \cos\left(\frac{16\pi}{5}\right). \end{aligned}$$

Recalling by (2.4) that the entries of the Gram matrices are the inner products between the corresponding vectors, we then have

$$\boldsymbol{\Psi}_1^*\boldsymbol{\Psi}_1 = \begin{bmatrix} 1 & c & d & d & c\\ c & 1 & c & d & d\\ d & c & 1 & c & d\\ d & d & c & 1 & c\\ c & d & d & c & 1\end{bmatrix}, \quad \boldsymbol{\Psi}_2^*\boldsymbol{\Psi}_2 = \begin{bmatrix} 1 & d & c & c & d\\ d & 1 & d & c & c\\ c & d & 1 & d & c\\ c & c & d & 1 & d\\ d & c & c & d & 1\end{bmatrix}.$$

The entrywise average of the entrywise modulus squared of these two Gram matrices is

$$\frac{1}{2}\Big(|\boldsymbol{\Psi}_1^*\boldsymbol{\Psi}_1|^2 + |\boldsymbol{\Psi}_2^*\boldsymbol{\Psi}_2|^2\Big) = \begin{bmatrix} 1 & e & e & e & e\\ e & 1 & e & e & e\\ e & e & 1 & e & e\\ e & e & e & 1 & e\\ e & e & e & e & 1\end{bmatrix}, \quad e = \frac{1}{2}(c^2+d^2).$$

Thus, $\big((\boldsymbol{\psi}_{r,n})_{n\in\mathbb{Z}_5}\big)_{r=1}^2$ is stratified, satisfying (3.1) for any distinct $n_1, n_2 \in \mathbb{Z}_5$, where

$$\Lambda = e = \frac{1}{2}(c^2+d^2) = \frac{1}{2}\left[\Big(\frac{-1+\sqrt{5}}{4}\Big)^2 + \Big(\frac{-1-\sqrt{5}}{4}\Big)^2\right] = \frac{3}{8}. \tag{3.3}$$

Remark 3.3. As discussed in Section 2.1, since each of the Gram matrices in the aforementioned example are real-valued, we could have chosen our frames to reside in $\mathbb{R}^2$ instead of $\mathbb{C}^2$ via the application of a suitable unitary transformation.

We now provide the fundamental contribution of this thesis: a method of constructing an ECTFF from any SUNTF.

Theorem 3.4. Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$, let $D \geq 1$, $N \geq 2$ and $R \geq 1$ be integers, and let $\mathcal{N}$ and $\mathcal{R}$ be finite sets of cardinality N and R, respectively. For each $r \in \mathcal{R}$, let $(\boldsymbol{\psi}_{r,n})_{n\in\mathcal{N}}$ be a $\mathrm{UNTF}_{\mathbb{F}}(D,N)$ in some Euclidean space $\mathcal{E}_r$. For each $n \in \mathcal{N}$ and $r \in \mathcal{R}$, define

$$\boldsymbol{\varphi}_{n,r} \in \bigoplus_{r'\in\mathcal{R}} \mathcal{E}_{r'}, \quad \boldsymbol{\varphi}_{n,r} := \begin{cases} \boldsymbol{\psi}_{r',n}, & r' = r, \\ \mathbf{0}, & r' \neq r. \end{cases}$$

For each $n \in \mathcal{N}$, let $\boldsymbol{\Phi}_n$ be the synthesis map of $(\boldsymbol{\varphi}_{n,r})_{r\in\mathcal{R}}$ and $\mathcal{V}_n := \operatorname{span}(\boldsymbol{\varphi}_{n,r})_{r\in\mathcal{R}}$. Then:

(a) For each $n_1, n_2 \in \mathcal{N}$, $\boldsymbol{\Phi}_{n_1}^*\boldsymbol{\Phi}_{n_2}$ is diagonal, with $\langle \boldsymbol{\psi}_{r,n_1}, \boldsymbol{\psi}_{r,n_2}\rangle$ as its rth diagonal entry.

(b) For each $n \in \mathcal{N}$, $\boldsymbol{\Phi}_n$ is an isometry, that is, $(\boldsymbol{\varphi}_{n,r})_{r\in\mathcal{R}}$ is orthonormal.

(c) $(\mathcal{V}_n)_{n\in\mathcal{N}}$ is a $\mathrm{TFF}_{\mathbb{F}}(DR, R, N)$.

(d) $(\mathcal{V}_n)_{n\in\mathcal{N}}$ is equichordal if and only if $\big((\boldsymbol{\psi}_{r,n})_{n\in\mathcal{N}}\big)_{r\in\mathcal{R}}$ is stratified.

Our formal proof of this result is given in Section 3.2. It relies on some technical lemmas that we provide in Section 3.1. We now help motivate these ideas with an example.

Example 3.5. To illustrate the method behind Theorem 3.4, we will construct an $\mathrm{ECTFF}_{\mathbb{C}}(4,2,5)$ from the SUNTF $\big((\boldsymbol{\psi}_{r,n})_{n\in\mathbb{Z}_5}\big)_{r=1}^2$ given in Example 3.2. Recalling the synthesis maps $\boldsymbol{\Psi}_1$ and $\boldsymbol{\Psi}_2$, we let $\boldsymbol{\Upsilon}$ be the 4×10 block diagonal matrix with $\boldsymbol{\Psi}_1$ and $\boldsymbol{\Psi}_2$ as its diagonal blocks:

$$\boldsymbol{\Upsilon} := \begin{bmatrix} \boldsymbol{\Psi}_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Psi}_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \left[\begin{array}{ccccc|ccccc} 1 & \omega & \omega^2 & \omega^3 & \omega^4 & 0 & 0 & 0 & 0 & 0 \\ 1 & \overline{\omega} & \overline{\omega^2} & \overline{\omega^3} & \overline{\omega^4} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & \omega^2 & \omega^4 & \omega & \omega^3 \\ 0 & 0 & 0 & 0 & 0 & 1 & \overline{\omega^2} & \overline{\omega^4} & \overline{\omega} & \overline{\omega^3} \end{array}\right].$$

Since each column of both $\boldsymbol{\Psi}_1$ and $\boldsymbol{\Psi}_2$ are unit norm, the same is true for each column of $\boldsymbol{\Upsilon}$. Moreover, since the rows of $\boldsymbol{\Psi}_1$ are orthogonal and have norm $\sqrt{\frac{5}{2}}$, and since the same is true for the rows $\boldsymbol{\Psi}_2$, the rows of $\boldsymbol{\Upsilon}$ have these same properties. As such, $\boldsymbol{\Upsilon}$ is the synthesis map of a $\mathrm{UNTF}_{\mathbb{C}}(4,10)$ for $\mathbb{C}^4$. The Gram matrix $\boldsymbol{\Upsilon}^*\boldsymbol{\Upsilon}$ of this UNTF is block diagonal, having $\boldsymbol{\Psi}_1^*\boldsymbol{\Psi}_1$ and $\boldsymbol{\Psi}_2^*\boldsymbol{\Psi}_2$ as its two 5×5 diagonal blocks:

$$\boldsymbol{\Upsilon}^*\boldsymbol{\Upsilon} = \begin{bmatrix}\boldsymbol{\Psi}_1^*\boldsymbol{\Psi}_1 & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\Psi}_2^*\boldsymbol{\Psi}_2\end{bmatrix} = \left[\begin{array}{ccccc|ccccc}
1 & c & d & d & c & 0 & 0 & 0 & 0 & 0\\
c & 1 & c & d & d & 0 & 0 & 0 & 0 & 0\\
d & c & 1 & c & d & 0 & 0 & 0 & 0 & 0\\
d & d & c & 1 & c & 0 & 0 & 0 & 0 & 0\\
c & d & d & c & 1 & 0 & 0 & 0 & 0 & 0\\
\hline
0 & 0 & 0 & 0 & 0 & 1 & d & c & c & d\\
0 & 0 & 0 & 0 & 0 & d & 1 & d & c & c\\
0 & 0 & 0 & 0 & 0 & c & d & 1 & d & c\\
0 & 0 & 0 & 0 & 0 & c & c & d & 1 & d\\
0 & 0 & 0 & 0 & 0 & d & c & c & d & 1
\end{array}\right]. \tag{3.4}$$

To see how $\boldsymbol{\Upsilon}$ gives rise to an ECTFF, we now perfectly shuffle its columns, permuting its two sequences of five columns apiece into five sequences of two columns apiece, yielding

$$\boldsymbol{\Phi} := \frac{1}{\sqrt{2}}\left[\begin{array}{cc|cc|cc|cc|cc}
1 & 0 & \omega & 0 & \omega^2 & 0 & \omega^3 & 0 & \omega^4 & 0\\
1 & 0 & \overline{\omega} & 0 & \overline{\omega^2} & 0 & \overline{\omega^3} & 0 & \overline{\omega^4} & 0\\
0 & 1 & 0 & \omega^2 & 0 & \omega^4 & 0 & \omega & 0 & \omega^3\\
0 & 1 & 0 & \overline{\omega^2} & 0 & \overline{\omega^4} & 0 & \overline{\omega} & 0 & \overline{\omega^3}
\end{array}\right].$$

Since such permutations preserve both norms of columns and inner products of rows, this matrix $\boldsymbol{\Phi}$ is itself the synthesis map of a $\mathrm{UNTF}_{\mathbb{C}}(4,10)$ for $\mathbb{C}^4$. The Gram matrix of $\boldsymbol{\Phi}$ is obtained by perfectly shuffling both the rows and columns of that of $\boldsymbol{\Upsilon}$ in this same manner; doing so converts the 2×2 block-diagonal array of 5×5 blocks given in (3.4) into the following

5×5 array of 2×2 diagonal blocks:

$$\boldsymbol{\Phi}^*\boldsymbol{\Phi} = \left[\begin{array}{cc|cc|cc|cc|cc} 1 & 0 & c & 0 & d & 0 & d & 0 & c & 0 \\ 0 & 1 & 0 & d & 0 & c & 0 & c & 0 & d \\ \hline c & 0 & 1 & 0 & c & 0 & d & 0 & d & 0 \\ 0 & d & 0 & 1 & 0 & d & 0 & c & 0 & c \\ \hline d & 0 & c & 0 & 1 & 0 & c & 0 & d & 0 \\ 0 & c & 0 & d & 0 & 1 & 0 & d & 0 & c \\ \hline d & 0 & d & 0 & c & 0 & 1 & 0 & c & 0 \\ 0 & c & 0 & c & 0 & d & 0 & 1 & 0 & d \\ \hline c & 0 & d & 0 & d & 0 & c & 0 & 1 & 0 \\ 0 & d & 0 & c & 0 & c & 0 & d & 0 & 1 \end{array}\right]. \tag{3.5}$$

Notably, each diagonal block of $\boldsymbol{\Phi}^*\boldsymbol{\Phi}$ is a 2×2 identity matrix. This means that each of these five pairs of columns of $\boldsymbol{\Phi}$ is an orthonormal pair, namely that $\boldsymbol{\Phi}$ is in fact the synthesis map of the concatenation of orthonormal bases for a finite sequence $(\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4)$ of five two-dimensional subspaces of $\mathbb{C}^4$. Each of these subspaces is the image of one of the following five corresponding 4×2 isometries:

$$\boldsymbol{\Phi}_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \ \boldsymbol{\Phi}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} \omega & 0 \\ \overline{\omega} & 0 \\ 0 & \omega^2 \\ 0 & \overline{\omega^2} \end{bmatrix}, \ \boldsymbol{\Phi}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} \omega^2 & 0 \\ \overline{\omega^2} & 0 \\ 0 & \omega^4 \\ 0 & \overline{\omega^4} \end{bmatrix}, \ \boldsymbol{\Phi}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} \omega^3 & 0 \\ \overline{\omega^3} & 0 \\ 0 & \omega \\ 0 & \overline{\omega} \end{bmatrix}, \ \boldsymbol{\Phi}_4 = \frac{1}{\sqrt{2}}\begin{bmatrix} \omega^4 & 0 \\ \overline{\omega^4} & 0 \\ 0 & \omega^3 \\ 0 & \overline{\omega^3} \end{bmatrix}.$$

Since $\boldsymbol{\Phi}$ is the synthesis map of a $\mathrm{UNTF}_{\mathbb{C}}(4,10)$ for $\mathbb{C}^4$, this finite sequence $(\mathcal{V}_n)_{n\in\mathbb{Z}_5}$ is a $\mathrm{TFF}_{\mathbb{C}}(4,2,5)$ for $\mathbb{C}^4$. As we now further explain, $(\mathcal{V}_n)_{n\in\mathbb{Z}_5}$ is moreover equichordal, and so is in fact an $\mathrm{ECTFF}_{\mathbb{C}}(4,2,5)$ for $\mathbb{C}^4$. To see this, note that for any $n_1, n_2 \in \mathbb{Z}_5$, the corresponding 2×2 cross-Gram matrix $\boldsymbol{\Phi}_{n_1}^*\boldsymbol{\Phi}_{n_2}$ appears as the (n_1, n_2)th block of (3.5). It thus suffices to show that the off-diagonal 2×2 blocks of (3.5) have constant Frobenius norm. Notably, all such blocks are diagonal; as detailed below, this is a feature of the construction of Theorem 3.4 in general. As such, this simplifies to showing that the "diagonals" of these off-diagonal blocks have constant Euclidean norm. In this example, this is particularly obvious since $c^2 + d^2 = d^2 + c^2$. In general, as we shall explain, this is due to the fact that $\bigl((\boldsymbol{\psi}_{r,n})_{n\in\mathcal{N}}\bigr)_{r\in\mathcal{R}}$ is stratified in the sense of Definition 3.1. Along these lines, the chordal constant here is identical to the stratification constant determined in Example 3.2, namely $\frac{NR-C}{C(N-1)} = \frac{5(2)-4}{4(5-1)} = \frac{3}{8}$.

3.1 Striped Matrices

In this section, we discuss some technical results that we will use to prove Theorem 3.4. Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$, let $\mathcal{N}$ and $\mathcal{R}$ be finite nonempty sets, and let $\boldsymbol{G} \in \mathbb{F}^{(\mathcal{N}\times\mathcal{R})\times(\mathcal{N}\times\mathcal{R})}$. For any $n_1, n_2 \in \mathcal{N}$ and any $r, r_1, r_2 \in \mathcal{R}$, we define $\boldsymbol{G}_{n_1,n_2} \in \mathbb{F}^{\mathcal{R}\times\mathcal{R}}$ and $\boldsymbol{G}_r \in \mathbb{F}^{\mathcal{N}\times\mathcal{N}}$ by

$$\boldsymbol{G}_{n_1,n_2}(r_1, r_2) := \boldsymbol{G}((n_1, r_1), (n_2, r_2)), \quad \boldsymbol{G}_r(n_1, n_2) := \boldsymbol{G}((n_1, r), (n_2, r)). \tag{3.6}$$

We say a matrix $\boldsymbol{G} \in \mathbb{F}^{(\mathcal{N}\times\mathcal{R})\times(\mathcal{N}\times\mathcal{R})}$ is *striped* if $\boldsymbol{G}_{n_1,n_2}$ is diagonal for all $n_1, n_2 \in \mathcal{N}$, that is, if $\boldsymbol{G}((n_1, r_1), (n_2, r_2)) = 0$ for any $n_1, n_2 \in \mathcal{N}$ and any distinct $r_1, r_2 \in \mathcal{R}$. When this occurs, $\boldsymbol{G}$ is fully characterized by $(\boldsymbol{G}_r)_{r\in\mathcal{R}}$. Properties of striped matrices are discussed below.

Lemma 3.6. If $\boldsymbol{G} \in \mathbb{F}^{(\mathcal{N}\times\mathcal{R})\times(\mathcal{N}\times\mathcal{R})}$ is striped then so is $\boldsymbol{G}^*$.

Proof. Since $\boldsymbol{G}$ is striped, for any $n_1, n_2 \in \mathcal{N}$ and any distinct $r_1, r_2 \in \mathcal{R}$, we have

$$\boldsymbol{G}^*((n_1, r_1), (n_2, r_2)) = \overline{\boldsymbol{G}((n_2, r_2), (n_1, r_1))} = \overline{0} = 0. \qquad \square$$

Lemma 3.7. If $\boldsymbol{G} \in \mathbb{F}^{(\mathcal{N}\times\mathcal{R})\times(\mathcal{N}\times\mathcal{R})}$ is striped then so is $\boldsymbol{G}^2$ and $(\boldsymbol{G}^2)_r = (\boldsymbol{G}_r)^2$ for each $r \in \mathcal{R}$.

Proof. Since $\boldsymbol{G}$ is striped, for any $n_1, n_2 \in \mathcal{N}$ and any distinct $r_1, r_2 \in \mathcal{R}$, we note

$$\begin{aligned}
\boldsymbol{G}^2((n_1, r_1), (n_2, r_2)) &= \sum_{(n,r)\in(\mathcal{N}\times\mathcal{R})} \boldsymbol{G}((n_1, r_1), (n, r))\boldsymbol{G}((n, r), (n_2, r_2)) \\
&= \sum_{n\in\mathcal{N}} \sum_{r\in\mathcal{R}} \boldsymbol{G}((n_1, r_1), (n, r))\boldsymbol{G}((n, r), (n_2, r_2)).
\end{aligned}$$

We note by the striped property, $\boldsymbol{G}((n_1, r_1), (n, r))$ can only be nonzero if $r = r_1$. Yet, it similarly follows that $\boldsymbol{G}((n, r), (n_2, r_2))$ can only be nonzero if $r = r_2$. Since $r_1 \neq r_2$ by assumption, the above sum is always zero, since each summand is a product of at least one zero term. Thus, $\boldsymbol{G}^2((n_1, r_1), (n_2, r_2)) = 0$ for any distinct $r_1, r_2 \in \mathcal{R}$, and so $\boldsymbol{G}^2$ is striped.

We next note that for any $r \in \mathcal{R}$ and $n_1, n_2 \in \mathcal{N}$ that

$$
\begin{aligned}
(\boldsymbol{G}^2)_r(n_1, n_2) &= \boldsymbol{G}^2((n_1, r), (n_2, r)) \\
&= \sum_{(n_3, r_3) \in (\mathcal{N} \times \mathcal{R})} \boldsymbol{G}((n_1, r), (n_3, r_3)) \boldsymbol{G}((n_3, r_3), (n_2, r)) \\
&= \sum_{n_3 \in \mathcal{N}} \sum_{r_3 \in \mathcal{R}} \boldsymbol{G}((n_1, r), (n_3, r_3)) \boldsymbol{G}((n_3, r_3), (n_2, r)).
\end{aligned}
$$

Again by the striped property, we note both $\boldsymbol{G}((n_1, r), (n_3, r_3))$ and $\boldsymbol{G}((n_3, r_3), (n_2, r))$ can only be nonzero if $r = r_3$. As such, all other summands $r_3 \neq r$ are zero, so our sum becomes

$$
\begin{aligned}
(\boldsymbol{G}^2)_r(n_1, n_2) &= \sum_{n_3 \in \mathcal{N}} \boldsymbol{G}((n_1, r), (n_3, r)) \boldsymbol{G}((n_3, r), (n_2, r)) \\
&= \sum_{n_3 \in \mathcal{N}} \boldsymbol{G}_r(n_1, n_3) \boldsymbol{G}_r(n_3, n_2) \\
&= (\boldsymbol{G}_r)^2(n_1, n_2). \qquad \square
\end{aligned}
$$

The following lemma captures several properties of striped matrices in a manner similar to the statement of Theorem 3.4.

Lemma 3.8. Suppose $\boldsymbol{G} \in \mathbb{F}^{(\mathcal{N} \times \mathcal{R}) \times (\mathcal{N} \times \mathcal{R})}$ is striped. Then:

(a) $\boldsymbol{G} = \boldsymbol{G}^*$ if and only if $\boldsymbol{G}_r = \boldsymbol{G}_r^*$ for each $r \in \mathcal{R}$.

(b) For any $A > 0$, $\boldsymbol{G}^2 = A\boldsymbol{G}$ if and only if $(\boldsymbol{G}_r)^2 = A\boldsymbol{G}_r$ for each $r \in \mathcal{R}$.

(c) $\boldsymbol{G}_{n,n} = \mathbf{I}_{\mathcal{R}}$ for each $n \in \mathcal{N}$ if and only if $\boldsymbol{G}_r(n, n) = 1$ for each $r \in \mathcal{R}$, $n \in \mathcal{N}$.

(d) For any distinct $n_1, n_2 \in \mathcal{N}$,

$$
\frac{1}{R} \sum_{r_1 \in \mathcal{R}} \sum_{r_2 \in \mathcal{R}} \left| \boldsymbol{G}_{n_1, n_2}(r_1, r_2) \right|^2 = \frac{1}{R} \sum_{r \in \mathcal{R}} \left| \boldsymbol{G}_r(n_1, n_2) \right|^2.
$$

Proof. Let $\boldsymbol{G} \in \mathbb{F}^{(\mathcal{N} \times \mathcal{R}) \times (\mathcal{N} \times \mathcal{R})}$ be striped.

(a) ($\Rightarrow$) Further suppose that $\boldsymbol{G} = \boldsymbol{G}^*$. For any $r \in \mathcal{R}$, $n_1, n_2 \in \mathcal{N}$, Lemma 3.6 gives

$$\boldsymbol{G}_r(n_1, n_2) = \boldsymbol{G}((n_1, r), (n_2, r)) = \boldsymbol{G}^*((n_1, r), (n_2, r)).$$

Since the adjoint of a matrix is its conjugate transpose, this implies that

$$\boldsymbol{G}_r(n_1, n_2) = \overline{\boldsymbol{G}((n_2, r), (n_1, r))} = \overline{\boldsymbol{G}_r(n_2, n_1)} = \boldsymbol{G}_r^*(n_1, n_2),$$

so $\boldsymbol{G}_r = \boldsymbol{G}_r^*$. ($\Leftarrow$) Conversely, suppose $\boldsymbol{G}_r = \boldsymbol{G}_r^*$ for each $r \in \mathcal{R}$. We show that $\boldsymbol{G}((n_1, r_1), (n_2, r_2)) = \boldsymbol{G}^*((n_1, r_1), (n_2, r_2))$ for all $n_1, n_2 \in \mathcal{N}$ and $r_1, r_2 \in \mathcal{R}$. By Lemma 3.6, $\boldsymbol{G}^*$ is striped, thus for any $n_1, n_2 \in \mathcal{N}$ and $r_1, r_2 \in \mathcal{R}$ such that $r_1 \neq r_2$, we have $\boldsymbol{G}^*((n_1, r_1), (n_2, r_2)) = 0 = \boldsymbol{G}((n_1, r_1), (n_2, r_2))$. In the remaining case where r_1 and r_2 are equal, letting $r := r_1 = r_2$, we have

$$\boldsymbol{G}^*((n_1, r), (n_2, r)) = \boldsymbol{G}_r^*(n_1, n_2) = \boldsymbol{G}_r(n_1, n_2) = \boldsymbol{G}((n_1, r_1), (n_2, r_2)).$$

(b) Fix $A > 0$. ($\Rightarrow$) Suppose that $\boldsymbol{G}^2 = A\boldsymbol{G}$. By Lemma 3.7, for each $r \in \mathcal{R}$,

$$(\boldsymbol{G}_r)^2 = (\boldsymbol{G}^2)_r = (A\boldsymbol{G})_r = A\boldsymbol{G}_r.$$

($\Leftarrow$) Suppose that $(\boldsymbol{G}_r)^2 = A\boldsymbol{G}_r$ for each $r \in \mathcal{R}$. It suffices to show that $\boldsymbol{G}^2((n_1, r_1), (n_2, r_2)) = A\boldsymbol{G}((n_1, r_1), (n_2, r_2))$ for all $n_1, n_2 \in \mathcal{N}$ and $r_1, r_2 \in \mathcal{R}$. This is immediate when $r_1 \neq r_2$ since Lemma 3.7 gives that both of these quantities vanish. When instead $r_1 = r_2$,

$$\boldsymbol{G}^2((n_1, r), (n_2, r)) = (\boldsymbol{G}^2)_r(n_1, n_2) = A\boldsymbol{G}_r(n_1, n_2) = A\boldsymbol{G}((n_1, r), (n_2, r)).$$

(c) ($\Rightarrow$) Suppose that $\boldsymbol{G}_{n,n} = \mathbf{I}_{\mathcal{R}}$ for each $n \in \mathcal{N}$. For any $r \in \mathcal{R}$,

$$\boldsymbol{G}_r(n,n) = \boldsymbol{G}((n,r),(n,r)) = \boldsymbol{G}_{n,n}(r,r) = \mathbf{I}_{\mathcal{R}}(r,r) = 1.$$

($\Leftarrow$) Conversely, suppose that $\boldsymbol{G}_r(n,n) = 1$ for each $r \in \mathcal{R}$ and $n \in \mathcal{N}$. Since $\boldsymbol{G}$ is striped, we have that $\boldsymbol{G}_{n,n}(r_1,r_2) = \boldsymbol{G}((n,r_1),(n,r_2)) = 0$ for distinct $r_1, r_2 \in \mathcal{R}$. When $r_1 = r_2$, we instead have $\boldsymbol{G}_{n,n}(r,r) = \boldsymbol{G}((n,r),(n,r)) = \boldsymbol{G}_r(n,n) = 1$. Thus, for any $n \in \mathcal{N}$, $\boldsymbol{G}_{n,n} = \mathbf{I}_{\mathcal{R}}$.

(d) For any distinct $n_1, n_2 \in \mathcal{N}$, since $\boldsymbol{G}$ is striped, we know $\boldsymbol{G}_{n_1,n_2}$ is diagonal, so

$$\begin{aligned}
\frac{1}{R}\sum_{r_1\in\mathcal{R}}\sum_{r_2\in\mathcal{R}}\bigl|\boldsymbol{G}_{n_1,n_2}(r_1,r_2)\bigr|^2 &= \frac{1}{R}\sum_{r\in\mathcal{R}}\bigl|\boldsymbol{G}_{n_1,n_2}(r,r)\bigr|^2\\
&= \frac{1}{R}\sum_{r\in\mathcal{R}}\bigl|\boldsymbol{G}\bigl((n_1,r),(n_2,r)\bigr)\bigr|^2\\
&= \frac{1}{R}\sum_{r\in\mathcal{R}}\bigl|\boldsymbol{G}_r(n_1,n_2)\bigr|^2.
\end{aligned}$$

□

3.2 Formally Validating our Construction Technique

Proof of Theorem 3.4. Recalling the conditions and definitions presented in the statement of Theorem 3.4, we proceed with the proof of each claim:

(a) For any $n_1, n_2 \in \mathcal{N}$ and $r_1, r_2 \in \mathcal{R}$, the (r_1,r_2)th entry of the (n_1,n_2)th cross-Gram matrix $\boldsymbol{\Phi}_{n_1}^*\boldsymbol{\Phi}_{n_2} \in \mathbb{F}^{\mathcal{R}\times\mathcal{R}}$ is

$$(\boldsymbol{\Phi}_{n_1}^*\boldsymbol{\Phi}_{n_2})(r_1,r_2) = \bigl\langle \boldsymbol{\varphi}_{n_1,r_1}, \boldsymbol{\varphi}_{n_2,r_2}\bigr\rangle = \sum_{r\in\mathcal{R}}\left\langle \left\{\begin{array}{ll}\boldsymbol{\psi}_{r_1,n_1}, & r = r_1\\ 0, & r \neq r_1\end{array}\right\}, \left\{\begin{array}{ll}\boldsymbol{\psi}_{r_2,n_2}, & r = r_2\\ 0, & r \neq r_2\end{array}\right\}\right\rangle.$$

Clearly, any summand for which $r \neq r_1$ or $r \neq r_2$ is 0, thus

$$\begin{aligned}
(\boldsymbol{\Phi}_{n_1}^*\boldsymbol{\Phi}_{n_2})(r_1,r_2) &= \sum_{r\in\mathcal{R}}\left\{\begin{array}{cc}\langle\boldsymbol{\psi}_{r_1,n_1},\boldsymbol{\psi}_{r_2,n_2}\rangle, & r = r_1 = r_2\\ 0, & \text{else}\end{array}\right\}\\
&= \left\{\begin{array}{cc}\langle\boldsymbol{\psi}_{r_1,n_1},\boldsymbol{\psi}_{r_2,n_2}\rangle, & r_1 = r_2\\ 0, & \text{else}\end{array}\right\}.
\end{aligned}$$

Thus, $\boldsymbol{\Phi}_{n_1}^*\boldsymbol{\Phi}_{n_2} \in \mathbb{F}^{\mathcal{R}\times\mathcal{R}}$ is diagonal with $(\boldsymbol{\Phi}_{n_1}^*\boldsymbol{\Phi}_{n_2})(r,r) = \langle \boldsymbol{\psi}_{r,n_1}, \boldsymbol{\psi}_{r,n_2}\rangle$ for any $r \in \mathcal{R}$.

(b) For any $n \in \mathcal{N}$, we note by part (a) that $\boldsymbol{\Phi}_n^*\boldsymbol{\Phi}_n$ is a diagonal matrix whose rth diagonal entry is $\langle \boldsymbol{\psi}_{r,n}, \boldsymbol{\psi}_{r,n}\rangle = \|\boldsymbol{\psi}_{r,n}\|^2 = 1$. Thus, $\boldsymbol{\Phi}_n^*\boldsymbol{\Phi}_n = \mathbf{I}$, so $\boldsymbol{\Phi}_n$ is an isometry.

(c) We show that $(\mathcal{V}_n)_{n\in\mathcal{N}}$ is a $\mathrm{TFF}_{\mathbb{F}}(DR, R, N)$ by applying Proposition 2.4 with $\boldsymbol{G} = \boldsymbol{\Phi}^*\boldsymbol{\Phi}$. Under the notation outlined in (3.6), for any $n_1, n_2 \in \mathcal{N}$, $r_1, r_2 \in \mathcal{R}$,

$$(\boldsymbol{\Phi}^*\boldsymbol{\Phi})_{n_1,n_2}(r_1,r_2) = (\boldsymbol{\Phi}^*\boldsymbol{\Phi})((n_1,r_1),(n_2,r_2)) = (\boldsymbol{\Phi}_{n_1}^*\boldsymbol{\Phi}_{n_2})(r_1,r_2).$$

Thus, $(\boldsymbol{\Phi}^*\boldsymbol{\Phi})_{n_1,n_2} = \boldsymbol{\Phi}_{n_1}^*\boldsymbol{\Phi}_{n_2}$. Since each $\boldsymbol{\Phi}_{n_1}^*\boldsymbol{\Phi}_{n_2}$ is diagonal by (a), $\boldsymbol{\Phi}^*\boldsymbol{\Phi}$ is striped. We now show that $\boldsymbol{\Phi}^*\boldsymbol{\Phi}$ satisfies properties (i), (ii) and (iii) of Proposition 2.4:

(i) Any Gram matrix is self-adjoint, so $\boldsymbol{\Phi}^*\boldsymbol{\Phi}$ satisfies Proposition 2.4(i).

(ii) For any $r \in \mathcal{R}$ and $n_1, n_2 \in \mathcal{N}$,

$$\begin{aligned}(\boldsymbol{\Phi}^*\boldsymbol{\Phi})_r(n_1,n_2) &= (\boldsymbol{\Phi}^*\boldsymbol{\Phi})((n_1,r),(n_2,r))\\ &= \langle \boldsymbol{\psi}_{r,n_1}, \boldsymbol{\psi}_{r,n_2}\rangle\\ &= (\boldsymbol{\Psi}_r^*\boldsymbol{\Psi}_r)(n_1,n_2). \end{aligned} \tag{3.7}$$

Thus, for any $r \in \mathcal{R}$, $(\boldsymbol{\Phi}^*\boldsymbol{\Phi})_r = \boldsymbol{\Psi}_r^*\boldsymbol{\Psi}_r$ is the Gram matrix of a $\mathrm{UNTF}_{\mathbb{F}}(D,N)$. By Proposition 2.1, it follows that $(\boldsymbol{\Psi}_r^*\boldsymbol{\Psi}_r)^2 = A\boldsymbol{\Psi}_r^*\boldsymbol{\Psi}_r$ where $A = \frac{N}{D}$. By Lemma 3.8(ii), this implies $(\boldsymbol{\Phi}^*\boldsymbol{\Phi})^2 = A\boldsymbol{\Phi}^*\boldsymbol{\Phi}$, satisfying Proposition 2.4(ii).

(iii) For each $n \in \mathcal{N}$, part (b) gives that $\boldsymbol{\Phi}_n^*\boldsymbol{\Phi}_n = \mathbf{I}$, satisfying Proposition 2.4(iii).

Thus, by Proposition 2.4, $\boldsymbol{\Phi}^*\boldsymbol{\Phi}$ is the fusion Gram matrix of a $\mathrm{TFF}_{\mathbb{F}}(C,R,N)$ with $C = \frac{1}{A}NR = \frac{D}{N}NR = DR$.

(d) For any distinct $n_1, n_2 \in \mathcal{N}$, it follows by part (d) of Lemma 3.8 and by (3.7) that

$$\begin{aligned}\frac{1}{R}\sum_{r_1\in\mathcal{R}}\sum_{r_2\in\mathcal{R}}\big|\big(\boldsymbol{\Phi}^*\boldsymbol{\Phi}\big)\big((n_1,r_1),(n_2,r_2)\big)\big|^2 &= \frac{1}{R}\sum_{r_1\in\mathcal{R}}\sum_{r_2\in\mathcal{R}}\big|(\boldsymbol{\Phi}^*\boldsymbol{\Phi})_{n_1,n_2}(r_1,r_2)\big|^2\\ &= \frac{1}{R}\sum_{r\in\mathcal{R}}\big|(\boldsymbol{\Phi}^*\boldsymbol{\Phi})_r(n_1,n_2)\big|^2\\ &= \frac{1}{R}\sum_{r\in\mathcal{R}}\big|(\boldsymbol{\Psi}_r^*\boldsymbol{\Psi}_r)(n_1,n_2)\big|^2\\ &= \frac{1}{R}\sum_{r\in\mathcal{R}}\big|\langle\boldsymbol{\psi}_{r,n_1},\boldsymbol{\psi}_{r,n_2}\rangle\big|^2.\end{aligned}$$

By Proposition 2.4(iv), equichordality in $(\mathcal{V}_n)_{n\in\mathcal{N}}$ ensures the first term is constant over all distinct $n_1, n_2 \in \mathcal{N}$, implying $\big((\boldsymbol{\psi}_{r,n})_{n\in\mathcal{N}}\big)_{r\in\mathcal{R}}$ is stratified. On the contrary, $\big((\boldsymbol{\psi}_{r,n})_{n\in\mathcal{N}}\big)_{r\in\mathcal{R}}$ being stratified ensures the last term is constant for any distinct $n_1, n_2 \in \mathcal{N}$, implying $(\mathcal{V}_n)_{n\in\mathcal{N}}$ is equichordal. $\square$

IV. Constructions of Stratified Unit Norm Tight Frames

In the previous chapter, we introduced the concept of a SUNTF (Definition 3.1) and then showed how to construct an ECTFF from any such object (Theorem 3.4). In this chapter, we demonstrate the strength of that approach by using it to construct infinitely many apparently new ECTFFs. Along the way, we moreover use it to unify and streamline two known ECTFF construction methods that were previously thought to be unrelated. Specifically, in Theorem 4.3, we exploit SUNTFs to streamline a previously known way to construct an ECTFF from a difference family for a finite abelian group [31, 25]. In Theorem 4.10, we then use SUNTFs to recover Zauner's way [43] of constructing ECTFFs in the special case where the underlying BIBD is resolvable. In Theorem 4.14, we then generalize this special case of Zauner's method in a distinct way, constructing a SUNTF from any suitable combination of a resolvable BIBD and an ETF. As detailed in Corollary 4.16, choosing this ETF to be a regular simplex yields a novel method for converting any resolvable BIBD into a real ECTFF.

4.1 SUNTFs from Difference Families

We begin by recalling established facts about difference families. Let $\mathcal{G}$ be a finite abelian group of order G, and let $\mathcal{R}$ be a finite set of cardinality $R \geq 1$. For each $r \in \mathcal{R}$, we let $\mathcal{D}_r$ be a subset of $\mathcal{G}$ of fixed cardinality $D \geq 1$. We say $(\mathcal{D}_r)_{r\in\mathcal{R}}$ is a *difference family for* $\mathcal{G}$, with abbreviated notation $\mathrm{DF}(G, D, \lambda)$, if there exists an integer $\lambda \geq 1$ such that

$$\sum_{r\in\mathcal{R}} \#\big\{(d_1, d_2) \in \mathcal{D}_r \times \mathcal{D}_r : d_1 - d_2 = g\big\} = \lambda \tag{4.1}$$

for all $g \in \mathcal{G} \setminus \{0\}$.

Example 4.1. Letting $\mathcal{G}$ be the group $\mathbb{Z}_5$ and $\mathcal{R} := [2]$, we consider the sequence of subsets $(\mathcal{D}_r)_{r=1}^2$ of $\mathbb{Z}_5$ given by $\mathcal{D}_1 = \{1, 4\}$ and $\mathcal{D}_2 = \{2, 3\}$. To demonstrate that $(\mathcal{D}_r)_{r=1}^2$ indeed is a difference family for $\mathbb{Z}_5$, we construct the difference tables for $\mathcal{D}_1$ and $\mathcal{D}_2$:

$$\begin{array}{c|cc} - & 1 & 4 \\ \hline 1 & 0 & 2 \\ 4 & 3 & 0 \end{array}, \qquad \begin{array}{c|cc} - & 2 & 3 \\ \hline 2 & 0 & 4 \\ 3 & 1 & 0 \end{array}.$$

Since every nonzero element of $\mathcal{G}$ appears in these tables the same number of times, $(\mathcal{D}_1, \mathcal{D}_2)$ is indeed a difference family for $\mathcal{G}$. In fact, since the number of times that any such element appears is $\lambda = 1$, it is a DF$(5, 2, 1)$.

In looking to expand upon (4.1), we make note of the following relationship, where for any $r \in \mathcal{R}$ and $g \in \mathcal{G}$, it follows by Proposition E.1 that

$$\#\big\{(d_1, d_2) \in \mathcal{D}_r \times \mathcal{D}_r : d_1 - d_2 = g\big\} = \#\big(\mathcal{D}_r \cap (g + \mathcal{D}_r)\big).$$

This leads to the following expression in terms of the corresponding characteristic functions:

$$\#\big\{(d_1, d_2) \in \mathcal{D}_r \times \mathcal{D}_r : d_1 - d_2 = g\big\} = \sum_{g' \in \mathcal{G}} \big(\boldsymbol{\chi}_{\mathcal{D}_r \cap (g+\mathcal{D}_r)}\big)(g') = \sum_{g' \in \mathcal{G}} \big(\boldsymbol{\chi}_{\mathcal{D}_r}\big)(g')\big(\boldsymbol{\chi}_{g+\mathcal{D}_r}\big)(g').$$

By (2.29), it then follows that

$$\#\big\{(d_1, d_2) \in \mathcal{D}_r \times \mathcal{D}_r : d_1 - d_2 = g\big\} = \sum_{g' \in \mathcal{G}} \overline{\big(\boldsymbol{T}^g \boldsymbol{\chi}_{\mathcal{D}_r}\big)(g')}\big(\boldsymbol{\chi}_{\mathcal{D}_r}\big)(g') = \langle \boldsymbol{T}^g \boldsymbol{\chi}_{\mathcal{D}_r}, \boldsymbol{\chi}_{\mathcal{D}_r}\rangle,$$

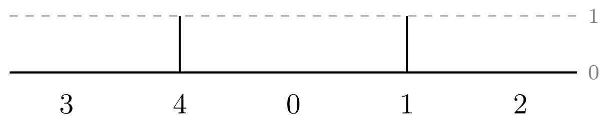
which, recalling (2.25), implies that

$$\#\big\{(d_1, d_2) \in \mathcal{D}_r \times \mathcal{D}_r : d_1 - d_2 = g\big\} = (\boldsymbol{\chi}_{\mathcal{D}_r} * \widetilde{\boldsymbol{\chi}}_{\mathcal{D}_r})(g). \tag{4.2}$$

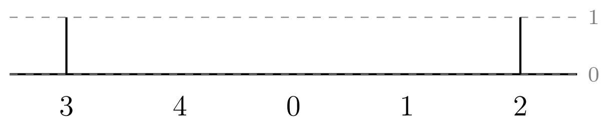
Thus, $(\mathcal{D}_r)_{r \in \mathcal{R}}$ forms a difference family for $\mathcal{G}$ if and only if the sum $\sum_{r \in \mathcal{R}} \boldsymbol{\chi}_{\mathcal{D}_r} * \widetilde{\boldsymbol{\chi}}_{\mathcal{D}_r}$ of the autocorrelations of the characteristic functions of $(\mathcal{D}_r)_{r \in \mathcal{R}}$ is constant over all nonzero $g \in \mathcal{G}$.

Example 4.2. Considering once again the subsets $\mathcal{D}_1 := \{1, 4\}$, $\mathcal{D}_2 := \{2, 3\}$ of $\mathbb{Z}_5$, we now represent their characteristic functions as a sum of standard basis vectors, which we might regard as a signal over the cyclic group $\mathbb{Z}_5$:

$\boldsymbol{\chi}_{\mathcal{D}_1} = \boldsymbol{\delta}_1 + \boldsymbol{\delta}_4 :$

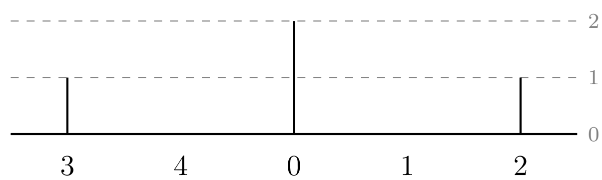


$\boldsymbol{\chi}_{\mathcal{D}_2} = \boldsymbol{\delta}_2 + \boldsymbol{\delta}_3 :$

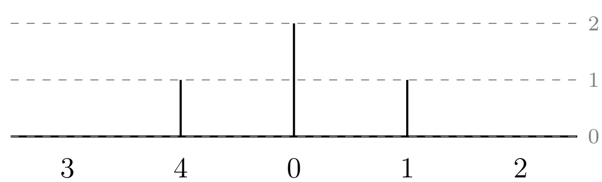


By (4.2), we note that the autocorrelations of $\boldsymbol{\chi}_{\mathcal{D}_1}$ and $\boldsymbol{\chi}_{\mathcal{D}_2}$ are equivalently expressed in terms of the difference tables determined in Example 4.1, yielding the following representations:

$\boldsymbol{\chi}_{\mathcal{D}_1} * \widetilde{\boldsymbol{\chi}}_{\mathcal{D}_1} = 2\boldsymbol{\delta}_0 + \boldsymbol{\delta}_2 + \boldsymbol{\delta}_3 :$



$\boldsymbol{\chi}_{\mathcal{D}_2} * \widetilde{\boldsymbol{\chi}}_{\mathcal{D}_2} = 2\boldsymbol{\delta}_0 + \boldsymbol{\delta}_1 + \boldsymbol{\delta}_4 :$

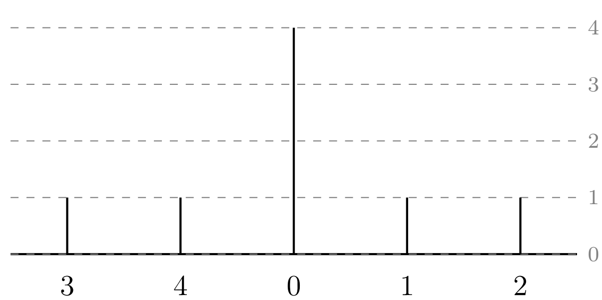


We note that the sum of the two autocorrelations is then

$$\boldsymbol{\chi}_{\mathcal{D}_1} * \widetilde{\boldsymbol{\chi}}_{\mathcal{D}_1} + \boldsymbol{\chi}_{\mathcal{D}_2} * \widetilde{\boldsymbol{\chi}}_{\mathcal{D}_2} = 4\boldsymbol{\delta}_0 + \boldsymbol{\delta}_1 + \boldsymbol{\delta}_2 + \boldsymbol{\delta}_3 + \boldsymbol{\delta}_4, \tag{4.3}$$

which can similarly be represented as

$\boldsymbol{\chi}_{\mathcal{D}_1} * \widetilde{\boldsymbol{\chi}}_{\mathcal{D}_1} + \boldsymbol{\chi}_{\mathcal{D}_2} * \widetilde{\boldsymbol{\chi}}_{\mathcal{D}_2} :$



We observe that the sum of the autocorrelation has a clear spike at 0, and is otherwise flat, with a constant value of 1. It turns out this *flat-plus-spike* signal shape is critical to our construction of SUNTFs via difference families.

Theorem 4.3. Let $(\mathcal{D}_r)_{r\in\mathcal{R}}$ be a finite sequence of subsets of a finite abelian group $\mathcal{G}$ where, for each $r \in \mathcal{R}$, $(\boldsymbol{\psi}_{r,\gamma})_{\gamma\in\widehat{\mathcal{G}}}$ is the associated harmonic frame. That is, for each $\boldsymbol{\gamma} \in \widehat{\mathcal{G}}$, let $\boldsymbol{\psi}_{r,\gamma} \in \mathbb{C}^{\mathcal{D}_r}$ be given by $\boldsymbol{\psi}_{r,\gamma}(d) := \frac{1}{\sqrt{D}}\boldsymbol{\gamma}(d)$. Then, $\big((\boldsymbol{\psi}_{r,\gamma})_{\gamma\in\widehat{\mathcal{G}}}\big)_{r\in\mathcal{R}}$ is stratified if and only if $(\mathcal{D}_r)_{r\in\mathcal{R}}$ is a difference family for $\mathcal{G}$, yielding an $\mathrm{ECTFF}_{\mathbb{C}}(DR, R, G)$ via Theorem 3.4.

Before proving this result in generality, we consider an example.

Example 4.4. We again consider the subsets of $\mathbb{Z}_5$ given by $\mathcal{D}_1 := \{1, 4\}$ and $\mathcal{D}_2 := \{2, 3\}$, each consisting of $D = 2$ elements. For each $r \in [2]$, we let $(\boldsymbol{\psi}_{r,n})_{n\in\mathbb{Z}_5}$ be the harmonic frame associated to $\mathcal{D}_r$. For these two frames to be stratified, we note by (2.31) there would need to exist some constant $\lambda > 0$ such that for any distinct $n_1, n_2 \in \mathbb{Z}_5$,

$$\begin{aligned}
\Lambda &= \frac{1}{2}\sum_{r=1}^{2}\big|\langle\boldsymbol{\psi}_{r,n_1}, \boldsymbol{\psi}_{r,n_2}\rangle\big|^2 \\
&= \frac{1}{2}\big|\langle\boldsymbol{\psi}_{1,n_1}, \boldsymbol{\psi}_{1,n_2}\rangle\big|^2 + \frac{1}{2}\big|\langle\boldsymbol{\psi}_{2,n_1}, \boldsymbol{\psi}_{2,n_2}\rangle\big|^2 \\
&= \frac{1}{8}\big[\boldsymbol{\Gamma}^*(\boldsymbol{\chi}_{\mathcal{D}_1} * \widetilde{\boldsymbol{\chi}}_{\mathcal{D}_1})\big](n_1 - n_2) + \frac{1}{8}\big[\boldsymbol{\Gamma}^*(\boldsymbol{\chi}_{\mathcal{D}_2} * \widetilde{\boldsymbol{\chi}}_{\mathcal{D}_2})\big](n_1 - n_2) \\
&= \frac{1}{8}\big[\boldsymbol{\Gamma}^*(\boldsymbol{\chi}_{\mathcal{D}_1} * \widetilde{\boldsymbol{\chi}}_{\mathcal{D}_1} + \boldsymbol{\chi}_{\mathcal{D}_2} * \widetilde{\boldsymbol{\chi}}_{\mathcal{D}_2})\big](n_1 - n_2).
\end{aligned}$$

That is, the DFT of the sum of autocorrelations would need to be constant over all nonzero indices. By (4.3), we note the sum of the autocorrelations of $\boldsymbol{\chi}_{\mathcal{D}_1}$ and $\boldsymbol{\chi}_{\mathcal{D}_2}$ can be equivalently expressed in the following manner:

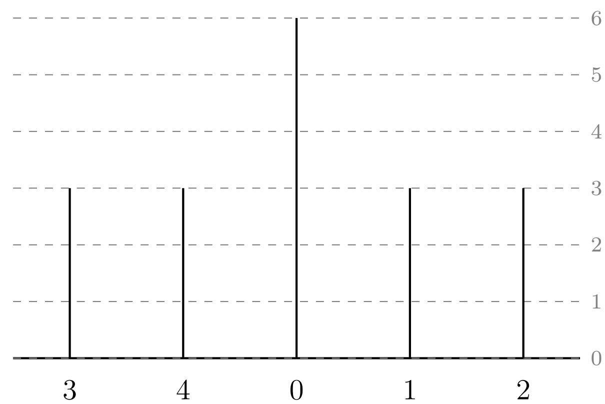
$$\boldsymbol{\chi}_{\mathcal{D}_1} * \widetilde{\boldsymbol{\chi}}_{\mathcal{D}_1} + \boldsymbol{\chi}_{\mathcal{D}_2} * \widetilde{\boldsymbol{\chi}}_{\mathcal{D}_2} = 3\boldsymbol{\delta}_0 + \sum_{n=0}^{4}\boldsymbol{\delta}_n = 3\boldsymbol{\delta}_0 + \mathbf{1}.$$

By (2.20) and (2.21), this is equivalent to having

$$\boldsymbol{\Gamma}^*\big(\boldsymbol{\chi}_{\mathcal{D}_1} * \widetilde{\boldsymbol{\chi}}_{\mathcal{D}_1} + \boldsymbol{\chi}_{\mathcal{D}_2} * \widetilde{\boldsymbol{\chi}}_{\mathcal{D}_2}\big) = \boldsymbol{\Gamma}^*\big(3\boldsymbol{\delta}_0 + \mathbf{1}\big) = 3\mathbf{1} + 5\boldsymbol{\delta}_0.$$

We then consider the representation of the DFT of the sum of the autocorrelations:

$\boldsymbol{\Gamma}^*(\boldsymbol{\chi}_{\mathcal{D}_1} * \widetilde{\boldsymbol{\chi}}_{\mathcal{D}_1} + \boldsymbol{\chi}_{\mathcal{D}_2} * \widetilde{\boldsymbol{\chi}}_{\mathcal{D}_2})$:



As evident above, the DFT of the sum of the autocorrelations is itself flat-plus-spike, implying it is constant over all nonzero indices. Thus, $((\boldsymbol{\psi}_{r,n})_{n\in\mathbb{Z}_5})_{r=1}^2$ is indeed stratified.

Proof of Theorem 4.3. For any $r \in \mathcal{R}$, we recall that since every harmonic frame is a unit norm tight frame, $(\boldsymbol{\psi}_{r,\gamma})_{\gamma\in\widehat{\mathcal{G}}}$ is a $\mathrm{UNTF}_{\mathbb{C}}(D,G)$ for $\mathbb{C}^{\mathcal{D}_r}$. We further recall that for any $\gamma_1, \gamma_2 \in \widehat{\mathcal{G}}$, it follows by (2.30) that

$$\langle \boldsymbol{\psi}_{r,\gamma_1}, \boldsymbol{\psi}_{r,\gamma_2} \rangle = \frac{1}{D}(\boldsymbol{\Gamma}^* \boldsymbol{\chi}_{\mathcal{D}_r})(\gamma_1\gamma_2^{-1}).$$

Thus, $\big((\boldsymbol{\psi}_{r,\gamma})_{\gamma\in\widehat{\mathcal{G}}}\big)_{r\in\mathcal{R}}$ is stratified if and only if the following quantity is constant over all distinct $\gamma_1, \gamma_2 \in \widehat{\mathcal{G}}$:

$$\begin{aligned}\frac{1}{R}\sum_{r\in\mathcal{R}} \big|\langle \boldsymbol{\psi}_{r,\gamma_1}, \boldsymbol{\psi}_{r,\gamma_2} \rangle\big|^2 &= \frac{1}{R}\sum_{r\in\mathcal{R}} \bigg|\frac{1}{D}(\boldsymbol{\Gamma}^* \boldsymbol{\chi}_{\mathcal{D}_r})(\gamma_1\gamma_2^{-1})\bigg|^2 \\ &= \frac{1}{D^2 R}\sum_{r\in\mathcal{R}} \big[\boldsymbol{\Gamma}^*(\widetilde{\boldsymbol{\chi}}_{\mathcal{D}_r} * \boldsymbol{\chi}_{\mathcal{D}_r})\big](\gamma_1\gamma_2^{-1}) \\ &= \frac{1}{D^2 R}\bigg[\boldsymbol{\Gamma}^*\bigg(\sum_{r\in\mathcal{R}} \widetilde{\boldsymbol{\chi}}_{\mathcal{D}_r} * \boldsymbol{\chi}_{\mathcal{D}_r}\bigg)\bigg](\gamma_1\gamma_2^{-1}).\end{aligned}$$

That is, $\big((\boldsymbol{\psi}_{r,\gamma})_{\gamma\in\widehat{\mathcal{G}}}\big)_{r\in\mathcal{R}}$ is stratified if and only if there exists $\lambda_1 \in \mathbb{C}$ such that

$$\bigg[\boldsymbol{\Gamma}^*\bigg(\sum_{r\in\mathcal{R}} \widetilde{\boldsymbol{\chi}}_{\mathcal{D}_r} * \boldsymbol{\chi}_{\mathcal{D}_r}\bigg)\bigg](\gamma) = \lambda_1$$

for each $\gamma \in \widehat{\mathcal{G}} \setminus \{\mathbf{1}\}$. Equivalently, $\big((\boldsymbol{\psi}_{r,\gamma})_{\gamma\in\widehat{\mathcal{G}}}\big)_{r\in\mathcal{R}}$ is stratified if and only if

$$\boldsymbol{\Gamma}^*\Bigg(\sum_{r\in\mathcal{R}} \widetilde{\boldsymbol{\chi}}_{\mathcal{D}_r} * \boldsymbol{\chi}_{\mathcal{D}_r}\Bigg) = \lambda_1 \mathbf{1} + \lambda_2 \boldsymbol{\delta}_1$$

for some $\lambda_1, \lambda_2 \in \mathbb{C}$. Applying $\frac{1}{G}\boldsymbol{\Gamma}$, this is equivalent to having

$$\sum_{r\in\mathcal{R}} \boldsymbol{\chi}_{\mathcal{D}_r} * \widetilde{\boldsymbol{\chi}}_{\mathcal{D}_r} = \lambda_1 \boldsymbol{\delta}_0 + \frac{\lambda_2}{G}\mathbf{1} \tag{4.4}$$

for some $\lambda_1, \lambda_2 \in \mathbb{C}$. In letting $\lambda = \frac{\lambda_2}{G}$, it follows by (4.2) that (4.4) is precisely the criterion that $(\mathcal{D}_r)_{r\in\mathcal{R}}$ is a $\mathrm{DF}(G, D, \lambda)$ for $\mathcal{G}$. □

Remark 4.5. We note that for any $\mathrm{DF}(G, D, \lambda)$, Corollary 4.3 of [25] gives a construction of an $\mathrm{ECTFF}_{\mathbb{C}}(DR, R, G)$, namely an ECTFF with precisely the same parameters as we give in Theorem 4.3. That said, the two construction techniques seem distinct, at least superficially. Moreover, our method might lend itself to the construction of ECTFFs that happen to be real, cf. Remark 3.3. We leave deeper investigations of these ideas for future work.

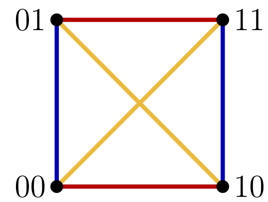
4.2 SUNTFs from Resolvable Balanced Incomplete Block Designs

We begin this section with a review of commonly known results from combinatorial design theory regarding block designs. Let V and K be integers such that $V > K \geq 2$. Let $\mathcal{V}$ be a set of cardinality V whose members we call *vertices*. Let $(\mathcal{K}_b)_{b\in\mathcal{B}}$ be a finite sequence of K-element subsets of $\mathcal{V}$ whose members we call *blocks*. The pair $(\mathcal{V}, (\mathcal{K}_b)_{b\in\mathcal{B}})$ is a *balanced incomplete block design* (BIBD) if every pair of distinct vertices appears in exactly the same number of blocks, that is, if there exists $\lambda \geq 1$ such that $\#\{b \in \mathcal{B} : v_1, v_2 \in \mathcal{K}_b\} = \lambda$ for any distinct $v_1, v_2 \in \mathcal{V}$.

Example 4.6. The affine plane of order 2 has $\mathcal{V} = \mathbb{F}_2 \times \mathbb{F}_2 = \{00, 01, 10, 11\}$ as its vertex set, while its blocks are affine lines in this finite vector space:

$$(\mathcal{K}_b)_{b\in\mathcal{B}} := \Big\{\{00, 10\}, \{01, 11\}, \{00, 11\}, \{10, 01\}, \{00, 01\}, \{10, 11\}\Big\}.$$

We visualize it as follows:



We elect to index our blocks by slope-intercept pairs, using y-intercepts for nonvertical lines and x-intercepts for vertical ones:

$$\mathcal{B} := \Big\{(0,0),(0,1),(1,0),(1,1),(\infty,0),(\infty,1)\Big\}.$$

Since any pair of distinct vertices in $\mathcal{V}$ determine a unique line, they belong to precisely one block, and so $(\mathcal{V},(\mathcal{K}_b)_{b\in\mathcal{B}})$ is a BIBD with $\lambda = 1$.

We are often interested in two other parameters of BIBDs: the number of blocks containing any given vertex and the total number of blocks, which we denote R and B, respectively. Given any BIBD with parameters V, K and λ, there is in fact no choice for the values R and B, since as demonstrated in Proposition E.2, they are given by

$$R = \frac{\lambda(V-1)}{K-1}, \quad B = \frac{VR}{K}. \tag{4.5}$$

Accordingly, we often use the shorthand notation BIBD(V,K,λ,R,B). These parameters are often made apparent by viewing the corresponding *incidence matrix* $\boldsymbol{X} \in \mathbb{R}^{\mathcal{B}\times\mathcal{V}}$, given by

$$\boldsymbol{X}(b,v) = \begin{cases} 1, & v \in \mathcal{K}_b, \\ 0, & v \notin \mathcal{K}_b. \end{cases}$$

Since each column of $\boldsymbol{X}$ corresponds to a vertex $v \in \mathcal{V}$, while each row corresponds to a block index $b \in \mathcal{B}$, it has V columns and B rows. Moreover $\boldsymbol{X}$ has K ones in each of its rows, R ones in each of its columns, and the dot product of any two of its columns is λ.

Example 4.7. We consider the BIBD from Example 4.6 and note that by (4.5), the corresponding values for R and B are

$$R = \frac{1(4-1)}{2-1} = 3, \quad B = \frac{4(3)}{2} = 6.$$

We then form the incidence matrix $\boldsymbol{X}$ corresponding to this BIBD$(4, 2, 1, 3, 6)$:

$$\boldsymbol{X} = \begin{array}{c} \\ (0,0) \\ (0,1) \\ (1,0) \\ (1,1) \\ (\infty,0) \\ (\infty,1) \end{array} \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}.$$

Clearly $\boldsymbol{X}$ has $V = 4$ columns and $B = 6$ rows. It has exactly $K = 2$ ones in each row and $R = 3$ ones in each column. The dot product between any two of its columns is $\lambda = 1$.

Letting $\mathcal{R}$ be a finite set of cardinality R, we further say a BIBD(V, K, λ, R, B) is *resolvable* if $\mathcal{B}$ can be partitioned as $\mathcal{B} = \sqcup_{r\in\mathcal{R}}\mathcal{B}_r$ where, for each $r \in \mathcal{R}$, we have $\mathcal{V} = \sqcup_{b\in\mathcal{B}_r}\mathcal{K}_b$. In this event, we call $(\mathcal{V}, (\mathcal{K}_b)_{b\in\mathcal{B}})$ a *resolvable balanced incomplete block design* (RBIBD), and we adopt the notation RBIBD(V, K, λ, R, B). For each $r \in \mathcal{R}$, we refer to $(\mathcal{K}_b)_{b\in\mathcal{B}_r}$ as a *parallel class*, and since each $\mathcal{K}_b$ is of cardinality K, it follows that $\#(\mathcal{B}_r) = \frac{V}{K}$.

Example 4.8. We again recall the BIBD$(4, 2, 1, 3, 6)$ from the previous two examples. We now partition the blocks according to their slopes:

$$\mathcal{B}_0 := \{(0,0), (0,1)\}, \quad \mathcal{B}_1 := \{(1,0), (1,1)\}, \quad \mathcal{B}_\infty := \{(\infty,0), (\infty,1)\}.$$

Letting $\mathcal{R} := \{0, 1, \infty\}$ be the set of slopes, we note $(\mathcal{B}_r)_{r\in\mathcal{R}}$ indeed partitions $\mathcal{B}$ into parallel classes. To visualize this partitioning, we form the incidence matrix, equipped with horizontal lines to indicate the 3 subsets:

$$
\boldsymbol{X} = \begin{array}{c} \\ (0,0) \\ (0,1) \\ (1,0) \\ (1,1) \\ (\infty,0) \\ (\infty,1) \end{array}
\begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ \hline 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \end{array}.
$$

It is then apparent that in any subset $\mathcal{B}_r$, each vertex of $\mathcal{V}$ appears exactly once. Therefore, $(\mathcal{V}, (\mathcal{K}_b)_{b\in\mathcal{B}})$ is in fact an RBIBD$(4, 2, 1, 3, 6)$.

Lemma 4.9. Let $(\mathcal{K}_b)_{b\in\mathcal{B}}$ be the blocks of an RBIBD(V, K, λ, R, B) over a vertex set $\mathcal{V}$. Further partition $\mathcal{B}$ into $(\mathcal{B}_r)_{r\in\mathcal{R}}$ such that $(\mathcal{K}_b)_{b\in\mathcal{B}_r}$ is a parallel class for each $r \in \mathcal{R}$. Then, for any $r \in \mathcal{R}$ and $v_1, v_2 \in \mathcal{V}$, it follows that

$$
\#\{b \in \mathcal{B}_r : v_1, v_2 \in \mathcal{K}_b\} \leq 1. \tag{4.6}
$$

Moreover, for any $v_1, v_2 \in \mathcal{V}$, it further follows that

$$
\#\{r \in \mathcal{R} : \exists\, b \in \mathcal{B}_r \text{ such that } v_1, v_2 \in \mathcal{K}_b\} = \lambda. \tag{4.7}
$$

Proof. Assume to the contrary that the left-hand quantity in (4.6) is at least 2. Then, there exists distinct $b_1, b_2 \in \mathcal{B}_r$ such that v_1, v_2 are members of both $\mathcal{K}_{b_1}$ and $\mathcal{K}_{b_2}$, implying $\mathcal{K}_{b_1} \cap \mathcal{K}_{b_2}$ is nonempty. This contradicts the notion that $(\mathcal{K}_b)_{b\in\mathcal{B}_r}$ forms a parallel class. Thus, (4.6) holds. Next, we claim that

$$
\#\{b \in \mathcal{B}_r : v_1, v_2 \in \mathcal{K}_b\} = \begin{cases} 1, & \exists\, b \in \mathcal{B}_r \text{ such that } v_1, v_2 \in \mathcal{K}_b, \\ 0, & \text{else.} \end{cases} \tag{4.8}
$$

To demonstrate (4.8), note that both sides of it vanish if $v_1, v_2 \notin \mathcal{K}_b$ for every $b \in \mathcal{B}_r$. When instead $v_1, v_2 \in \mathcal{K}_b$ for some $b \in \mathcal{B}_r$, the quantity $\#\{b \in \mathcal{B}_r : v_1, v_2 \in \mathcal{K}_b\}$ is at least 1, and is moreover at most 1 by (4.6).

Having demonstrated (4.8), we then have

$$\begin{aligned}
\#\{r \in \mathcal{R} : \exists\, b \in \mathcal{B}_r \text{ such that } v_1, v_2 \in \mathcal{K}_b\} &= \sum_{r \in \mathcal{R}} \left\{ \begin{array}{ll} 1, & \exists\, b \in \mathcal{B}_r \text{ such that } v_1, v_2 \in \mathcal{K}_b \\ 0, & \text{else} \end{array} \right\} \\
&= \sum_{r \in \mathcal{R}} \#\{b \in \mathcal{B}_r : v_1, v_2 \in \mathcal{K}_b\} \\
&= \#\left(\bigsqcup_{r \in \mathcal{R}} \{b \in \mathcal{B}_r : v_1, v_2 \in \mathcal{K}_b\} \right) \\
&= \#\{b \in \mathcal{B} : v_1, v_2 \in \mathcal{K}_b\} \\
&= \lambda. \qquad \square
\end{aligned}$$

We now present the main result of this section, which gives one way to construct a SUNTF from any RBIBD.

Theorem 4.10. Let $(\mathcal{K}_b)_{b \in \mathcal{B}}$ be the blocks of an RBIBD(V, K, λ, R, B) over a vertex set $\mathcal{V}$, and partition $\mathcal{B}$ into $(\mathcal{B}_r)_{r \in \mathcal{R}}$ such that $(\mathcal{K}_b)_{b \in \mathcal{B}_r}$ is a parallel class for each $r \in \mathcal{R}$. For each $r \in \mathcal{R}$, define $(\boldsymbol{\psi}_{r,v})_{v \in \mathcal{V}}$ in $\mathbb{R}^{\mathcal{B}_r}$ by

$$\boldsymbol{\psi}_{r,v}(b) := \left\{ \begin{array}{ll} 1, & v \in \mathcal{K}_b, \\ 0, & v \notin \mathcal{K}_b. \end{array} \right.$$

Then, each $(\boldsymbol{\psi}_{r,v})_{v \in \mathcal{V}}$ is a UNTF$_{\mathbb{R}}(\frac{V}{K}, V)$ for $\mathbb{R}^{\mathcal{B}_r}$. Moreover, these UNTFs are stratified, and so by Theorem 3.4 yield an ECTFF$_{\mathbb{R}}(B, R, V)$.

Before proving this result in generality, we consider an example.

Example 4.11. Recall the RBIBD$(4, 2, 1, 3, 6)$ from the previous examples, and form matrices $(\boldsymbol{\Psi}_r)_{r \in \mathcal{R}}$ by extracting the rows indexed by $\mathcal{B}_r$ from the incidence matrix $\boldsymbol{X}$:

$$\boldsymbol{\Psi}_0 := \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{\Psi}_1 := \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{\Psi}_\infty := \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Each $\boldsymbol{\Psi}_r$ is the synthesis map of a UNTF$_{\mathbb{R}}(2, 4)$. Indeed, each column of each $\boldsymbol{\Psi}_r$ is clearly unit norm, and $\boldsymbol{\Psi}_0 \boldsymbol{\Psi}_0^* = \boldsymbol{\Psi}_1 \boldsymbol{\Psi}_1^* = \boldsymbol{\Psi}_\infty \boldsymbol{\Psi}_\infty^* = 2\mathbf{I}$. To demonstrate that these UNTFs form a

SUNTF, we examine their corresponding Gram matrices:

$$\boldsymbol{\Psi}_0^*\boldsymbol{\Psi}_0 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{\Psi}_1^*\boldsymbol{\Psi}_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{\Psi}_\infty^*\boldsymbol{\Psi}_\infty = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

The entrywise sum of the entrywise modulus squared of the Gram matrices is thus

$$|\boldsymbol{\Psi}_0^*\boldsymbol{\Psi}_0|^2 + |\boldsymbol{\Psi}_1^*\boldsymbol{\Psi}_1|^2 + |\boldsymbol{\Psi}_\infty^*\boldsymbol{\Psi}_\infty|^2 = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix}.$$

Since its off-diagonal entries are constant, we conclude that $((\boldsymbol{\psi}_{r,v})_{v\in\mathcal{V}})_{r\in\mathcal{R}}$ is a SUNTF.

Proof of Theorem 4.10. Take any $r \in \mathcal{R}$. To demonstrate that $(\boldsymbol{\psi}_{r,v})_{v\in\mathcal{V}}$ is indeed a $\mathrm{UNTF}_{\mathbb{R}}(\frac{V}{K}, V)$, we note that since $(\mathcal{K}_b)_{b\in\mathcal{B}_r}$ partitions $\mathcal{V}$, for any $v \in \mathcal{V}$, there is precisely one $b' \in \mathcal{B}_r$ such that $v \in \mathcal{K}_b$. Accordingly,

$$\|\boldsymbol{\psi}_{r,v}\|^2 = \sum_{b\in\mathcal{B}_r} |\boldsymbol{\psi}_{r,v}(b)|^2 = |1|^2 + \sum_{b\in\mathcal{B}_r\setminus\{b'\}} |0|^2 = 1.$$

Thus, each $\boldsymbol{\psi}_{r,v}$ is unit norm. To prove tightness, note that for any $b_1, b_2 \in \mathcal{B}$, the (b_1, b_2)th entry of the frame operator $\boldsymbol{\Psi}_r\boldsymbol{\Psi}_r^* \in \mathbb{R}^{\mathcal{B}_r\times\mathcal{B}_r}$ is

$$\begin{aligned}(\boldsymbol{\Psi}_r\boldsymbol{\Psi}_r^*)(b_1, b_2) &= \sum_{v\in\mathcal{V}} \boldsymbol{\Psi}_r(b_1, v)\overline{\boldsymbol{\Psi}_r^*(b_2, v)} \\ &= \sum_{v\in\mathcal{V}} \boldsymbol{\psi}_{r,v}(b_1)\overline{\boldsymbol{\psi}_{r,v}(b_2)} \\ &= \sum_{v\in\mathcal{V}} \left\{ \begin{array}{ll} 1, & v \in \mathcal{K}_{b_1} \\ 0, & v \notin \mathcal{K}_{b_1} \end{array} \right\} \left\{ \begin{array}{ll} 1, & v \in \mathcal{K}_{b_2} \\ 0, & v \notin \mathcal{K}_{b_2} \end{array} \right\}.\end{aligned}$$

Since each summand is one only if v is in both blocks and zero otherwise, this implies

$$(\boldsymbol{\Psi}_r\boldsymbol{\Psi}_r^*)(b_1, b_2) = \#(\mathcal{K}_{b_1} \cap \mathcal{K}_{b_2}) = \left\{ \begin{array}{ll} K, & b_1 = b_2, \\ 0, & b_1 \neq b_2. \end{array} \right.$$

Thus, $\boldsymbol{\Psi}_r\boldsymbol{\Psi}_r^* = K\mathbf{I}$. Since $\#(\mathcal{B}_r) = \frac{V}{K}$, we indeed have that $(\boldsymbol{\psi}_{r,v})_{v\in\mathcal{V}}$ is a $\mathrm{UNTF}_{\mathbb{R}}(\frac{V}{K}, V)$ for $\mathbb{R}^{\mathcal{B}_r}$. Next, we show that these UNTFs are stratified. For any $r \in \mathcal{R}$ and $v_1, v_2 \in \mathcal{V}$,

$$\langle \boldsymbol{\psi}_{r,v_1}, \boldsymbol{\psi}_{r,v_2}\rangle = \sum_{b\in\mathcal{B}_r} \overline{\boldsymbol{\psi}_{r,v_1}(b)}\boldsymbol{\psi}_{r,v_2}(b) = \sum_{b\in\mathcal{B}_r} \left\{ \begin{array}{ll} 1, & v_1 \in \mathcal{K}_b \\ 0, & v_1 \notin \mathcal{K}_b \end{array} \right\} \left\{ \begin{array}{ll} 1, & v_2 \in \mathcal{K}_b \\ 0, & v_2 \notin \mathcal{K}_b \end{array} \right\}.$$

Since each summand is one when both v_1, v_2 are in the bth block and is otherwise zero,

$$\langle \boldsymbol{\psi}_{r,v_1}, \boldsymbol{\psi}_{r,v_2}\rangle = \#\{b \in \mathcal{B}_r : v_1, v_2 \in \mathcal{K}_b\}.$$

By (4.6), this then implies $\langle \boldsymbol{\psi}_{r,v_1}, \boldsymbol{\psi}_{r,v_2}\rangle = |\langle \boldsymbol{\psi}_{r,v_1}, \boldsymbol{\psi}_{r,v_2}\rangle|^2$. Thus,

$$\begin{aligned}
\frac{1}{R}\sum_{r\in\mathcal{R}}\left|\langle \boldsymbol{\psi}_{r,v_1}, \boldsymbol{\psi}_{r,v_2}\rangle\right|^2 &= \frac{1}{R}\sum_{r\in\mathcal{R}}\langle \boldsymbol{\psi}_{r,v_1}, \boldsymbol{\psi}_{r,v_2}\rangle \\
&= \frac{1}{R}\sum_{r\in\mathcal{R}}\#\{b \in \mathcal{B}_r : v_1, v_2 \in \mathcal{K}_b\} \\
&= \frac{1}{R}\#\left(\bigsqcup_{r\in\mathcal{R}}\{b \in \mathcal{B}_r : v_1, v_2 \in \mathcal{K}_b\}\right) \\
&= \frac{1}{R}\#\{b \in \mathcal{B} : v_1, v_2 \in \mathcal{K}_b\} \\
&= \left\{ \begin{array}{ll} 1, & v_1 = v_2, \\ \frac{\lambda}{R}, & v_1 \neq v_2. \end{array} \right.
\end{aligned}$$

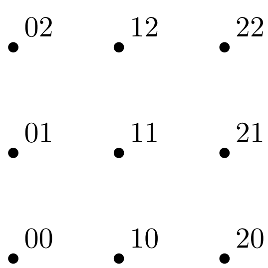
In particular, the quantity (3.1) is indeed constant over all distinct $v_1, v_2 \in \mathcal{V}$. Applying Theorem 3.4 to this SUNTF yields an $\mathrm{ECTFF}(DR, R, V)$ where $DR = \frac{VR}{K} = B$. $\square$

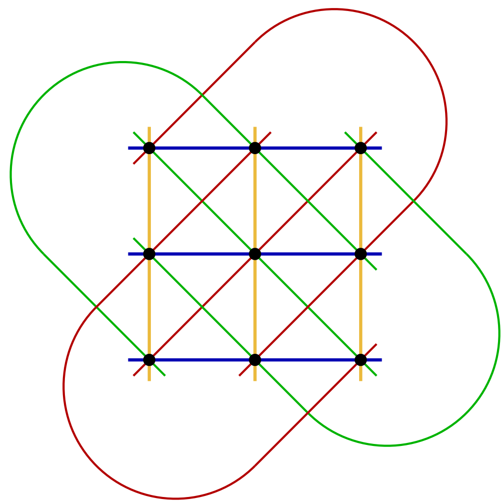
Remark 4.12. Zauner [43] constructs an $\mathrm{ECTFF}_{\mathbb{R}}(B, R, V)$ from any $\mathrm{BIBD}(V, K, \lambda, R, B)$. Though we strongly suspect that Zauner's construction reduces to that of our combined Theorems 3.4 and 4.10 in the special case where the requisite BIBD is resolvable, we leave a formal investigation of this idea for future work.

4.3 A New Infinite Family of ECTFFs

In this section, we detail a method for constructing SUNTFs similar to our RBIBD method from the previous section; however, we introduce a method of perturbing the parallel classes of the RBIBD via an ETF which, through Theorem 3.4, leads to the construction of new ECTFFs. We begin by introducing another example of an RBIBD which better lends itself to the illustration of this method.

Example 4.13. Consider the affine plane of order 3, where we regard each of its vertices xy as members of $\mathcal{V} := \mathbb{F}_3 \times \mathbb{F}_3$:





We again index each line by a slope-intercept pair:

$$\mathcal{B} := \Big\{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2),(\infty,0),(\infty,1),(\infty,2)\Big\}.$$

And again, we consider the block design $(\mathcal{V}, (\mathcal{K}_b)_{b\in\mathcal{B}})$ where for each $b \in \mathcal{B}$, the block $\mathcal{K}_b$ is the subset of $\mathcal{V}$ containing the three vertices of the bth line:

$$\begin{aligned}(\mathcal{K}_b)_{b\in\mathcal{B}} := \Big\{&\{00,10,20\},\{01,11,21\},\{02,12,22\},\\ &\{00,11,22\},\{01,12,20\},\{02,10,21\},\\ &\{00,12,21\},\{01,10,22\},\{02,11,20\},\\ &\{00,01,02\},\{10,11,12\},\{20,21,22\}\Big\}.\end{aligned}$$

Since any two vertices $v_1, v_2 \in \mathcal{V}$ uniquely determine a line, they belong to precisely one block $\mathcal{K}_b$, and so $\lambda = 1$. We note that by (4.5), the corresponding values for R and B are 4 and 12, respectively. Thus, $(\mathcal{V}, (\mathcal{K}_b)_{b\in\mathcal{B}})$ is a BIBD$(9, 3, 1, 4, 12)$. As before, we partition the blocks into parallel classes according to their slopes $\mathcal{R} := \{0, 1, 2, \infty\}$:

$$\mathcal{B}_0 := \{(0,0), (0,1), (0,2)\}, \quad \mathcal{B}_1 := \{(1,0), (1,1), (1,2)\},$$
$$\mathcal{B}_2 := \{(2,0), (2,1), (2,2)\}, \quad \mathcal{B}_\infty := \{(\infty,0), (\infty,1), (\infty,2)\}.$$

The incidence matrix of this RBIBD$(9, 3, 1, 4, 12)$ is

$$\boldsymbol{X} = \begin{array}{c} \\ (0,0) \\ (0,1) \\ (0,2) \\ (1,0) \\ (1,1) \\ (1,2) \\ (2,0) \\ (2,1) \\ (2,2) \\ (\infty,0) \\ (\infty,1) \\ (\infty,2) \end{array} \begin{array}{c} \begin{array}{ccccccccc} 00 & 01 & 02 & 10 & 11 & 12 & 20 & 21 & 22 \end{array} \\ \left[\begin{array}{ccccccccc} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right] \end{array}.$$

Theorem 4.14. Let $(\mathcal{K}_b)_{b\in\mathcal{B}}$ be the blocks of an RBIBD(V, K, λ, R, B) over a vertex set $\mathcal{V}$. Further partition $\mathcal{B}$ into $(\mathcal{B}_r)_{r\in\mathcal{R}}$ such that $(\mathcal{K}_b)_{b\in\mathcal{B}_r}$ is a parallel class for each $r \in \mathcal{R}$. Also let $D \geq 1$ and suppose, for each $r \in \mathcal{R}$, that $(\boldsymbol{\theta}_{r,b})_{b\in\mathcal{B}_r}$ is an ETF$_{\mathbb{F}}(D, \frac{V}{K})$ for $\mathbb{F}^D$. For each $r \in \mathcal{R}$, define $(\boldsymbol{\xi}_{r,v})_{v\in\mathcal{V}}$ in $\mathbb{F}^D$ by

$$\boldsymbol{\xi}_{r,v} := \boldsymbol{\theta}_{r,b(r,v)},$$

where $b(r,v)$ is the unique member of $\mathcal{B}_r$ such that $v \in \mathcal{K}_b$. Then, each $(\boldsymbol{\xi}_{r,v})_{v\in\mathcal{V}}$ is a $\mathrm{UNTF}_{\mathbb{F}}(D,V)$. Moreover, these UNTFs are stratified, and yield an $\mathrm{ECTFF}_{\mathbb{F}}(DR,R,V)$ via Theorem 3.4.

Once again, we consider an example which illustrates the main ideas of this result before proving it in full generality.

Example 4.15. Continuing with the RBIBD$(9,3,1,4,12)$ of the previous example, we construct matrices $(\boldsymbol{\Psi}_r)_{r\in\mathcal{R}}$ by extracting the rows indexed by $\mathcal{B}_r$ from its incidence matrix:

$$\boldsymbol{\Psi}_0 = \left[\begin{array}{ccc|ccc|ccc} 1&0&0&1&0&0&1&0&0\\ 0&1&0&0&1&0&0&1&0\\ 0&0&1&0&0&1&0&0&1 \end{array}\right], \qquad \boldsymbol{\Psi}_1 = \left[\begin{array}{ccc|ccc|ccc} 1&0&0&0&1&0&0&0&1\\ 0&1&0&0&0&1&1&0&0\\ 0&0&1&1&0&0&0&1&0 \end{array}\right],$$

$$\boldsymbol{\Psi}_2 = \left[\begin{array}{ccc|ccc|ccc} 1&0&0&0&0&1&0&1&0\\ 0&1&0&1&0&0&0&0&1\\ 0&0&1&0&1&0&1&0&0 \end{array}\right], \qquad \boldsymbol{\Psi}_\infty = \left[\begin{array}{ccc|ccc|ccc} 1&1&1&0&0&0&0&0&0\\ 0&0&0&1&1&1&0&0&0\\ 0&0&0&0&0&0&1&1&1 \end{array}\right].$$

Let $\boldsymbol{\Theta}$ be the synthesis map of the Mercedes-Benz frame given in Example 2.2. For each $r \in \mathcal{R}$, we define $\boldsymbol{\Xi}_r \in \mathbb{R}^{2\times 9}$ by $\boldsymbol{\Xi}_r := \boldsymbol{\Theta}\boldsymbol{\Psi}_r$. This gives the following matrices:

$$\boldsymbol{\Xi}_0 = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \left[\begin{array}{ccc|ccc|ccc} 1&0&0&1&0&0&1&0&0\\ 0&1&0&0&1&0&0&1&0\\ 0&0&1&0&0&1&0&0&1 \end{array}\right] = \left[\begin{array}{ccc|ccc|ccc} 1 & -\frac{1}{2} & -\frac{1}{2} & 1 & -\frac{1}{2} & -\frac{1}{2} & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{array}\right],$$

$$\boldsymbol{\Xi}_1 = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \left[\begin{array}{ccc|ccc|ccc} 1&0&0&0&1&0&0&0&1\\ 0&1&0&0&0&1&1&0&0\\ 0&0&1&1&0&0&0&1&0 \end{array}\right] = \left[\begin{array}{ccc|ccc|ccc} 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & 0 & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & 0 \end{array}\right],$$

$$\boldsymbol{\Xi}_2 = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \left[\begin{array}{ccc|ccc|ccc} 1&0&0&0&0&1&0&1&0\\ 0&1&0&1&0&0&0&0&1\\ 0&0&1&0&1&0&1&0&0 \end{array}\right] = \left[\begin{array}{ccc|ccc|ccc} 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & 0 & -\frac{\sqrt{3}}{2} & 0 & \frac{\sqrt{3}}{2} \end{array}\right],$$

$$\boldsymbol{\Xi}_\infty = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \left[\begin{array}{ccc|ccc|ccc} 1&1&1&0&0&0&0&0&0\\ 0&0&0&1&1&1&0&0&0\\ 0&0&0&0&0&0&1&1&1 \end{array}\right] = \left[\begin{array}{ccc|ccc|ccc} 1 & 1 & 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{array}\right].$$

We note that each $\boldsymbol{\Xi}_r$ is obtained by permuting the columns of a horizontal concatenation of three copies of $\boldsymbol{\Theta}$. Since such a concatenation is tight, and since column permutations preserve norms of columns and inner products of rows, each $\boldsymbol{\Xi}_r$ is the synthesis map of a

$\mathrm{UNTF}_{\mathbb{R}}(2,9)$. We further recall by (2.7) that the inner product of distinct columns of $\boldsymbol{\Theta}$ is $-\frac{1}{2}$, from which it follows that a given entry of $\boldsymbol{\Xi}_r^*\boldsymbol{\Xi}_r$ is 1 or $h := -\frac{1}{2}$ when the corresponding entry of $\boldsymbol{\Psi}_r^*\boldsymbol{\Psi}_r$ is 1 or 0, respectively:

$$\boldsymbol{\Xi}_0^*\boldsymbol{\Xi}_0 = \begin{bmatrix} 1 & h & h & 1 & h & h & 1 & h & h \\ h & 1 & h & h & 1 & h & h & 1 & h \\ h & h & 1 & h & h & 1 & h & h & 1 \\ 1 & h & h & 1 & h & h & 1 & h & h \\ h & 1 & h & h & 1 & h & h & 1 & h \\ h & h & 1 & h & h & 1 & h & h & 1 \\ 1 & h & h & 1 & h & h & 1 & h & h \\ h & 1 & h & h & 1 & h & h & 1 & h \\ h & h & 1 & h & h & 1 & h & h & 1 \end{bmatrix}, \qquad \boldsymbol{\Xi}_1^*\boldsymbol{\Xi}_1 = \begin{bmatrix} 1 & h & h & h & 1 & h & h & h & 1 \\ h & 1 & h & h & h & 1 & 1 & h & h \\ h & h & 1 & 1 & h & h & h & 1 & h \\ h & h & 1 & 1 & h & h & h & 1 & h \\ 1 & h & h & h & 1 & h & h & h & 1 \\ h & 1 & h & h & h & 1 & 1 & h & h \\ h & 1 & h & h & h & 1 & 1 & h & h \\ h & h & 1 & 1 & h & h & h & 1 & h \\ 1 & h & h & h & 1 & h & h & h & 1 \end{bmatrix},$$

$$\boldsymbol{\Xi}_2^*\boldsymbol{\Xi}_2 = \begin{bmatrix} 1 & h & h & h & h & 1 & h & 1 & h \\ h & 1 & h & 1 & h & h & h & h & 1 \\ h & h & 1 & h & 1 & h & 1 & h & h \\ h & 1 & h & 1 & h & h & h & h & 1 \\ h & h & 1 & h & 1 & h & 1 & h & h \\ 1 & h & h & h & h & 1 & h & 1 & h \\ h & h & 1 & h & 1 & h & 1 & h & h \\ 1 & h & h & h & h & 1 & h & 1 & h \\ h & 1 & h & 1 & h & h & h & h & 1 \end{bmatrix}, \qquad \boldsymbol{\Xi}_\infty^*\boldsymbol{\Xi}_\infty = \begin{bmatrix} 1 & 1 & 1 & h & h & h & h & h & h \\ 1 & 1 & 1 & h & h & h & h & h & h \\ 1 & 1 & 1 & h & h & h & h & h & h \\ h & h & h & 1 & 1 & 1 & h & h & h \\ h & h & h & 1 & 1 & 1 & h & h & h \\ h & h & h & 1 & 1 & 1 & h & h & h \\ h & h & h & h & h & h & 1 & 1 & 1 \\ h & h & h & h & h & h & 1 & 1 & 1 \\ h & h & h & h & h & h & 1 & 1 & 1 \end{bmatrix}.$$

The sum of the entrywise modulus squared of these Gram matrices is

$$\sum_{r\in\mathcal{R}} \big|\boldsymbol{\Xi}_r^*\boldsymbol{\Xi}_r\big|^2 = \begin{bmatrix} 4 & f & f & f & f & f & f & f & f \\ f & 4 & f & f & f & f & f & f & f \\ f & f & 4 & f & f & f & f & f & f \\ f & f & f & 4 & f & f & f & f & f \\ f & f & f & f & 4 & f & f & f & f \\ f & f & f & f & f & 4 & f & f & f \\ f & f & f & f & f & f & 4 & f & f \\ f & f & f & f & f & f & f & 4 & f \\ f & f & f & f & f & f & f & f & 4 \end{bmatrix}, \qquad f = 1 + 3h^2.$$

Again, since the off-diagonal entries are constant, we conclude that $(\boldsymbol{\Xi}_r)_{r\in\mathcal{R}}$ is a SUNTF.

Proof of Theorem 4.14. For each $r \in \mathcal{R}$, let $\boldsymbol{\Psi}_r : \mathbb{F}^{\mathcal{V}} \to \mathbb{F}^{\mathcal{B}_r}$ be the synthesis map of the finite sequence $(\boldsymbol{\psi}_{r,v})_{v\in\mathcal{V}}$ defined in Theorem 4.10. For each $r \in \mathcal{R}$, further let $\boldsymbol{\Theta}_r : \mathbb{F}^{\mathcal{B}_r} \to \mathbb{F}^{D}$ be the synthesis map of $(\boldsymbol{\theta}_{r,b})_{b\in\mathcal{B}_r}$, and define $\boldsymbol{\Xi}_r : \mathbb{F}^{\mathcal{V}} \to \mathbb{F}^{\mathcal{D}}$ by $\boldsymbol{\Xi}_r := \boldsymbol{\Theta}_r\boldsymbol{\Psi}_r$. For any $v \in \mathcal{V}$,

$$\boldsymbol{\Xi}_r\boldsymbol{\delta}_v = \boldsymbol{\Theta}_r\boldsymbol{\Psi}_r\boldsymbol{\delta}_v = \boldsymbol{\Theta}_r\boldsymbol{\psi}_{r,v}.$$

Furthermore, for any $d \in [D]$,

$$(\boldsymbol{\Theta}_r\boldsymbol{\psi}_{r,v})(d) = \sum_{b\in\mathcal{B}_r}\boldsymbol{\Theta}_r(d,b)\boldsymbol{\psi}_{r,v}(b) = \sum_{b\in\mathcal{B}_r}\boldsymbol{\theta}_{r,b}(d)\left\{\begin{array}{ll} 1, & v\in\mathcal{K}_b \\ 0, & v\notin\mathcal{K}_b \end{array}\right\} = \boldsymbol{\theta}_{r,b(r,v)}(d) = \boldsymbol{\xi}_{r,v}(d).$$

Thus, $\boldsymbol{\Xi}_r$ is the synthesis map of $(\boldsymbol{\xi}_{r,v})_{v\in\mathcal{V}}$. We then note

$$\boldsymbol{\Xi}_r\boldsymbol{\Xi}_r^* = (\boldsymbol{\Theta}_r\boldsymbol{\Psi}_r)(\boldsymbol{\Theta}_r\boldsymbol{\Psi}_r)^* = \boldsymbol{\Theta}_r\boldsymbol{\Psi}_r\boldsymbol{\Psi}_r^*\boldsymbol{\Theta}_r^* = \boldsymbol{\Theta}_r K\mathbf{I}\boldsymbol{\Theta}_r^* = K\boldsymbol{\Theta}_r\boldsymbol{\Theta}_r^* = K\frac{V}{KD}\mathbf{I} = \frac{V}{D}\mathbf{I},$$

so $(\boldsymbol{\xi}_{r,v})_{v\in\mathcal{V}}$ is tight. Furthermore, for each $r\in\mathcal{R}$, $(\boldsymbol{\theta}_{r,b})_{b\in\mathcal{B}_r}$ achieves equality in (2.17) since it is an $\mathrm{ETF}_{\mathbb{F}}(D,\frac{V}{K})$. Accordingly, for any distinct $b_1, b_2 \in \mathcal{B}_r$, we have that

$$\left|\langle\boldsymbol{\theta}_{r,b_1},\boldsymbol{\theta}_{r,b_2}\rangle\right|^2 = \frac{\frac{V}{K}-D}{D(\frac{V}{K})-1} = \frac{V-DK}{D(V-K)}.$$

When $b_1 = b_2$, the modulus squared of the inner product is instead 1, since each $\boldsymbol{\theta}_{r,b}$ is unit norm. Thus, for any $v_1, v_2 \in \mathcal{V}$,

$$\begin{aligned}\left|\langle\boldsymbol{\xi}_{r,v_1},\boldsymbol{\xi}_{r,v_2}\rangle\right|^2 &= \left|\langle\boldsymbol{\theta}_{r,b(r,v_1)},\boldsymbol{\theta}_{r,b(r,v_2)}\rangle\right|^2 \\ &= \left\{\begin{array}{ll} 1, & b(r,v_1) = b(r,v_2), \\ \frac{V-DK}{D(V-K)}, & b(r,v_1)\neq b(r,v_2). \end{array}\right. \\ &= \left\{\begin{array}{ll} 1, & \exists\, b\in\mathcal{B}_r \text{ such that } v_1,v_2\in\mathcal{K}_b, \\ \frac{V-DK}{D(V-K)}, & \text{else.} \end{array}\right.\end{aligned}$$

Each $\boldsymbol{\xi}_{r,v}$ is then unit norm, so $(\boldsymbol{\xi}_{r,v})_{v\in\mathcal{V}}$ is a $\mathrm{UNTF}_{\mathbb{F}}(D,V)$. Moreover, for any distinct $v_1, v_2 \in \mathcal{V}$, it follows by (4.7) that

$$\begin{aligned}\frac{1}{R}\sum_{r\in\mathcal{R}}\bigl|\langle\boldsymbol{\xi}_{r,v_1},\boldsymbol{\xi}_{r,v_2}\rangle\bigr|^2 &= \frac{1}{R}\sum_{r\in\mathcal{R}}\left\{\begin{array}{cl} 1, & \exists\, b\in\mathcal{B}_r \text{ such that } v_1,v_2\in\mathcal{K}_b \\ \frac{V-DK}{D(V-K)}, & \text{else}\end{array}\right\} \\ &= \frac{1}{R}\left[\lambda(1)+(R-\lambda)\frac{V-DK}{D(V-K)}\right].\end{aligned}$$

Thus, $\bigl((\boldsymbol{\xi}_{r,v})_{v\in\mathcal{V}}\bigr)_{r\in\mathcal{R}}$ is stratified, and so yields an $\mathrm{ECTFF}_{\mathbb{F}}(DR,R,V)$ via Theorem 3.4. $\square$

For any $\mathrm{RBIBD}(V,K,\lambda,R,B)$, note that (4.5) gives

$$\Bigl(\frac{V}{K}-1\Bigr)R = \frac{VR}{K} - R = B - R.$$

Recalling from Chapter 2 that real regular simplices exist in every dimension, we can always apply the previous result with $\mathbb{F}=\mathbb{R}$ and $D=\frac{V}{K}-1$ to obtain:

Corollary 4.16. For any $\mathrm{RBIBD}(V,K,\lambda,R,B)$, there exists an $\mathrm{ECTFF}_{\mathbb{R}}(B-R,R,V)$.

Any one of several known infinite families of RBIBDs yields an infinite family of ECTFFs via Corollary 4.16. Yet, unlike our RBIBD-based construction from Section 4.2, the resultant ECTFFs here have parameters that are distinct from those given by Zauner's construction. Furthermore, we know of no other method for constructing an ECTFF with these parameters from an RBIBD in general. For these reasons, we believe that the ECTFFs given by Corollary 4.16 are novel, in general. We further remark that more generally, one can also apply Theorem 4.14 with any $\mathrm{ETF}_{\mathbb{F}}(D,N)$ by choosing the corresponding RBIBD to be a round-robin design on $2N$ vertices, for example, namely an $\mathrm{RBIBD}(2N,2,1,2N-1,N(2N-1))$.

V. Conclusions and Future Work

An ECTFF is a finite sequence of subspaces of a Euclidean space that achieves equality in the simplex bound (1.2). Every ECTFF is thus a type of optimal Grassmannian code. They arise in applications such as quantum information theory [43] and compressed sensing [14, 6]. That said, despite several decades of active research, the problem of characterizing their existence remains largely open [24].

To help address this deficit, we have developed a novel way to construct ECTFFs that yields infinitely many apparently new instances of them. Our key idea, as detailed in Theorem 3.4, is to construct an ECTFF from a more fundamental object that we call a SUNTF, as introduced and codified in Definition 3.1. This general theory, given in Chapter 3, prompted us to search for SUNTFs, the results of which are summarized in Chapter 4. In Theorem 4.3 for instance, we construct a SUNTF from any difference family for a finite abelian group. When combined with Theorem 3.4, this apparently generalizes a construction of [31, 25] in a way that sometimes permits the resulting ECTFFs to be real. Meanwhile, in Theorem 4.10, we construct a SUNTF from any RBIBD. When combined with Theorem 3.4, this apparently recovers Zauner's ECTFF construction [43] in the special case where the underlying BIBD is resolvable. Notably, the formalism of Chapter 3 unifies the constructions of [43] with those of [31, 25]; the existence of such a unification is itself an apparently novel contribution to the literature. Most significantly, our way of constructing SUNTFs from certain pairs of ETFs and RBIBDs in Theorem 4.14 yields infinite families of apparently new ECTFFs via this same machinery.

This work has led to numerous avenues for follow-up research. Chief among these is a conjectured converse of Theorem 3.4: our preliminary work indicates that if an ECTFF arises from a sequence of isometries whose corresponding cross-Gram matrices commute then it is necessarily equivalent to one that arises from a SUNTF via Theorem 3.4. If true, this would seem to imply that a *filter bank fusion frame* [8], for example, could only be an ECTFF if its filters satisfied a certain property that is suitably equivalent to Definition 3.1. Our preliminary work moreover indicates that a need for such “filter bank” ECTFFs naturally arises in certain real-world applications involving waveform design. In essence, SUNTFs seem to be optimal for certain real-world scenarios, and we hope to formally codify this idea.

We have a number of other conjectures that would, if proven true, flesh out the general framework established in Chapter 3; due to time constraints, these laid outside the scope of this thesis. For instance, we believe that an ECTFF that arises via Theorem 3.4 is *equi-isoclinic* if and only if each “layer” of the corresponding SUNTF is an ETF. We further conjecture that a “layerwise” Naimark complement of a SUNTF is itself a SUNTF, and that the two resulting ECTFFs that arise via Theorem 3.4 are themselves Naimark complementary. Our results in Chapter 4 inspired yet more potential avenues for future work. For example, it seems relatively straightforward to exploit the multiplicative group of a finite field to construct difference families in its additive group, and this might lead to a host of ECTFF parameters via Theorem 4.3. In a similar vein, Theorems 4.10 and 4.14 suggest that some SUNTFs have the property that each of their layers is obtained by permuting the vectors of some common UNTF.

A number of follow-up problems of lesser importance also remain open. For instance, in the interest of due diligence, one should probably verify that the constructions of Theorems 4.3 and 4.10 are indeed equivalent to those of [31, 25] and the resolvable case of [43], respectively. One should probably also perform a thorough review of the literature of difference families that happen to be *reversible*, as each such family will seemingly yield a real ECTFF via Theorems 3.4 and 4.3. In the special case where $R = 1$, this reduces to the known fact that harmonic ETFs arising from reversible difference sets are real [26]. One may also wish to use the Naimark-spatial machinery of [24, 25] to determine the extent to which any putatively new ECTFF is truly novel.

Appendix A. Linear Operator Theory

In this appendix, we discuss several known results from linear operator theory. Throughout it, let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$, and let $\mathcal{E}$ and $\mathcal{F}$ be Euclidean spaces over $\mathbb{F}$.

Proposition A.1. If $\boldsymbol{L} : \mathcal{E} \to \mathcal{F}$ is linear then $\ker(\boldsymbol{L}^*\boldsymbol{L}) = \ker(\boldsymbol{L})$.

Proof. If $\boldsymbol{x} \in \ker(\boldsymbol{L})$ then $\boldsymbol{x} \in \ker(\boldsymbol{L}^*\boldsymbol{L})$ since $\boldsymbol{L}^*\boldsymbol{L}\boldsymbol{x} = \boldsymbol{L}^*\boldsymbol{0} = \boldsymbol{0}$. Conversely, if $\boldsymbol{x} \in \ker(\boldsymbol{L}^*\boldsymbol{L})$ then $\boldsymbol{x} \in \ker(\boldsymbol{L})$ since $\|\boldsymbol{L}\boldsymbol{x}\|^2 = \langle \boldsymbol{L}\boldsymbol{x}, \boldsymbol{L}\boldsymbol{x}\rangle = \langle \boldsymbol{x}, \boldsymbol{L}^*\boldsymbol{L}\boldsymbol{x}\rangle = \langle \boldsymbol{x}, \boldsymbol{0}\rangle = 0$. □

Proposition A.2. If $\boldsymbol{L} : \mathcal{E} \to \mathcal{F}$ is linear then $\boldsymbol{L}$, $\boldsymbol{L}^*$, $\boldsymbol{L}^*\boldsymbol{L}$ and $\boldsymbol{L}\boldsymbol{L}^*$ have equal rank.

Proof. Combining the rank-nullity theorem with Proposition A.1 gives

$$\operatorname{rank}(\boldsymbol{L}) = \dim(\mathcal{E}) - \dim(\ker(\boldsymbol{L})) = \dim(\mathcal{E}) - \dim(\ker(\boldsymbol{L}^*\boldsymbol{L})) = \operatorname{rank}(\boldsymbol{L}^*\boldsymbol{L}).$$

Applying this fact with "$\boldsymbol{L}$" being $\boldsymbol{L}$ and $\boldsymbol{L}^*$, respectively, gives:

$$\operatorname{rank}(\boldsymbol{L}) = \operatorname{rank}(\boldsymbol{L}^*\boldsymbol{L}) \le \operatorname{rank}(\boldsymbol{L}^*) = \operatorname{rank}(\boldsymbol{L}\boldsymbol{L}^*) \le \operatorname{rank}(\boldsymbol{L}),$$

from which the result immediately follows. □

Proposition A.3. $\operatorname{rank}\bigl(\sum_{n\in\mathcal{N}} \boldsymbol{L}_n\bigr) \le \sum_{n\in\mathcal{N}} \operatorname{rank}(\boldsymbol{L}_n)$ for any finite sequence $(\boldsymbol{L}_n)_{n\in\mathcal{N}}$ of linear maps from $\mathcal{E}$ into $\mathcal{F}$.

Proof. For every $\boldsymbol{y} \in \operatorname{im}\bigl(\sum_{n\in\mathcal{N}} \boldsymbol{L}_n\bigr)$, there exists $\boldsymbol{x} \in \mathcal{E}$ such that

$$\boldsymbol{y} = \Bigl(\sum_{n\in\mathcal{N}} \boldsymbol{L}_n\Bigr)\boldsymbol{x} = \sum_{n\in\mathcal{N}} \boldsymbol{L}_n\boldsymbol{x} \in \sum_{n\in\mathcal{N}} \operatorname{im}(\boldsymbol{L}_n).$$

Thus, $\mathrm{im}\big(\sum_{n\in\mathcal{N}} \boldsymbol{L}_n\big) \subseteq \sum_{n\in\mathcal{N}} \mathrm{im}(\boldsymbol{L}_n)$. As such,

$$\begin{aligned}
\mathrm{rank}\Big(\sum_{n\in\mathcal{N}} \boldsymbol{L}_n\Big) &= \dim\bigg(\mathrm{im}\Big(\sum_{n\in\mathcal{N}} \boldsymbol{L}_n\Big)\bigg) \\
&\leq \dim\bigg(\sum_{n\in\mathcal{N}} \mathrm{im}(\boldsymbol{L}_n)\bigg) \\
&\leq \sum_{n\in\mathcal{N}} \dim\big(\mathrm{im}(\boldsymbol{L}_n)\big) \\
&= \sum_{n\in\mathcal{N}} \mathrm{rank}(\boldsymbol{L}_n). \qquad \square
\end{aligned}$$

Proposition A.4. If $\boldsymbol{L} : \mathcal{E} \to \mathcal{F}$ is linear then $[\mathrm{im}(\boldsymbol{L})]^\perp = \ker(\boldsymbol{L}^*)$.

Proof. Recall that the orthogonal complement of any subspace $\mathcal{U}$ of an inner product space $\mathcal{V}$ is $\mathcal{U}^\perp := \{\boldsymbol{v} \in \mathcal{V} : \langle \boldsymbol{u}, \boldsymbol{v}\rangle = 0, \forall \boldsymbol{u} \in \mathcal{U}\}$. Thus,

$$\begin{aligned}
[\mathrm{im}(\boldsymbol{L})]^\perp &= \{\boldsymbol{y} \in \mathcal{F} : \langle \boldsymbol{y}, \boldsymbol{L}\boldsymbol{x}\rangle = 0,\ \forall\, \boldsymbol{x} \in \mathcal{E}\} \\
&= \{\boldsymbol{y} \in \mathcal{F} : \langle \boldsymbol{L}^*\boldsymbol{y}, \boldsymbol{x}\rangle = 0,\ \forall\, \boldsymbol{x} \in \mathcal{E}\} \\
&= \big\{\boldsymbol{y} \in \mathcal{F} : \boldsymbol{L}^*\boldsymbol{y} \in \mathcal{E}^\perp\ = \{\boldsymbol{0}\}\big\} \\
&= \ker(\boldsymbol{L}^*). \qquad \square
\end{aligned}$$

Proposition A.5. If $\boldsymbol{L} : \mathcal{E} \to \mathcal{F}$ is linear then $\mathrm{im}(\boldsymbol{L}\boldsymbol{L}^*) = \mathrm{im}(\boldsymbol{L})$.

Proof. Combining Propositions A.1 and A.4 with the fact that $(\mathcal{U}^\perp)^\perp = \mathcal{U}$ for any subspace of a Euclidean space gives

$$\mathrm{im}(\boldsymbol{L}) = \Big\{\big[\mathrm{im}(\boldsymbol{L})\big]^\perp\Big\}^\perp = \big[\ker(\boldsymbol{L}^*)\big]^\perp = \big[\ker(\boldsymbol{L}\boldsymbol{L}^*)\big]^\perp = \Big\{\big[\mathrm{im}(\boldsymbol{L}\boldsymbol{L}^*)\big]^\perp\Big\}^\perp = \mathrm{im}(\boldsymbol{L}\boldsymbol{L}^*). \quad \square$$

Proposition A.6. Let $\boldsymbol{L} : \mathcal{E} \to \mathcal{F}$ and $\boldsymbol{K} : \mathcal{F} \to \mathcal{E}$ be linear. If $\dim(\mathcal{E}) = \dim(\mathcal{F})$ and $\boldsymbol{K}\boldsymbol{L} = \mathbf{I}_\mathcal{E}$ then $\boldsymbol{L}\boldsymbol{K} = \mathbf{I}_\mathcal{F}$.

Proof. Note $\dim(\mathcal{F}) = \dim(\mathcal{E}) = \operatorname{rank}(\mathbf{I}_{\mathcal{E}}) = \operatorname{rank}(\boldsymbol{KL}) \leq \operatorname{rank}(\boldsymbol{L}) \leq \dim(\mathcal{F})$. Thus, $\operatorname{rank}(\boldsymbol{L}) = \dim(\mathcal{F}) = \dim(\mathcal{E})$. Thus, $\operatorname{im}(\boldsymbol{L})$ is a subspace of $\mathcal{F}$ of dimension $\dim(\mathcal{F})$ and, by the rank-nullity theorem, $\ker(\boldsymbol{L})$ has dimension $\dim(\mathcal{E}) - \operatorname{rank}(\boldsymbol{L}) = 0$, and so $\boldsymbol{L}$ is invertible. As such, $\boldsymbol{K} = \boldsymbol{KLL}^{-1} = \mathbf{I}_{\mathcal{E}}\boldsymbol{L}^{-1} = \boldsymbol{L}^{-1}$ and so $\boldsymbol{LK} = \mathbf{I}_{\mathcal{F}}$. $\square$

Proposition A.7. Let $\mathcal{E}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ be Euclidean spaces over $\mathbb{F}$ and let $\boldsymbol{L}_1 : \mathcal{E} \to \mathcal{F}_1$ and $\boldsymbol{L}_2 : \mathcal{E} \to \mathcal{F}_2$ be linear and surjective. Then, $\boldsymbol{L}_1^*\boldsymbol{L}_1 = \boldsymbol{L}_2^*\boldsymbol{L}_2$ if and only if there exists a unitary map $\boldsymbol{U} : \mathcal{F}_1 \to \mathcal{F}_2$ such that $\boldsymbol{L}_2 = \boldsymbol{UL}_1$.

Proof. ($\Leftarrow$) If there exists a unitary map $\boldsymbol{U} : \mathcal{F}_1 \to \mathcal{F}_2$ such that $\boldsymbol{L}_2 = \boldsymbol{UL}_1$ then

$$\boldsymbol{L}_2^*\boldsymbol{L}_2 = (\boldsymbol{UL}_1)^*\boldsymbol{UL}_1 = \boldsymbol{L}_1^*\boldsymbol{U}^*\boldsymbol{UL}_1 = \boldsymbol{L}_1^*\boldsymbol{L}_1.$$

($\Rightarrow$) Suppose $\boldsymbol{L}_1^*\boldsymbol{L}_1 = \boldsymbol{L}_2^*\boldsymbol{L}_2$. Since $\boldsymbol{L}_1$ is surjective, Proposition A.2 gives $\operatorname{im}(\boldsymbol{L}_1\boldsymbol{L}_1^*) = \operatorname{im}(\boldsymbol{L}_1) = \mathcal{F}_1$. By Proposition A.4, $\ker(\boldsymbol{L}_1\boldsymbol{L}_1^*) = \left[\operatorname{im}(\boldsymbol{L}_1\boldsymbol{L}_1^*)\right]^\perp = \mathcal{F}_1^\perp = \{\mathbf{0}\}$. Thus, $\boldsymbol{L}_1\boldsymbol{L}_1^*$ is invertible. Define $\boldsymbol{U} : \mathcal{F}_1 \to \mathcal{F}_2$ by $\boldsymbol{U} := \boldsymbol{L}_2\boldsymbol{L}_1^*(\boldsymbol{L}_1\boldsymbol{L}_1^*)^{-1}$. Since $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ are surjective, the domain and codomain of $\boldsymbol{U}$ have equal dimension:

$$\dim(\mathcal{F}_1) = \operatorname{rank}(\boldsymbol{L}_1) = \operatorname{rank}(\boldsymbol{L}_1^*\boldsymbol{L}_1) = \operatorname{rank}(\boldsymbol{L}_2^*\boldsymbol{L}_2) = \operatorname{rank}(\boldsymbol{L}_2) = \dim(\mathcal{F}_2).$$

Moreover,

$$\boldsymbol{U}^*\boldsymbol{U} = (\boldsymbol{L}_1\boldsymbol{L}_1^*)^{-1}\boldsymbol{L}_1\boldsymbol{L}_2^*\boldsymbol{L}_2\boldsymbol{L}_1^*(\boldsymbol{L}_1\boldsymbol{L}_1^*)^{-1} = (\boldsymbol{L}_1\boldsymbol{L}_1^*)^{-1}\boldsymbol{L}_1\boldsymbol{L}_1^*\boldsymbol{L}_1\boldsymbol{L}_1^*(\boldsymbol{L}_1\boldsymbol{L}_1^*)^{-1} = \mathbf{I}.$$

Together these facts imply via Proposition A.6 that $\boldsymbol{UU}^* = \mathbf{I}$. Thus, $\boldsymbol{U}$ is unitary. Also,

$$\boldsymbol{U}^*\boldsymbol{L}_2 = (\boldsymbol{L}_1\boldsymbol{L}_1^*)^{-1}\boldsymbol{L}_1\boldsymbol{L}_2^*\boldsymbol{L}_2 = (\boldsymbol{L}_1\boldsymbol{L}_1^*)^{-1}\boldsymbol{L}_1\boldsymbol{L}_1^*\boldsymbol{L}_1 = \boldsymbol{L}_1,$$

and so $\boldsymbol{L}_2 = \boldsymbol{UL}_1$, as claimed. $\square$

Proposition A.8. Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$, and let $\mathcal{N}$ be a finite nonempty set. A matrix $\boldsymbol{G} \in \mathbb{F}^{\mathcal{N}\times\mathcal{N}}$ is PSD if and only if it is the Gram matrix of a finite sequence of vectors $(\boldsymbol{\varphi}_n)_{n\in\mathcal{N}}$ that spans a Euclidean space of dimension $\operatorname{rank}(\boldsymbol{G})$.

Proof. ($\Leftarrow$) Suppose $\boldsymbol{G} = \boldsymbol{\Phi}^*\boldsymbol{\Phi}$ where $\boldsymbol{\Phi} : \mathbb{F}^{\mathcal{N}} \to \mathcal{E}$ is the synthesis map of a finite sequence $(\varphi_n)_{n\in\mathcal{N}}$ of vectors that spans a Euclidean space $\mathcal{E}$ of dimension $\operatorname{rank}(\boldsymbol{G})$. Then $\boldsymbol{\Phi}^*\boldsymbol{\Phi}$ is self-adjoint, and is moreover PSD since $\langle \boldsymbol{x}, \boldsymbol{\Phi}^*\boldsymbol{\Phi}\boldsymbol{x}\rangle = \langle \boldsymbol{\Phi}\boldsymbol{x}, \boldsymbol{\Phi}\boldsymbol{x}\rangle = \|\boldsymbol{\Phi}\boldsymbol{x}\|^2 \geq 0$ for all $x \in \mathbb{F}^N$.

($\Rightarrow$) Now suppose that $\boldsymbol{G} \in \mathbb{F}^{\mathcal{N}\times\mathcal{N}}$ is PSD. Since $\boldsymbol{G}$ is self adjoint and therefore normal, it is unitarily diagonalizable, that is, $\boldsymbol{G} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^*$ for some unitary matrix $\boldsymbol{U} \in \mathbb{F}^{\mathcal{N}\times[N]}$ and diagonal matrix $\boldsymbol{\Lambda} \in \mathbb{F}^{N\times N}$. Thus, $\boldsymbol{U}$ is the synthesis map of an orthonormal basis $(\boldsymbol{u}_n)_{n=1}^N$ for $\mathbb{F}^{\mathcal{N}}$, where each $\boldsymbol{u}_n$ is an eigenvector for $\boldsymbol{G}$ whose corresponding eigenvalue λ_n is the nth diagonal entry of $\boldsymbol{\Lambda}$. Since $\boldsymbol{G}$ is PSD, $\lambda_n \geq 0$ for all $n \in [N]$. By reindexing if necessary, we further assume without loss of generality that $(\lambda_n)_{n=1}^N$ is nonincreasing. Letting $D := \operatorname{rank}(\boldsymbol{G}) = \#\{n \in [N] : \lambda_n > 0\}$, note that for any $\boldsymbol{x} \in \mathbb{F}^{\mathcal{N}}$,

$$\boldsymbol{G}\boldsymbol{x} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^*\boldsymbol{x} = \sum_{n=1}^N \lambda_n \langle \boldsymbol{u}_n, \boldsymbol{x}\rangle \boldsymbol{u}_n = \sum_{d=1}^D \lambda_d \langle \boldsymbol{u}_d, \boldsymbol{x}\rangle \boldsymbol{u}_d = \boldsymbol{U}_D\boldsymbol{\Lambda}_D\boldsymbol{U}_D^*\boldsymbol{x},$$

where $\boldsymbol{U}_D \in \mathbb{F}^{\mathcal{N}\times[D]}$ is the synthesis map of $(\boldsymbol{u}_d)_{d=1}^D$ and $\boldsymbol{\Lambda}_D \in \mathbb{F}^{D\times D}$ is the diagonal matrix whose dth diagonal entry is λ_d. Letting $\boldsymbol{\Sigma}_D \in \mathbb{F}^{D\times D}$ be the diagonal matrix whose dth diagonal entry is $\sqrt{\lambda_d}$, it follows that $\boldsymbol{\Lambda}_D = \boldsymbol{\Sigma}_D^2 = \boldsymbol{\Sigma}_D\boldsymbol{\Sigma}_D^*$. Accordingly,

$$\boldsymbol{G} = \boldsymbol{U}_D\boldsymbol{\Lambda}_D\boldsymbol{U}_D^* = \boldsymbol{U}_D\boldsymbol{\Sigma}_D\boldsymbol{\Sigma}_D^*\boldsymbol{U}_D^* = (\boldsymbol{U}_D\boldsymbol{\Sigma}_D)(\boldsymbol{U}_D\boldsymbol{\Sigma}_D)^* = \boldsymbol{\Phi}^*\boldsymbol{\Phi},$$

where $\boldsymbol{\Phi} := (\boldsymbol{U}_D\boldsymbol{\Sigma}_D)^*$. Since $\boldsymbol{\Phi} : \mathbb{F}^{\mathcal{N}} \to \mathbb{F}^D$ is linear, it is the synthesis map of some finite sequence $(\boldsymbol{\varphi}_n)_{n\in\mathcal{N}}$ of vectors in $\mathbb{F}^D$. Moreover,

$$\dim(\operatorname{span}(\boldsymbol{\varphi}_n)_{n\in\mathcal{N}}) = \dim(\operatorname{im}\boldsymbol{\Phi}) = \operatorname{rank}(\boldsymbol{\Phi}) = \operatorname{rank}(\boldsymbol{\Phi}^*\boldsymbol{\Phi}) = \operatorname{rank}(\boldsymbol{G}) = D,$$

so $(\boldsymbol{\varphi}_n)_{n\in\mathcal{N}}$ spans $\mathbb{F}^D$. $\square$

Appendix B. Properties of the Trace

In this appendix, we discuss various known properties of the trace.

Proposition B.1. The trace of a linear operator $\boldsymbol{L}$ over a Euclidean Space $\mathcal{E}$ is well-defined, that is, for any two orthonormal bases $(\boldsymbol{v}_n)_{n\in\mathcal{N}}$ and $(\boldsymbol{u}_m)_{m\in\mathcal{M}}$ of $\mathcal{E}$,

$$\sum_{n\in\mathcal{N}}\langle\boldsymbol{v}_n,\boldsymbol{L}\boldsymbol{v}_n\rangle=\sum_{m\in\mathcal{M}}\langle\boldsymbol{u}_m,\boldsymbol{L}\boldsymbol{u}_m\rangle.$$

Proof. Let both $(\boldsymbol{v}_n)_{n\in\mathcal{N}}$ and $(\boldsymbol{u}_m)_{m\in\mathcal{M}}$ be orthonormal bases for $\mathcal{E}$, and let $\boldsymbol{L}$ be any linear operator on $\mathcal{E}$. For any $m'\in\mathcal{M}$ and $n'\in\mathcal{N}$,

$$\boldsymbol{v}_{n'}=\sum_{m\in\mathcal{M}}\langle\boldsymbol{u}_m,\boldsymbol{v}_{n'}\rangle\boldsymbol{u}_m,\quad\text{and}\quad\boldsymbol{u}_{m'}=\sum_{n\in\mathcal{N}}\langle\boldsymbol{v}_n,\boldsymbol{u}_{m'}\rangle\boldsymbol{v}_n.$$

Thus,

$$\begin{aligned}
\operatorname{Tr}(\boldsymbol{L})&=\sum_{n\in\mathcal{N}}\langle\boldsymbol{v}_n,\boldsymbol{L}\boldsymbol{v}_n\rangle\\
&=\sum_{n\in\mathcal{N}}\left\langle\boldsymbol{v}_n,\boldsymbol{L}\left(\sum_{m\in\mathcal{M}}\langle\boldsymbol{u}_m,\boldsymbol{v}_n\rangle\boldsymbol{u}_m\right)\right\rangle\\
&=\sum_{n\in\mathcal{N}}\left\langle\boldsymbol{v}_n,\sum_{m\in\mathcal{M}}\langle\boldsymbol{u}_m,\boldsymbol{v}_n\rangle\boldsymbol{L}\boldsymbol{u}_m\right\rangle\\
&=\sum_{n\in\mathcal{N}}\sum_{m\in\mathcal{M}}\big\langle\boldsymbol{v}_n,\langle\boldsymbol{u}_m,\boldsymbol{v}_n\rangle\boldsymbol{L}\boldsymbol{u}_m\big\rangle\\
&=\sum_{n\in\mathcal{N}}\sum_{m\in\mathcal{M}}\langle\boldsymbol{u}_m,\boldsymbol{v}_n\rangle\big\langle\boldsymbol{v}_n,\boldsymbol{L}\boldsymbol{u}_m\big\rangle\\
&=\sum_{m\in\mathcal{M}}\sum_{n\in\mathcal{N}}\big\langle\boldsymbol{v}_n,\boldsymbol{L}\boldsymbol{u}_m\big\rangle\langle\boldsymbol{u}_m,\boldsymbol{v}_n\rangle\\
&=\sum_{m\in\mathcal{M}}\langle\boldsymbol{u}_m,\sum_{n\in\mathcal{N}}\big\langle\boldsymbol{v}_n,\boldsymbol{L}\boldsymbol{u}_m\big\rangle\boldsymbol{v}_n\rangle\\
&=\sum_{m\in\mathcal{M}}\langle\boldsymbol{u}_m,\boldsymbol{L}\boldsymbol{u}_m\rangle.
\end{aligned}$$

□

Proposition B.2. If $\mathcal{E}$ and $\mathcal{F}$ are Euclidean spaces and $\boldsymbol{L} : \mathcal{E} \to \mathcal{F}$ and $\boldsymbol{L} : \mathcal{F} \to \mathcal{E}$ are linear then $\operatorname{Tr}(\boldsymbol{KL}) = \operatorname{Tr}(\boldsymbol{LK})$.

Proof. Let $(\boldsymbol{u}_m)_{m\in\mathcal{M}}$ and $(\boldsymbol{v}_n)_{n\in\mathcal{N}}$ be orthonormal bases for $\mathcal{E}$ and $\mathcal{F}$, respectively. Then

$$
\begin{aligned}
\operatorname{Tr}(\boldsymbol{KL}) &= \sum_{m\in\mathcal{M}} \langle \boldsymbol{u}_m, \boldsymbol{KLu}_m \rangle \\
&= \sum_{m\in\mathcal{M}} \left\langle \boldsymbol{u}_m, \boldsymbol{K}\left(\sum_{n\in\mathcal{N}} \langle \boldsymbol{v}_n, \boldsymbol{Lu}_m \rangle \boldsymbol{v}_n \right) \right\rangle \\
&= \sum_{m\in\mathcal{M}} \langle \boldsymbol{u}_m, \sum_{n\in\mathcal{N}} \langle \boldsymbol{v}_n, \boldsymbol{Lu}_m \rangle \boldsymbol{Kv}_n \rangle \\
&= \sum_{m\in\mathcal{M}} \sum_{n\in\mathcal{N}} \langle \boldsymbol{v}_n, \boldsymbol{Lu}_m \rangle \langle \boldsymbol{u}_m, \boldsymbol{Kv}_n \rangle \\
&= \sum_{n\in\mathcal{N}} \sum_{m\in\mathcal{M}} \langle \boldsymbol{u}_m, \boldsymbol{Kv}_n \rangle \langle \boldsymbol{v}_n, \boldsymbol{Lu}_m \rangle \\
&= \sum_{n\in\mathcal{N}} \langle \boldsymbol{v}_n, \sum_{m\in\mathcal{M}} \langle \boldsymbol{u}_m, \boldsymbol{Kv}_n \rangle \boldsymbol{Lu}_m \rangle \\
&= \sum_{n\in\mathcal{N}} \left\langle \boldsymbol{v}_n, \boldsymbol{L}\left(\sum_{m\in\mathcal{M}} \langle \boldsymbol{u}_m, \boldsymbol{Kv}_n \rangle \boldsymbol{u}_m \right) \right\rangle \\
&= \sum_{n\in\mathcal{N}} \langle \boldsymbol{v}_n, \boldsymbol{LKv}_n \rangle \\
&= \operatorname{Tr}(\boldsymbol{LK}).
\end{aligned}
$$

□

Proposition B.3. The trace is linear, that is, $\operatorname{Tr}(c_1\boldsymbol{L}_1 + c_2\boldsymbol{L}_2) = c_1 \operatorname{Tr}(\boldsymbol{L}_1) + c_2 \operatorname{Tr}(\boldsymbol{L}_2)$ for any $c_1, c_2 \in \mathbb{F}$ and linear operators $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ over some Euclidean space $\mathcal{E}$.

Proof. Let $(\boldsymbol{v}_n)_{\{n\in\mathcal{N}\}}$ be an orthonormal basis for $\mathcal{E}$. Then

$$\begin{aligned}\operatorname{Tr}(c_1\boldsymbol{L}_1 + c_2\boldsymbol{L}_2) &= \sum_{n\in\mathcal{N}}\langle \boldsymbol{v}_n, (c_1\boldsymbol{L}_1 + c_2\boldsymbol{L}_2)\boldsymbol{v}_n\rangle \\ &= \sum_{n\in\mathcal{N}}\langle \boldsymbol{v}_n, c_1\boldsymbol{L}_1\boldsymbol{v}_n + c_2\boldsymbol{L}_2\boldsymbol{v}_n\rangle \\ &= c_1\sum_{n\in\mathcal{N}}\langle \boldsymbol{v}_n, \boldsymbol{L}_1\boldsymbol{v}_n\rangle + c_2\sum_{n\in\mathcal{N}}\langle \boldsymbol{v}_n, \boldsymbol{L}_2\boldsymbol{v}_n\rangle \\ &= c_1\operatorname{Tr}(\boldsymbol{L}_1) + c_2\operatorname{Tr}(\boldsymbol{L}_2). \qquad \square\end{aligned}$$

Proposition B.4. For any linear operator $\boldsymbol{L}$ on $\mathcal{E}$, we have $\operatorname{Tr}(\boldsymbol{L}^*) = \overline{\operatorname{Tr}(\boldsymbol{L})}$.

Proof. Letting $(\boldsymbol{v}_n)_{n\in\mathcal{N}}$ be an orthonormal basis for $\mathcal{E}$,

$$\operatorname{Tr}(\boldsymbol{L}^*) = \sum_{n\in\mathcal{N}}\langle \boldsymbol{v}_n, \boldsymbol{L}^*\boldsymbol{v}_n\rangle = \sum_{n\in\mathcal{N}}\langle \boldsymbol{L}\boldsymbol{v}_n, \boldsymbol{v}_n\rangle = \sum_{n\in\mathcal{N}}\overline{\langle \boldsymbol{v}_n, \boldsymbol{L}\boldsymbol{v}_n\rangle} = \overline{\sum_{n\in\mathcal{N}}\langle \boldsymbol{v}_n, \boldsymbol{L}\boldsymbol{v}_n\rangle} = \overline{\operatorname{Tr}(\boldsymbol{L})}. \qquad \square$$

Proposition B.5. The trace of a projection over a Euclidean space $\mathcal{E}$ is equal to its rank.

Proof. Let $\mathcal{V}$ be a subspace of $\mathcal{E}$, and let $\boldsymbol{\Phi} : \mathbb{F}^R \to \mathcal{V}$ be the synthesis map of some orthonormal basis $(\boldsymbol{\varphi}_r)_{r=1}^R$ for $\mathcal{V}$. Note $\boldsymbol{\Phi}^*\boldsymbol{\Phi} = \mathbf{I}_R$ since $(\boldsymbol{\varphi}_r)_{r=1}^R$ is orthonormal. Moreover, $\boldsymbol{\Phi}\boldsymbol{\Phi}^*$ is the projection onto $\mathcal{V}$ since it is self-adjoint with $(\boldsymbol{\Phi}\boldsymbol{\Phi}^*)^2 = \boldsymbol{\Phi}\boldsymbol{\Phi}^*\boldsymbol{\Phi}\boldsymbol{\Phi}^* = \boldsymbol{\Phi}\boldsymbol{\Phi}^*$ and $\operatorname{im}(\boldsymbol{\Phi}\boldsymbol{\Phi}^*) = \operatorname{im}(\boldsymbol{\Phi}) = \operatorname{span}(\boldsymbol{\varphi}_r)_{r\in\mathcal{R}} = \mathcal{V}$. To see that the trace of $\boldsymbol{\Phi}\boldsymbol{\Phi}^*$ equals its rank, note

$$\operatorname{Tr}(\boldsymbol{\Phi}\boldsymbol{\Phi}^*) = \operatorname{Tr}(\boldsymbol{\Phi}^*\boldsymbol{\Phi}) = \operatorname{Tr}(\mathbf{I}_R) = R = \operatorname{rank}(\mathbf{I}_R) = \operatorname{rank}(\boldsymbol{\Phi}^*\boldsymbol{\Phi}) = \operatorname{rank}(\boldsymbol{\Phi}\boldsymbol{\Phi}^*). \qquad \square$$

Appendix C. Harmonic Analysis over Finite Abelian Groups

In this appendix, we demonstrate that any finite abelian group is isomorphic to its Pontryagin dual, discussing other known results in the process.

Proposition C.1. The Pontryagin dual $\widehat{\mathcal{G}}$ of a finite abelian group $\mathcal{G}$, given by

$$\widehat{\mathcal{G}} := \{\gamma : \mathcal{G} \to \mathbb{T} : \gamma(g_1 + g_2) = \gamma(g_1)\gamma(g_2),\ \forall\, g_1, g_2 \in \mathcal{G}\},$$

is an abelian group under the Hadamard product.

Proof. Throughout this proof, let γ_1, γ_2 and γ_3 be characters of $\mathcal{G}$. For any $g_1, g_2 \in \mathcal{G}$,

$$\begin{aligned}(\gamma_1\gamma_2)(g_1 + g_2) &= \gamma_1(g_1 + g_2)\gamma_2(g_1 + g_2)\\ &= \gamma_1(g_1)\gamma_1(g_2)\gamma_2(g_1)\gamma_2(g_2)\\ &= \gamma_1(g_1)\gamma_2(g_1)\gamma_1(g_2)\gamma_2(g_2)\\ &= (\gamma_1\gamma_2)(g_1)(\gamma_1\gamma_2)(g_2).\end{aligned}$$

Thus, the product of any two characters is itself a character. Moreover, for any $g \in \mathcal{G}$,

$$\begin{aligned}(\gamma_1(\gamma_2\gamma_3))(g) &= \gamma_1(g)(\gamma_2\gamma_3)(g)\\ &= \gamma_1(g)(\gamma_2(g)\gamma_3(g))\\ &= (\gamma_1(g)\gamma_2(g))\gamma_3(g)\\ &= (\gamma_1\gamma_2)(g)\gamma_3(g)\\ &= ((\gamma_1\gamma_2)\gamma_3)(g).\end{aligned}$$

Thus, the Hadamard product is associative. Next, the identity character 1 is a multiplicative identity for this product since for any $\gamma \in \widehat{\mathcal{G}}$ and $g \in \mathcal{G}$,

$$(1\gamma)(g) = 1(g)\gamma(g) = 1\gamma(g) = \gamma(g) = \gamma(g)1 = \gamma(g)1(g) = (\gamma 1)(g).$$

Next, for any $\gamma \in \widehat{\mathcal{G}}$, defining $\gamma^{-1} : \mathcal{G} \to \mathbb{T}$ by $\gamma^{-1}(g) := \overline{\gamma(g)} = (\gamma(g))^{-1}$, we have

$$\begin{aligned}(\gamma\gamma^{-1})(g) &= \gamma(g)\gamma^{-1}(g) = \gamma(g)(\gamma(g))^{-1} = 1, \\ (\gamma^{-1}\gamma)(g) &= \gamma^{-1}(g)\gamma(g) = (\gamma(g))^{-1}\gamma(g) = 1,\end{aligned}$$

for any $g \in \mathcal{G}$. Therefore, $(\widehat{\mathcal{G}}, \odot)$ is a group. It is abelian since for any $g \in \mathcal{G}$,

$$(\gamma_1\gamma_2)(g) = \gamma_1(g)\gamma_2(g) = \gamma_2(g)\gamma_1(g) = (\gamma_2\gamma_1)(g). \qquad \square$$

Proposition C.2. If $\mathcal{G}$ is a finite abelian group of order G then $(\frac{1}{\sqrt{G}}\boldsymbol{\gamma})_{\gamma \in \widehat{\mathcal{G}}}$ is orthonormal.

Proof. For any $\gamma_1, \gamma_2 \in \widehat{\mathcal{G}}$,

$$\langle \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \rangle = \sum_{g \in \mathcal{G}} \overline{\boldsymbol{\gamma}_1(g)}\boldsymbol{\gamma}_2(g) = \sum_{g \in \mathcal{G}} \boldsymbol{\gamma}_1^{-1}(g)\boldsymbol{\gamma}_2(g) = \sum_{g \in \mathcal{G}} (\boldsymbol{\gamma}_1^{-1}\boldsymbol{\gamma}_2)(g).$$

When $\gamma := \gamma_1 = \gamma_2$, we have $\boldsymbol{\gamma}^{-1}\boldsymbol{\gamma} = \mathbf{1}$, and this becomes $\|\boldsymbol{\gamma}\|^2 = \langle \boldsymbol{\gamma}, \boldsymbol{\gamma} \rangle = G$. When instead γ_1 and γ_2 are distinct, taking $g' \in \mathcal{G}$ such that $\boldsymbol{\gamma}_1(g') \neq \boldsymbol{\gamma}_2(g')$, we have

$$\begin{aligned}\big[(\boldsymbol{\gamma}_1^{-1}\boldsymbol{\gamma}_2)(g')\big]\langle \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \rangle &= \big[(\boldsymbol{\gamma}_1^{-1}\boldsymbol{\gamma}_2)(g')\big] \sum_{g \in \mathcal{G}} (\boldsymbol{\gamma}_1^{-1}\boldsymbol{\gamma}_2)(g) \\ &= \sum_{g \in \mathcal{G}} (\boldsymbol{\gamma}_1^{-1}\boldsymbol{\gamma}_2)(g')(\boldsymbol{\gamma}_1^{-1}\boldsymbol{\gamma}_2)(g) \\ &= \sum_{g \in \mathcal{G}} (\boldsymbol{\gamma}_1^{-1}\boldsymbol{\gamma}_2)(g + g') \\ &= \sum_{g'' \in \mathcal{G}} (\boldsymbol{\gamma}_1^{-1}\boldsymbol{\gamma}_2)(g'') \\ &= \langle \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \rangle.\end{aligned}$$

Thus, $0 = \big[1 - (\boldsymbol{\gamma}_1^{-1}\boldsymbol{\gamma}_2)(g')\big]\langle \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \rangle$. Since $\boldsymbol{\gamma}_1(g') \neq \boldsymbol{\gamma}_2(g')$, we have that $(\boldsymbol{\gamma}_1^{-1}\boldsymbol{\gamma}_2)(g') \neq 1$, so $\langle \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \rangle = 0$ whenever $\gamma_1 \neq \gamma_2$. $\square$

Proposition C.3. For any positive integer N, $\mathbb{Z}_N$ is isomorphic to its Pontryagin dual.

Proof. From Example 2.5, recall that $\phi : \mathbb{Z}_N \to \widehat{\mathbb{Z}}_N$, $\phi(n) := \mathbf{e}_n$ is an injective homomorphism. Moreover, $\#(\widehat{\mathbb{Z}}_N) \leq \#(\mathbb{Z}_N)$ since the members of $\widehat{\mathbb{Z}}_N$ are orthogonal and nonzero by Proposition C.2. Thus, ϕ is also surjective, and so is an isomorphism. $\square$

Proposition C.4. Let η and κ be characters of finite abelian groups $\mathcal{H}$ and $\mathcal{K}$, respectively. Then $\eta \otimes \kappa$ is a character of $\mathcal{H} \times \mathcal{K}$. Moreover, $\eta_1 \otimes \kappa_1 \neq \eta_2 \otimes \kappa_2$ if either $\eta_1 \neq \eta_2$ or $\kappa_1 \neq \kappa_2$.

Proof. For any $(h_1, k_1), (h_2, k_2) \in \mathcal{H} \times \mathcal{K}$,

$$\begin{aligned}
(\eta \otimes \kappa)(h_1 + h_2, k_1 + k_2) &= \eta(h_1 + h_2)\kappa(k_1 + k_2) \\
&= \eta(h_1)\eta(h_2)\kappa(k_1)\kappa(k_2) \\
&= \eta(h_1)\kappa(k_1)\eta(h_2)\kappa(k_2) \\
&= (\eta \otimes \kappa)(h_1, k_1)(\eta \otimes \kappa)(h_2, k_2).
\end{aligned}$$

Thus, $\eta \otimes \kappa$ is a character of $\mathcal{H} \times \mathcal{K}$. Next, if $\eta_1 \otimes \kappa_1 = \eta_2 \otimes \kappa_2$ for some $\eta_1, \eta_2 \in \widehat{\mathcal{H}}$ and $\kappa_1, \kappa_2 \in \widehat{\mathcal{K}}$ then $\eta_1 = \eta_2$ and $\kappa_1 = \kappa_2$ since for any $h \in \mathcal{H}$, $k \in \mathcal{K}$,

$$\begin{aligned}
\eta_1(h) &= (\eta_1 \otimes \kappa_1)(h, 0) = (\eta_2 \otimes \kappa_2)(h, 0) = \eta_2(h), \\
\kappa_1(k) &= (\eta_1 \otimes \kappa_1)(0, k) = (\eta_2 \otimes \kappa_2)(0, k) = \kappa_2(k).
\end{aligned}$$

$\square$

Proposition C.5. $(\mathcal{H} \times \mathcal{K})^\wedge \cong \widehat{\mathcal{H}} \times \widehat{\mathcal{K}}$ for any finite abelian groups $\mathcal{H}$ and $\mathcal{K}$.

Proof. Consider the mapping $(\eta, \kappa) \mapsto \eta \otimes \kappa$ from $\widehat{\mathcal{H}} \times \widehat{\mathcal{K}}$ into $(\mathcal{H} \times \mathcal{K})^\wedge$. By Proposition C.4, this mapping is both well-defined and injective. It is moreover surjective since for any $\gamma \in (\mathcal{H} \times \mathcal{K})^\wedge$, letting $\eta(h) := \gamma(h, 0)$ and $\kappa(k) := \gamma(0, k)$ defines $\eta \in \widehat{\mathcal{H}}$ and $\kappa \in \widehat{\mathcal{K}}$ such that

$$\gamma(h, k) = \gamma(h, 0)\gamma(0, k) = \eta(h)\kappa(k) = (\eta \otimes \kappa)(h, k).$$

This mapping is a homomorphism since for any $(\eta_1, \kappa_1), (\eta_2, \kappa_2) \in \widehat{\mathcal{H}} \times \widehat{\mathcal{K}}$ and $(h, k) \in \mathcal{H} \times \mathcal{K}$,

$$\begin{aligned}
\big[(\eta_1, \kappa_1)(\eta_2, \kappa_2)\big](h, k) &= \big[(\eta_1 \otimes \kappa_1)(\eta_2 \otimes \kappa_2)\big](h, k) \\
&= \big[(\eta_1 \otimes \kappa_1)(h, k)\big]\big[(\eta_2 \otimes \kappa_2)(h, k)\big] \\
&= \eta_1(h)\kappa_1(k)\eta_2(h)\kappa_2(k) \\
&= \eta_1(h)\eta_2(h)\kappa_1(k)\kappa_2(k) \\
&= (\eta_1\eta_2)(h)(\kappa_1\kappa_2)(k) \\
&= (\eta_1\eta_2 \otimes \kappa_1\kappa_2)(h, k).
\end{aligned}$$

□

Proposition C.6. If $\mathcal{H}$ and $\mathcal{K}$ are finite abelian groups and $\mathcal{H} \cong \mathcal{K}$ then $\widehat{\mathcal{H}} \cong \widehat{\mathcal{K}}$.

Proof. Let $\phi : \mathcal{H} \to \mathcal{K}$ be an isomorphism. Define $\hat{\phi} : \widehat{\mathcal{H}} \to \widehat{\mathcal{K}}$ by $\hat{\phi}(\eta) := \eta \circ \phi^{-1}$. To see that $\hat{\phi}$ is a well-defined, note that for any $k_1, k_2 \in \mathcal{K}$,

$$\begin{aligned}
(\hat{\phi}(\eta))(k_1 + k_2) &= (\eta \circ \phi^{-1})(k_1 + k_2) \\
&= \eta(\phi^{-1}(k_1 + k_2)) \\
&= \eta(\phi^{-1}(k_1) + \phi^{-1}(k_2)) \\
&= \eta(\phi^{-1}(k_1))\eta(\phi^{-1}(k_2)) \\
&= (\eta \circ \phi^{-1})(k_1)(\eta \circ \phi^{-1})(k_2) \\
&= (\hat{\phi}(\eta))(k_1)(\hat{\phi}(\eta))(k_2).
\end{aligned}$$

To see that $\hat{\phi}$ is a homomorphism, note that by the defintion of the termwise product,

$$\hat{\phi}(\eta_1\eta_2) = (\eta_1\eta_2) \circ \phi^{-1} = (\eta_1 \circ \phi^{-1})(\eta_2 \circ \phi^{-1}) = \hat{\phi}(\eta_1)\hat{\phi}(\eta_2)$$

for any $\eta_1, \eta_2 \in \widehat{\mathcal{H}}$. To show that $\hat{\phi}$ is an isomorphism, note that $\hat{\phi}^{-1} : \widehat{\mathcal{K}} \to \widehat{\mathcal{H}}$, $\hat{\phi}^{-1}(\kappa) := \kappa \circ \phi$ is an inverse of $\hat{\phi}$ since for any $\eta \in \widehat{\mathcal{H}}$ and $\kappa \in \widehat{\mathcal{K}}$,

$$\hat{\phi}(\hat{\phi}^{-1}(\kappa)) = \hat{\phi}(\kappa \circ \phi) = (\kappa \circ \phi) \circ \phi^{-1} = \kappa,$$
$$\hat{\phi}^{-1}(\hat{\phi}(\eta)) = \hat{\phi}^{-1}(\eta \circ \phi^{-1}) = (\eta \circ \phi^{-1}) \circ \phi = \eta. \qquad \square$$

Proposition C.7. Any finite abelian group $\mathcal{G}$ is isomorphic to its Pontryagin dual $\widehat{\mathcal{G}}$.

Proof. By the fundamental theorem of finite abelian groups,

$$\mathcal{G} \cong \bigtimes_{i \in \mathcal{I}} \mathbb{Z}_{N_i},$$

for some finite sequence $(N_i)_{i \in \mathcal{I}}$ of positive integers. By Propositions C.6, C.5 and C.3,

$$\widehat{\mathcal{G}} \cong \left(\bigtimes_{i \in \mathcal{I}} \mathbb{Z}_{N_i} \right)^{\wedge} \cong \bigtimes_{i \in \mathcal{I}} \widehat{\mathbb{Z}}_{N_i} \cong \bigtimes_{i \in \mathcal{I}} \mathbb{Z}_{N_i} \cong \mathcal{G}. \qquad \square$$

Appendix D. Filters and Convolution

In this appendix, we discuss a variety of known results regarding linear filters and convolution. Throughout it, let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$ and let $\mathcal{G}$ be a finite abelian group.

Proposition D.1. $(\boldsymbol{T}^g)^* = \boldsymbol{T}^{-g}$ for any $g \in \mathcal{G}$.

Proof. For any $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{F}^{\mathcal{G}}$ and $g \in \mathcal{G}$,

$$\begin{aligned}
\langle \boldsymbol{x}_1, \boldsymbol{T}^g \boldsymbol{x}_2 \rangle &= \sum_{g' \in \mathcal{G}} \overline{\boldsymbol{x}_1(g')} (\boldsymbol{T}^g \boldsymbol{x}_2)(g') \\
&= \sum_{g' \in \mathcal{G}} \overline{\boldsymbol{x}_1(g')} \boldsymbol{x}_2(g' - g) \\
&= \sum_{g'' \in \mathcal{G}} \overline{\boldsymbol{x}_1(g'' + g)} (\boldsymbol{x}_2)(g'') \\
&= \langle \boldsymbol{T}^{-g} \boldsymbol{x}_1, \boldsymbol{x}_2 \rangle. \qquad \square
\end{aligned}$$

Proposition D.2. The translation operators $(\boldsymbol{T}^g)_{g \in \mathcal{G}}$ are linear, unitary, and commutative.

Proof. To see that each $\boldsymbol{T}^g$ is linear, note that for any $c_1, c_2 \in \mathbb{F}$, $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{F}^{\mathcal{G}}$ and $g \in \mathcal{G}$,

$$\begin{aligned}
\boldsymbol{T}^g (c_1 \boldsymbol{x}_1 + c_2 \boldsymbol{x}_2)(g') &= (c_1 \boldsymbol{x}_1 + c_2 \boldsymbol{x}_2)(g' - g) \\
&= c_1 \boldsymbol{x}_1(g' - g) + c_2 \boldsymbol{x}_2(g' - g) \\
&= c_1 \boldsymbol{T}^g(\boldsymbol{x}_1)(g') + c_2 \boldsymbol{T}^g(\boldsymbol{x}_2)(g').
\end{aligned}$$

Next, for any $g_1, g_2, g \in \mathcal{G}$ and $\boldsymbol{x} \in \mathbb{F}^{\mathcal{G}}$,

$$(\boldsymbol{T}^{g_1} \boldsymbol{T}^{g_2} \boldsymbol{x})(g) = \boldsymbol{T}^{g_2} \boldsymbol{x}(g - g_1) = \boldsymbol{x}(g - g_1 - g_2) = \boldsymbol{x}(g - (g_1 + g_2)) = (\boldsymbol{T}^{g_1 + g_2} \boldsymbol{x})(g).$$

Since $\mathcal{G}$ is abelian, this implies $\boldsymbol{T}^{g_1} \boldsymbol{T}^{g_2} = \boldsymbol{T}^{g_1 + g_2} = \boldsymbol{T}^{g_2 + g_1} = \boldsymbol{T}^{g_2} \boldsymbol{T}^{g_1}$ for all $g_1, g_2 \in \mathcal{G}$. It moreover implies that $\boldsymbol{T}^{-g} \boldsymbol{T}^g = \boldsymbol{T}^g \boldsymbol{T}^{-g} = \boldsymbol{T}^{g + (-g)} = \boldsymbol{T}^0 = \mathbf{I}$ for any $g \in \mathcal{G}$. Thus, for any $g \in \mathcal{G}$, we find that $\boldsymbol{T}^g$ is invertible with $(\boldsymbol{T}^g)^{-1} = \boldsymbol{T}^{-g} = (\boldsymbol{T}^g)^*$, so $\boldsymbol{T}^g$ is unitary. $\square$

Proposition D.3. $\boldsymbol{T}^g \boldsymbol{\delta}_{g'} = \boldsymbol{\delta}_{g+g'}$ for any $g, g' \in \mathcal{G}$.

Proof. For any $g, g', g'' \in \mathcal{G}$,

$$(\boldsymbol{T}^g \boldsymbol{\delta}_{g'})(g'') = \boldsymbol{\delta}_{g'}(g'' - g') = \left\{ \begin{array}{ll} 1, & g'' - g = g', \\ 0, & g'' - g \neq g'. \end{array} \right\} = \left\{ \begin{array}{ll} 1, & g'' = g' + g, \\ 0, & g'' \neq g' + g. \end{array} \right\} = \boldsymbol{\delta}_{g'+g}(g''). \quad \square$$

Proposition D.4. Under convolution, $\mathbb{F}^{\mathcal{G}}$ is a commutative algebra with unity.

Proof. Throughout this proof, let $\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3, \boldsymbol{f}$ and $\boldsymbol{x}$ be any elements of $\mathbb{F}^{\mathcal{G}}$ and let c_1, c_2, c_3 be any scalars in $\boldsymbol{F}$. Let $\boldsymbol{F}_1, \boldsymbol{F}_2, \boldsymbol{F}_3$ be the filters corresponding to the impulse responses $\boldsymbol{f}_1, \boldsymbol{f}_2$ and $\boldsymbol{f}_3$, respectively. Since matrix multiplication is associative,

$$\boldsymbol{f}_1 * (\boldsymbol{f}_2 * \boldsymbol{f}_3) = \boldsymbol{F}_1(\boldsymbol{f}_2 * \boldsymbol{f}_3) = \boldsymbol{F}_1((\boldsymbol{F}_2\boldsymbol{F}_3)\boldsymbol{\delta}_0) = \boldsymbol{F}_1((\boldsymbol{F}_2(\boldsymbol{F}_3\boldsymbol{\delta}_0)) = (\boldsymbol{F}_1\boldsymbol{F}_2)\boldsymbol{f}_3 = (\boldsymbol{f}_1 * \boldsymbol{f}_2) * \boldsymbol{f}_3.$$

Thus, convolution is associative. To see that it is commutative, note

$$(\boldsymbol{f}_1 * \boldsymbol{f}_2)(g) = \sum_{g' \in \mathcal{G}} \boldsymbol{f}_1(g')\boldsymbol{f}_2(g - g') = \sum_{g'' \in \mathcal{G}} \boldsymbol{f}_2(g'')\boldsymbol{f}_1(g - g'') = (\boldsymbol{f}_2 * \boldsymbol{f}_1)(g).$$

Next, note that $\boldsymbol{\delta}_0$ is a multiplicative identity since $\boldsymbol{\delta}_0 * \boldsymbol{f} = \boldsymbol{f} * \boldsymbol{\delta}_0 = \boldsymbol{F}\boldsymbol{\delta}_0 = \boldsymbol{f}$. Next, since filters are linear,

$$\boldsymbol{f}_1 * (c_2 * \boldsymbol{f}_2 + c_3\boldsymbol{f}_3) = \boldsymbol{F}_1(c_2\boldsymbol{f}_2 + c_3\boldsymbol{f}_3) = c_2\boldsymbol{F}_1\boldsymbol{f}_2 + c_3\boldsymbol{F}_1\boldsymbol{f}_3 = c_2\boldsymbol{f}_1 * \boldsymbol{f}_2 + c_3\boldsymbol{f}_1 * \boldsymbol{f}_3.$$

In particular, $\boldsymbol{f} * (c\boldsymbol{x}) = c(\boldsymbol{f} * \boldsymbol{x}) = (c\boldsymbol{f}) * \boldsymbol{x}$. $\square$

Proposition D.5. $\boldsymbol{F}_1\boldsymbol{F}_2$ is a filter, and has $\boldsymbol{f}_1 * \boldsymbol{f}_2$ as its impulse response.

Proof. $\boldsymbol{F}_1\boldsymbol{F}_2$ is a filter since it belongs to $\mathrm{span}(\boldsymbol{T}^g)_{g \in \mathcal{G}}$:

$$\boldsymbol{F}_1\boldsymbol{F}_2 = \left(\sum_{g_1 \in \mathcal{G}} \boldsymbol{f}_1(g_1)\boldsymbol{T}_1^g\right)\left(\sum_{g_2 \in \mathcal{G}} \boldsymbol{f}_2(g_2)\boldsymbol{T}_2^g\right) = \sum_{g_1 \in \mathcal{G}} \sum_{g_2 \in \mathcal{G}} \boldsymbol{f}_1(g_1)\boldsymbol{f}_2(g_2)\boldsymbol{T}^{g_1+g_2}.$$

Its impulse response is $(\boldsymbol{F}_1\boldsymbol{F}_2)\boldsymbol{\delta}_0 = \boldsymbol{F}_1(\boldsymbol{F}_2\boldsymbol{\delta}_0) = \boldsymbol{F}_1\boldsymbol{f}_2 = \boldsymbol{f}_1 * \boldsymbol{f}_2$. □

Proposition D.6. For any $\boldsymbol{y}, \boldsymbol{y}_1, \boldsymbol{y}_2 \in \mathbb{F}^{\mathcal{G}}$ and $\gamma \in \widehat{\mathcal{G}}$,

$$\begin{aligned}
(\boldsymbol{\Gamma}^*(\boldsymbol{y}_1 * \boldsymbol{y}_2))(\gamma) &= (\boldsymbol{\Gamma}^*\boldsymbol{y}_1)(\gamma)(\boldsymbol{\Gamma}^*\boldsymbol{y}_2)(\gamma), \\
(\boldsymbol{\Gamma}^*\widetilde{\boldsymbol{y}})(\gamma) &= \overline{(\boldsymbol{\Gamma}^*\boldsymbol{y})(\gamma)}, \\
(\boldsymbol{\Gamma}^*(\boldsymbol{y} * \widetilde{\boldsymbol{y}}))(\gamma) &= \left|(\boldsymbol{\Gamma}^*\boldsymbol{y})(\gamma)\right|^2.
\end{aligned}$$

Proof. For the first claim,

$$\begin{aligned}
(\boldsymbol{\Gamma}^*(\boldsymbol{y}_1 * \boldsymbol{y}_2))(\gamma) &= \langle \boldsymbol{\gamma}, \boldsymbol{y}_1 * \boldsymbol{y}_2 \rangle \\
&= \sum_{g \in \mathcal{G}} \overline{\boldsymbol{\gamma}(g)}(\boldsymbol{y}_1 * \boldsymbol{y}_2)(g) \\
&= \sum_{g \in \mathcal{G}} \overline{\boldsymbol{\gamma}(g)} \sum_{g' \in \mathcal{G}} \boldsymbol{y}_1(g')\boldsymbol{y}_2(g - g') \\
&= \sum_{g \in \mathcal{G}} \sum_{g' \in \mathcal{G}} \overline{\boldsymbol{\gamma}(g)}\boldsymbol{y}_1(g')\boldsymbol{y}_2(g - g') \\
&= \sum_{g' \in \mathcal{G}} \boldsymbol{y}_1(g') \sum_{g \in \mathcal{G}} \overline{\boldsymbol{\gamma}(g)}\boldsymbol{y}_2(g - g') \\
&= \sum_{g' \in \mathcal{G}} \boldsymbol{y}_1(g') \sum_{g'' \in \mathcal{G}} \overline{\boldsymbol{\gamma}(g' + g'')}\boldsymbol{y}_2(g'') \\
&= \sum_{g' \in \mathcal{G}} \boldsymbol{y}_1(g') \sum_{g'' \in \mathcal{G}} \overline{\boldsymbol{\gamma}(g')\boldsymbol{\gamma}(g'')}\boldsymbol{y}_2(g'') \\
&= \sum_{g' \in \mathcal{G}} \overline{\boldsymbol{\gamma}(g')}\boldsymbol{y}_1(g') \sum_{g'' \in \mathcal{G}} \overline{\boldsymbol{\gamma}(g'')}\boldsymbol{y}_2(g'') \\
&= \langle \boldsymbol{\gamma}, \boldsymbol{y}_1 \rangle \langle \boldsymbol{\gamma}, \boldsymbol{y}_2 \rangle \\
&= (\boldsymbol{\Gamma}^*\boldsymbol{y}_1)(\gamma)(\boldsymbol{\Gamma}^*\boldsymbol{y}_2)(\gamma).
\end{aligned}$$

For the second claim, note that letting $g' = g$ gives

$$(\boldsymbol{\Gamma}^*\widetilde{\boldsymbol{y}})(\gamma) = \langle \boldsymbol{\gamma}, \widetilde{\boldsymbol{y}} \rangle = \sum_{g \in \mathcal{G}} \overline{\boldsymbol{\gamma}(g)}\widetilde{\boldsymbol{y}}(g) = \overline{\sum_{g \in \mathcal{G}} \boldsymbol{\gamma}(g)\boldsymbol{y}(-g)} = \overline{\sum_{g' \in \mathcal{G}} \boldsymbol{\gamma}(-g')\boldsymbol{y}(g')}.$$

Since $\overline{\boldsymbol{\gamma}(g)} = \boldsymbol{\gamma}(-g)$, this implies

$$(\boldsymbol{\Gamma}^* \widetilde{\boldsymbol{y}})(\gamma) = \overline{\sum_{g \in \mathcal{G}} \boldsymbol{\gamma}(-g') \boldsymbol{y}(g')} = \overline{\sum_{g \in \mathcal{G}} \overline{\boldsymbol{\gamma}(g')} \boldsymbol{y}(g')} = \overline{\langle \boldsymbol{\gamma}, \boldsymbol{y} \rangle} = \overline{(\boldsymbol{\Gamma}^* \boldsymbol{y})(\gamma)}.$$

Finally, combining the first and second claim gives the third:

$$(\boldsymbol{\Gamma}^*(\boldsymbol{y} * \widetilde{\boldsymbol{y}}))(\gamma) = (\boldsymbol{\Gamma}^* \boldsymbol{y} \odot \boldsymbol{\Gamma}^* \widetilde{\boldsymbol{y}})(\gamma) = (\boldsymbol{\Gamma}^* \boldsymbol{y})(\gamma) \overline{(\boldsymbol{\Gamma}^* \boldsymbol{y})(\gamma)} = \left| (\boldsymbol{\Gamma}^* \boldsymbol{y})(\gamma) \right|^2. \qquad \square$$

Appendix E. Combinatorial Design

In this appendix, we discuss some well-known facts about difference families and BIBDs.

Proposition E.1. If $\mathcal{D}$ is a subset of a finite abelian group of order G, then for any $g \in \mathcal{G}$,

$$\#\big\{(d_1, d_2) \in \mathcal{D} \times \mathcal{D} : d_1 - d_2 = g\big\} = \#\big(\mathcal{D} \cap (g + \mathcal{D})\big).$$

Proof. It suffices to show that the following function is a well-defined bijection:

$$\boldsymbol{f} : \big\{(d_1, d_2) \in \mathcal{D} \times \mathcal{D} : d_1 - d_2 = g\big\} \to \mathcal{D} \cap (g + \mathcal{D}), \quad \boldsymbol{f}(d_1, d_2) = d_1.$$

To see that f is well-defined, note that for any (d_1, d_2) in $\mathcal{D} \times \mathcal{D}$ such that $d_1 - d_2 = g$, we have that $d_1 \in \mathcal{D}$ and $d_1 = g + d_2 \in g + \mathcal{D}$. To see that f is injective, note that if $(d_1, d_2), (d_3, d_4) \in \mathcal{D} \times \mathcal{D}$ satisfy $d_1 - d_2 = g = d_3 - d_4$ and $d_1 = \boldsymbol{f}(d_1, d_2) = \boldsymbol{f}(d_3, d_4) = d_3$, then $d_2 = d_1 + g = d_3 + g = d_4$, implying $(d_1, d_2) = (d_3, d_4)$. To see that $\boldsymbol{f}$ is surjective, note that for any $d_1 \in \mathcal{D} \cap (g + \mathcal{D})$, the fact that $d_1 \in g + \mathcal{D}$ implies there exists $d_2 \in \mathcal{D}$ such that $d_1 = g + d_2$. Thus, $\boldsymbol{f}(d_1, d_2) = d_1$ where $(d_1, d_2) \in \mathcal{D} \times \mathcal{D}$ satisfies $d_1 - d_2 = g$. ☐

Proposition E.2. Let $(\mathcal{V}, (\mathcal{K}_b)_{b \in \mathcal{B}})$ be a BIBD(V, K, λ). Then each vertex $v \in \mathcal{V}$ is contained in exactly $R = \frac{\lambda(V-1)}{K-1}$ blocks, and the total number of blocks is $B = \frac{VR}{K}$.

Proof. For any $v \in \mathcal{V}$, let $\mathcal{B}_v := \{b \in \mathcal{B} : v \in \mathcal{K}_b\}$ and $R_v := \#(\mathcal{B}_v)$. Note that

$$\begin{aligned}
\#\big\{(b, v') \in \mathcal{B} \times \mathcal{V} : v, v' \in \mathcal{K}_b, v' \neq v\big\} &= \#\big\{(b, v') \in \mathcal{B} \times \mathcal{V} : v \in \mathcal{K}_b, v' \in \mathcal{K}_b \setminus \{v\}\big\} \\
&= \sum_{b \in \mathcal{B}_v} \#\Big\{v' \in \mathcal{V} : v' \in \mathcal{K}_b \setminus \{v\}\Big\} \\
&= \sum_{b \in \mathcal{B}_v} \big(\#(\mathcal{K}_b) - 1\big) \\
&= \sum_{b \in \mathcal{B}_v} (K - 1) \\
&= (K - 1) R_v.
\end{aligned}$$

It is also true that

$$\begin{aligned}
\#\big\{(b,v') \in \mathcal{B} \times \mathcal{V} : v, v' \in \mathcal{K}_b, v' \neq v\big\} &= \#\big\{(b,v') \in \mathcal{B} \times \mathcal{V} : v, v' \in \mathcal{K}_b, v' \neq v\big\} \\
&= \sum_{v' \in \mathcal{V} \setminus \{v\}} \#\big\{b \in \mathcal{B} : v, v' \in \mathcal{K}_b\big\} \\
&= \sum_{v' \in \mathcal{V} \setminus \{v\}} \lambda \\
&= \lambda(V-1).
\end{aligned}$$

Therefore, each vertex is indeed contained in exactly $R := R_v = \frac{\lambda(V-1)}{K-1}$ blocks. Since R is independent of $v \in \mathcal{V}$, we moreover have $B = \frac{VR}{K}$ since

$$\begin{aligned}
VR &= \sum_{v \in \mathcal{V}} \#\{b \in \mathcal{B} : v \in \mathcal{K}_b\} \\
&= \#\{(b,v) \in \mathcal{B} \times \mathcal{V} : v \in \mathcal{K}_b\} \\
&= \sum_{b \in \mathcal{B}} \#\{v \in \mathcal{V} : v \in \mathcal{K}_b\} \\
&= BK.
\end{aligned}$$

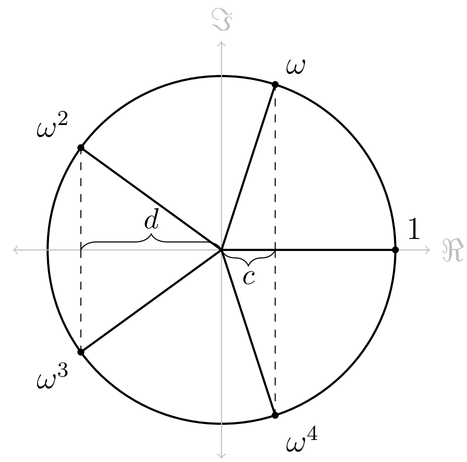
□

Appendix F. Fifth Roots of Unity and the Golden Ratio

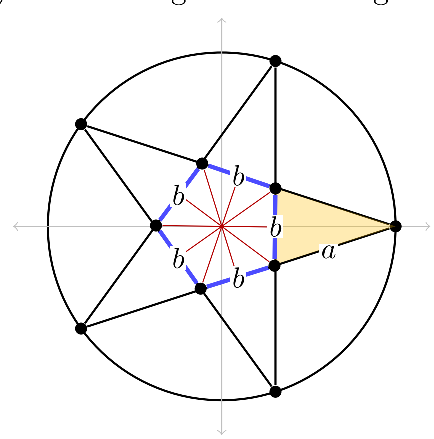
Recall the constants c and d from Example 3.2. A keen observer may have noted the following relations between these constants and the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$:

$$c = \frac{1}{2\phi}, \quad d = -\frac{\phi}{2}.$$

In this appendix, we give a classical geometric argument that explains this connection. By (3.2), c and d are the real parts of the first and second primitive fifth roots of unity:

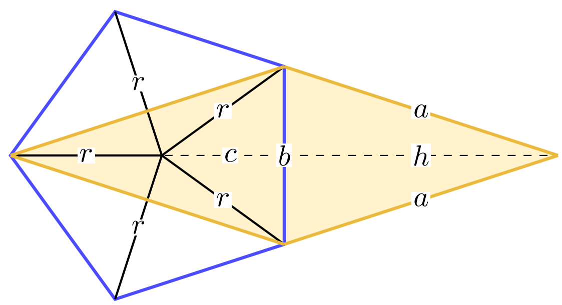


In order to determine the values of c and d, we first connect every 5th root of unity with its two opposite (or nonadjacent) roots. Doing so forms a regular pentagram:



It has been well-known, even as early as the Pythagoreans, that the spikes of a regular pentagram are golden triangles, meaning that the ratio of their duplicated sides a to their base b is the golden ratio. The gold triangle in the diagram above captures this relation.

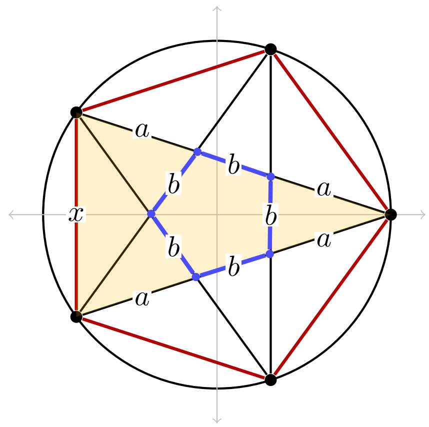
Moreover, inscribed within the regular pentagram is a regular pentagon of side length b, with its center at the origin. It is also well known that the diagonals of a regular pentagon are in golden ratio with its side. Thus, within the inscribed pentagon is a copy of the gold triangle, as shown below:



Letting h be the height of the golden triangle and r be the circumradius of the regular pentagon, we note that $h = r + c$, since the two golden triangles are congruent. Moreover, since the center of the pentagon is the center of the unit circle, we have that $c + h = 1$. Thus, by the golden ratio and the Pythagorean theorem, we have the following equations in terms of a, b and c:

$$\begin{aligned} \frac{a}{b} &= \phi = \frac{1+\sqrt{5}}{2}, \\ \sqrt{c^2 + \frac{b^2}{4}} &= \sqrt{a^2 - \frac{b^2}{4}} - c, \\ 1 &= \sqrt{a^2 - \frac{b^2}{4}} + c. \end{aligned}$$

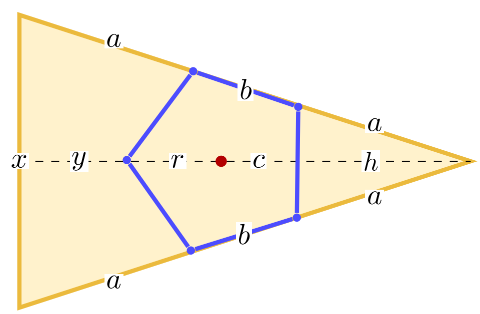
With these three equations and three unknowns, we can solve for c to arrive at the expected value of $\frac{-1+\sqrt{5}}{4}$. We then turn our attention to the constant d and note that we could have formed a regular pentagon between the fifth roots of unity:



As discussed earlier, the diagonals of a regular pentagon are in golden ratio with its sides, so the inscribed gold triangle above is indeed another golden triangle. Accordingly, we can then obtain an expression for x in terms of a and b:

$$\frac{2a+b}{x} = \frac{a}{b} \quad \Rightarrow \quad x = \frac{2ab+b^2}{a}.$$

Since we have the values for r, c and h by our earlier work, the problem of determining d amounts to solving for y in the following diagram:



Recalling that $r+c = h$, we have the following expressions again by the Pythagorean theorem, of which the only unknown value is y:

$$(2a+b)^2 = \frac{x^2}{4} + (2h+y)^2.$$

Finally, $y + r$ then yields the expected value for d of $\frac{-1-\sqrt{5}}{4}$.

Bibliography

[1] R. J. R. Abel, M. Buratti, Difference families, in: C. J. Colbourn, J. H. Dinitz (Eds.), Handbook of Combinatorial Designs, Second Edition (2007) 392–410.

[2] W. U. Bajwa, R. Calderbank, D. G. Mixon, Two are better than one: fundamental parameters of frame coherence, Appl. Comput. Harmon. Anal. 33 (2012) 58–78.

[3] A. S. Bandeira, M. Fickus, D. G. Mixon, P. Wong, The road to deterministic matrices with the Restricted Isometry Property, J. Fourier Anal. Appl. 19 (2013) 1123–1149.

[4] R. Calderbank, P. G. Casazza, A. Heinecke, G. Kutyniok, A. Pezeshki, Fusion frames: existence and construction, arXiv:0906.5606 (2009).

[5] A. R. Calderbank, R. H. Hardin, E. M. Rains, P. W. Shor, N. J. A. Sloane, A group-theoretic framework for the construction of packings in Grassmannian spaces, J. Algebraic Combin. 9 (1999) 129–140.

[6] R. Calderbank, A. Thompson, Y. Xie, On block coherence of frames, Appl. Comput. Harmon. Anal. 38 (2015) 50–71.

[7] P. G. Casazza, M. Fickus, D. G. Mixon, Y. Wang, Z. Zhou, Constructing tight fusion frames, Appl. Comput. Harmon. Anal. 30 (2011) 175–187.

[8] A. Chebira, M. Fickus, D. G. Mixon, Filter bank fusion frames, IEEE Trans. Signal Process. 59 (2011) 953–963.

[9] H. Cohn, A. Kumar, G. Minton, Optimal simplices and codes in projective spaces, Geom. Topol. 20 (2016) 1289–1357.

[10] J. H. Conway, R. H. Hardin, N. J. A. Sloane, Packing lines, planes, etc.: packings in Grassmannian spaces, Exp. Math. 5 (1996) 139–159.

[11] I. S. Dhillon, J. R. Heath, T. Strohmer, J. A. Tropp, Constructing packings in Grassmannian manifolds via alternating projection, Exp. Math. 17 (2008) 9–35.

[12] C. Ding, T. Feng, A generic construction of complex codebooks meeting the Welch bound, IEEE Trans. Inform. Theory 53 (2007) 4245–4250.

[13] S. Davis, Proving the existence of equichordal tight fusion frames using the Newton-Kantorovich theorem, M.S. Thesis, Air Force Institute of Technology, 2024.

[14] Y. C. Eldar, P. Kuppinger, H. Bölcskei, Block-sparse signals: uncertainty relations and efficient recovery, IEEE Trans. Signal Process. 58 (2010) 3042–3054.

[15] M. Fickus, D. G. Mixon, J. Jasper, Equiangular tight frames from hyperovals, IEEE Trans. Inform. Theory. 62 (2016) 5225–5236.

[16] M. Fickus, E. Gomez-Leos, J. W. Iverson, Radon–Hurwitz Grassmannian codes, to appear in IEEE Trans. Inform. Theory.

[17] M. Fickus, J. W. Iverson, J. Jasper, E. J. King, Grassmannian codes from paired difference sets, arXiv:2010.06639 (2020).

[18] M. Fickus, J. W. Iverson, J. Jasper, D. G. Mixon, A note on totally symmetric equi-isoclinic tight fusion frames, Proc. IEEE Int. Conf. Acoust. Speech Signal Process. (2022) 8987–8991.

[19] M. Fickus, J. W. Iverson, J. Jasper, D. G. Mixon, Harmonic Grassmannian codes, Appl. Comput. Harmon. Anal. 65 (2023) 1–39.

[20] M. Fickus, J. W. Iverson, J. Jasper, D. G. Mixon, Equi-isoclinic subspaces, covers of the complete graph, and complex conference matrices, Linear Algebra Appl. 702 (2024) 240–249.

[21] M. Fickus, J. W. Iverson, J. Jasper, D. G. Mixon, Equi-isoclinic subspaces from symmetry, arXiv:2406.19542 (2024).

[22] M. Fickus, J. Jasper, E. J. King, D. G. Mixon, Equiangular tight frames that contain regular simplices, Linear Algebra Appl. 555 (2018) 98–138.

[23] M. Fickus, B. R. Mayo, Mutually unbiased equiangular tight frames, IEEE Trans. Inform. Theory 67 (2021) 1656–1667.

[24] M. Fickus, B. R. Mayo, C. E. Watson, Certifying the novelty of equichordal tight fusion frames, arXiv:2103.03192 (2021).

[25] M. Fickus, B. R. Mayo, C. E. Watson, Naimark-spatial families of equichordal tight fusion frames, Appl. Comput. Harmon. Anal. 74 (2025) 101720/1–24.

[26] M. Fickus, D. G. Mixon, Tables of the existence of equiangular tight frames, arXiv:1504.00253 (2016).

[27] M. Fickus, C. A. Schmitt, Harmonic equiangular tight frames comprised of regular simplices, Linear Algebra Appl. 586 (2020) 130–169.

[28] C. A. Fuchs, M. C. Hoang, B. C. Stacey, The SIC question: history and state of play, Axioms 6 (2017) 21:1–20.

[29] C. Godsil, A. Roy, Equiangular lines, mutually unbiased bases, and spin models, European J. Combin. 30 (2009) 246–262.

[30] S. G. Hoggar, New sets of equi-isoclinic n-planes from old, Proc. Edinb. Math. Soc. 20 (1977) 287–291.

[31] E. J. King, New constructions and characterizations of flat and almost flat Grassmannian fusion frames, arXiv:1612.05784 (2016).

[32] H. König, Cubature formulas on spheres, Math. Res. 107 (1999) 201–212.

[33] G. Kutyniok, A. Pezeshki, R. Calderbank, T. Liu, Robust dimension reduction, fusion frames, and Grassmannian packings, Appl. Comput. Harmon. Anal. 26 (2009) 64–76.

[34] P. W. H. Lemmens, J. J. Seidel, Equi-isoclinic subspaces of Euclidean spaces, Indag. Math. 76 (1973) 98–107.

[35] A. Pott, Finite geometry and character theory, Lecture Notes in Math. 1601 (1995).

[36] J. M. Renes, R. Blume-Kohout, A. J. Scott, C. M. Caves, Symmetric informationally complete quantum measurements, J. Math. Phys. 45 (2004) 2171–2180.

[37] T. Strohmer, R. W. Heath, Grassmannian frames with applications to coding and communication, Appl. Comput. Harmon. Anal. 14 (2003) 257–275.

[38] M. A. Sustik, J. A. Tropp, I. S. Dhillon, R. W. Heath, On the existence of equiangular tight frames, Linear Algebra Appl. 426 (2007) 619–635.

[39] S. Waldron, Tight frames over the quaternions and equiangular lines, arXiv:2006.06126 (2020).

[40] L. R. Welch, Lower bounds on the maximum cross correlation of signals, IEEE Trans. Inform. Theory 20 (1974) 397–399.

[41] R. M. Wilson, Cyclotomy and difference families in elementary abelian groups, J. Number Theory 4 (1972) 17–47.

[42] P. Xia, S. Zhou, G. B. Giannakis, Achieving the Welch bound with difference sets, IEEE Trans. Inform. Theory 51 (2005) 1900–1907.

[43] G. Zauner, Quantum designs: Foundations of a noncommutative design theory, Ph.D. Thesis, University of Vienna, 1999.

REPORT DOCUMENTATION PAGE

1. REPORT DATE	2. REPORT TYPE	3. DATES COVERED START DATE	END DATE
March 27, 2025	M.S. Thesis	Sep 2023	Mar 2025

4. TITLE AND SUBTITLE

Grassmannian codes from stratified frames

5a. CONTRACT NUMBER	5b. GRANT NUMBER	5c. PROGRAM ELEMENT NUMBER
5d. PROJECT NUMBER	**5e. TASK NUMBER**	**5f. WORK UNIT NUMBER**

6. AUTHOR(S)

William J. Brinkley, Second Lieutenant, USAF

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)	8. PERFORMING ORGANIZATION REPORT NUMBER
Department of Mathematics and Statistics Air Force Institute of Technology Wright-Patterson Air Force Base, OH 45433	AFIT-ENC-MS-25-M-240

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSOR/MONITOR'S ACRONYM(S)	11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT

Distribution Statement A. Approved for public release: distribution is unlimited.

13. SUPPLEMENTARY NOTES

14. ABSTRACT

An equichordal tight fusion frame (ECTFF) is a finite sequence of equi-dimensional subspaces of a Euclidean space that achieves equality in Conway, Hardin and Sloane's simplex bound. Every ECTFF is an optimal Grassmannian code with respect to the chordal distance. We introduce a method for constructing an ECTFF from any finite sequence of unit norm tight frames that happen to be "stratified" in a certain sense. We moreover show how to construct stratified unit norm tight frames from a difference family for a finite abelian group, as well as from a suitable combination of a resolvable balanced incomplete block design and an equiangular tight frame. These results streamline and unify several known constructions that were previously regarded as disparate, and moreover yield infinitely many apparently new ECTFFs.

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF: a. REPORT	b. ABSTRACT	c. THIS PAGE	17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES
U	U	U	UU	86

19a. NAME OF RESPONSIBLE PERSON	19b. PHONE NUMBER (*Include area code*)
Dr. Matthew Fickus, AFIT/ENC	(937) 255-6277

INSTRUCTIONS FOR COMPLETING SF 298

1. REPORT DATE.
Full publication date, including day, month, if available. Must cite at least the year and be Year 2000 compliant, e.g. 30-06-1998; xx-06-1998; xx-xx-1998.

2. REPORT TYPE.
State the type of report, such as final, technical, interim, memorandum, master's thesis, progress, quarterly, research, special, group study, etc.

3. DATES COVERED.
Indicate the time during which the work was performed and the report was written.

4. TITLE.
Enter title and subtitle with volume number and part number, if applicable. On classified documents, enter the title classification in parentheses.

5a. CONTRACT NUMBER.
Enter all contract numbers as they appear in the report, e.g. F33615-86-C-5169.

5b. GRANT NUMBER.
Enter all grant numbers as they appear in the report, e.g. AFOSR-82-1234.

5c. PROGRAM ELEMENT NUMBER.
Enter all program element numbers as they appear in the report, e.g. 61101A

5d. PROJECT NUMBER.
Enter all project numbers as they appear in the report, e.g. 1F665702D1257; ILIR.

5e. TASK NUMBER. Enter all task numbers as they appear in the report, e.g. 05; RF0330201; T4112.

5f. WORK UNIT NUMBER.
Enter all work unit numbers as they appear in the report, e.g. 001; AFAPL30480105.

6. AUTHOR(S). Enter name(s) of person(s) responsible for writing the report, performing the research, or credited with the content of the report. The form of entry is the last name, first name, middle initial, and additional qualifiers separated by commas, e.g. Smith, Richard, J, Jr.

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES). Self-explanatory.

8. PERFORMING ORGANIZATION REPORT NUMBER. Enter all unique alphanumeric report numbers assigned by the performing organization, e.g. BRL-1234; AFWL-TR-85-4017-Vol-21-PT-2.

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES). Enter the name and address of the organization(s) financially responsible for and monitoring the work.

10. SPONSOR/MONITOR'S ACRONYM(S). Enter, if available, e.g. BRL, ARDEC, NADC.

11. SPONSOR/MONITOR'S REPORT NUMBER(S). Enter report number as assigned by the sponsoring/monitoring agency, if available, e.g. BRL-TR-829; -215.

12. DISTRIBUTION/ AVAILABILITY STATEMENT. Use agency-mandated availability statements to indicate the public availability or distribution limitations of the report. If additional limitations/ restrictions or special markings are indicated, follow agency authorization procedures, e.g. RD/FRD, PROPIN, ITAR, etc. Include copyright information.

13. SUPPLEMENTARY NOTES. Enter information not included elsewhere such as: prepared in cooperation with; translation of; report supersedes; old edition number, etc.

14. ABSTRACT. A brief (approximately 200 words) factual summary of the most significant information.

15. SUBJECT TERMS. Key words or phrases identifying major concepts in the report.

16. SECURITY CLASSIFICATION. Enter security classification in accordance with security classification regulations, e.g. U, C, S, etc. If this form contains classified information, stamp classification level on the top and bottom of this page.

17. LIMITATION OF ABSTRACT. This block must be completed to assign a distribution limitation to the abstract. Enter UU (Unclassified Unlimited) or SAR (Same as Report). An entry in this block is necessary if the abstract is to be limited.

PREVIOUS EDITION IS OBSOLETE.

STANDARD FORM 298 (REV. 5/2020)
Prescribed by ANSI Std. Z39.18