Lab1 : back-propagation

110550095 王皓平

1 Introduction

In this assignment, we implemented a simple neural network to classify a set of 2D points based on their position relative to a decision boundary only using Numpy. The model was trained using gradient descent and different activation functions. This report will show the detail of the implementation and discuss about the performance of the model with different training parameters.

2 Implementation Details

2.1 Sigmoid Function

We use Sigmoid as the activate function of the hidden layers.

$$\sigma(x) = \frac{1}{1+e^{-x}}, \quad 0 < \sigma(x) < 1, \quad \forall x \in \mathbb{R} \tag{1}$$

The sigmoid function is a widely used activation function in neural networks , particularly in the output layer for binary classification tasks. The function maps any value to a range between 0 and 1.

2.2 Neural Network Architecture

The neural network consists of:

- **Input layer**: 2 neurons (x_1, x_2)
- **Hidden layers**: 2 layers, 4 neurons each
- **Output layer**: 1 neuron

2.3 Backpropagation

Using the chain rule, we compute the gradients for each layer:

$$dA_3 = A_3 - y \tag{2}$$

$$dW_3 = \frac{1}{m} A_2^T (dA_3 \cdot \sigma'(A_3)) \tag{3}$$

$$dW_2 = \frac{1}{m} A_1^T (dA_2 \cdot \sigma'(A_2)) \tag{4}$$

$$dW_1 = \frac{1}{m} X^T (dA_1 \cdot \sigma'(A_1)) \tag{5}$$

where $\sigma'(x) = \sigma(x)(1 - \sigma(x))$.

2.4 Weight Update (Gradient Descent)

The weight matrices are updated using gradient descent:

$$W_i = W_i - \alpha dW_i \tag{6}$$

where: α is the learning rate, dW_i is the gradient of the weight matrix.

3 Experimental Results

3.1 Screenshot and comparison figure

```
Epoch     0     Loss: 0.249764
Epoch  5000     Loss: 0.006943
Epoch 10000     Loss: 0.004522
Epoch 15000     Loss: 0.003242
Epoch 20000     Loss: 0.002395
Epoch 25000     Loss: 0.001813
Epoch 30000     Loss: 0.001405
Epoch 35000     Loss: 0.001113
Epoch 40000     Loss: 0.000900
Epoch 45000     Loss: 0.000741
Epoch 50000     Loss: 0.000621
Epoch 55000     Loss: 0.000527
Epoch 60000     Loss: 0.000454
Epoch 65000     Loss: 0.000395
Epoch 70000     Loss: 0.000347
Epoch 75000     Loss: 0.000308
Epoch 80000     Loss: 0.000275
Epoch 85000     Loss: 0.000248
Epoch 90000     Loss: 0.000225
Epoch 95000     Loss: 0.000205
```

(a) Linear Data

```
Epoch     0     Loss: 0.310752
Epoch  5000     Loss: 0.246487
Epoch 10000     Loss: 0.208476
Epoch 15000     Loss: 0.057058
Epoch 20000     Loss: 0.012353
Epoch 25000     Loss: 0.004461
Epoch 30000     Loss: 0.002455
Epoch 35000     Loss: 0.001638
Epoch 40000     Loss: 0.001210
Epoch 45000     Loss: 0.000952
Epoch 50000     Loss: 0.000780
Epoch 55000     Loss: 0.000659
Epoch 60000     Loss: 0.000569
Epoch 65000     Loss: 0.000500
Epoch 70000     Loss: 0.000445
Epoch 75000     Loss: 0.000401
Epoch 80000     Loss: 0.000364
Epoch 85000     Loss: 0.000334
Epoch 90000     Loss: 0.000308
Epoch 95000     Loss: 0.000285
```

(b) XOR Data

Figure 1: Comparison of Training: Linear vs. XOR

3.2 Accuracy of Predictions

```
Iter 80 |     Ground Truth: 0 |     Prediction: 0.000000 |
Iter 81 |     Ground Truth: 0 |     Prediction: 0.000000 |
Iter 82 |     Ground Truth: 0 |     Prediction: 0.000000 |
Iter 83 |     Ground Truth: 0 |     Prediction: 0.000005 |
Iter 84 |     Ground Truth: 0 |     Prediction: 0.000000 |
Iter 85 |     Ground Truth: 1 |     Prediction: 0.999998 |
Iter 86 |     Ground Truth: 0 |     Prediction: 0.000000 |
Iter 87 |     Ground Truth: 0 |     Prediction: 0.000000 |
Iter 88 |     Ground Truth: 0 |     Prediction: 0.000000 |
Iter 89 |     Ground Truth: 0 |     Prediction: 0.000000 |
Iter 90 |     Ground Truth: 1 |     Prediction: 1.000000 |
Iter 91 |     Ground Truth: 1 |     Prediction: 1.000000 |
Iter 92 |     Ground Truth: 0 |     Prediction: 0.000000 |
Iter 93 |     Ground Truth: 1 |     Prediction: 0.999998 |
Iter 94 |     Ground Truth: 1 |     Prediction: 1.000000 |
Iter 95 |     Ground Truth: 0 |     Prediction: 0.000000 |
Iter 96 |     Ground Truth: 1 |     Prediction: 1.000000 |
Iter 97 |     Ground Truth: 1 |     Prediction: 1.000000 |
Iter 98 |     Ground Truth: 0 |     Prediction: 0.000000 |
Iter 99 |     Ground Truth: 1 |     Prediction: 1.000000 |
Test Accuracy: 99.00%
```

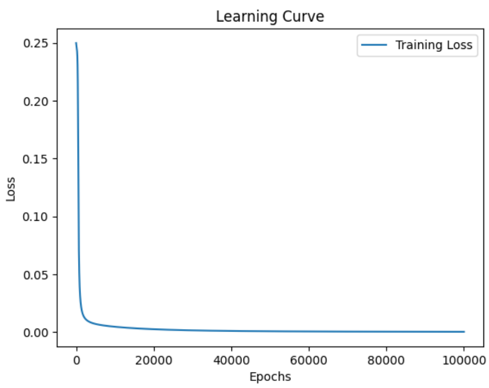
(a) Linear Data

```
Iter  0 |     Ground Truth: 0 |     Prediction: 0.015459 |
Iter  1 |     Ground Truth: 1 |     Prediction: 0.996463 |
Iter  2 |     Ground Truth: 0 |     Prediction: 0.015357 |
Iter  3 |     Ground Truth: 1 |     Prediction: 0.996380 |
Iter  4 |     Ground Truth: 0 |     Prediction: 0.015284 |
Iter  5 |     Ground Truth: 1 |     Prediction: 0.996073 |
Iter  6 |     Ground Truth: 0 |     Prediction: 0.015247 |
Iter  7 |     Ground Truth: 1 |     Prediction: 0.994657 |
Iter  8 |     Ground Truth: 0 |     Prediction: 0.015251 |
Iter  9 |     Ground Truth: 1 |     Prediction: 0.960788 |
Iter 10 |     Ground Truth: 0 |     Prediction: 0.015299 |
Iter 11 |     Ground Truth: 0 |     Prediction: 0.015395 |
Iter 12 |     Ground Truth: 1 |     Prediction: 0.965762 |
Iter 13 |     Ground Truth: 0 |     Prediction: 0.015543 |
Iter 14 |     Ground Truth: 1 |     Prediction: 0.992516 |
Iter 15 |     Ground Truth: 0 |     Prediction: 0.015744 |
Iter 16 |     Ground Truth: 1 |     Prediction: 0.994235 |
Iter 17 |     Ground Truth: 0 |     Prediction: 0.016002 |
Iter 18 |     Ground Truth: 1 |     Prediction: 0.994612 |
Iter 19 |     Ground Truth: 0 |     Prediction: 0.016320 |
Iter 20 |     Ground Truth: 1 |     Prediction: 0.994688 |
Test Accuracy: 100.00%
```

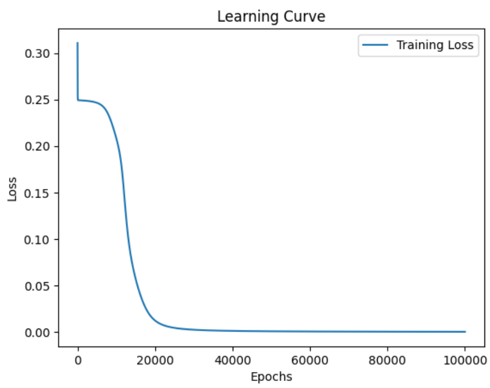
(b) XOR Data

Figure 2: Comparison of Accuracy: Linear vs. XOR

3.3 Learning Curve



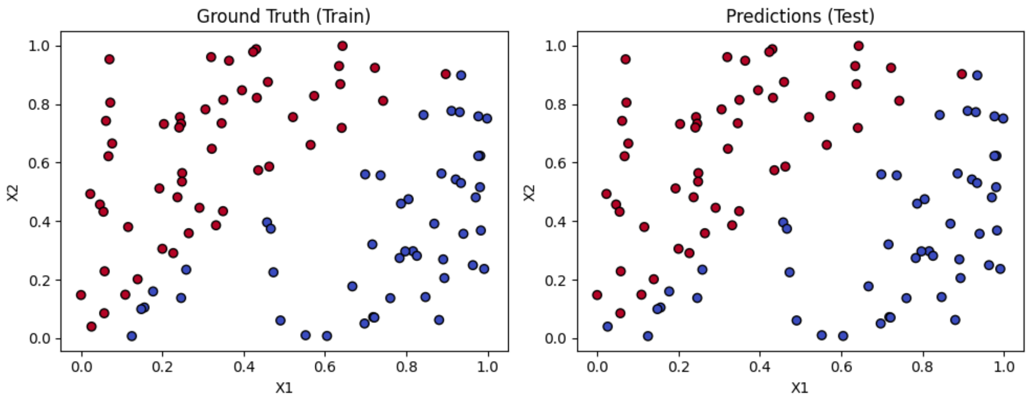
(a) Linear Data



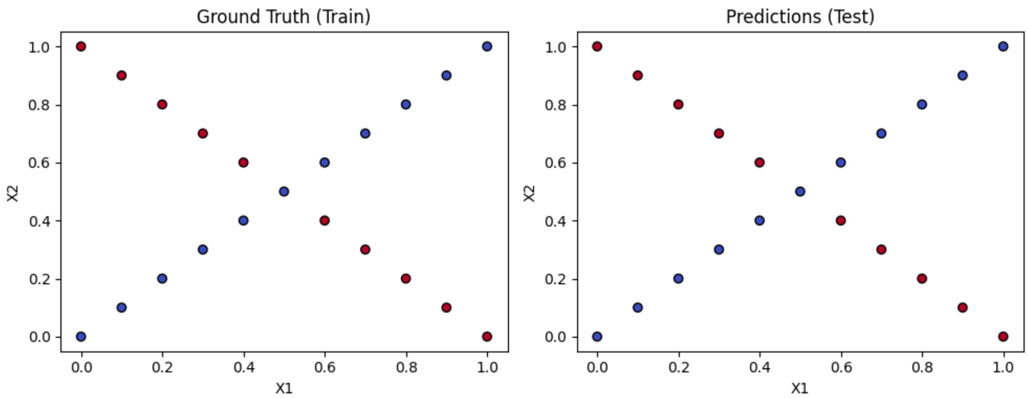
(b) XOR Data

Figure 3: Comparison of Learning Curve: Linear vs. XOR

3.4 Visualization



(a) Linear Data

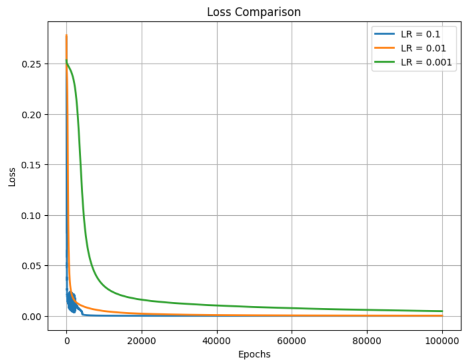


(b) XOR Data

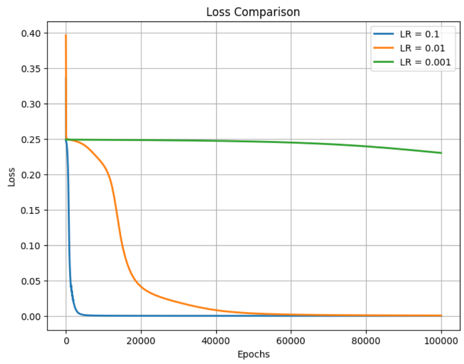
Figure 4: Visualization Plot: Linear vs. XOR

4 Discussion

4.1 Different Learning Rates

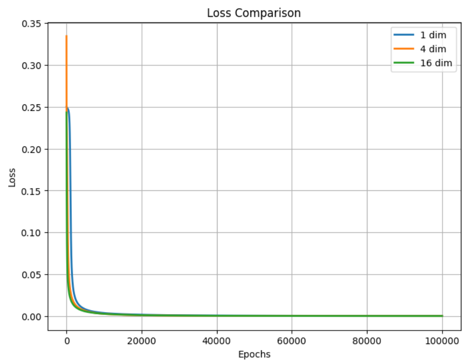


(a) Linear Data

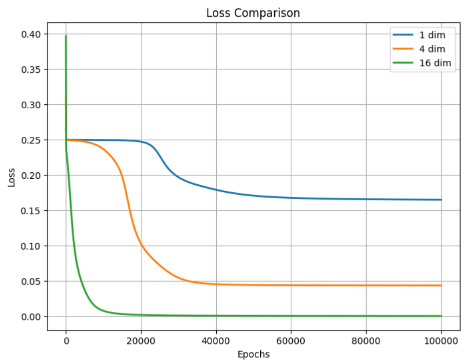


(b) XOR Data

4.2 Different Numbers of Hidden Units

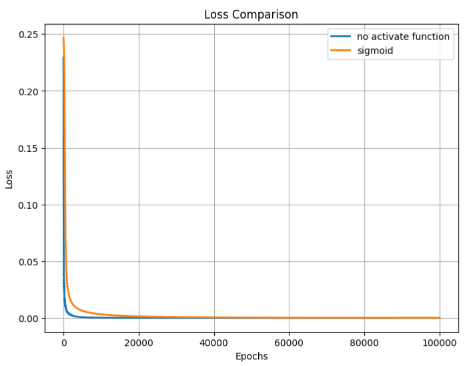


(a) Linear Data

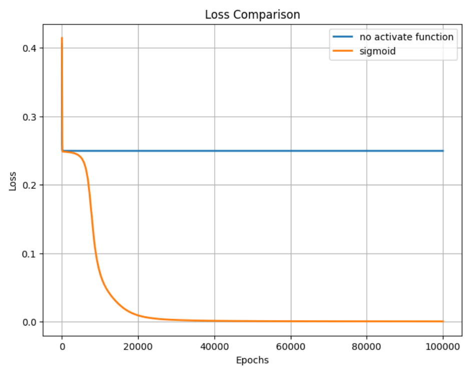


(b) XOR Data

4.3 With and Without Activation Functions



(a) Linear Data



(b) XOR Data

5 Questions

5.1 What is the purpose of activation functions?

Activation functions introduce **non-linearity** into the neural network, allowing it to learn complex patterns beyond simple linear relationships. Without an activation function, a neural network would behave like a linear regression model, limiting its ability to solve complex tasks.

5.2 What might happen if the learning rate is too large or too small?

- **if the learning rate is too large:**
  - The model may fail to converge or oscillate around the minimum.
  - It may skip over the optimal solution, preventing proper learning.
- **If the learning rate is too small:**
  - Training will be very slow, requiring many iterations to reach a good solution.
  - The model may get stuck in a local minimum, failing to generalize well.

5.3 What is the purpose of weights and biases in a neural network?

Weights and biases are the key parameters that allow a neural network to learn from data.

- **Weights (W):**
  - Represent the strength of connections between neurons.
  - Higher weights indicate stronger influence of one neuron on another.
  - Are updated during training using gradient descent.
- **Biases (b):**
  - Allow the activation function to shift, helping the model fit better.
  - Ensure neurons activate even when all inputs are zero.

The neural network learns **optimal weights and biases** during training to minimize the loss function and improve predictions.

6 Extra

6.1 Implement Different Optimizers

To improve training performance, we implemented different optimization algorithms:

- **Stochastic Gradient Descent (SGD)**:

$$W = W - \alpha \frac{\partial L}{\partial W} \tag{7}$$

- **Adam (Adaptive Moment Estimation)**:

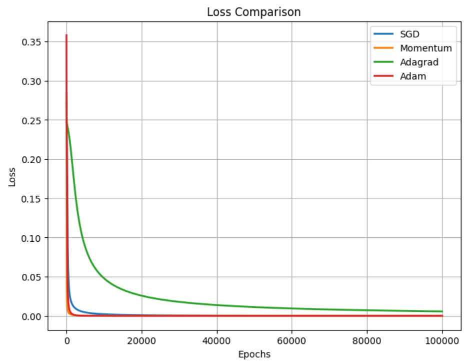
$$m_t = \beta_1 m_{t-1} + (1 - \beta_1) g_t \tag{8}$$

$$v_t = \beta_2 v_{t-1} + (1 - \beta_2) g_t^2 \tag{9}$$

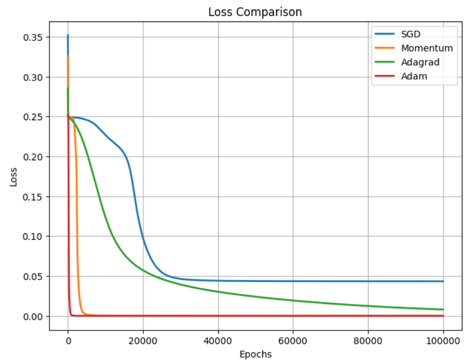
$$W = W - \alpha \frac{m_t}{\sqrt{v_t} + \epsilon} \tag{10}$$

- **RMSprop (Root Mean Square Propagation)**:

$$W = W - \frac{\alpha}{\sqrt{v_t} + \epsilon} g_t \tag{11}$$



(a) Linear Data

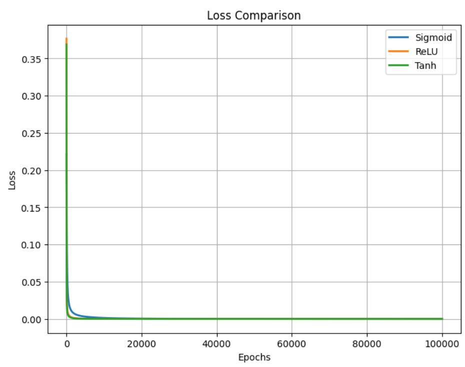


(b) XOR Data

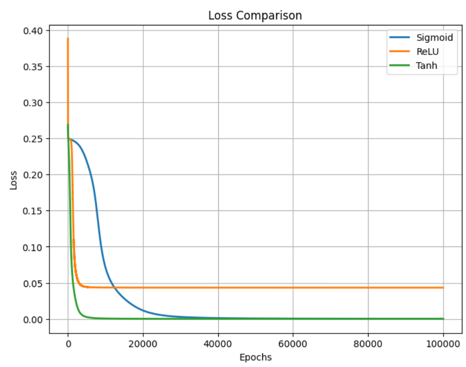
6.2 Implement Different Activation Functions

To compare the performance of different activation functions, we implemented:

- **Sigmoid**:
$$\sigma(x) = \frac{1}{1 + e^{-x}} \tag{12}$$
- **ReLU (Rectified Linear Unit)**:
$$f(x) = \max(0, x) \tag{13}$$
- **Tanh (Hyperbolic Tangent)**:
$$\tanh(x) = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} \tag{14}$$



(a) Linear Data



(b) XOR Data

6.3 Implement Convolutional Layers