Warning: Higher dimensions ahead !

A Fully Connected/Dense layer is insensitive to the order of features.

This is just a property of the dot product

$$\Theta^T \cdot \mathbf{x} = \Theta[\text{perm}]^T \cdot \mathbf{x}[\text{perm}]$$

where $\Theta[\text{perm}]^T$ and $\mathbf{x}[\text{perm}]$ are permutations of $\Theta, \mathbf{x}$.

$$\sum \begin{cases} \text{Machine} & \text{Learning} & \text{is} & \text{easy} & \text{not} & \text{hard} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \Theta_{\text{Machine}} & \Theta_{\text{Learning}} & \Theta_{\text{is}} & \Theta_{\text{easy}} & \Theta_{\text{not}} & \Theta_{\text{hard}} \end{cases}$$

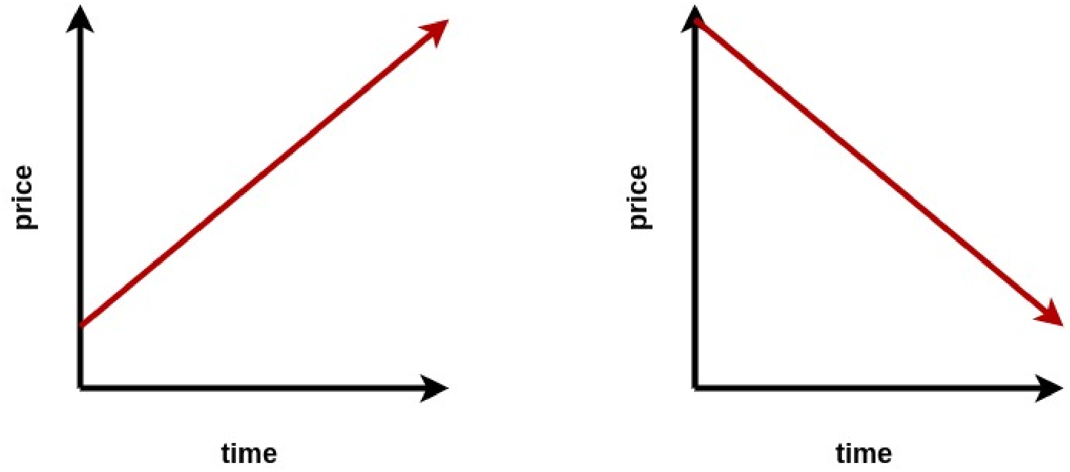
$$=$$

$$\sum \begin{cases} \text{Machine} & \text{Learning} & \text{is} & \text{hard} & \text{not} & \text{easy} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \Theta_{\text{Machine}} & \Theta_{\text{Learning}} & \Theta_{\text{is}} & \Theta_{\text{hard}} & \Theta_{\text{not}} & \Theta_{\text{easy}} \end{cases}$$

But there are many problems in which order is important.

Consider the following examples

Same prices



Same words

Machine	Learning	is	easy	not	difficult
Machine	Learning	is	difficult	not	easy

Same pixels

In this lecture, we will be dealing with examples that are *sequences*.

That is, we will add a new dimension to each example which we will call the

- *positional* dimension

and we will denote $\mathbf{x}_{(t)}$ as position t in sequence $\mathbf{x}$.

Often, the position is equated with time. In such cases we can also refer to the positional dimension as the *temporal* dimension*.

To make this concrete, consider a movie

- A movie is a sequence of snapshots
- Frame t of the movie corresponds to position t of the sequence.

Note the the snapshot has it's usual dimensions

- spatial dimensions
- feature dimension

Let $\mathbf{x}^{i}p$ be a example that happens to be a movie.

It is a sequence of items, at each of T positions

$$[\mathbf{x}^{(\mathbf{i})}_{(t)} \mid 1 \le t \le T]$$

where

- $\mathbf{x}^{(\mathbf{i})}$ is a movie: a sequence of frames
- $\mathbf{x}^{(\mathbf{i})}_{(t)}$ is the t^{th} frame in the movie
- $\mathbf{x}^{(\mathbf{i})}_{(t),j,j'}$ is a particular pixel within the frame $\mathbf{x}^{(\mathbf{i})}_{(t)}$
    - The positional dimension is indexed by (t) and the spatial dimensions by j, j'

There is an important difference between the positional and spatial dimensions

- spatial dimensions can often be permuted without changing meaning
    - shifting or flipping a frame
- positional dimensions often *can not* be permuted
    - causal relationships are encoded by order
    - frame t makes sense only if it occurs immediately after frame $(t-1)$ in the sequence

Functions on sequence

In the absence of a positional dimension, our multi-layer networks

- Computed functions from vectors to vectors

With a positional dimension, there are several variants of the function

- Many to one
    - Sequence as input, vector as output
    - Examples:
        - Predict next value in a time series (sequence of values)
        - Summarize the sentiment of a sentence (sequence of words)

- Many to many
    - Sequence as input, sequence of vectors as output
    - Examples
        - Translation of sentence in one language to sentence in second language
        - Caption a movie: sequence of frames to sequence of words

- One to many
    - Single input vector, sequence of vectors as output
    - Examples
        - Generating sentences from seed

Recurrent Neural Network (RNN) layer

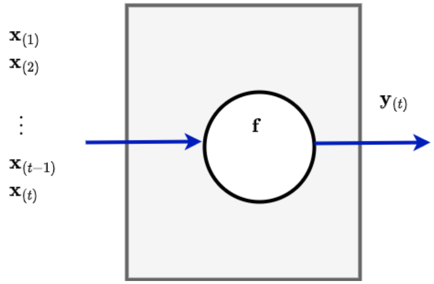
With a sequence $\mathbf{x}^{(\mathbf{i})}$ as input, and a sequence $\mathbf{y}$ as a potential output, the questions arises:

- How does an RNN produce $\mathbf{y}_{(t)}$, the t^{th} output ?

Some choices

- Predict $\mathbf{y}_{(t)}$ as a direct function of the prefix of $\mathbf{x}$ of length t:
$$p(\mathbf{y}_{(t)}|\mathbf{x}_{(1)} \ldots \mathbf{x}_{(t)})$$

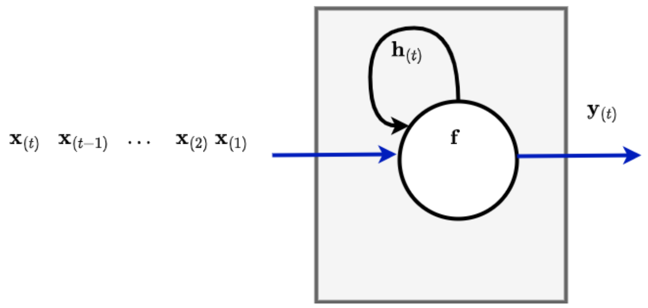
Direct function



- Loop
  - Uses a "latent state" that is updated with each element of the sequence, then predict the output

$p(\mathbf{h}_{(t)} \vert \mathbf{x}_{(t)}, \mathbf{h}_{(t-1)})$	latent variable $\mathbf{h}_{(t)}$ encodes $[\mathbf{x}_{(1)} \ldots \mathbf{x}_{(t)}]$
$p(\mathbf{y}_{(t)} \vert \mathbf{h}_{(t)})$	prediction contingent on latent variable

Loop with latent state



Since elements of the sequence are presented **one element at a time**

- the latent state $\mathbf{h}_{(t)}$ must act as a **summary** of all prior elements $\mathbf{x}_{(1)} \ldots \mathbf{x}_{(t)}$
$$\mathbf{h}_{(t)} = \text{summary}(\mathbf{x}_{([1:t])})$$

Note that $\mathbf{h}_{(t)}$ is a *vector* of fixed length.

Thus, it is a *fixed length* representation of the key aspects of a sequence $\mathbf{x}$ of potentially *unbounded* length.

Example

Let's use an RNN to compute the sum of a sequence numbers

- the latent state $\mathbf{h}_{(t)}$ can be maintained as
$$\mathbf{h}_{(t)} = \text{summary}(\mathbf{x}_{([1:t])}) = \sum_{t'=1}^{t} \mathbf{x}_{(t')}$$
- by updating $\mathbf{h}_{(t)}$ in the loop
$$\mathbf{h}_{(t)} = \mathbf{h}_{(t-1)} + \mathbf{x}_{(t)}$$

The Recurrent Neural Network (RNN) adopts the "latent state" approach.

A prime advantage of the latent state approach

- it can handle sequences of *unbounded* length

Here is some pseudo-code:

In [2]:

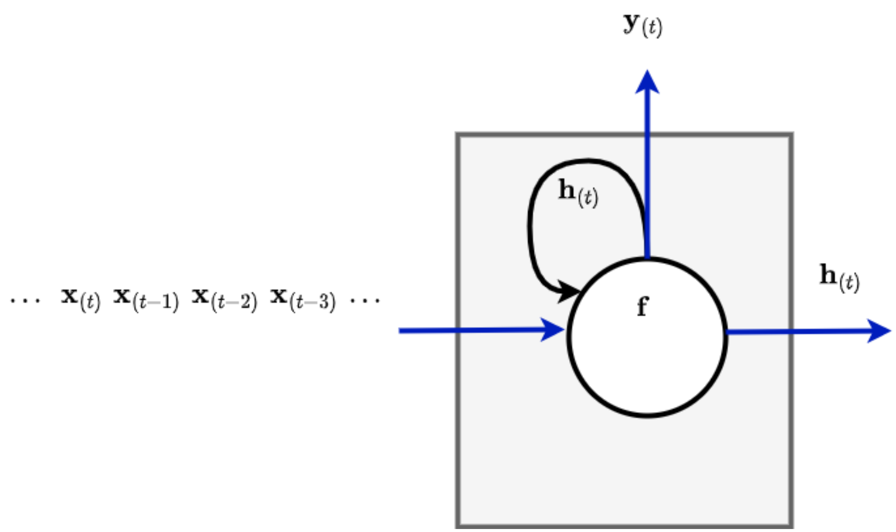
```python
def RNN( input_sequence, state_size ):
    state = np.random.uniform(size=state_size)

    for input in input_sequence:
        # Consume one input, update the state
        out, state = f(input, state)

    return out
```

And the corresponding diagram, showing the output `out` ($\mathbf{y}_{(t)}$)

RNN



Output $\hat{\mathbf{y}}_{(t)}$ of an RNN

According to our pseudo-code and diagram

$$\hat{\mathbf{y}}_{(t)} = \mathbf{h}_{(t)}$$

That is: the output is the same as the latent state.

It is easy to add another NN to transform $\mathbf{h}_{(t)}$ into a $\hat{\mathbf{y}}_{(t)}$ that is different

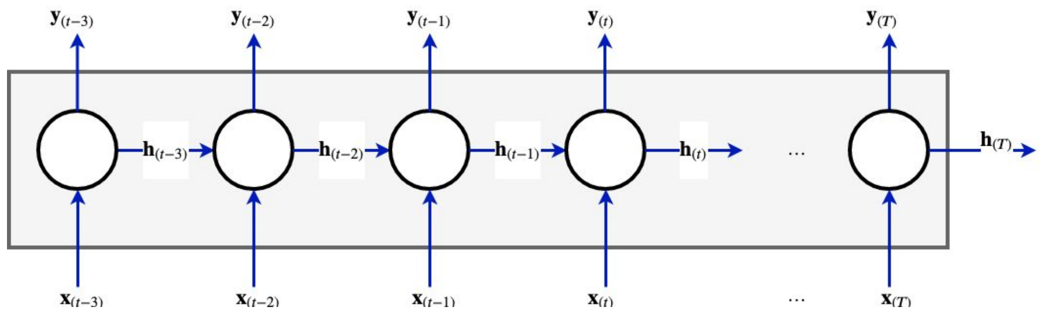
- we will omit this additional layer for clarity

Unrolled RNN diagram

We can "unroll" the loop into a kind of movie

- a sequence of steps
- step $t-1$ arranged to the left of step t

RNN many to many API



At each time step t

- Input $\mathbf{x}_{(t)}$ is processed
- Causes latent state $\mathbf{h}$ to update from $\mathbf{h}_{(t-1)}$ to $\mathbf{h}_{(t)}$
    - We use the same sequence notation to record the sequence of latent states $[\mathbf{h}_{(1)}, \ldots,]$
- Optionally outputs $\mathbf{y}_{(t)}$ (for outputs that are of type sequence)

When processing $\mathbf{x}_{(t)}$

- The function computed takes $\mathbf{h}_{(t-1)}$ as input
- Latent state $\mathbf{h}_{(t-1)}$ has been derived by having processed $[\mathbf{x}_{(1)} \ldots \mathbf{x}_{(t-1)}]$
- And is thus a *summary* of the prefix of the input encountered thus far

One can look at this unrolled graph as being a dynamically-created computation graph.

- A sequence of layers
- One layer per time step
- But with an **identical** computation for all layers

The unrolled version will be crucial in understanding how Gradient Descent works when RNN layers are present.

- Just conceptualize the unrolled loop as a sequence of layers
- All our logic and intuition carries over

Note that $\mathbf{x}, \mathbf{y}, \mathbf{h}$ are all vectors.

In particular, the state $\mathbf{h}$ *may have many* elements

- it is a vector of "synthesized" features
- to record information about the entire prefix of the input.

$\mathbf{h}_{(t)}$ is the latent state (sometimes called the *hidden state* as it is not visible outside the layer).

It is essentially a *fixed length* encoding of the variable length sequence $[\mathbf{x}_{(1)} \ldots \mathbf{x}_{(t)}]$

- All essential information about the prefix of $\mathbf{x}$ ending at step t is recorded in $\mathbf{h}_{(t)}$
- Hence, the size of $\mathbf{h}_{(t)}$ may need to be large

We will shortly attempt to gain some intuition as to what these synthesized features may be.

All "layers" in the unrolled graph share weights

One extremely important aspect that might not be apparent from the movie version:

- Each unrolled "frame" in the movie shares the *same weights* and computes the *same* function F
- In contrast to a true multi-layer network where each layer has its *own* weights

That is the unrolled RNN computes

$$\begin{aligned}
\mathbf{y}_{(t)} &= F(\mathbf{y}_{(t-1)}; \mathbf{W}) \\
&= F(\ F(\mathbf{y}_{(t-2)};\ \mathbf{W});\ \mathbf{W}\ ) \\
&= F(\ F(\ F(\mathbf{y}_{(t-3)};\ \mathbf{W});\ \mathbf{W}\ ); \mathbf{W}\ ) \\
&= \vdots
\end{aligned}$$

rather than

$$\begin{aligned}
\mathbf{y}_{(l)} &= F_{(l)}(\mathbf{y}_{(l-1)}; \mathbf{W}_{(l)}) \\
&= F_{(l)}(\ F_{(l-1)}(\mathbf{y}_{(l-2)};\ \mathbf{W}_{(l-1)});\ \mathbf{W}_{(l)}\ ) \\
&= F_{(l)}(\ F_{(l-1)}(\ F_{(l-2)}(\mathbf{y}_{(l-3)};\ \mathbf{W}_{(l-2)});\ \mathbf{W}_{(l-1)}\ ); \mathbf{W}_{(l)}\ ) \\
&= \vdots
\end{aligned}$$

Note, in particular

- The repeated occurrence of the term $\mathbf{W}$ will complicate computing the derivative
- As we will see in a subsequent lecture

RNN's are sometimes drawn without separate outputs $\mathbf{y}_{(t)}$

- in that case, $\mathbf{h}_{(t)}$ may be considered the output.

The computation of $\mathbf{y}_{(t)}$ will just be a transformation of $\mathbf{h}_{(t)}$ so there is no loss in omitting it from the RNN and creating a separate node in the computation graph.

Geron does not distinguish between $\mathbf{y}_{(t)}$ and $\mathbf{h}_{(t)}$ and he uses the single $\mathbf{y}_{(t)}$ to denote the state.

I will use $\mathbf{h}$ rather than $\mathbf{y}$ to denote the "hidden state".

Typical uses of RNN

Many to one: Creating a fixed length summary of a variable length sequence

A typical Many to One task is predicting the next element in a sequence

For example

- Predict the next word in a sentence
- Predict the next price in a timeseries of prices

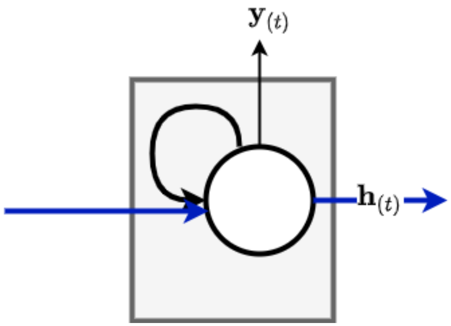
These are implemented by a NN (with RNN layers as components) followed by a Head Layer (Classifier or Regressor)

But the Head Layers take **fixed length** inputs and our sequences are of potentially unbounded length !

We first need to convert the variable length sequence into a fixed length representation.

Let's make this concrete with an example: a sequence of words

Machine Learning is easy not hard



$\mathbf{h}_{(t)}$ is a **fixed length** vector that "summarizes" the prefix of sequence $\mathbf{x}$ up to element t.

The sequence is processed element by element, so order matters.

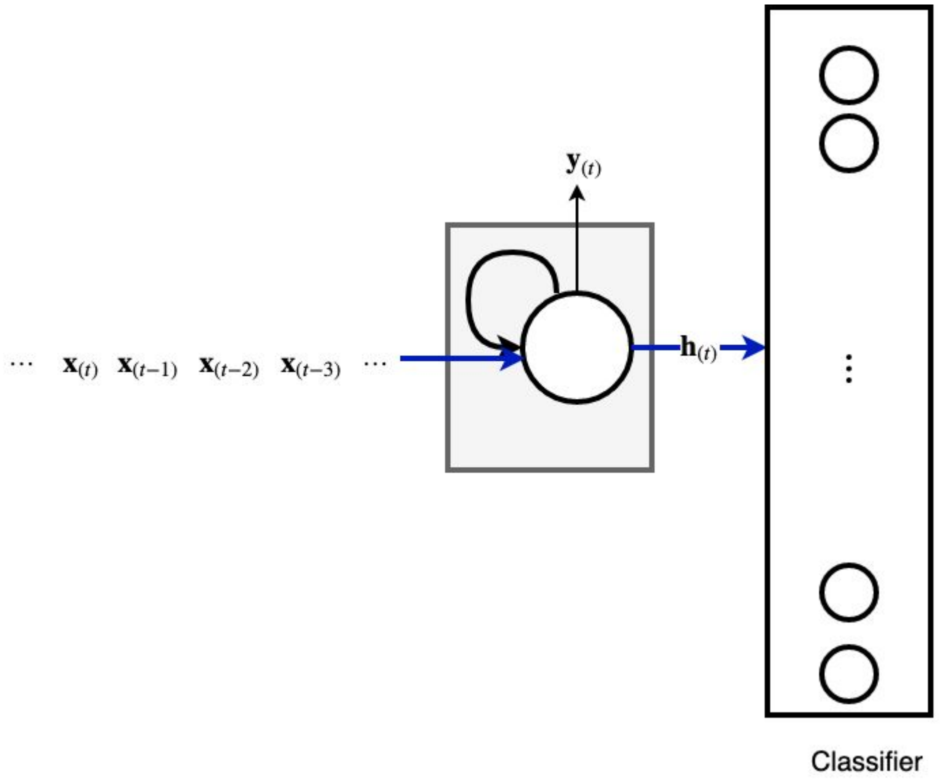
$$
\begin{aligned}
\mathbf{h}_{(0)} &= \text{summary}([\text{Machine}]) \\
\mathbf{h}_{(1)} &= \text{summary}([\text{Machine, Learning}]) \\
&\vdots \\
\mathbf{h}_{(t)} &= \text{summary}([\mathbf{x}_{(0)}, \ldots \mathbf{x}_{(t)}]) \\
&\vdots \\
\mathbf{h}_{(5)} &= \text{summary}([\text{Machine, Learning, is, easy, not, hard}])
\end{aligned}
$$

Turning an unbounded length sequence into a fixed length vector is very useful !

- All our other layer types take fixed length input

So we can feed $\mathbf{h}_{(5)}$ into a Classifier to decide on the sentiment of the sentence.

RNN Many to one; followed by classifier

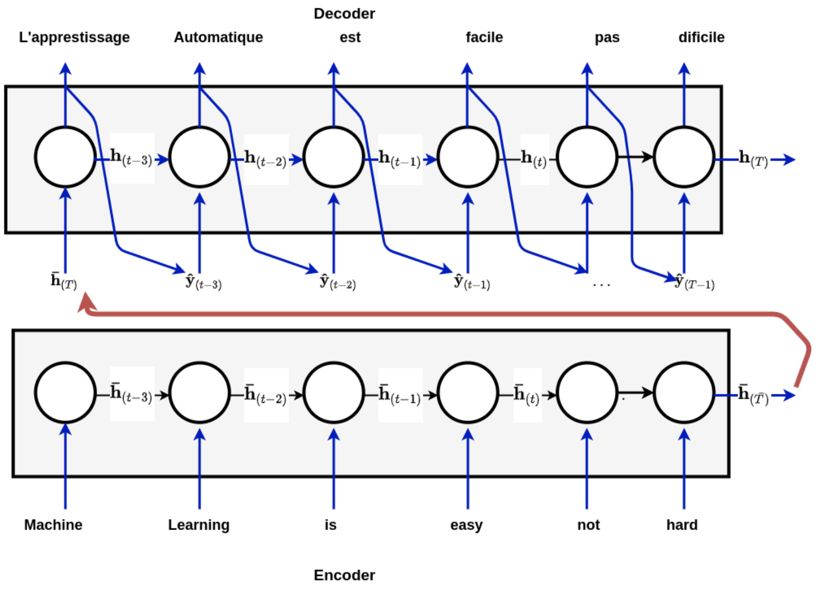


Many to many: Encoder-Decoder

Another common paradigm using RNN's that we will encounter is the *Encoder-Decoder* which is useful for tasks mapping sequences to sequences

- language translation
- Output and input sequence elements do not have a one to one correspondence
- The Encoder-Decoder decouples the sequences
    - Encoder summarizes the input sequence $\mathbf{x}_{([1:\bar{T}])}$ with $\bar{\mathbf{h}}_{(\bar{T})}$
    - Decoder generates output sequence $\hat{\mathbf{y}}_{([1:T])}$ from the summary

Encoder-Decoder for language translation



- The final latent state $\bar{\mathbf{h}}_{(T)}$ of the Encoder "summarizes" the source sentence (English)
- It initializes the latent state of the Decoder which produces the target sentence (French)
- The Decoder implements a one-to-many API
    - source language "summary" as seed

Decoupling means that the length of $\mathbf{x}$ (length $\bar{T}$) need not be equal to the length of $\hat{\mathbf{y}}$ (length T).

One to many: Generative ML (generating sequences from a seed)

The two main Machine Learning Tasks we have studied thus far (Regression, Classification) are called *discriminative* tasks

- they learn the relationship between features and targets of an example

We can also use Machine Learning for the *generative* task of creating new examples

- learns the distribution of features
- can sample from the learned distribution to construct new examples

RNN's are often used for generative tasks.

We generate a long sequence that is highly probable (from the learned distribution) given a short sequence as seed.

- The model is initially input a short "seed" sequence.
- The output is a prediction of the **next** element of the sequence
- The input sequence is extended by the prediction
- Repeat !

[Here (https://app.inferkit.com/demo)](https://app.inferkit.com/demo) is a demo of creating an entire story from an initial idea comprised of a few words.

Conclusion

We have introduced the key concepts of Recurrent Neural Networks.

- An unrolled RNN is just a multi-layer network
- In which *all the layers are identical*
- The latent state is a fixed length encoding of the prefix of the input

A more detailed view of sequences and RNN's will be our next topic.

In [3]:

```
print("Done")
```

```
Done
```