Introduction: Beyond the Feature dimension

Thus far, the data examples we have been using are vectors

- the only dimension is the "feature" dimension
- for example, the names of the features in the feature dimension are `Price, Volume, Open, Close`

The diagram shows (our typical, up to now: 0 non-feature dimensional) feature vector $\mathbf{x}$ matched against pattern $\mathbf{k}$

- where the feature dimension is length 3

Zero non-feature dimensions, length 3 feature dimension

In this module, we extend "pattern matching" to includes examples that have "shape"

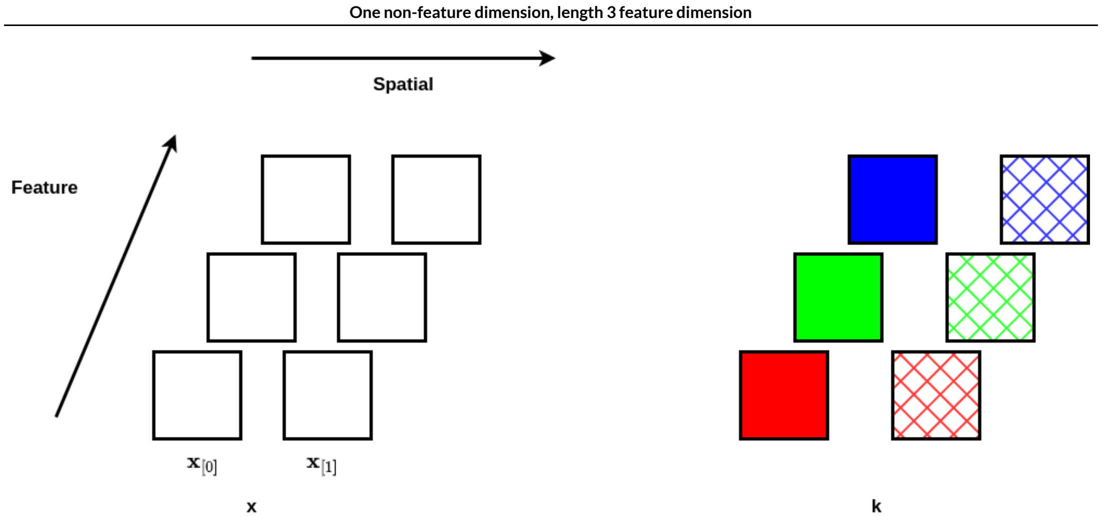
- dimensions beyond the feature dimension

The extended examples will have

- multiple elements, each having a vector of features
- For example
    - a timeseries: a sequence of elements each having the features `Price, Volume, Open, Close`
    - a two dimensional image: a grid of elements each have the features `Red, Green, Blue`

The diagram shows a

- 1 non-feature dimension (of length 2) vector $\mathbf{x}$
  - for clarity: we show the index of the non-feature dimensions in brackets: $\mathbf{x}_{[0]}, \mathbf{x}_{[1]}$
- with feature dimension length 3
- matched against pattern $\mathbf{k}$



How does pattern matching work in the presence of non-feature dimensions ?

We will generalize the dot-product

- producing a single output measuring the intensity of the match of the input with the pattern

Matching the one non-feature dimensional data $\mathbf{x}$ against pattern $\mathbf{k}$

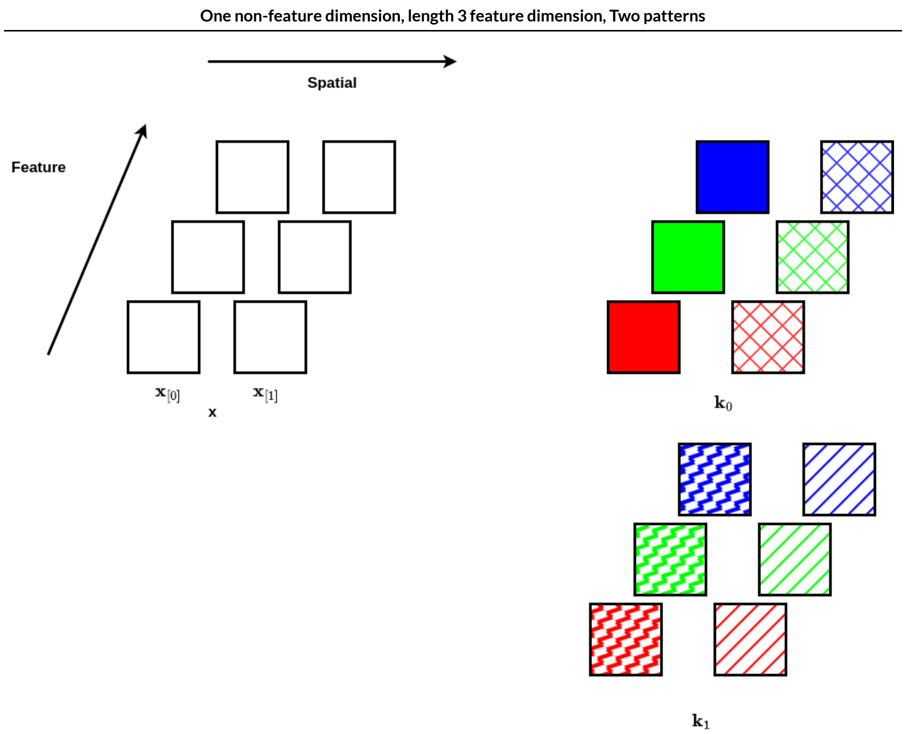
- we match (dot product) the first element $\mathbf{x}_{[0]}$ against the corresponding (first) element of the pattern
    - resulting in a scalar measuring the intensity of the match
- we match the second element $\mathbf{x}_{[1]}$ against the corresponding (second) element of the pattern
    - resulting in a second scalar
- add the two scalars together
- resulting in *one output feature* at a single location

We can create a *second* output feature by adding a second pattern

- similar to how a Fully Connected layer creates multiple features via multiple patterns

The diagram shows a

- 1 non-feature dimension (of length 2) vector $\mathbf{x}$
- with feature dimension length 3
- matched against 2 patterns $\mathbf{k}_0, \mathbf{k}_1$
- resulting in an output with 2 features
    - the first measuring the intensity of the match with the first pattern
    - the second measuring the intensity of the match with the second pattern



What can be achieved by adding non-feature dimensions ?

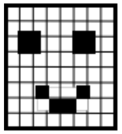
In the "smiling face" diagram below

- there are 2 non-feature dimensions, each of length 8 (called "row" and "col")
- a feature dimension of length 1

Note: there is **one** feature that appears over a region of size (8×8)

- **not** 64 features arranged in a 2D grid

Two non-feature (spatial) dimensions, length 1 feature dimension



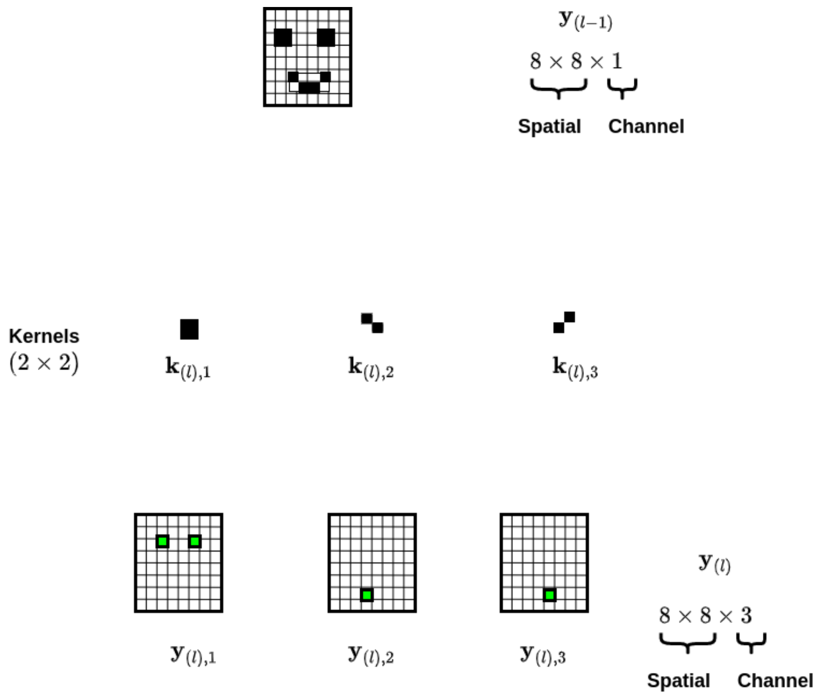
$$\mathbf{y}_{(l-1)}$$

$$\underbrace{8 \times 8}_{\textbf{Spatial}} \times \underbrace{1}_{\textbf{Channel}}$$

We can imagine using *multiple patterns* to try to identify the "smiling face"

- one pattern to match the "eyes"
- one pattern to match the left corner of the smile
- one pattern to match the right corner of the smile

Convolution: 1 input feature to 3 output features



In the above diagram

- the length of the patterns is smaller (in the non-feature dimensions) than the input
    - each pattern is (2×2)
- allowing us to find the presence of "small" patterns (sub-regions) within a larger (8×8) region
- by "sliding" the small pattern over the entire region
    - allowing us to associate an intensity of match at even location in the larger region
- creating 3 output *feature maps*, one each for each output feature
    - an eye
    - a left corner of a smile
    - a right corner of a smile

The above is an introduction to a method call a *Convolution*.

This method will be explained in greater detail below.

It will form the basis for a new layer type: the *Convolutional Neural Network (CNN)*

- that is a layer type typically used with data that has non-feature dimensions
    - e.g., images, timeseries

The non-feature dimensions are sometimes given names that identify their purpose

- temporal dimension
    - a timeseries with temporal dimension of length 1
        - where $\mathbf{x}_{[t],j}$ denotes feature j at time step t
    - a timeseries with a temporal dimension of length 2
        - $\mathbf{x}_{[d,t],j}$: temporal dimension of length 2 (date `d`, time `t`)
- spatial dimension
    - a 2D grid with spatial dimension of length 2
        - where $\mathbf{x}_{[\text{row},\text{col}],j}$ denotes feature j at row `row`, column `col` of the grid
- mixed dimension:
    - a timeseries of 2D grids (e.g., a movie) with a temporal dimension of length 1 and spatial dimension of length 2
        - where $\mathbf{x}_{[t,\text{row},\text{col}],j}$ denotes feature j at location (row, col) of frame t of the movie

For clarity

- we have surrounded the non-feature dimensions with brackets
    - writing $\mathbf{x}_{[t],j}$ rather than $\mathbf{x}_{t,j}$
        - rather than a vector as it has been up to this point
- Dropping the brackets
    - $\mathbf{x}_t$ is a vector with n dimensions
        - rather than a scalar (element t of the vector $\mathbf{x}$ whose length is n

Going forward:

- we may drop the brackets when the context is clear
- our convention is that the *last* dimension in a multi-dimensional object is the *feature* dimension
    - this is called the *channel last* convention
    - when the feature dimension is written as the first dimension: that is called *channel first*
    - TensorFlow layers assume channel last
        - but other toolkits and data providers may not
        - always check !

How is the Feature dimension different from non-feature dimensions ?

The feature dimension has some key differences from the non-feature dimensions

- the indices of the feature dimension are *unordered*
    - permuting the features from `Price, Volume, Open, Close` to `Open, Close, Price, Volume` does not change the meaning of the example
- the indices of the temporal dimension are totally ordered
    - reversing the indices makes time flow backwards rather than forwards
    - changing the meaning of the example
- the indices of the spatial dimension are (at least, partially) *ordered*
    - there is a spatial relationship between elements whose spatial indices differ by 1
    - row 5 occurs between rows 4 and 6
    - permuting the row order from 4, 5, 6 to 5, 4, 6 changes the meaning of the example

Because of this ordering, the behavior of certain layer types may not respect the order.

For example, consider a sequence of words and its permutation

- $\mathbf{x}$ = [Machine, Learning, is, easy, not, hard]
- $\mathbf{x}'$ = [Machine, Learning, is, hard, not, easy]

The example is a (6×1) vector with a one temporal dimension of length 6 and a single feature `word`.

- $\mathbf{x}_{[0]}$ is a feature vector of one element: 'Machine'
    - $\mathbf{x}_{[0],0}$ = 'Machine'

Clearly, the "meaning" of the two sequences are different.

But suppose we tried to represent this (6×1) vector as vector $\mathbf{x}''$ of length 6

- no non-feature dimensions
- a single feature dimension of length 6 with features `word1, word2, ..., word6`
    - $\mathbf{x}''$ is a vector of features of length $n = 6$
        - $\mathbf{x}''_0$ = 'Machine'

How would a Fully Connected (`Dense`) layer deal with the two permutations when they were represented without non-feature dimensions.

It would **not** be able to distinguish between the two permutations !

- A Fully Connected layer computes the dot product of features $\mathbf{x}$ and associated weights $\mathbf{w}$
    - where $\mathbf{x}, \mathbf{w}$ are vectors of length n containing *only* feature dimensions
- Let `perm` denote an ordering of indices that is a permutation of $[1, 2, \ldots, 6]$
- The dot product of the example and its permutation are the same

$$\mathbf{x} \cdot \mathbf{w} = \mathbf{x}[\text{perm}] \cdot \mathbf{w}[\text{perm}]$$

Our attempt at representing the (6×1) example as 6 features was not successful as it did not respect the ordering of the temporal dimension

Layers transform the *feature* dimension

- transforming raw features into synthetic features
- transforming synthetic features into new synthetic features of increasing complexity

Thus, we need new layer types that

- transform the feature dimension
- for *each element* of the non-feature dimensions

Why are the patterns smaller (in non-feature dimension) ?

The essence of Neural Networks (as evidenced by the Fully Connected Layer) is pattern matching

- a vector of features from the example
- is matched against a pattern (weights)
- via the dot product
- resulting in a scalar intensity measuring the degree of the match

But the dot product is defined on vectors of features

- no non-feature dimensions
- just a feature dimension of length n

We need to expand the definition of dot product to account for the non-feature dimensions.

We will do this in a subsequent section.

Given a generalized dot product to match features even in the presence of non-feature dimensions

- what would be the best way to use it ?

It would be natural

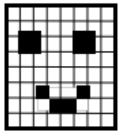
- to use patterns with non-feature dimensions that were identical in number and length to those of the input.

That is

- we seek to match a pattern against the entire non-feature dimensions of the input.

Consider the following pattern which is of identical dimension to the input

Pattern spanning entire non-feature dimensions, single feature



$$\mathbf{y}_{(l-1)}$$

$$\underbrace{8 \times 8}_{\textbf{Spatial}} \times \underbrace{1}_{\textbf{Channel}}$$

In fact: this pattern is identical to the input and matches it perfectly !

- the generalized dot product of the input and the pattern results in a high activation

But what about inputs similar to this one but

- shifted right/left or up/down
    - we week "translational invariance"
- a smaller smile
- different distance between the eyes

The pattern would not be as good a match

- lower activation
- even though the "meaning" of the similar input is the same as the original: "smiling face"

It would be more powerful to match patterns against a *sub-region* of the input

- we could look for a small pattern within the input
- while ignoring parts of the input beyond the size of the pattern

For example, suppose

- we want to create an output feature (i.e., match a pattern) that is active when the input contains a pattern matching
    - an "eye"
    - part of the "smile"

We can't use the dot product of the input and a pattern spanning the full input to create these output features

- These patterns are much smaller than the entire input
- The "eye" can occur in any sub-region of the input
- An "eye" can be present regardless of the presence/absence of any other parts of the input

The solution

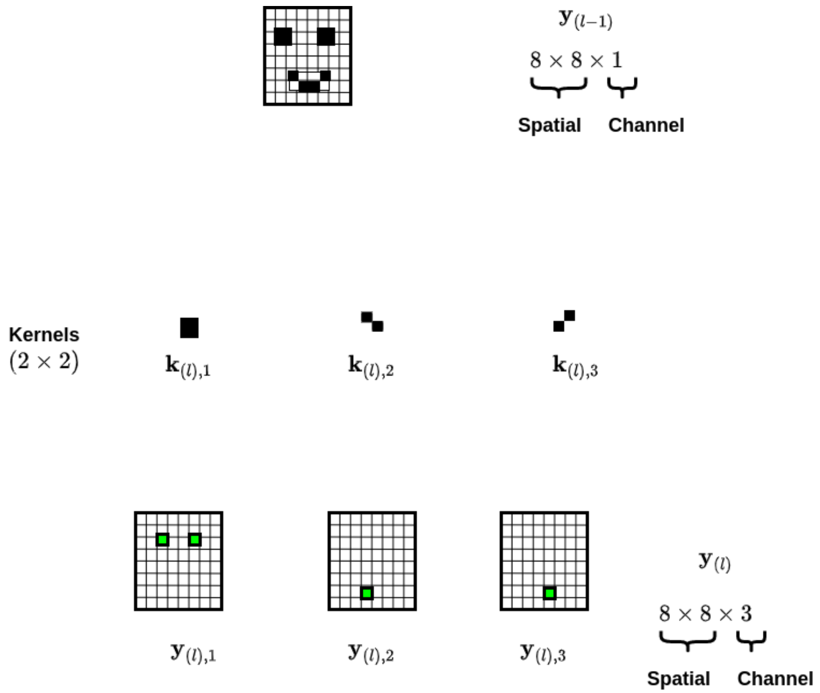
- define patterns whose non-feature dimensions are *smaller* than that of the entire input
- match these patterns against *every sub-region* of the input of equal size
- producing an output that is a *feature map*
    - an object with non-feature dimensions equal in number and length to that of the input
    - with a single feature
    - that is active at a particular location of the feature map
    - when the pattern matches a sub-region of the input centered at that location

The operation to produce the feature map is called *Convolution*

In the diagram below we show a Convolution that is performed by a CNN Layer type that is layer l of a Sequent NN

- the input is $\mathbf{y}_{(l-1)}$ (the output of layer $(l-1)$ in a multi-layer NN)
- there are 3 patterns with non-spatial dimensions (2×2)
    - $\mathbf{k}_{(l),1}$ is the pattern for an "eye"
    - $\mathbf{k}_{(l),2}$ and $\mathbf{k}_{(l),3}$ are patterns for the left/right corner of the smile
- the output feature map $\mathbf{y}_{(l)}$ (the layer output)
    - has non-feature dimensions equal in number and length to those of $\mathbf{y}_{(l-1)}$
    - shows the locations within input $\mathbf{y}_{(l-1)}$ where the pattern is matched

Convolution: 1 input feature to 3 output features



Generalizing the dot product to include non-feature dimensions

We can generalize the dot product to higher dimensions

- let the two arguments $\mathbf{x}, \mathbf{k}$ of the dot product have identical dimensions (both non-feature and feature)
- the point-wise product of the values in corresponding locations of the two arguments
- summed
- to yield a single value

The dot product is

$$\sum_{\mathbf{i} \in I} \mathbf{x_i} * \mathbf{k_i}$$

where I is an enumeration of the index set of $\mathbf{x}$ (and $\mathbf{k}$ since they are of the same dimension)

- each element $\mathbf{i}$ of I is in $D_0 \times \ldots D_{d-1} \times F$
    - $\mathbf{i}$ is a vector of length $d+1$
- where there are d non-feature dimensions
    - D_l are the set of indices in non-feature dimension l

This point-wise multiplication is mathematically equivalent to a multiplication described in a way that highlights the feature dimension of the arguments.

- taking sub-vector $\mathbf{x}_{[\mathbf{i}]}$ of example $\mathbf{x}$
    - located a single location $\mathbf{i}$ in the non-feature dimensions of example $\mathbf{x}$ $\mathbf{i}$ is a vector of length N
    - which is a vector of length n features
- and the sub-pattern $\mathbf{k}_{[\mathbf{i}]}$ of the pattern $\mathbf{k}$
    - located at the same location $\mathbf{i}$ in the non-feature dimensions of pattern $\mathbf{k}$
    - which is a vector of length n features
- taking the dot product

$$\mathbf{x}_{[\mathbf{i}]} \cdot \mathbf{k}_{[\mathbf{i}]}$$

- adding the dot products computed over *all* the locations i in the non-feature dimensions of $\mathbf{x}$ (and $\mathbf{k}$ since they are of same dimension)

That is, we re-write the point-wise multiplication and sum as

- the dot product at each location in the non-feature dimensions
    - each argument is a vector of n featurs
- summed

$$\sum_{\mathbf{i} \in D} \mathbf{x}_{[\mathbf{i}]} \cdot \mathbf{k}_{[\mathbf{i}]}$$

where

$$D = D_0 \times \ldots D_{d-1}$$

is an enumeration of the locations in the non-feature dimensions of $\mathbf{x}$ (and $\mathbf{k}$)

We are matching feature vectors (just as in the one-dimensional case)

- but there is one pair of feature vectors to match for each location in the non-feature dimensions
- so we need to average the pair-wise dot products over the multiple locations

With the addition of non-feature dimensions, we will generalize the pattern matching.

The patterns will now *also* have non-feature dimensions

- same number of dimensions as the example $\mathbf{x}$'s non-feature dimensions
- but the length f of each dimension in the pattern $\mathbf{k}$ will be *smaller* than the length of the corresponding dimension in the example

Convolution: definition

A convolution involves an example $\mathbf{x}$ and pattern $\mathbf{k}$

- The *number* of dimensions of $\mathbf{x}$ and $\mathbf{k}$ are the same
- The *length* of the feature dimension of $\mathbf{x}$ and $\mathbf{k}$ are the same
- The *length* of each *non-feature* dimensions of $\mathbf{k}$ is *less than or equal* to the length of the c corresponding non-feature dimension of $\mathbf{x}$
    - customarily, each non-feature dimension of $\mathbf{k}$ is the same length: f
        - but this is not necessary

The pattern $\mathbf{k}$, since the length of the non-feature dimensions may be smaller than the corresponding non-feature dimensions of $\mathbf{x}$

- can only be matched against a sub-region of $\mathbf{x}$ whose non-feature dimensions have length f
    - so that the dimensions of the sub-region match those of the pattern

For example suppose the non-feature dimensions $\mathbf{x}$ and $\mathbf{k}$ are (5×5) and (3×3) respectively.

One can take the dot product of $\mathbf{k}$ with any sub-region of $\mathbf{x}$ of size (3×3).

There are many such sub-regions of $\mathbf{x}$.

Convolution is an operation that computes the dot product of $\mathbf{k}$ and each sub-region of $\mathbf{x}$.

Let there be N non-feature dimensions in $\mathbf{x}$ and $\mathbf{k}$

- D_l are the set of indices in non-feature dimension l of $\mathbf{x}$
- D'_l are the set of indices in non-feature dimension l of $\mathbf{k}$
    - length of each D'_l is f
- F is the set of indices (size n) of the feature dimension of both $\mathbf{x}$ and $\mathbf{k}$

The index set of

- $\mathbf{x}$ is $(D_0 \times \ldots D_{d-1} \times F)$
- the non-feature dimensions of $\mathbf{x}$ is
$$D = D_0 \times \ldots D_{d-1}$$
- $\mathbf{k}$ is $(D'_0 \times \ldots D'_{d-1} \times F)$
    - length of each D'_l is f
    - $f \leq |D_l|$ for each $0 \leq l \leq (N-1)$

We will create sub-regions of $\mathbf{x}$.

Let $\mathbf{x}_{[\mathbf{i}]}$ be the sub-region of $\mathbf{x}$ centered at index $\mathbf{i} \in D$

- whose non-feature dimensions are of length f
- with feature dimension of length n

The dimensions of $\mathbf{x}_{[\mathbf{i}]}$ and $\mathbf{k}$ are identical

- so we can compute the dot product

Let

- $\text{Conv}(\mathbf{x}, \mathbf{k})$ denote the operation performing the convolution of example $\mathbf{x}$ and pattern $\mathbf{k}$
- let $\mathbf{y} = \text{Conv}(\mathbf{x}, \mathbf{k})$ be the output of the convolution operation
    - the non-feature dimensions of $\mathbf{y}$ and $\mathbf{x}$ are identical in number and length
    - the feature dimension of $\mathbf{y}$ is length 1

Convolution is defined as

$$\mathbf{y}_{[\mathbf{i}],0} = \mathbf{x}_{[\mathbf{i}]} \cdot \mathbf{k}$$

That is

- the non-feature dimensions of $\mathbf{y}$ and $\mathbf{x}$ are equal in number and length
- location $\mathbf{i}$ in the non-feature dimension of $\mathbf{y}$ is the result of matching
    - the pattern $\mathbf{k}$
    - against the sub-region of $\mathbf{x}$
        - centered at $\mathbf{i}$
        - whose length of non-feature dimensions matches those of $\mathbf{k}$

Just as a Fully Connected layer can create more than one output feature

- using a different pattern for each output feature
- we can define Convolution to output more than one feature
    - using a different pattern for each feature
    - the patterns are higher-dimensional for the Convolution compare to Fully Connected
        - since the Convolutional patters contain non-feature dimensions
        - as well as the feature dimension

Let $\mathbf{k}_j$ denote the pattern used for output feature j.

Then

$$\mathbf{y}_{[\mathbf{i}],j} = \mathbf{x}_{[\mathbf{i}]} \cdot \mathbf{k}_j$$

The output for feature j, denoted as $\mathbf{y}_{\ldots,j}$ is called *feature map* j

- shows how each location in the non-feature dimensions of $\mathbf{x}$
- is matched against pattern $\mathbf{k}_j$

Examples

It may be easier to grasp the working of a CNN layer

- by starting with simple cases of $\mathbf{x}$ and $\mathbf{k}$
- and working up to more complex cases
- using pictures to illustrate

We do this in the next module

Where do the patterns come from ? Training a CNN

Hopefully you understand how patterns (kernels) are "feature recognizers".

But you may be wondering: how do we determine the weights in each kernel ?

Answer: a Convolutional Layer is "just another" layer in a multi-layer network

- The kernels are just weights (like the weights in Fully Connected layers)
- We solve for all the weights $\mathbf{W}$ in the multi-layer network in the same way

The answer is: exactly as we did in Classical Machine Learning

- Define a loss function that is parameterized by $\mathbf{W}$:
$$\mathcal{L} = L(\hat{\mathbf{y}}, \mathbf{y}; \mathbf{W})$$
- The kernel weights are just part of $\mathbf{W}$
- Our goal is to find $\mathbf{W}^*$ the "best" set of weights
$$\mathbf{W}^* = \underset{W}{\operatorname{argmin}} \, L(\hat{\mathbf{y}}, \mathbf{y}; \mathbf{W})$$
- Using Gradient Descent !

In other words: their is nothing special about finding the "best" kernels.

In [ ]:

```
print("Done")
```

PREVIOUS VERSION OF THIS NOTEBOOK

The remainder of this notebook is a previous version

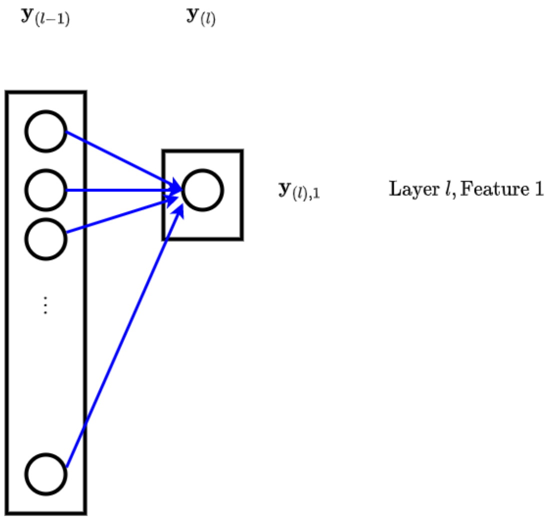
- the newer version (sections above) are hopefully improved
- the sections below are kept only for historic reference and will soon be delted

Convolutional Neural Networks

A Fully Connected/Dense Layer with a single unit producing a single feature at layer l computes

$$\mathbf{y}_{(l),1} = a_{(l)}(\mathbf{y}_{(l-1)} \cdot \mathbf{W}_{(l),1})$$

Fully connected, single feature



That is:

- It recognizes one new synthetic feature
- In the entirety ("fully" connected) of $\mathbf{y}_{(l-1)}$
- Using pattern $\mathbf{W}_{(l),1}$ (same size as $\mathbf{y}_{(l-1)}$)
- To reduce $\mathbf{y}_{(l-1)}$ to a single feature.

The pattern being matched spans the entirety of the input

- Might it be useful to recognize a smaller feature that spanned only *part* of the input ?
- What if this smaller feature could occur *anywhere* in the input rather than at a fixed location ?

For example

- A "spike" in a time series
- The eye in a face

A pattern whose length was that of the entire input could recognize the smaller feature only in a *specific* place

This motivates some of the key ideas behind a Convolutional Layer.

- Recognize smaller features within the whole
- Using small patterns
- That are "slid" over the entire input
- Localizing the specific part of the input containing the smaller feature

The spatial dimension

A small pattern (less than full length of input) can match a sub-section of input at any location.

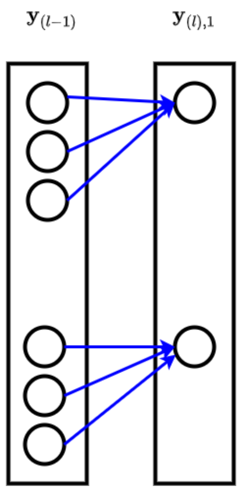
We can imagine centering the pattern on each input element and applying the match.

The output, therefore, will be a vector of length equal to the input length.

Here is the connectivity diagram of a Convolutional Layer producing a **single** feature at layer l

- Using a pattern of length 3
- Eventually we will show how to produce *multiple* features
- Hence the subscript "1" in $\mathbf{y}_{(l),1}$ to denote the first output feature

Convolutional layer, single feature



The output vector $\mathbf{y}_{(l),1}$ is called the first **feature map** as it attempts to match the first feature at each input location.

We refer to the indices of the feature map as the **spatial dimensions**

Thus, the output of layer l is 2 (or higher !) dimensional when layer l is a CNN

- a number of features
    - we have only shown a single feature thus far
- each feature producing a feature map
    - a feature map dimensions are called the spatial dimensions

The output of a CNN layer is a collection of

- $n_{(l)}$ feature maps (one per feature)
- each feature map having the same spatial dimension as its inputs

We can connect multiple CNN layers in sequence

- preserving the spatial dimensions across layers
- but creating more complex features as we get deeper

Technical note: special case

- The input to the first CNN layer l is often a one-dimensional vector of $n_{(l-1)}$ features
- A CNN treats this as a $(n_{(l-1)} \times 1)$ vector
  - 1 feature of a 1D spatial dimensions of shape $((n_{(l-1)},)$

Our convention will be that the **feature dimension** will appear as the **final** dimension of the output of layer l.

- all prior dimensions will be part of the **spatial dimension**

We need to distinguish which dimension is the feature dimension because

- **A convolution finds small patterns in the spatial dimension, not the feature dimension**

To be clear

- the vector of shape $(1 \times d)$ denotes d features at a single spatial location
- the vector of shape $(d \times 1)$ denotes a single features at d spatial locations

Notation

- the feature dimension will be the last index
- $n_{(l)}$ will always denote the *number of features* of a layer l
- $\mathbf{y}_{(l),j',j}$ denotes feature j of layer l at spatial location j'

We say that the above convolutional layer l

- Maps a single feature (defined over a 1D spatial dimension with $d_{(l)} = d_{(l-1)}$ locations) of layer $(l-1)$
- To a single feature, defined over an identical number of spatial locations in layer l

The importance of the spatial dimension

Let's contrast the CNN layer with a Fully Connected layer.

- The Fully Connected layer we depicted matches a pattern over the full *feature* dimension
    - There is no ordering (or spatial relationship) between features
- The CNN layer we depicted matches a pattern over the full *spatial* dimensions

Spatial dimensions are different than feature dimensions

- They have "order" (spatial relationships)

To see this, we show that a FC layer is insensitive to ordering of inputs

- Consider a vector $\mathbf{x}$ of n features (input to the Fully Connected layer)
- Let perm be permutation of the indices of $\mathbf{x}$: $[1 \ldots n]$.

If we permute both $\mathbf{x}$ and weights Θ, the dot product remains unchanged

$$\Theta^T \cdot \mathbf{x} = \Theta[\text{perm}]^T \cdot \mathbf{x}[\text{perm}]]$$

So shuffling inputs to a FC layer does not affect its outputs

- assuming they are shuffled the same way during training and inference

But for certain types of inputs (e.g. images) it is easy to imagine that spatial locality is important.

- Consider a 2D pixel grid depicting a face
- The relative ordering of pixels may be what **defines** a pattern to be recognized
  - The relative location of the pixels within the left eye are important
  - The relative location of the pixels constituting the left eye, right eye, nose and mouth are important

By using a small pattern (and restricting connectivity) we **emphasize the relative locations** of elements

- neighboring elements more important than far away elements.

The "spatial" dimension implies an ordering of elements

- but the ordering does not have to be in space
    - e.g., can be ordered in time

Consider the time series of prices of a single ticker over d days.

Two representations

- $(d \times 1)$: 1 feature ("price") over d spatial ("date") locations
- $(1 \times d)$: 1 ticker with d features $(\text{price } 1, \ldots, \text{price } d)$

The choice of where the singleton dimension appears is sometimes a matter of interpretation.

- but the last index always denotes the feature dimension

Mathematically, the One Dimensional Convolutional Layer (Conv1d) we have shown computes $\mathbf{y}_{(l)}$

$$\mathbf{y}_{(l),1} = \begin{pmatrix} a_{(l)} \left(\; N(\mathbf{y}_{(l-1)}, \mathbf{W}_{(l),1}, 1) \cdot \mathbf{W}_{(l),1} \; \right) \\ a_{(l)} \left(\; N(\mathbf{y}_{(l-1)}, \mathbf{W}_{(l),1}, 2) \cdot \mathbf{W}_{(l),1} \; \right) \\ \vdots \\ a_{(l)} \left(\; N(\mathbf{y}_{(l-1)}, \mathbf{W}_{(l),1}, d_{(l-1)} \cdot \mathbf{W}_{(l),1} \; \right) \end{pmatrix}$$

where $N(\; \mathbf{y}_{(l-1)}, \mathbf{W}_{(l),1}, j \;)$

- selects a subsequence of $\mathbf{y}_{(l-1),\ldots,1}$ centered at $\mathbf{y}_{(l-1),j,1}$
    - Note the extra spatial dimension in the subscripting; "$\ldots$" denotes the full spatial dimension
    - Centered at the j^{th} element in the spatial dimension of feature 1 of layer $(l-1)$

Note that

- The *same* weight matrix $\mathbf{W}_{(l),1}$ is used for the first feature at *all* locations j
- The size of $\mathbf{W}_{(l),1}$ is the same as the size of the subsequence $N(\; \mathbf{y}_{(l-1)}, \mathbf{W}_{(l),1}, j)$
    - Since dot product is element-wise multiplication
- The spatial dimension $d_{(l)}$ of $\mathbf{y}_{(l),1}$ is equal to $d_{(l-1)}$

Kernel, Filter

The vector $\mathbf{W}_{(l),1}$ above

- Is a smaller pattern
- That is applied to *each* spatial location j in $\mathbf{y}_{(l-1)}$
- $\mathbf{y}_{(l),j,1}$ recognizes the match/non-match of the smaller first feature at the spatial locations centered at $\mathbf{y}_{(l-1),j,1}$

$\mathbf{W}_{(l),1}$ is called the first convolutional *filter* or *kernel*

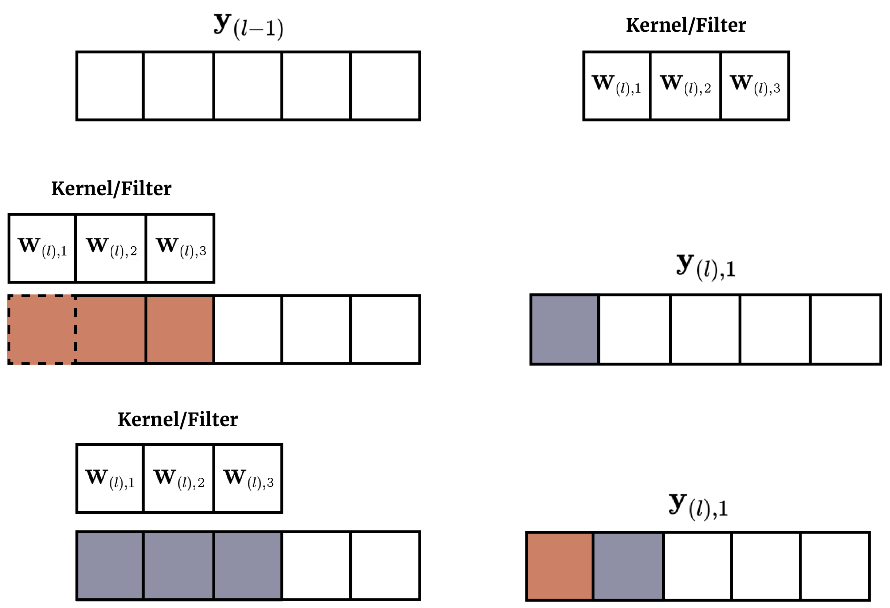
- We will often denote it $\mathbf{k}_{(l),1}$
- But it is just a part of the weights $\mathbf{W}$ of the multi-layer NN.
- We use $f_{(l)}$ to denote the size of the smaller pattern called the *filter size*

A *Convolution* is often depicted as

- A filter/kernel
- That is slid over each location in the input
- Producing a corresponding output for that location

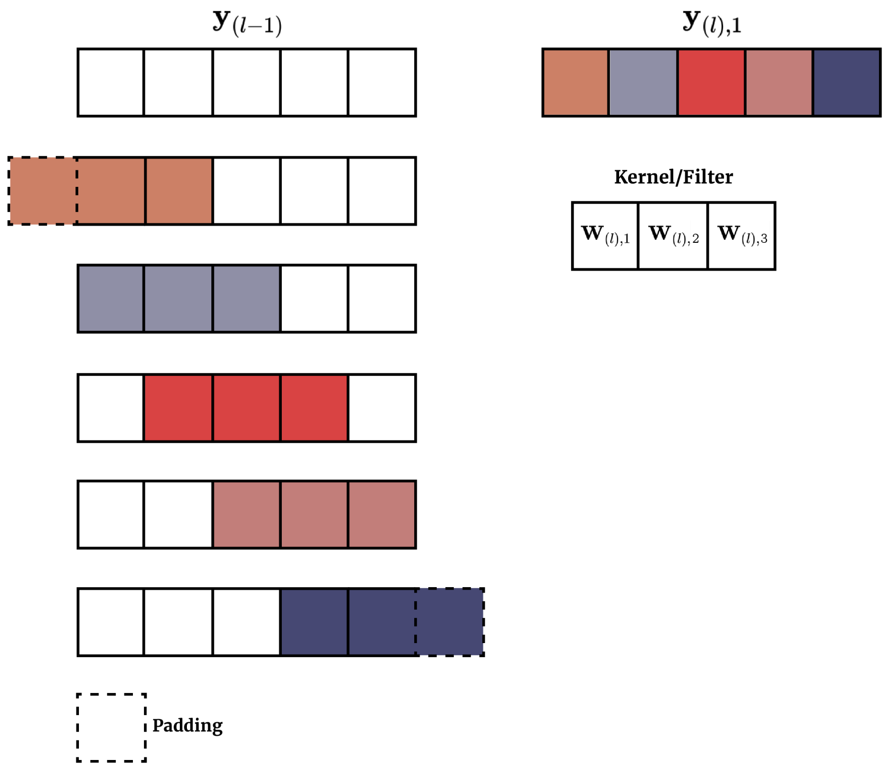
Here's a picture with a kernel of size $f_{(l)} = 3$

Conv 1D, single feature: sliding the filter



After sliding the Kernel over the whole $\mathbf{y}_{(l-1)}$ we get:

Conv 1D, single feature: output feature map



Element j of output $\mathbf{y}_{(l),\ldots,1}$ (i.e., $\mathbf{y}_{(l),j,1}$)

- Is colored (e.g., $j = 1$ is colored Red)
- Is computed by applying the *same* $\mathbf{W}_{(l),1}$ to
    - The $f_{(l)}$ elements of $\mathbf{y}_{(l-1),1}$, centered at $\mathbf{y}_{(l-1),j,1}$
    - Which have the same color as the output

Note however that, at the "ends" of $\mathbf{y}_{(l-1)}$ the kernel may extend beyond the input vector.

In that case $\mathbf{y}_{(l-1)}$ may be extended with *padding* (elements with 0 value typically)

Activation of a CNN layer

Just like the Fully Connected layer, a CNN layer is usually paired with an activation.

The default activation $a_{(l)}$ in Keras is "linear"

- That is: it returns the dot product input unchanged
- Always know what is the default activation for a layer; better yet: always specify !

Conv2d in action

We have thus far depicted a spatial dimension of length 1.

We can easily expand this into 2 spatial dimensions

- each feature map is 2 dimensional
- each location in the feature map corresponds to a position in a 2D grid

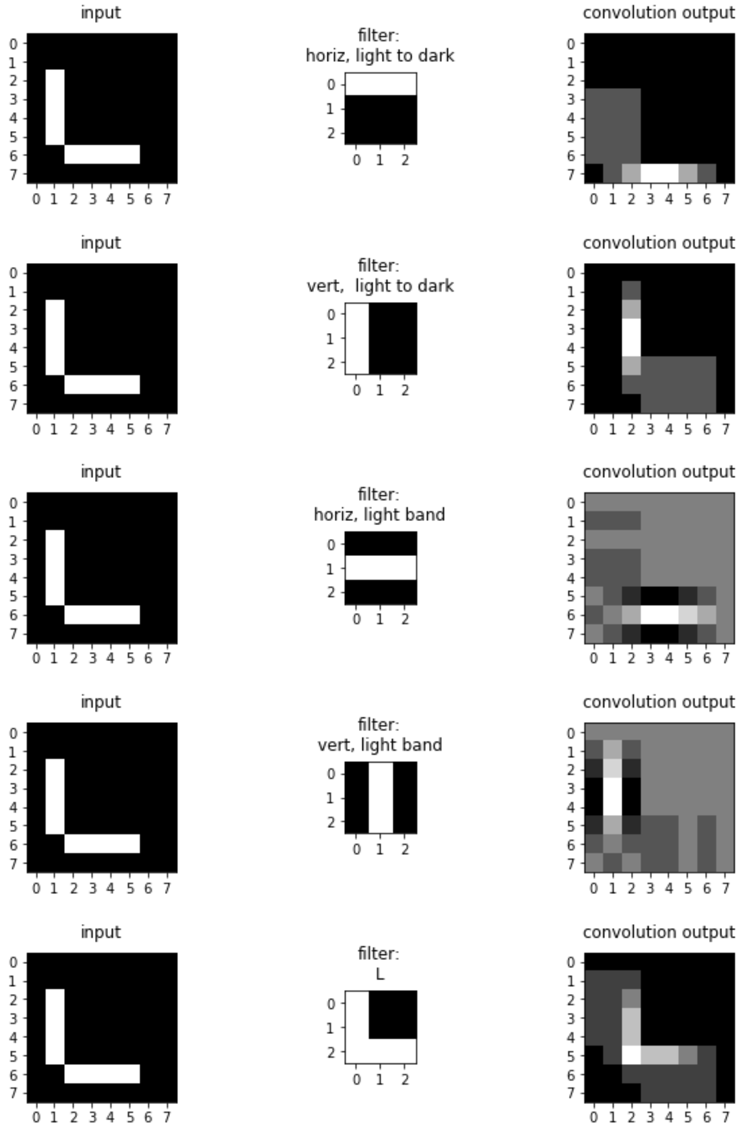
Pre-Deep Learning: manually specified filters have a rich history for image recognition.

Here is a list of manually constructed kernels (templates) that have proven useful

- list of filter matrices (https://en.wikipedia.org/wiki/Kernel_(image_processing))

Let's see some in action to get a better intuition.

In [4]:
```
_= cnnh.plot_convs()
```



- A bright element in the output indicates a high, positive dot product
- A dark element in the output indicates a low (or highly negative) dot product

In our example

- $N = 2$: Two spatial dimensions
- One input feature: $n_{(l-1)} = 1$
- One output feature $n_{(l)} = 1$
- $f_{(l)} = 3$
    - Kernel is $(3 \times 3 \times 1)$.

The template match will be maximized when

- high values in the input correspond to high values in the matching location of the template
- low values in the input correspond to low values in the matching locations of the template

We can have "spatial" dimensions of length even greater than 2

When we want to emphasize the number of features $n_{(l)}$ rather than the number of spatial dimensions, we will use ellipsis (dots)

$$\mathbf{y}_{\ldots,n_{(l)}}$$

where the ellipsis $(\ldots)$ is a place-holder for the spatial dimension shape.

In [5]:
```
print("Done")
```

```
Done
```