Numerical Analysis (CS 450)

Homework Set 5, Bill Karr

Problem 1: Interpolation, Newton and Cubic Splines (20 points)

(a) (9 points) Prove that the formula using divided differences:

$$f[t_1, t_2, ..., t_k] := \frac{f[t_2, t_3, ..., t_k] - f[t_1, t_2, ..., t_{k-1}]}{t_k - t_1}$$
$$f[t_j] := f(t_j)$$

indeed gives the coefficient of the jth basis function in the Newton interpolation polynomial.

Proof. Let p_n denote the polynomial that interpolates f at $t_1, ..., t_n$ for $n = 1, 2, ...$ Let q denote the polynomial that interpolates f at $t_2, ..., t_n$. Then,

$$p_n(x) = q(x) + \frac{x - t_n}{t_n - t_1}(q(x) - p_{n-1}(x)).$$

To see this, just evaluate both sides at t_k. If $k = n$, we obtain

$$f(t_n) = p(t_n) = q(t_n) + 0 = f(t_n).$$

If $k = 1$, we obtain $q(t_1) - (q(t_1) - p_{n-1}(t_1)) = p_{n-1}(t_1) = f(t_1) = p_n(t_n)$. Otherwise, $q(t_k) = p_{n-1}(t_k) = f(t_k)$ and the equation holds at all n points. Thus, the polynomials are equal on both sides.

The statement we want to prove is trivially true for degree zero polynomials. Suppose it holds for polynomials of degree $0, 1, ..., n-1$. I played around with this for a while and I'm not sure where to go from here. □

(b) (6 points) Given the three data points $(-1, 1), (0, 0), (1, 1)$, determine the interpolating polynomial of degree two using the monomial, Lagrange, and Newton bases. Show that the three representations give the same polynomial.

Solution & Proof. For the monomial basis, we have $m_0(t) = 1, m_1(t) = t, m_2(t) = t^2$. Thus, we can set up the system of equations

$$\begin{aligned} a \cdot 1 + b \cdot (-1) + c \cdot (-1)^2 &= a - b + c = 1 \\ a \cdot 1 + b \cdot (0) + c \cdot (0)^2 &= a = 0 \\ a \cdot 1 + b \cdot (1) + c \cdot (1)^2 &= a + b + c = 1 \end{aligned}$$

to obtain $a = 0, b = 0, c = 1$. Thus, the interpolating polynomial in the monomial basis is

$$p(t) = 0 \cdot 1 + 0 \cdot t + 1 \cdot t^2 = t^2.$$

For the Lagrange basis, we have

$$\begin{aligned} \ell_0(t) &= \frac{(t-0)(t-1)}{(-1-0)(-1-1)} = \frac{t^2 - t}{2} \\ \ell_1(t) &= \frac{(t-(-1))(t-1)}{(0-(-1))(0-1)} = \frac{t^2 - 1}{-1} \\ \ell_2(t) &= \frac{(t-(-1))(t-0)}{(1-(-1))(1-0)} = \frac{t^2 + t}{2} \end{aligned}$$

and the interpolating polynomial is given by

$$p(t) = 1 \cdot \ell_0(t) + 0 \cdot \ell_1(t) + 1 \cdot \ell_2(t) = \frac{t^2 - t}{2} + \frac{t^2 + t}{2} = t^2.$$

For the Newton basis, we have $n_0(t) = 1, n_1(x) = t - (-1) = t + 1, n_2(t) = (t+1)t = t^2 + t$. We conclude the divided differences and obtain

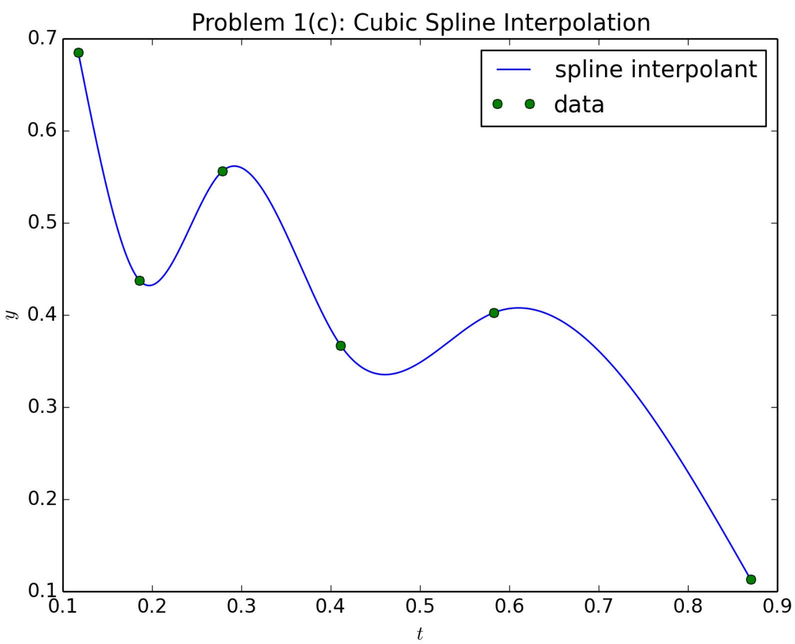
$$y[-1] = y(-1) = 1, \quad y[-1,0] = \frac{0-1}{0-(-1)} = -1, \quad y[-1,0,1] = \frac{y[1,0] - y[0,-1]}{1-(-1)} = \frac{1-(-1)}{2} = 1.$$

Thus, Newton interpolation gives

$$p(t) = 1 \cdot 1 + (-1) \cdot (t+1) + 1 \cdot (t^2 + t) = t^2$$

For each method, we obtain the same interpolating polynomial. $\square$

(c) (5 points) See `splines.py` for my code. Here is the plot I obtained.



Problem 2: Numerical Quadrature (25 points)

See `numerical_quad.py` for my code.

(i) My approximations were:

```
For Midpoint rule:
Approximations of pi:
h = 2**-10, pi = 3.141592733218101, EOC = none
h = 2**-11, pi = 3.141592673477424, EOC = 2.00140996086
h = 2**-12, pi = 3.141592658559272, EOC = 2.00070491332
h = 2**-13, pi = 3.141592654831860, EOC = 2.00035226347
h = 2**-14, pi = 3.141592653900270, EOC = 2.00018416024
h = 2**-15, pi = 3.141592653667407, EOC = 2.00009079913
```

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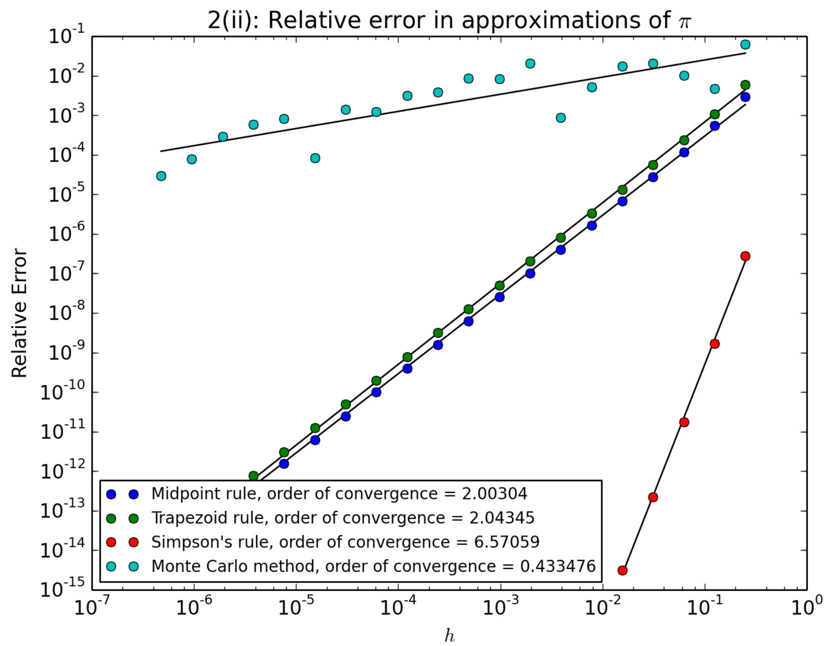
h = 2**-16, pi = 3.141592653609198, EOC = 1.9999339637
h = 2**-17, pi = 3.141592653594642, EOC = 2.0007596
h = 2**-18, pi = 3.141592653591006, EOC = 1.99920738298
h = 2**-19, pi = 3.141592653590098, EOC = 1.99314883259
h = 2**-20, pi = 3.141592653589872, EOC = 1.94633133521

For Trapezoid rule:
Approximations of pi:
h = 2**-10, pi = 3.141592494333178, EOC = none
h = 2**-11, pi = 3.141592613814530, EOC = 2.00140992462
h = 2**-12, pi = 3.141592643650833, EOC = 2.00070468768
h = 2**-13, pi = 3.141592651105661, EOC = 2.00035277923
h = 2**-14, pi = 3.141592652968836, EOC = 2.00017771168
h = 2**-15, pi = 3.141592653434568, EOC = 2.00012588161
h = 2**-16, pi = 3.141592653550985, EOC = 1.99993808964
h = 2**-17, pi = 3.141592653580094, EOC = 2.00046232817
h = 2**-18, pi = 3.141592653587369, EOC = 2.00026425406
h = 2**-19, pi = 3.141592653589195, EOC = 2.01995828714
h = 2**-20, pi = 3.141592653589640, EOC = 1.96819793991

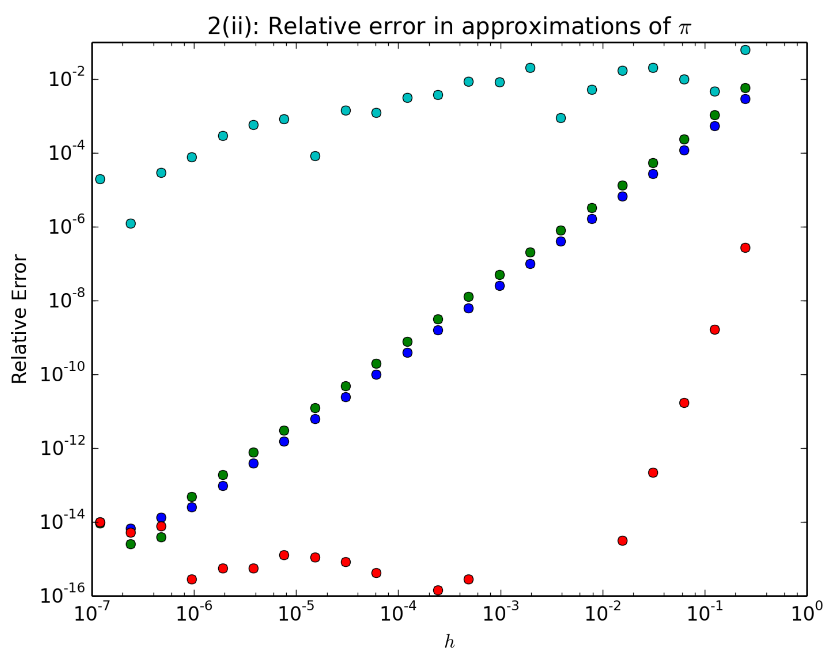
For Simpson's rule:
Approximations of pi:
h = 2**-2, pi = 3.141591780936043, EOC = none
h = 2**-3, pi = 3.141592648320655, EOC = 7.37169855865
h = 2**-4, pi = 3.141592653535359, EOC = 6.59692064869
h = 2**-5, pi = 3.141592653589095, EOC = 6.2839905107
h = 2**-6, pi = 3.141592653589783, EOC = 6.15987133678

For Monte Carlo method:
Approximations of pi:
h = 2**-2, pi = 3.336361607500348, EOC = none
h = 2**-3, pi = 3.126671118885677, EOC = 3.70629589889
h = 2**-4, pi = 3.173248073913034, EOC = -1.08505662515
h = 2**-5, pi = 3.205488380443046, EOC = -1.01326689955
h = 2**-6, pi = 3.087227368889564, EOC = 0.233033748107
h = 2**-7, pi = 3.158107112667869, EOC = 1.71895598763
h = 2**-8, pi = 3.138802232736366, EOC = 2.56517508377
h = 2**-9, pi = 3.077842286669497, EOC = -4.51387901358
h = 2**-10, pi = 3.167965867109579, EOC = 1.27336027432
h = 2**-11, pi = 3.168738485529880, EOC = -0.041657327689
h = 2**-12, pi = 3.153660735662613, EOC = 1.16953428578
h = 2**-13, pi = 3.131703274208692, EOC = 0.287244522115
h = 2**-14, pi = 3.137751019590208, EOC = 1.3641599089
h = 2**-15, pi = 3.146065699961296, EOC = -0.219537637645
h = 2**-16, pi = 3.141329062869102, EOC = 4.08488622618
h = 2**-17, pi = 3.139000109511650, EOC = -3.29799702931
h = 2**-18, pi = 3.143423662383446, EOC = 0.501729797907
h = 2**-19, pi = 3.142512928700046, EOC = 0.992501604829
h = 2**-20, pi = 3.141344166315301, EOC = 1.88889323942
h = 2**-21, pi = 3.141500851323004, EOC = 1.43657028818
```

(ii) See above for my empirical order of convergence values. Here is my plot of the relative errors with best fit curves for the four methods:



(iii) When I push my code to the limit with $h \approx 0$, the results stop improving in accuracy once the relative error gets close to machine precision. Below is a graph showing this. I omitted the legend, but the color coding is the same as above.



Problem 3: Gaussian Quadrature (20 points)

(a) Let p be a real polynomial of degree n such that:

$$\int_a^b p(x)x^k \, dx = 0, \quad k = 0, 1, ..., n-1.$$

(i) Show that the n zeros of p are real, simple, and lie on the open interval (a, b).

Proof. First, note that any root of $p(x)$ of odd order must be real. Let $q(x) = (x - x_1) \cdots (x - x_k)$ where $x_1, ..., x_k$ the distinct real odd-order roots of $p(x)$ (note: $k \leq n$). Then, $p(x)q(x)$ does not change sign on $[a, b]$. Thus, $\int_a^b p(x)q(x)\, dx \neq 0$ and $\deg(q(x)) = k \geq n \Rightarrow k = n$. Since $p(x)$ has degree n and $x_1, ..., x_n$ are real roots, it must be that they are all of the roots of p and must be simple. □

(ii) Show that the n-point interpolatory quadrature on $[a, b]$ whose nodes are the zeros of p has degree $2n - 1$.

Proof. Denote $Q(f) = \sum_{k=1}^n w_k f(x_k)$ as the interpolatory quadrature of a function f using the roots of p as nodes where w_j are the weights.

Consider a polynomial $f(x)$ of degree $\leq 2n - 1$. Perform polynomial long division to obtain $f(x) = p(x)q(x) + r(x)$ where p, r have degree $\leq n - 1$. We know interpolatory quadrature is exact for polynomials $< n$ since we have n nodes. Using the fact that p is orthogonal to q (by definition, p is orthogonal to any polynomial of degree $< n$),

$$\int_a^b f(x)\, dx = \int_a^b p(x)q(x) + r(x)\, dx = \int_a^b p(x)q(x)\, dx + \int_a^b r(x)\, dx = \int_a^b r(x)\, dx = Q(r).$$

However, since $p(x_k) = 0$,

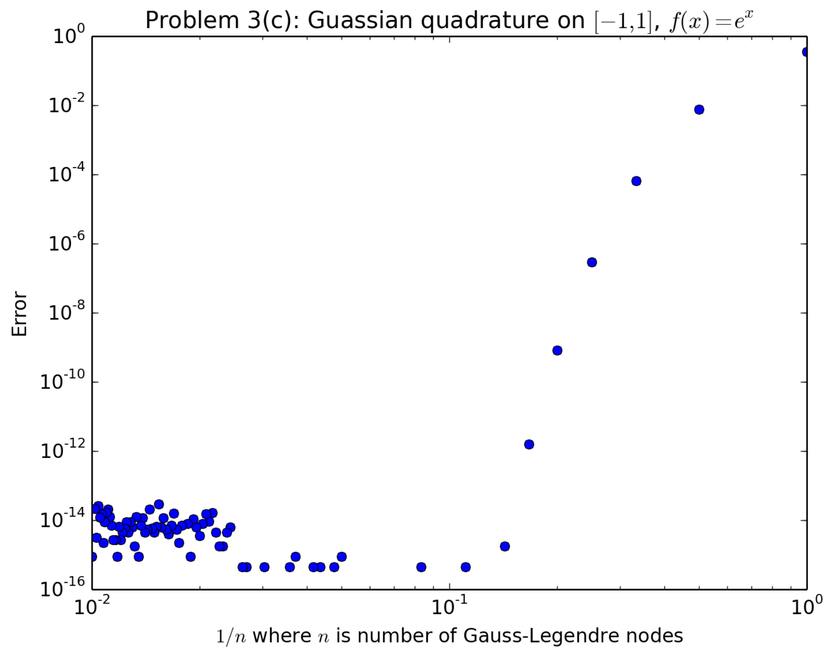
$$Q(f) = \sum_{k=1}^n w_k(p(x_k)q(x_k) + r(x_k)) = \sum_{k=1}^n w_k r(x_k) = Q(r) = \int_a^b f(x)\, dx.$$

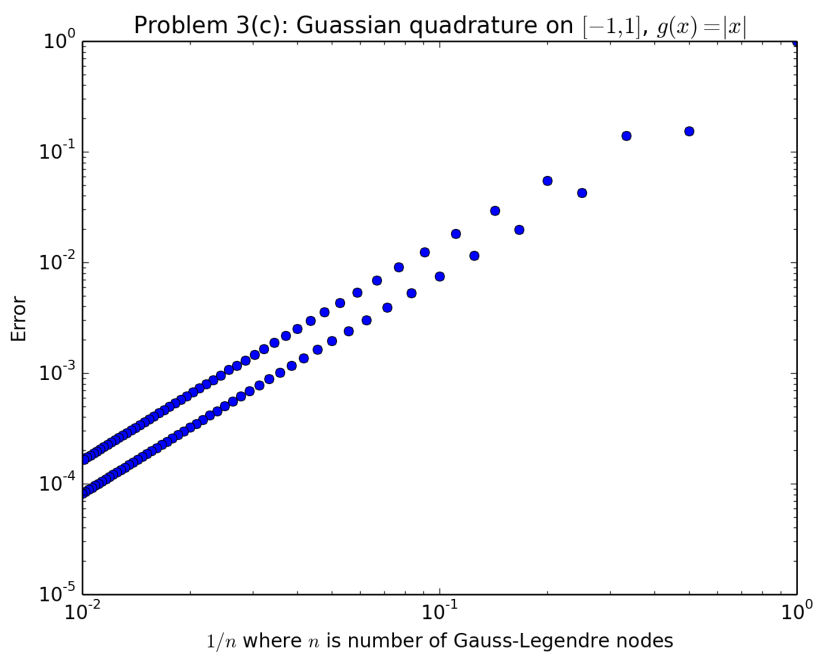
Thus, Q is exact for f, i.e. any polynomial of degree $\leq 2n - 1$. □

(b) See `guassian_quad.py` for my code.

(c) The first function does not obey $E(h) \approx Ch^p$, but the expected error does get small quickly as $h \to 0$. The latter obeys $E(h) \approx Ch^2$, although it changes C value depending on whether $n = 1/h$ is odd or even. This is because the Gauss-Legendre nodes are distributed in different ways for even and odd numbers.

Here are the plots of the errors for various h values of the two functions.


Problem 3(c): Guassian quadrature on $[-1,1]$, $f(x) = e^x$
Error
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
10^{-16}
10^{-2}
10^{-1}
10^{0}
$1/n$ where n is number of Gauss-Legendre nodes


Problem 3(c): Guassian quadrature on $[-1,1]$, $g(x) = |x|$
Error
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-2}
10^{-1}
10^{0}
$1/n$ where n is number of Gauss-Legendre nodes

Problem 4: Numerical Differentiation (15 points)

(a) Note that

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + O(h^3)$$

and thus,

$$f(x+2h) = f(x) + 2hf'(x) + 2h^2 f''(x) + O(h^3).$$

In order to get the $f(x)$ and $f''(x)$ terms to cancel, we can take $-3f(x)+4f(x+h)-f(x+2h)$ and obtain

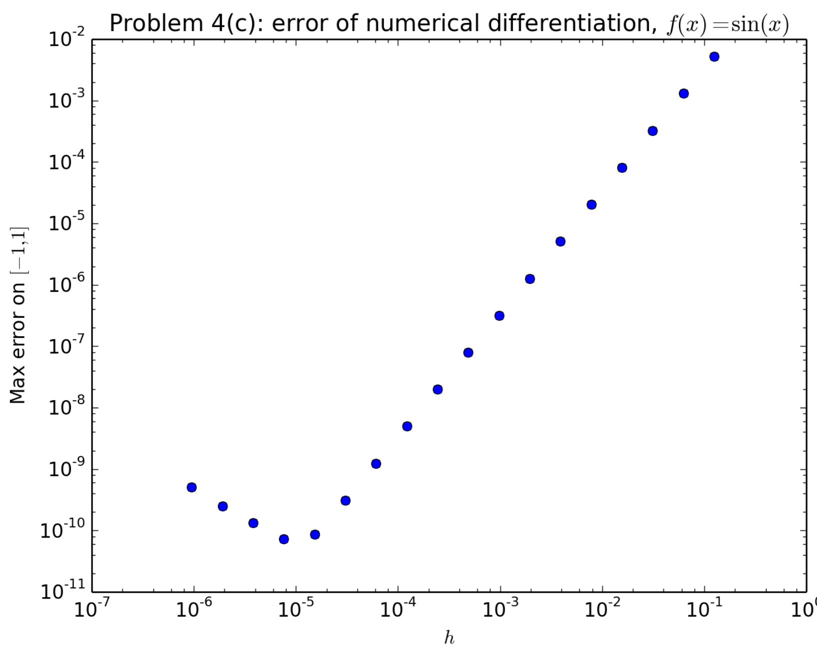
$$-3f(x)+4f(x+h)-f(x+2h) = 2hf'(x) + O(h^3).$$

Solving for $f'(x)$, we obtain

$$f'(x) = \frac{-3f(x)+4f(x+h)-f(x+2h)}{2h} + O(h^2).$$

(b) See `differentiation.py` for my code.

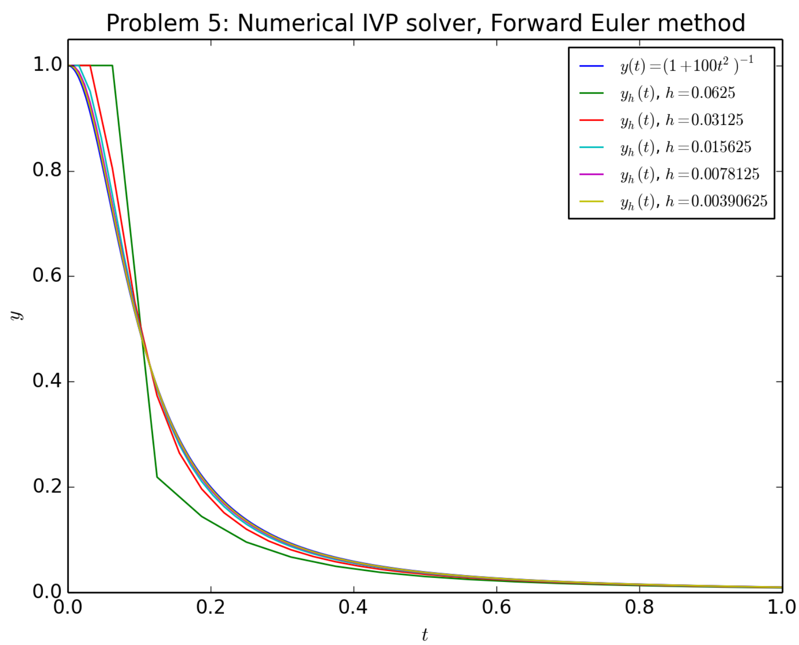
(c) The expected order of convergence is 2 by the above formula. I observe a convergence rate of 2 until $h \approx 10^{-5}$. Then the error starts increasing. This is likely due to errors in the way `numpy` computes $\cos(x)$ and $\sin(x)$ as well as computational errors. Here is my plot:

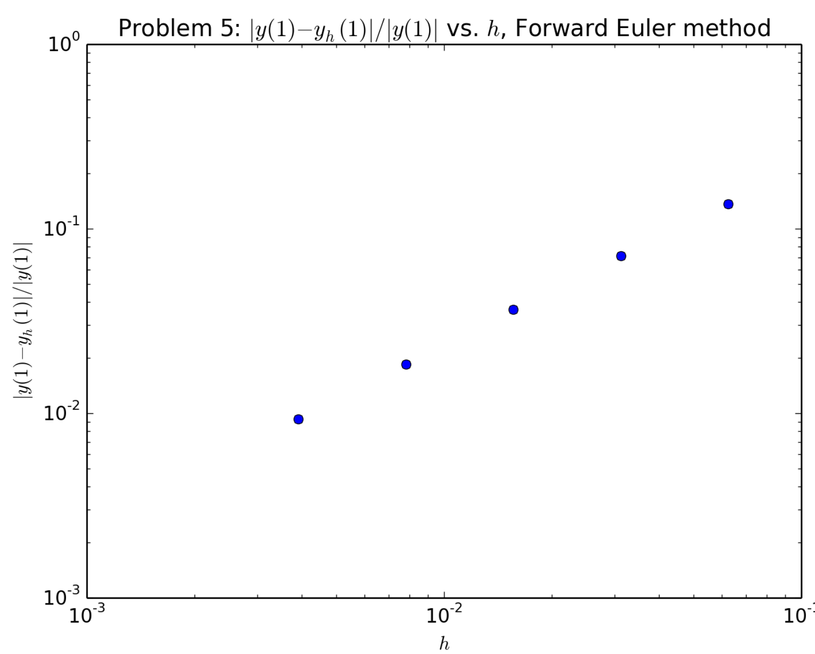


Problem 5: Initial Value Problems (20 points)

See `ivps.py` for my code.

(a) For Forward Euler method, Here are my plots.

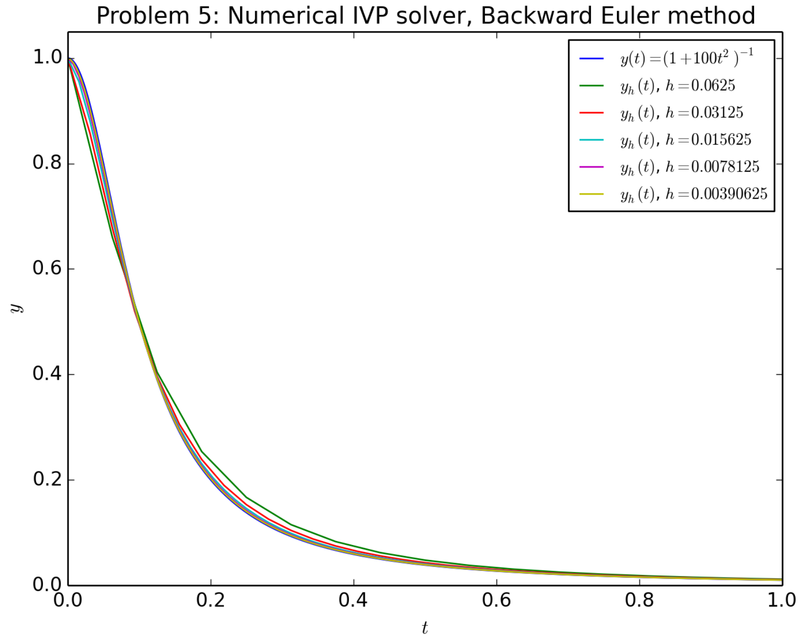


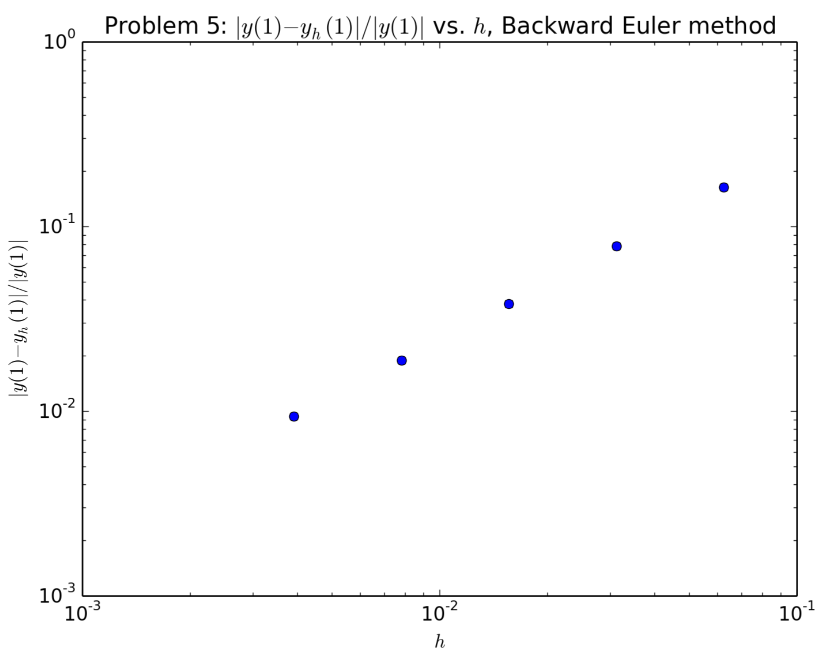


E.O.C. values: 0.93459598, 0.96506194, 0.98311564, 0.99166013.

Looks stable! As $h \to 0$, the approximate solutions are close to the true solution.

(b) For Backward Euler method, here are my plots.

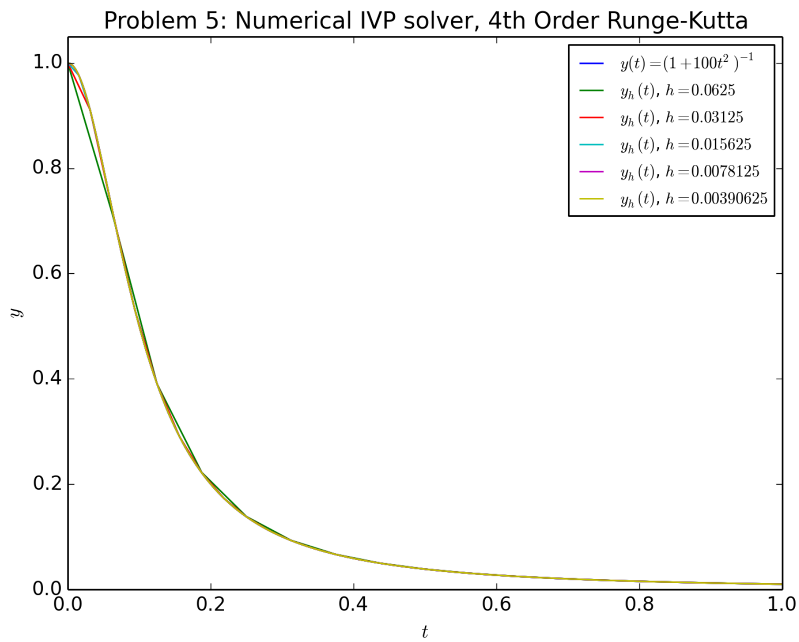


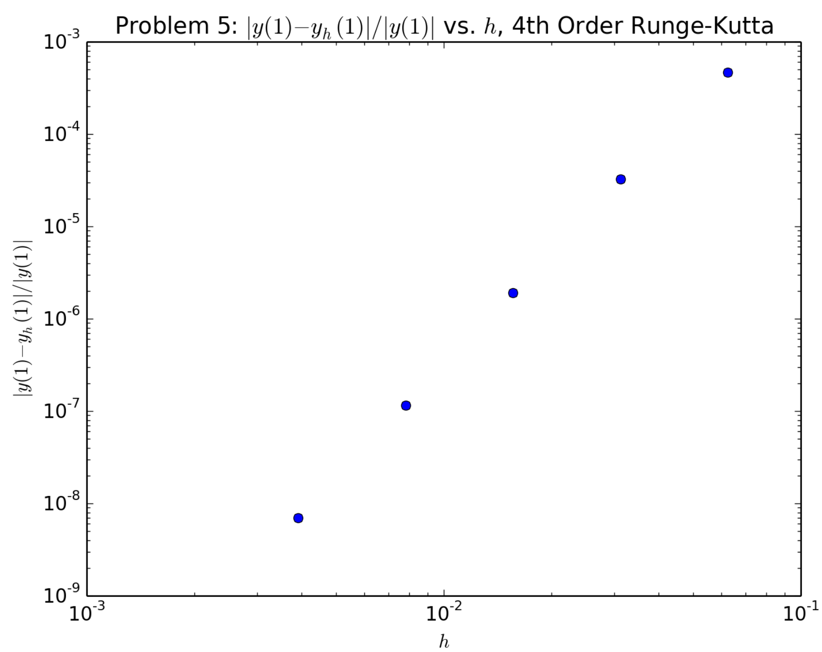


E.O.C. values: 1.06632339, 1.0327388, 1.01636717, 1.0082122.

Looks stable! As $h \to 0$, the approximate solutions are close to the true solution.

(c) For Fourth-order Runge-Kutta method, here are my plots.





E.O.C. values: 3.83391337, 4.09025602, 4.06263359, 4.03447133.

Looks stable! As $h \to 0$, the approximate solutions are close to the true solution.