# **Computer Graphic Note**

Jiechang Shi

Version: 1.1

Update: November 3, 2019

## 1 Class 1: Overview

- 1. Frame Buffer
  - Pixel: One element of a frame buffer
  - Pixel depth: Number of bytes per-pixel in the buffer
  - Resolution: Width × Height
  - Buffer size: Total memory allocated for frame buffer
  - Exam Question: Given Resolution and Pixel depth, Asked Buffer size
  - Z Value: For solving the hidden-surface removal(HSR) problem

## 2 Class 2,3: Rasterization

- 1. For each pixel on screen, the sample point of that pixel is the center point of that pixel, which have **integer** coordinates. For examples, (3, 2).
- 2. A simple problem: Rasterizing Lines

**Program Description**: Given two endpoints,  $P = (x_0, y_0)$ ,  $R = (x_1, y_1)$  find the pixels that make up the line.

Note that: Lines are infinitely thin so they rarely fall on pixel sample point.

A Feasible Description: Rasterize lines as closest pixels to actual lines, with 2 requirement

- No Gap
- Minimize error(distance to true line)

To make this question simplify: Only consider situation that  $|x_1 - x_0| \ge |y_1 - y_0| \ge 0 \land |x_1 - x_0| \ne 0$ , which means the  $-1 \le slope \le 1$ . Otherwise we just exchange x and y.

**A basic Algorithm**:  $k = \frac{y_1 - y_0}{x_1 - x_0}$ ,  $d = y_0 - kx_0$ , for each  $x_0 \le x \le x_1$ , y = ROUND(kx + d). This method by brute force is inefficient because of the multiplication and the function ROUND().

**Basic Incremental Algorithm**: for each  $x_0 \le x_i < x_{i+1} \le x_1$ ,  $y_{i+1} = y_i + k$ , However, the successsive addition of a real number can lead to a **cumulative error buildup**!

#### Midpoint Line Algorithm:

- For  $0 \le k \le 1$
- For one approximate point P = (x, y), we only have 2 choices for the next point E = (x + 1, y) and NE = (x + 1, y + 1), we should choose the one which is closer to k(x + 1) + d
- Calculate the middle point  $M = (x + 1, y + \frac{1}{2})$
- If the **Intersection point** Q is below M, take E as next, otherwise take NE as next.
- Note that: we consider this equation:

$$f(x, y) = ax + by + c = (y_1 - y_0)x - (x_1 - x_0)y + (x_1y_0 - y_1x_0)$$

We assume a > 0

• For a point (x, y)

if f(x, y) = 0, (x, y) lies on the line.

if f(x, y) < 0, (x, y) lies upon the line.

if f(x, y) > 0, (x, y) lies below the line.

- So we have to test  $f(M) = a(x+1) + b(y+\frac{1}{2}) + c = f(Former) + a + \frac{b}{2}$
- Assum a > 0, if f(M) > 0 choose NE otherwise choose E
- Update f(Former):

If we choose E, f(Former) = f(Former) + a

If we choose NE, f(Former) = f(Former) + a + b

Note that: a and b are constant integer, so here is no cumulative error issuse

3. A harder problem: Triangles Rasterization

### Why Triangle:

- Triangles (tris) are a simple explicit 3D surface representation.
- Convex and concave polygons (polys) can be decomposed into triangles.
- Tris are planar and unambiguously defined by three vertex(verts) coordinates (coords).

**Definition**: Find and draw pixel samples inside tri edges and interpolate parameters defined at verts

- 4. Rasterizxation and Hidden Surface Removal(HSR) Algorithm Classes:
  - Image order rasterization: ray tracing/ ray casting traverse pixel, process each in world-space

transform rays from image-space to world-space

Object order rasterization: scan-line / LEE
 <u>traverse</u> triangles, process each in image-space
 transform objects from model-space to image-space

#### 5. LEE Linear Expression Evaluation Algorithm:

• We already discussed in *Midpoint Line Algorithm* that how to determine a point is on the left(up) or right(below) the line, just *a quick review here*:

Assume the lien have a positive slope

For an Edge Equation E, for point (x, y): E(x, y) = dY(x - X) - dX(y - Y)

if E(x, y) = 0, (x, y) lies on the line.

if E(x, y) < 0, (x, y) lies right(below) the line.

if E(x, y) > 0, (x, y) lies left(up) the line.

#### • For **Rasterization**:

Compute LEE result for all three edges.

Pixels with **consistent sign** for all three edges are inside the *tri*.

Include edge pixels on left or right edges.

- LEE need to check every pixel in the bounding box.
- LEE is very good in parallel(SIMD) system.
- Furthermore: Given 3 random *verts* how to find CW edge cycle Determine L/R and Top/Bot edges for edge-pixel ownership

#### 6. Scan Line Rasterizer:

- Sort vets by Y
- Setup edge DDAs for edges
- Sort edges by L or R(The long edge on left or right)
- Start from Top Vertice, and switch DDA when hit the middle vertice.

#### 7. Interpolate Z:

A general 3D plane equation has 4 terms: Ax + By + Cz + D = 0

(A, B, C) is the normal of that plane, so  $(X, Y, Z)_0 \times (X, Y, Z)_1 = (A, B, C)$ 

Then plug any vertex coord into equation and solve for D.

Given (A, B, C, D) and any point (x, y) can solve z

#### 8. Used *Z*-buffer to remove hidden surfaces

Initial Z-buffer to MAXINT at the begining of every frame

Interpolate vertex Z values to get  $Z_{pix}$ 

Only write new pixel to the buffer if  $Z_{pix} < Z_{buffer}$ 

Notice that Z should always **bigger or equal to zero!** 

#### 9. Hidden Line Removal(HLR):

Simple z-buffer does not work when the render only draws edge(outlines of polygons).

Need edge-crossing and object sorting methods.

### 10. Painter's Algorithm: render in order front to back

A object is in front of another object means:

Z of all verts of one object is less than the other.

This algorithm not work if Z-sort is ambiguous.

#### 11. Warnock Algorithm:

Subdivide screen untril a leaf region has a simple front/back relationship.

Leaf regions have one or zero surfaces visible, and the smallest region is usually a pixel

Usually use quad tree subdivision

#### 12. BSP-Tree:

View-Independent binary tree(pre-calculated) allows a view-dependent front-to-back or back-to-front traversal of surfaces.

Use Painter Algorithm to do back-to-front traversal.

Useful for **transparency** - full depth-sort of all surface.

#### 13. Culling:

- Culling with portals: pre-compute the invisible part.
- Culling by View Frustum: Skip a triangle iff all its vertices are beyond the same screen edge!
   Pitfall: If the vertices are beyond different edge, some part of the *tri* might still in the screen. Image a giant *tri* that cover the whole screen.
- Backface Culling: For **closed(water-tight)** objects, surfaces with **oriented-normals facing away** from the camera are never visible.

Pitfall: BF Culling only work for water-tight object!

• Frustum: Only visible triangles are drawn into the frame buffer.

## 3 Class 4,5,6,7:Transformations

• Linear transformations (*Xforms*) define a mapping of coordinates (*coords*) in one coordinate frame to another.

$$V_b = X_{ba}V_a$$

- **Homogeneous Vector** (V) is  $4 \times 1$  columns  $(x, y, z, w)^T$
- **Homogeneous Transforms** (X) is  $4 \times 4$  matrix
- From Homogeneous Vector to 3D Vector:

$$x = \frac{x}{w}, y = \frac{y}{w}, z = \frac{z}{w}$$

• Why we use Homogeneous Vector?

We want to uniform the transform matrix including translation, scaling, rotation in the same form of matrix.

#### 3.1 Transformation Matrix

1. Translation:

$$T(t_x, t_y, t_z) \Rightarrow \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}, T^{-1}(t_x, t_y, t_z) = T(-t_x, -t_y, -t_z)$$

2. Scaling:

$$S(s_x, s_y, s_z) \Rightarrow \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, S^{-1}(s_x, s_y, s_z) = S(\frac{1}{s_x}, \frac{1}{s_y}, \frac{1}{s_z})$$

3. Rotation, CCW:

$$R_{x}(\theta) \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, R_{x}^{-1}(\theta) = R_{x}^{T}(\theta)$$

$$R_{y}(\theta) \Rightarrow \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, R_{y}^{-1}(\theta) = R_{y}^{T}(\theta)$$

$$R_{z}(\theta) \Rightarrow \begin{bmatrix} cos\theta & -sin\theta & 0 & 0 \\ sin\theta & cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, R_{z}^{-1}(\theta) = R_{z}^{T}(\theta)$$

4. Pitfall: commutative property is for S,R only.

Here assume uniform scaling in all dimensions.

If S is not an uniform scaling matrix. S,R don't have commutative property.

### 3.2 Spaces Transformation

#### 1. NDC to Output Device

$$X_{sp} \Rightarrow \begin{bmatrix} \frac{xs}{2} & 0 & 0 & \frac{xs}{2} \\ 0 & -\frac{ys}{2} & 0 & \frac{ys}{2} \\ 0 & 0 & MAXINT & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note that:

Output Device is **RH coords** and origin in **upper left**. $X \in [0, xs), Y \in [0, ys), Z \in [0, MAXINT]$ 

NDC is **LH coords** and origin at screen center.  $X, Y \in [-1, 1], Z \in [0, 1]$ 

### 2. Perspective Projection

$$X_{pi} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{d} & 0 \\ 0 & 0 & \frac{1}{d} & 1 \end{bmatrix}$$

**What** is d And **Why** there are two  $\frac{1}{d}$ ?

- Assume camera is on (0,0,-d), perspective(also image) plane is z=0, a object in world space is (X,Y,Z)
- Defined FOV(field of view) as the angle the camera can see.
- Note:  $X \in [-1, 1]$ , so the distant(d) from Forcus point to view plane can be calculate by this equation:

$$\frac{1}{d} = tan(\frac{FOV}{2})$$

• Futher More: The object project to view plane can be calculate by these equations:

$$\frac{X}{Z+d} = \frac{x}{d} \Rightarrow x = \frac{X}{\frac{Z}{d}+1}$$

$$\frac{Y}{Z+d} = \frac{y}{d} \Rightarrow y = \frac{Y}{\frac{Z}{d}+1}$$

$$\frac{Z}{Z+d} = \frac{z}{d} \Rightarrow z = \frac{Z}{\frac{Z}{d}+1}$$

$$(x,y,z) = (\frac{X}{\frac{Z}{d}+1}, \frac{Y}{\frac{Z}{d}+1}, \frac{Z}{\frac{Z}{d}+1})$$

• Write this 3D vector to Homogeneous Vector:

$$(x, y, z, w) = (X, Y, Z, \frac{Z}{d} + 1)$$

6

• Futher: We forcus on the **range** of Z now is  $z \in (-\infty, d)$ .

But in NDC we hope  $z \in [0, 1)$  So:

We delete all vector that Z < 0, because they cannot project to the view plane.

For  $z \ge 0$ , we define  $z' = \frac{z}{d}$ 

•  $(x, y, z', w) = (X, Y, \frac{Z}{d}, \frac{Z}{d} + 1) = X_{pi} * (X, Y, Z, 1)$ 

Pitfall: Do Z interpolation in Perspective Plane!

Why we need the farest plane?

Asymptotic curve of Z vs. z, that z increase slower when Z is large.

It might map different Z to the same z

## 3.3 Camera Matrix

- 1. Assume camera position is c, camera look-at point is l, here c and l are both in world coordinate. And the world up vector is  $\vec{up}$
- 2. Camera Z-axis in world coordinate is

$$\vec{Z} = \frac{\vec{cl}}{||\vec{cl}||}$$

3. Camera Y-axis in world coordinate is the orthogonal(vertical) part of world-up vector to Z-axis which is

$$\vec{up'} = \vec{up} - (\vec{up} \cdot \vec{Z})\vec{Z}$$

$$\vec{Y} = \frac{\vec{up'}}{||\vec{up'}||}$$

4. Camera X-axis in world coordinate is orthogonal to both Y and Z axises. So:

$$\vec{X} = \vec{Y} \times \vec{Z}$$

5. Build the  $X_{wi}$  from camera space to world space: X-axis vector [1,0,0] in camera space should be  $\vec{X}$  in world space, also for Y,Z-axis vectors, So:

$$X_{wi} \Rightarrow \begin{bmatrix} X_x & Y_x & Z_x & 0 \\ X_y & Y_y & Z_y & 0 \\ X_z & Y_z & Z_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

6. Also we need to add the translation of the camera to the Matrix:

$$X_{wi} \Rightarrow \begin{bmatrix} X_x & Y_x & Z_x & c_x \\ X_y & Y_y & Z_y & c_y \\ X_z & Y_z & Z_z & c_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

7. Now we can get the inverst matrix  $X_{iw}$ :

$$X_{iw} \Rightarrow egin{bmatrix} X_x & X_y & X_z & -X \cdot c \\ Y_x & Y_y & Y_z & -Y \cdot c \\ Z_x & Z_y & Z_z & -Z \cdot c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

8. In this proof we know that: If we know the X,Y,Z-axis in world coordinate for a specific space, we can easily build and inverst the translation from or to that space.

This method can also be used to **proof the general rotation matrix**.

9. Orbit a Model about a Point

The idea is the same as place camera.

Need to care about which space you current in!

## 4 Class8-10 Illumination and Shading

1. Global vs. Local Illumination

Global: Models indirect illumination and occlusions Local: Only models direct illumination

2. Irradiance

$$E = \int_{\Omega} I(x, \omega) cos\theta dx$$

Note:  $I(x,\omega)$  is the light intensity arriving from all directions and entering the hemisphere  $\Omega$  over unit serface area.

Also we only care the vertical(normal) part of the light, we dismiss all lights parallel to the surface by using  $\cos \theta$ 

3. Simplified lighting:

Assume all lights are distant-point light.

- Source have uniform intensity distribution
- Neglect distance fallout
- Direction to source is constant within scene
- Using 2 parameters to define a light:
   direction(x, y, z) vector from surface to light source
   intensity(r, g, b) of the light
- 4. specular reflection and diffuse reflection
  - Color shift by attenuation of RPG components for all reflection
  - Specular Reflection Model(View-Dependent):

$$L_{j}(V) = L_{e} \cdot K_{s} \cdot (V \cdot R)^{spec}$$

$$R = 2(N \cdot L)N - L$$

Note: *V* and *R* should be normalized.

Direction: Reflection occurs mainly in the "mirror" R direction, but there is some spread in similar directions V.

*spec* controls the distribution of intensity about R. Higher value of *spec* make the surface smoother  $K_s$  controls the color attenuation of Surface.

• Diffuse Reflection Models(View-Independent):

$$L_i = L_e \cdot K_d(L \cdot N)$$

Direction: All output directions are the same. But we only care vertical input light.

L and N should be normalized.

 $K_d$  is the surfface attenuation component.

Ambient Light

$$L_i = L_a \cdot K_a$$

Direction: All input and output directions are the same.

Only one ambient light is needed and allowed.

• Complete Shading Equation:

$$Color = (K_s \sum L_e \cdot (V \cdot R)^{spec}) + (K_d \sum L_e \cdot (L \cdot N)) + (L_a \cdot K_a)$$

5. Detail about HW4(Lighting Implementation)

- $\vec{L}$  denotes the direction to a infinity-far point-light source
- $\vec{E}$  denotes the camera direction. If camera is far away,  $\vec{E}$  is constant(In HW4.)
- $\vec{N}$  is specified at triangle vertices.
- $\vec{R}$  must be computed for each lighting calculation (at **a point**). Calculation of  $\vec{R}$ :

$$\vec{R} = 2(\vec{N} \cdot \vec{L})\vec{N} - \vec{L}$$

Avoiding sqrt-root in this calculation

• Choosing a Shading Space: Wee need all  $\vec{L}, \vec{E}, \vec{N}, \vec{R}$  in some affine(pre-perspective) space Suggest use **Image Space** for HW4.

**Model space** is also a reasonable choice since Normal vectors are already in that space. This is most **efficient**!

• Image Space Lighting (ISL)

Create a Transformation stack from model space to image space.

Need to normalized the Scale and delete translation for each matrix, only maintain the rotation, before push into this stack!

• Check the sign of  $\vec{N} \cdot \vec{E}$  and  $\vec{N} \cdot \vec{L}$ :

Both positive: Compute lighting model.

Both negative: **Flip normal**( $\vec{N}$ ) and compute lighting model.

Different sign: Skip it.

- Check the sign of  $\vec{R} \cdot \vec{E}$ : If negative, set to 0.
- Check color overflow(> 1.0): Set to 1.
- Compute Color at all pixels:

Per Face - flat shading

Per Vertex - interpolate vertex colors, Gouraud Shading(specular highlights are undersampled, aliased).

**Per Pixel** - interpolate normals, Phong Shading (Expensive computation, but better sampling)

Set Shading Modes Parameter for different lighting calculation.

• Pitfall in Phong Interpolation:

Need to **normalize** the interpolation normal vector.

### 4.1 Class10: Something More About Shading

1. Non-Uniform Scaling:

A non-uniform scaling alters the relationship between the surface orientation and the Normal Vector.

So we **cannot** use the same matrix M for transformation of the Normals and the vertex coordinates.

We can fix this by using a different transformation Q = f(M) for transforming the Normals.

2. How to create a matrix for Normals:

In HW4, We create a matrix dismiss all scale matrix.

For Detail:

As the definition of Normals:

$$\vec{N}^T \cdot \vec{P} = 0$$

After include the transform matrix:

$$(Q\vec{N})^T\cdot (M\vec{P})=0$$

By Definition of Matrix Multiplyer:

$$\vec{N}^T \cdot Q^T \cdot M \cdot \vec{P} = 0$$

Since we already know  $\vec{N}^T \cdot \vec{P} = 0$  we only need the inner part equal to identity matrix:

$$Q^T \cdot M = I, Q = (M^{-1})^T$$

Note that: If we only used uniform scaling: S = I after normalization.

If we compute Q for each M pushed on the  $X_{im}$  transform stack, the resulting  $X_n$  stack has Q and therefore allows non-uniform scaling.

3. Model Space Lighting(MSL):

Only need to transform Global lighting parameters once per models.

Also need to transform Eye/camera direction into model space.

## 5 Class 11-13: Texture Mapping

### 5.1 Screen-Space Parameter Interpolation

- 1. In Z-buffer interpolation, we know that linear interpolation for z is **wrong in image space**, we need to interpolate in **perspective space**.
- 2. Accurate interpolation of RGB color or Normal vectors should also take perspective into account. But we can ignore the color and normal interpolation error.
- 3. Interpolation for **Texture Function**: checkerboard Example: Using Linear Interpolation for u&v is also wrong!
- 4. How to compute perspective-correct interpolation of u, v at each pixel.
  - For each parameter P, we used  $P^s$  to denote the value in perspective space.
  - Note that: For Z interpolation  $V_z^s = \frac{V_z}{\frac{V_z}{T+1}} = \frac{V_z \cdot d}{V_z + d}$
  - Rescale  $V_z^s$  to  $V_z^s \in [0, Z_{max}]$

$$V_z^s = \frac{V_z \cdot d}{V_z + d} \cdot \left(\frac{Z_{max}}{d}\right) = \frac{V_z \cdot Z_{max}}{V_z + d}$$

• We can also get the invert equation:

$$V_z = \frac{V_z^s \cdot d}{Z_{max} - V_z^s}$$

• For parameter from image space to perspective space:

$$P^{s} = \frac{P}{\frac{V_{z}}{d} + 1} = \frac{Pd}{V_{z} + d}$$

• Also we can get inver equation:

$$P = \frac{P^s(V_z + d)}{d}$$

• We don't have  $V_z$  but we already calculated  $V_z^s$  in HW2, so we can used that:

$$P^{s} = \frac{P}{(\frac{V_{z}^{s}}{Z_{max} - V_{z}^{s}} + 1)}$$

$$P = P^s \cdot \frac{V_z^s}{Z_{max} - V_z^s} + 1$$

- Note that we only have  $V_z^s$  and  $Z_{max}$  in this equation that we already know the value, we don't need to care d and some other parameter.
- We used  $V_z' = \frac{V_z^s}{Z_{max} V_z^s}$  to simplify the equation:

$$P^s = \frac{P}{V_z' + 1}$$

$$P = P^s \cdot (V_7' + 1)$$

5. The Step for Parameter interpolation:

Get  $V_7^p$  for each vertex.

Transform P to perspective space  $P^s$  for each vertex.

Interpolate  $V_7^p$  for each pixel.

Interpolate  $P^s$  for each pixel.

Transform  $P^z$  back to P by using  $V_z^p$  for each pixel.

#### 5.2 Texture

1. Scale *u*, *v* to Texture Image Size:

$$(u, v)$$
 coords range over  $[0, 1]$ 

2D Image is a pixel array of 
$$xs - 1$$
,  $ys - 1$ 

But u \* (xs - 1) might not be Integer so we need to interpolate the color for non-Integer (u, v) coordinate from nearest 4 Integer point.

$$Color(p) = (1 - s)(1 - t)A + s(1 - t)B + stC + (1 - s)tD$$

- 2. For Phong Shading, using texture function f(u, v) to replace  $k_d$  and  $k_a$
- 3. For Gouraud Shading, using f(u, v) to replace all  $k_s$ ,  $k_d$  and  $k_a$
- 4. Procedural Texture
- 5. Bump Texture: Alter normals at each pixel to create bump.
- 6. Noise Texture: 3D Noise Volume, Tubulence
- 7. Evironment(Reflection) Mapping: Cube Map

Problems: Sampling problems and high curvature problems

## **5.3** Implementation Of Texutre(HW5)

1. Step1: Texture coordinates: surface point  $\rightarrow (u, v)$ 

Input: vertex in image space

Output: (u, v)

2. Step2:  $(u, v) \rightarrow RGB$  color

Input: (u, v) Output: RGB color from image LUT

- 3. Interpolation of (u, v) need to be in perspective space.
- 4. Interpolation of 4-corner for non-Integer (u, v) is needed.

## 6 Class14-15 Antialiasing

## 6.1 The Source of Aliasing

- 1. Quantization error arise from insufficient accuracy of sample
- 2. Aliasing error arise from insufficient samples
- 3. Nyquist Theorem: Sample at least twice the rate of highest frequency present in the signal. f(t) filtered for cutoff freq  $\omega_F$  (Remove high frequencies before sampling) Sample Rate  $\frac{1}{T_0}$  is greater than  $2\omega_F$  Reconstruct(interpolate) with sinc function
- 4. Solution: Band-limit the input signal before sampling.

## **6.2** Implement Antialiasing(HW6)

- 1. Antialiasing by jitter supersampling
- 2. Sample a pixel several with different center and weight