Increasing Relative Angular Velocity in Unmanned Air Combat

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Angular velocity plays a critical role in determining the outcome of a close-range aerial engagement between two fighter aircraft pitching at full deflection. If the aircraft are identical, then a pursuer may exploit its ability to roll in order to increase its relative angular velocity against a pitching opponent. In this paper, we present a repeatable maneuver for an unmanned fighter aircraft that increases its relative angular velocity. Additionally, we provide maneuvers for aligning an aircraft's trajectory with a desired target trajectory.

Keywords: Flight control; air combat; trajectory planning.

1. Introduction

As the capabilities of autonomous flight control systems become more advanced, studies of optimal strategies and policies that unmanned aircraft can implement in air combat become increasingly relevant. In particular, an aerial engagement between two unmanned fighter aircraft may involve extended periods of pitching at full deflection, since considerations for g-force tolerance are limited to only those for the physical integrity of the aircraft.

In this paper, we consider a pursuit scenario involving two physically identical fighter aircraft. We develop a maneuver that enables the pursuer to increase its relative angular velocity, assuming the other aircraft maintains a policy of pitching at full deflection. A desirable property of our maneuver is its *repeatability*; i.e. the resulting state upon completion of the maneuver is one in which the maneuver can be performed again.

Prior examples of work in this area such as [1–3] present analytic models of pursuit and evasion. Various agents for autonomous air combat have also been implemented. The work of Solinger in [4] uses rule-based systems and a decision tree to create an agent that can recognize various situations and choose suitable maneuvers. The agent in [5] uses fuzzy logic and genetic algorithms to control an aircraft in a simulation. The work [6] by McGrew implements a neuro-dynamic programming method to control fixed-wing aircraft under experimental conditions. In [7], a reduced-order mathematical model is used to study a similar problem in two-dimensional space. However, the performance characteristics of the two aircraft are asymmetric in [7]. Furthermore, the absence of the third dimension does not allow for lateral movements, which will be essential to the maneuvers we derive in this paper. The work by Olsder [8] also uses a coplanar model, with the aim of classifying roles of two aircraft given initial starting conditions. In our paper, we begin with the assumption that the roles have been definitively established, and from there we determine a way for the attacker to gain advantage. These starting conditions are similar to those in the work by López [9], with the difference being that we have one possible initial scenario instead of four. However, the work [9] is concerned with the application of a discrete set of maneuvers using a heuristic scoring function. In our paper, we are interested in the development of the maneuvers themselves.

To the best of the authors' knowledge, optimization approaches to the problem of gaining advantage have not yielded any solutions. The work of Raivio [10] applies approximate optimization techniques to an aerial combat scenario. However, the initial conditions are such that the scenario is resolved with an intercepting trajectory from the pursuer. In our scenario, the trajectories have stabilized so that given no external input both aircraft will continue along circular paths indefinitely. Therefore, we will approach the problem heuristically.

In this paper, we will use a kinematic model to describe the combat scenario. We assume that the trajectories of the turning aircraft form circles, which allows us to efficiently represent them in terms of trigonometric functions. This also gives us symmetric differential equations that have solutions with elegant properties which will be used to derive our maneuver. Our kinematic model is derived from the analysis of the trajectory of an aircraft under the forces of lift, drag, and thrust, without gravity. We then utilise the symmetric properties of our model to construct a maneuver that an aircraft can perform to gain advantage. This maneuver is composed of spiral segments that result from the application of a continuous roll whilst pitching. We then further develop our theory to include methods for manipulating arbitrary initial trajectories into a scenario where our maneuver can be performed. Finally, we extend our results to account for situations in which the aircraft are yawing.

2. Kinematic Model and Assumptions

In this section, we introduce our model and assumptions. We shall disregard gravity and other physical variables, and approach the problem from a kinematic standpoint. In our model, we have two identical fighter aircraft. They are each represented by a position and orientation in three-dimensional Euclidean space. Their forward speeds are equal and will always remain constant. The initial positions and orientations of the aircraft are such that one has a distinct advantage over the other; it is pursuing the other and wishes to increase its advantage. In our discussion, we assume the perspective of the pursuer. This will typically be referred to as *our* aircraft. Consequently, the other aircraft will be referred to as the *enemy* aircraft.

In addition to their forward speeds being constant, both aircraft will always be pitching upwards at a constant rate. In our model, distance is measured in *units* and time is measured in *seconds*. For simplicity, we set the speed to one unit per second, and the pitch rate q to one radian per second. The variable t will be used to represent time.

Given our initial conditions, it is clear that the trajectories of the aircraft will each be a circle. We will be using the idea of *turn circles* extensively in the following sections. Turn circles have been used in other works such as [11–13] in order to calculate various maneuvers in unmanned systems.

Definition 2.1. A *turn circle* is the locus of an aircraft pitching upwards at a constant rate and moving with constant forward speed. A *parameterized turn circle* is a function $C\colon \mathbb{R} \to \mathbb{R}^3$ such that $C(t)$ is the position of the aircraft at time t. The *polarity* of an aircraft is the direction it is traveling with respect to its turn circle; i.e. either clockwise or anticlockwise.

We can now specify our initial conditions as follows: the turn circles of our aircraft and the enemy aircraft are initially identical, with the position of our aircraft being slightly behind the enemy aircraft.

Without loss of generality, we can set the parameterized turn circles of the enemy aircraft and our aircraft respectively to $(\cos(t+\gamma), \sin(t+\gamma), 0)$ and $(\cos(t), \sin(t), 0)$, for some small $\gamma > 0$.

We seek a maneuver that brings our aircraft closer to the enemy aircraft. We will impose the additional restriction that upon completion of the maneuver our aircraft must have the same turn circle as it did previously, with the same polarity. Additionally, we shall only permit adjustments of roll rate in the execution of the maneuver. We will also limit the maximum roll rate to M where $M > 0$.

Observe that since we are required to return to our original turn circle, we can measure the advantage we have gained at the end of the maneuver in terms of the parameterized turns circles of our aircraft and the enemy aircraft.

The vectors that describe the orientation of our aircraft are shown in Figure 1. We will be using the Frenet-Serret frame. The unit vector $\mathbf{T}(t)$ is the velocity at time t, $\mathbf{N}(t)$ is the normal unit vector, and $\mathbf{B}(t)$ is the binormal unit vector. We also introduce the vector $\mathbf{k} = p\mathbf{T} + q\mathbf{B}$, which will play an important role in the following sections, as well as the vector $\mathbf{N} = \mathbf{B} \times \mathbf{T}$, which will be used when we extend our results to include yaw.

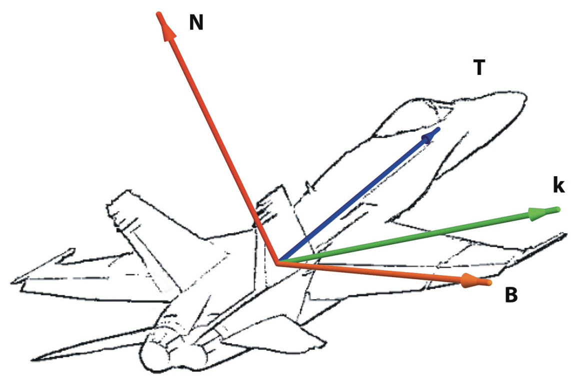

Fig. 1. Vectors describing the orientation of our aircraft. $\mathbf{T}$ is the forward velocity, $\mathbf{B}$ is the direction outwards from the aircraft to the right, parallel to the pitch axis of the aircraft. Note that $\mathbf{N} = \mathbf{B} \times \mathbf{T}$ and $\mathbf{k} = p\mathbf{T} + q\mathbf{B}$.

Definition 2.2. A *roll* is a rotation of $\mathbf{B}$ about $\mathbf{T}$. An *advantage maneuver* (or simply *maneuver*) is a function $f\colon [0,T] \to [-M, M]$ where $T > 0$ such that if our aircraft rolls at $f(t)$radians per second at time t then the turn circle of our aircraft at $t = T$ is identical to the turn circle of our aircraft at $t = 0$.

Definition 2.3. If the parameterized turn circle of our aircraft before applying an advantage maneuver is $(\cos(t), \sin(t), 0)$, and afterwards is $(\cos(t+x), \sin(t+x), 0)$ for some $0 \le x < 2\pi$, then we refer to the value x as the *advantage* or *advantage gained* from the maneuver.

3. Instantaneous Roll Example

In this section, we will consider how instantaneous rolls affect the movement of our aircraft in order to illustrate the geometric mechanism behind the advantage maneuver introduced in the following section.

Let our aircraft's parameterized turn circle be

$$C_0(t) = (\cos(t), \sin(t), 0) \tag{1}$$

and let the components of C_0 be denoted by $C_{0,i}$, i.e.

$$C_{0,1} = \cos(t), C_{0,2} = \sin(t), C_{0,3} = 0. \tag{2}$$

Definition 3.1. An *instantaneous roll* $R_{t_1,\mu}$ is a transformation that rotates a parameterized turn circle around its tangent vector at a particular time t_1 by μradians.

Let $W(\mathbf{T}, \mu)$ be the Rodrigues rotation matrix around a unit vector $\mathbf{T}$ by an angle of μ.

Then

$$\begin{aligned} C_1 &= R_{t_1,\mu}(C_0) \\ &= W\left(C_0'(t_1), \mu\right)\left(C_0(t) - C_0(t_1)\right) + C_0(t_1) \end{aligned} \tag{3}$$

is the parameterized turn circle of our aircraft after applying an instantaneous roll of μradians at t_1.

Definition 3.2. Let C_0 be a parameterized turn circle. An *instantaneous advantage maneuver* (or simply *instant maneuver*) is a sequence of instantaneous rolls $R_{t_0,\mu_0}, R_{t_1,\mu_1}, ..., R_{t_n,\mu_n}$ such that $C_{n+1}(t) = C_0(t + x)$ for some $x \geq 0$ where $C_{n+1} = R_{t_n,\mu_n}(R_{t_{n-1},\mu_{n-1}}(...R_{t_0,\mu_0}(C_0)...))$. We extend the definition of *advantage gained* to include instant maneuvers.

We now give an example of an instantaneous advantage maneuver $R_{t_0,\mu_0}, R_{t_1,\mu_1}, ..., R_{t_5,\mu_5}$.

Let

$$\begin{aligned} C_1 &= R_{t_0,\mu_0}(C_0) = R_{0,\frac{\pi}{8}}(C_0), \\ C_2 &= R_{t_1,\mu_1}(C_1) = R_{\frac{\pi}{2},\frac{\pi}{8}}(C_1). \end{aligned} \tag{4}$$

Solving $C_{2,3}'(t_\beta) = 0$ we obtain two solutions for t_β. We will choose

$$t_\beta = \pi + \arctan\left(\cos\frac{\pi}{8}\right) \tag{5}$$

for reasons that will soon become clear.

Let

$$\begin{aligned} \delta &= \arctan\frac{7 - 4\sqrt{2}}{2\sqrt{52 + 14\sqrt{2}}}, \\ \mu_\beta &= \arctan\sqrt{23 - 16\sqrt{2}}, \\ C_3 &= R_{t_2,\mu_2}(C_2) = R_{t_\beta,\mu_\beta}(C_2), \\ C_4 &= R_{t_3,\mu_3}(C_3) = R_{3\pi - x, -\frac{\pi}{8}}(C_3), \\ C_5 &= R_{t_4,\mu_4}(C_4) = R_{3\pi - x + \frac{\pi}{2}, -\frac{\pi}{8}}(C_4), \\ C_6 &= R_{t_5,\mu_5}(C_5) = R_{3\pi - x + t_\beta, -\mu_\beta}(C_5). \end{aligned} \tag{6}$$

This instant maneuver describes an aircraft initially rolling right by an angle of $\frac{\pi}{8}$, then $\frac{\pi}{8}$ again. The next roll is chosen such that it occurs either at the maximum (with respect to the z-axis; i.e. $C_{2,3}$, the third component of C_2) or minimum point of C_2. In this example, we have chosen the maximum point $C_2(t_\beta)$. This then allowed us to choose μ_β so that $C_{3,3}$ is constant; i.e. our aircraft is now moving horizontally in the image of C_3, as it did in C_0. Hence by symmetry, we can choose appropriate values for t_3, t_4, t_5 and apply the instantaneous rolls $R_{t_3,-\frac{\pi}{8}}, R_{t_4,-\frac{\pi}{8}}, R_{t_5,-\mu_\beta}$ to bring our aircraft back to its original turn circle; i.e. $C_6(t) = C_0(t + x)$.

In this case, our advantage gained is given by

$$x = 2\delta. \tag{7}$$

Thus, through a sequence of instantaneous rolls, our aircraft has effectively traveled with higher angular velocity than it otherwise would have if it remained in its original turn circle C_0.

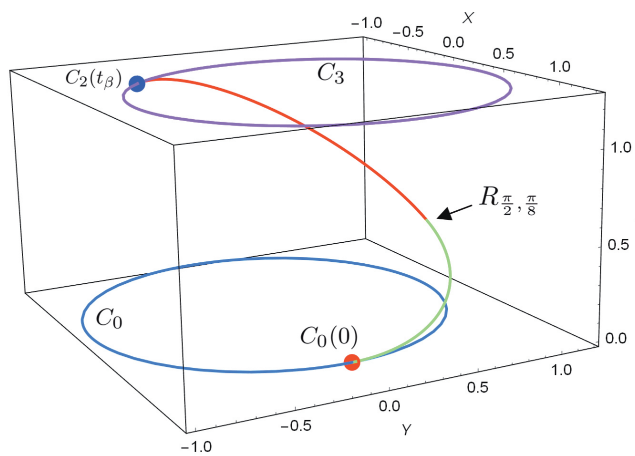

Fig. 2. The curve connecting $C_0(0)$ (red point) and $C_2(t_\beta)$ (blue point) is the trajectory of our aircraft from $t = 0$ to $t = t_\beta$ during the instant maneuver. The circles C_0 and C_3 are shown in blue and purple respectively. The circular arcs of C_1 and C_2 are shown in green and red respectively.

3.1. *Explanation of advantage gained*

We now give a geometric explanation for the origin of this advantage gained.

The value of δ was computed by taking

$$\delta = \arctan\left(C_3(0) - Q(C_3)\right), \tag{8}$$

where

$$Q(C_3) = \frac{C_3(0) + C_3(\pi)}{2} \tag{9}$$

is the center of C_3.

This is because $C_3(t) - Q(C_3) = C_0(t+\delta)$; i.e. the parameter of C_3 has shifted by δ. This implies that our aircraft has gained advantage. We now explain intuitively the cause of this shift. This shift is caused by the second roll $R_{\frac{\pi}{2},\frac{\pi}{8}}$. At time $t = \frac{\pi}{2}$, our aircraft is in the circle C_1. Observe that the circle C_1 is oblique to C_0. This implies that the velocity vector $C_1'(\frac{\pi}{2})$ does not lie in the horizontal plane. Hence a rotation of the circle C_1 about the vector $C_1'(\frac{\pi}{2})$ will also rotate C_1 by some amount about the vertical axis $(0,0,1)$. After adjusting for positional offsets, we find that this amount is δ.

Next, our aircraft reaches the highest point on C_2 at t_β. At this point, $C_{2,3}'(t_\beta) = 0$; i.e. the velocity vector lies in the horizontal plane. Thus, after applying a rotation of μ_β to C_2 we obtain a horizontal circle C_3, where

$$C_3(t) = C_0(t+\delta) + Q(C_3). \tag{10}$$

(A turn circle is *horizontal* if the plane containing it is parallel to the plane spanned by $\{(1,0,0),(0,1,0)\}$.)

By symmetry, R_{t_4,μ_4} also shifts t by δ. Then the end result is that the advantage gained is

$$x = 2\delta. \tag{11}$$

4. Continuous Roll Maneuver

In this section, we will derive a step function advantage maneuver. This derivation follows from the natural inclination to integrate the discrete sequences of instantaneous rolls discussed above. In order to do this, we will first derive equations that give the precise position and orientation of our aircraft over time if we apply a constant roll rate to it.

Let q be the pitch rate of our aircraft. Suppose we now also apply a constant roll rate to our aircraft. Then let p be the roll rate.

If we consider an infinitesimal change in time dt, then the changes in $\mathbf{T}$ and $\mathbf{B}$ are given by

$$\begin{aligned} d\mathbf{T} = q\mathbf{B}\times\mathbf{T}dt &\implies \frac{d\mathbf{T}}{dt} = q\mathbf{B}\times\mathbf{T}, \\ d\mathbf{B} = p\mathbf{T}\times\mathbf{B}dt &\implies \frac{d\mathbf{B}}{dt} = p\mathbf{T}\times\mathbf{B}. \end{aligned} \tag{12}$$

Observe that

$$\frac{d\mathbf{T}}{dt} = -\frac{q}{p}\frac{d\mathbf{B}}{dt} \text{ and } \frac{d\mathbf{B}}{dt} = -\frac{p}{q}\frac{d\mathbf{T}}{dt}. \tag{13}$$

Integrating (see section 7.1.1) both sides with respect to t and solving for the constant of integration, we obtain

$$\frac{d\mathbf{T}}{dt} = \mathbf{k}\times\mathbf{T}, \tag{14}$$

$$\frac{d\mathbf{B}}{dt} = \mathbf{k}\times\mathbf{B}, \tag{15}$$

where $\mathbf{k} = p\mathbf{T}(0) + q\mathbf{B}(0)$.

We will refer to $\mathbf{k}$ as the *orientation vector*. By geometric arguments [14], we see that equation (14) has solution

$$\begin{aligned} \mathbf{T}(t) = \mathbf{k}\frac{\mathbf{k}\cdot\mathbf{T}(0)}{\mathbf{k}\cdot\mathbf{k}} + \left(\mathbf{T}(0) - \mathbf{k}\frac{\mathbf{k}\cdot\mathbf{T}(0)}{\mathbf{k}\cdot\mathbf{k}}\right)\cos(|\mathbf{k}|\,t) \\ + \frac{\mathbf{k}}{|\mathbf{k}|}\times\mathbf{T}(0)\sin(|\mathbf{k}|\,t) \end{aligned} \tag{16}$$

and that $\mathbf{T}$ moves at constant speed in a circle around $\mathbf{k}$ with period $\frac{2\pi}{|\mathbf{k}|}$, as does $\mathbf{B}$. Integrating $\mathbf{T}$ over time yields a spiral in the direction of $\mathbf{k}$ (or $-\mathbf{k}$ if $p<0$) with period $\frac{2\pi}{|\mathbf{k}|}$. This implies that our aircraft resets to the same orientation it had when it started rolling every $\frac{2\pi}{|\mathbf{k}|}$ seconds. This means that at the points of reset, our aircraft is in a *parallel* turn circle, and furthermore that it is in the same position on the turn circle as it was when we first started rolling. (Two turn circles are said to be *parallel* if they are identical under translation.) Hence by symmetry if we stop rolling at one of these points of reset and continue along the parallel turn circle for exactly half its period, then start rolling at the same rate we initially did but in the opposite direction, we will return to our original turn circle, opposite the point where we first left it. Thus, our advantage maneuver is given by

$$f_p(t) = \begin{cases} p & \text{if } 0 \le t < \frac{2\pi}{|\mathbf{k}|} \\ 0 & \text{if } \frac{2\pi}{|\mathbf{k}|} \le t < \frac{2\pi}{|\mathbf{k}|} + \frac{\pi}{q} \\ -p & \text{if } \frac{2\pi}{|\mathbf{k}|} + \frac{\pi}{q} \le t \le \frac{4\pi}{|\mathbf{k}|} + \frac{\pi}{q} \end{cases}. \tag{17}$$

We will often refer to the three intervals of f and their images by their geometric forms. For example, we may call the interval $[0, \frac{2\pi}{|\mathbf{k}|}]$ the initial *spiral* or initial *spiral period.*

Geometrically, our maneuver is a spiral, followed by a semicircle, followed by a spiral. Thus, we have traveled 2.5revolutions with respect to our original turn circle. However, we have only taken $\frac{4\pi}{|\mathbf{k}|} + \frac{\pi}{q}$ seconds to do so. Subtracting this amount from the duration of 2.5 revolutions $\frac{5\pi}{q}$ and simplifying, we find that the advantage gained is

$$4\pi\left(1 - \sqrt{1 - \frac{p^2}{q^2+p^2}}\right). \tag{18}$$

4.1. *Example of an advantage maneuver*

Let

$$C(t) = (\cos(t - \pi/2), \sin(t - \pi/2) + 1, 0) \tag{19}$$

be our aircraft's parameterized turn circle and

$$f_{p_0} \tag{20}$$

be our advantage maneuver where $p_0 = 0.15$.

Then the advantage gained is

$$4\pi \left(1 - \sqrt{1 - \frac{0.15^2}{q^2 + 0.15^2}}\right) \approx 0.14 \tag{21}$$

and the trajectory of our aircraft during the maneuver is shown in Figure 3.

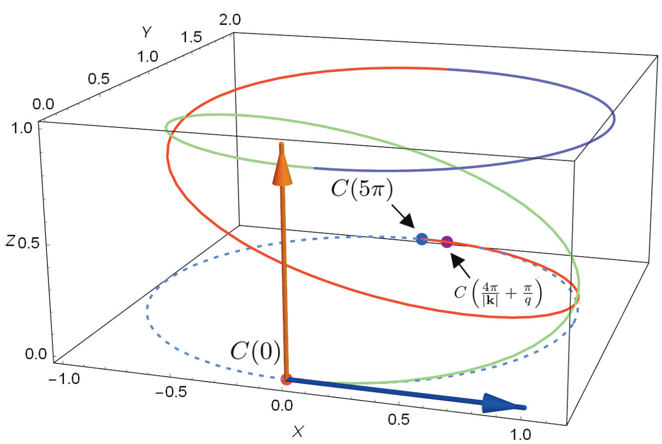

Fig. 3. The curve connecting $C(0)$ (red point) and $C(5\pi)$ (blue point) is the trajectory of our aircraft from $t = 0$ to $t = \frac{4\pi}{|\mathbf{k}|} + \frac{\pi}{q}$ during the maneuver. The initial spiral is shown in green, the final spiral is shown in red. The semicircle where $f_{p_0}(t) = 0$ is shown in purple. The purple point is $C\left(\frac{4\pi}{|\mathbf{k}|} + \frac{\pi}{q}\right)$; i.e. the position our aircraft would have been at if it stayed in C. $\mathbf{B}(0)$ is the orange vector, $\mathbf{T}(0)$ is the blue vector. The dashed circle is our original turn circle C.

4.2. *Displacement of the advantage maneuver*

We can measure the displacement $\mathbf{h}$ of the advantage maneuver by the distance between the original turn circle and the turn circle of our aircraft when $f(t) = 0$.

In the $\{\mathbf{T}, \mathbf{B}\}$ basis, $\mathbf{k} = (p, q)$. The rate of displacement of our aircraft's turn circle is given by the projection of $\mathbf{T}$ onto $\mathbf{k}$, i.e.

$$\frac{\mathbf{T} \cdot \mathbf{k}}{\mathbf{k} \cdot \mathbf{k}} \mathbf{k}. \tag{22}$$

Since our maneuver begins with a spiral of exactly one period, our displacement is given by multiplying the rate by the period $\frac{2\pi}{|\mathbf{k}|}$. This simplifies to the following expression:

$$\mathbf{h} = (h_1(p), h_2(p)) = \left(\frac{2\pi p^2}{(p^2 + q^2)^{\frac{3}{2}}}, \frac{2\pi pq}{(p^2 + q^2)^{\frac{3}{2}}} \right), \tag{23}$$

which has magnitude $\frac{2\pi p}{p^2 + q^2}$.

5. Connecting Two Parallel Turn Circles

We have assumed thus far that our aircraft and the enemy aircraft start in the same turn circle. We now consider the case where both aircraft start in parallel turn circles spaced arbitrarily far apart. Trajectory generation methods are given in [15,16], however the symmetry of our model allows for a geometric method to be obtained. Furthermore, our method maintains full pitch deflection, which is desirable as it does not decrease our relative angular velocity. The formula for $\mathbf{h}$ allows us to derive a method of translating our turn circle towards any given parallel turn circle. Without loss of generality, we can assume that the turn circles are horizontal (i.e. the third component of $C(t)$ is constant for all t), and that the target turn circle is centered at the origin. Additionally, assume that both aircraft have the same polarity.

Consider the vertical plane P centered at the origin and intersecting the center of our aircraft's turn circle. If there is more than one such P, choose one arbitrarily. Working in coordinates with respect to P, label the center of our aircraft's turn circle $\mathbf{g}_O$ and the center of the target turn circle $\mathbf{g}_T$. Our goal is to shift $\mathbf{g}_O$ to $\mathbf{g}_T$. Let H be the set of the four functions

$$\begin{aligned} \mathbf{h}_1(p) &= (h_1(p), h_2(p)), 0 \le p \le M \\ \mathbf{h}_2(p) &= (-h_1(p), h_2(p)), 0 \le p \le M \\ \mathbf{h}_3(p) &= (h_1(-p), -h_2(-p)), -M \le p \le 0 \\ \mathbf{h}_4(p) &= (-h_1(-p), -h_2(-p)), -M \le p \le 0. \end{aligned} \tag{24}$$

Let $\mathbf{k}(p, t) = q\mathbf{B}(t) + p\mathbf{T}(t)$ so that $|\mathbf{k}(p)| = \sqrt{p^2 + q^2}$. Let S_O be the set of the four functions $\{\mathbf{g}_O + \mathbf{h}_i(p)\}, 1 \le i \le 4$. Let S_T be defined similarly. Then the union of the images of all the functions in S_O are all the *admissible* centers within one spiral period for our aircraft in P; i.e. it is possible for our aircraft to translate the center of its turn circle to any point in the union by choosing an appropriate p and rolling for $\frac{2\pi}{|\mathbf{k}(p)|}$ seconds, whilst ensuring the turn circle it ends up in is parallel to the original starting turn circle. Then the procedure is as follows:

(1) If possible, choose $\mathbf{s}_O \in S_O$ and $\mathbf{s}_T \in S_T$ such that $\mathrm{Im}\ \mathbf{s}_O \cap \mathrm{Im}\ \mathbf{s}_T \neq \emptyset$.

  (a) If an intersection exists, arbitrarily choose one and label it $\mathbf{x}$ and proceed to Step 2.

(b) Otherwise choose $\mathbf{s}_O$ and p_0 such that $|\mathbf{s}_O(p_0) - \mathbf{g}_T|$ is minimized. Apply the roll rate p_0 to our aircraft for $\frac{2\pi}{|\mathbf{k}(p_0)|}$ seconds, beginning when $\mathbf{T}(t)$ intersects P. Note that $\mathbf{T}(t)$ intersects P at two points in time; choose the point t_0 such that the angle between $\mathbf{s}_O(p_0) - \mathbf{g}_O$ and $\mathbf{T}(t_0)$ is smaller. After *applying* the roll rate, $\mathbf{T}$ and $\mathbf{B}$ now refer to our aircraft's orientation in its new turn circle. Redefine S_O and S_T for our aircraft's new position, then go back to Step 1.

(2) Let p_1, p_2 be such that $\mathbf{s}_O(p_1) = \mathbf{s}_T(p_2) = \mathbf{x}$. Choose t_1 such that $\mathbf{T}(t_1)$ intersects P and the angle between $\mathbf{s}_O(p_1) - \mathbf{g}_O$ and $\mathbf{T}(t_1)$ is minimized. Apply the roll rate p_1 to our aircraft for $\frac{2\pi}{|\mathbf{k}(p_1)|}$ seconds starting at t_1. If $\mathbf{x} = g_T$, we are done. Otherwise proceed to Step 3.

(3) Since an intersection $\mathbf{x}$ exists, by symmetry of the set H there exists $\mathbf{s} \in \{\mathbf{s}_O(p_1) + \mathbf{h}_i(p)\}$ such that $\mathbf{s}(-p_2) = \mathbf{g}_T$. Choose t_2 such that $\mathbf{T}(t_2)$ intersects P and the angle between $\mathbf{s}(-p_2) - \mathbf{s}_O(p_1)$ and $\mathbf{T}(t_2)$ is minimized. Apply the roll rate $-p_2$ to our aircraft for $\frac{2\pi}{|\mathbf{k}(p_2)|}$ seconds starting at t_2.

Our aircraft's turn circle now coincides with the target turn circle.

5.1. *Example of connecting two parallel turn circles*

Let $M = 2$, and

$$\begin{aligned} C(t) &= (\cos(t) + 4, \sin(t) + 1, 4)\,, \\ T(t) &= (\cos(t), \sin(t) + 1, 0)\,, \end{aligned} \tag{25}$$

be our aircraft's parameterized turn circle and the target parameterized turn circle respectively.

Then

$$\begin{aligned} t_1 &= \frac{\pi}{2}, \\ \mathbf{s}_T(p) &= \mathbf{h}_1(p), \\ \mathbf{s}_O(p) &= (4,4) + \mathbf{h}_4(p), \\ \mathbf{s}(p) &= \mathbf{s}_O(p_1) + \mathbf{h}_4(p), \\ \mathbf{x} &\approx (2.41, 1.59)\,, \\ p_1 &\approx -0.66, \\ p_2 &\approx 1.52, \\ t_2 &= t_1 + \frac{2\pi}{|\mathbf{k}(p_1)|}. \end{aligned} \tag{26}$$

The procedure is then applying p_1 for $\frac{2\pi}{|\mathbf{k}(p_1)|}$ seconds, followed by immediately applying $-p_2$ for $\frac{2\pi}{|\mathbf{k}(p_2)|}$ seconds. The intersection $\mathbf{x}$ is shown in Figure 4. The trajectory of our aircraft's position throughout the procedure is shown in Figure 5.

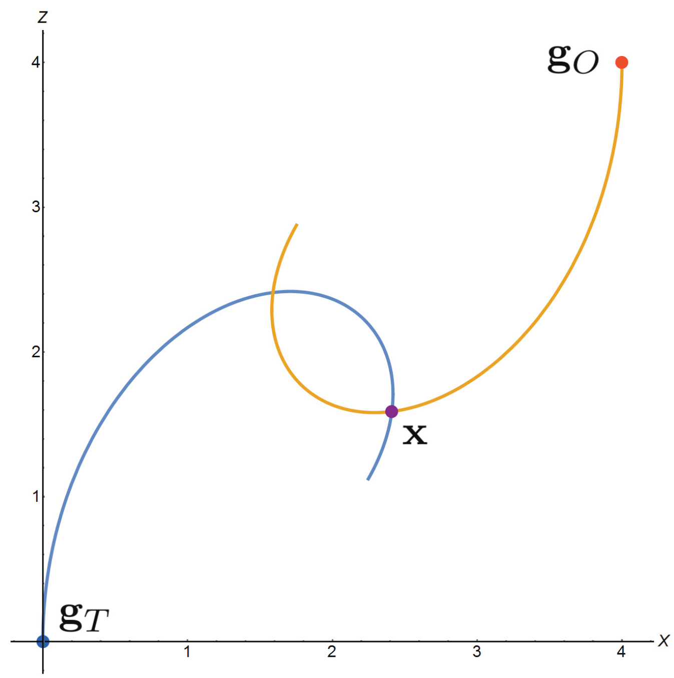

Fig. 4. A subset of admissible centers for our turn circle starting from turn circles centered at the three points via the advantage maneuver. This is the image of the curves of $\mathbf{s}_T(p)$ (blue) and $\mathbf{s}_O(p)$ (yellow) for $0 \le p \le M$. $\mathbf{g}_O$ is the red point, $\mathbf{g}_T$ is the blue point. $\mathbf{x}$ is the purple point.

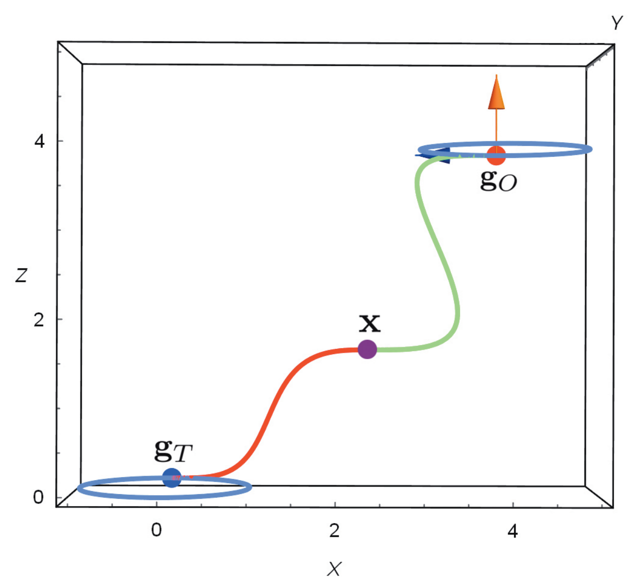

Fig. 5. The corresponding maneuver to move our aircraft between the points shown in Figure 4. Position of our aircraft over time. Initial point (red), final point (blue). Applying p_1 (green curve), then applying $-p_2$ (red curve). The circles $T(t)$ and $C(t)$ are shown in blue. $\mathbf{B}(t_1)$ is the orange vector, $\mathbf{T}(t_1)$ is the blue vector. $\mathbf{g}_O, \mathbf{g}_T$, and $\mathbf{x}$ are shown as they would appear when projected in the direction of the Y-axis.

6. Rotating a Non-Parallel Turn Circle

In section 5, we established a procedure for translating our aircraft's turn circle into a parallel turn circle. If we consider the case where the target turn circle is not parallel, then we can reduce the problem of making the turn circles coincide to the parallel case if we are able to rotate our aircraft's turn circle to be parallel. In this section, we describe a procedure to perform such a rotation.

Let our aircraft's parameterized turn circle be $C(t)$. We desire to be parallel to the turn circle $T(t)$. Observe that we only need to rotate our aircraft's $\mathbf{B}$ vector until

$$\mathbf{B} = T(0) \times T\left(\frac{\pi}{2}\right), \tag{27}$$

where $T(0) \times T\left(\frac{\pi}{2}\right)$ is the corresponding vector $\mathbf{B}_T$ of $T(t)$. Choose t_0 such that $\mathbf{B}_T$ lies in the plane P spanned by $\{\mathbf{B}(t_0), \mathbf{T}(t_0)\}$ and the angle between $\mathbf{T}(t_0)$ and $\mathbf{B}_T$ is minimized. Let α be the angle between $\mathbf{B}(t_0)$ and $\mathbf{B}_T$. Observe that if we roll at $p > 0$radians per second then we are rotating $\mathbf{B}(t_0)$ around $\mathbf{k}(p, t_0)$ so that $\mathbf{B}(t_1) = \mathbf{B}\left(t_0 + \frac{\pi}{|\mathbf{k}(p)|}\right) \in P$. Then the angle between $\mathbf{B}(t_1)$ and $\mathbf{B}_T$ is given by

$$\alpha - 2\arctan\frac{p}{q}. \tag{28}$$

Then the procedure is as follows:

(1) Choose the smallest $n \in \mathbb{Z}^+$ giving $p_0 \in [0, M]$ such that

$$2\arctan\frac{p_0}{q} = \frac{\alpha}{n}. \tag{29}$$

(2) Apply p_0 for $\frac{\pi}{|\mathbf{k}(p_0)|}$ seconds.
(3) Repeat the following steps $n-1$ times.
  (a)Continue on the current turn circle for $\frac{\pi}{q}$ seconds.
  (b)Apply p_0 for $\frac{\pi}{|\mathbf{k}(p_0)|}$ seconds.

Our $\mathbf{B}$ vector is now equal to $\mathbf{B}_T$. Geometrically, the trajectory of our aircraft during this procedure alternates between half-spirals and semicircles, repeatedly rotating $\mathbf{B}$ by $\frac{\alpha}{n}$.

6.1. *Example of rotating a non-parallel turn circle*

Let

$$\begin{aligned} M &= 1.5, \\ t_0 &= 0, \\ \mathbf{T}(t_0) &= (1,0,0)\,, \\ \mathbf{B}(t_0) &= (0,0,1)\,, \\ \mathbf{B}_T &= (0.8, 0, -0.6)\,. \end{aligned} \tag{30}$$

Then

$$\begin{aligned} \alpha &= 2\arctan 2, \\ n &= 2, \\ p_0 &= \tan\left(\frac{\arctan 2}{2}\right). \end{aligned} \tag{31}$$

Then we apply p_0 for $\frac{\pi}{|\mathbf{k}(p_0)|}$ seconds, continue on the resulting turn circle for $\frac{\pi}{q}$ seconds, and apply p_0 for $\frac{\pi}{|\mathbf{k}(p_0)|}$ seconds again. The trajectory of our aircraft's position throughout the procedure is shown in Figure 7, and the trajectory of our aircraft's $\mathbf{B}$ vector is shown in Figure 6.

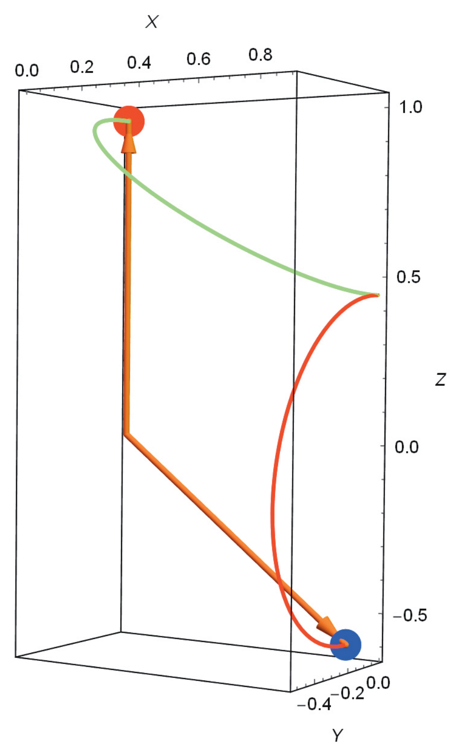

Fig. 6. The direction of the $\mathbf{B}$ vector (orange) of our aircraft over time under the rotation procedure. Initial direction (vector on the red point), final direction (vector on the blue point). Intermediate points when applying p_0 (green curve), then applying p_0 again (red curve).

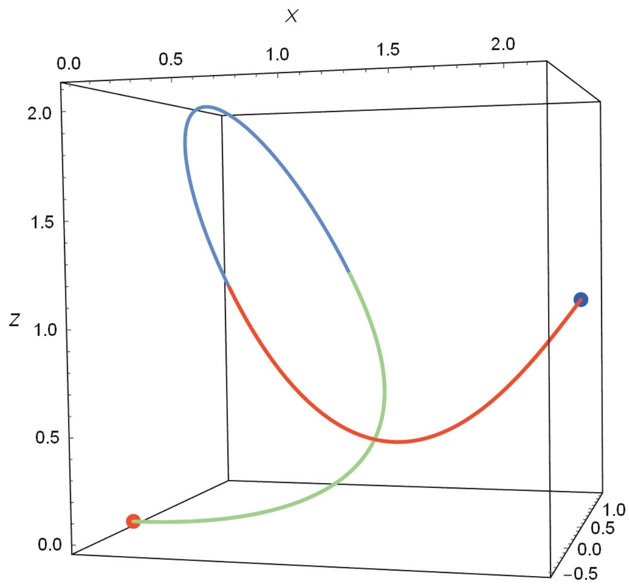

Fig. 7. The curve connecting the red point and the blue point is the position of our aircraft over time under the rotation procedure. Initial point (red), final point (blue). Applying p_0 (green curve), then continuing on the circle (blue curve), then applying p_0 again (red curve).

7. Generalizing to Include Yaw

The maneuvers described in this paper are still applicable if both aircraft are yawing at a constant rate, in addition to pitching at a constant rate. We find that if our aircraft is yawing at a constant rate, the resulting trajectory is still a circle. We can then extend the definition of a turn circle to include "yawing at a constant rate." Yawing is preferable as it reduces the period and radius of the turn circle.

Let r be the yaw rate, and $\mathbf{N}$ be the unit vector $\mathbf{B} \times \mathbf{T}$.

7.1. *Derivations of formulas*

We now give derivations that include yaw for the formulas presented in this paper.

7.1.1. *Integrating to obtain* $\mathbf{k} \times \mathbf{T}$

The following differential equations describe the motion of our aircraft:

$$\frac{d\mathbf{T}}{dt} = q\mathbf{B} \times \mathbf{T} + r\mathbf{N} \times \mathbf{T}, \tag{32}$$

$$\begin{aligned} \frac{d\mathbf{B}}{dt} &= p\mathbf{T} \times \mathbf{B} + r\mathbf{N} \times \mathbf{B}, \\ \frac{d\mathbf{N}}{dt} &= q\mathbf{B} \times \mathbf{N} + p\mathbf{T} \times \mathbf{N}. \end{aligned} \tag{33}$$

Observe that due to the anticommutativity of the cross product, we have

$$p\frac{d\mathbf{T}}{dt} + q\frac{d\mathbf{B}}{dt} + r\frac{d\mathbf{N}}{dt} = 0. \tag{34}$$

Integrating both sides with respect to t, we obtain

$$p\mathbf{T}(t) + q\mathbf{B}(t) + r\mathbf{N}(t) = \mathbf{k}, \tag{35}$$

$$\mathbf{k} = p\mathbf{T}(0) + q\mathbf{B}(0) + r\mathbf{N}(0). \tag{36}$$

Rearrange (35) to obtain

$$\mathbf{B} = -\frac{p}{q}\mathbf{T} - \frac{r}{q}\mathbf{N} + \frac{\mathbf{k}}{q} \tag{37}$$

and substitute into (32) to obtain

$$\begin{aligned} \frac{d\mathbf{T}}{dt} &= q(-\frac{p}{q}\mathbf{T} - \frac{r}{q}\mathbf{N} + \frac{\mathbf{k}}{q}) \times \mathbf{T} + r\mathbf{N} \times \mathbf{T} \\ &= (-p\mathbf{T} - r\mathbf{N} + \mathbf{k}) \times \mathbf{T} + r\mathbf{N} \times \mathbf{T} \\ &= \mathbf{k} \times \mathbf{T} \end{aligned} \tag{38}$$

as desired.

Hence by symmetry we also have

$$\frac{d\mathbf{B}}{dt} = \mathbf{k} \times \mathbf{B} \text{ and } \frac{d\mathbf{N}}{dt} = \mathbf{k} \times \mathbf{N}. \tag{39}$$

Thus we can still apply the solution given in (16), which will still yield a circle.

7.1.2. *Yawing reduces the size of the turn circle*

If $p = 0$ and $r \neq 0$, then

$$\begin{aligned} \mathbf{T}(0) \cdot \mathbf{k} &= \mathbf{T}(0) \cdot (p\mathbf{T}(0) + q\mathbf{B}(0) + r\mathbf{N}(0)) \\ &= 0 \end{aligned} \tag{40}$$

since $\mathbf{T}, \mathbf{B}, \mathbf{N}$ are always orthogonal. Hence $\mathbf{T}(0)$ is orthogonal to $\mathbf{k}$. But we know from (16) that $\mathbf{T}$ orbits $\mathbf{k}$; i.e. the angle between $\mathbf{T}$ and $\mathbf{k}$ is constant. Then the projection of $\mathbf{T}$ onto $\mathbf{k}$ is always 0 and thus the locus of our aircraft is still a circle, even with constant yaw. Without yaw, the period of our turn circle is given by $\frac{2\pi}{q}$. However, with yaw the period is given by

$$\frac{2\pi}{|\mathbf{k}|} = \frac{2\pi}{\sqrt{q^2 + r^2}} < \frac{2\pi}{q} \tag{41}$$

so our turn circle is tighter than it would be if we did not yaw.

7.2. *Non-reversal of yaw*

Although we are constantly yawing in one direction, our advantage maneuver function does not change. Furthermore we do not need to define an additional function to control yaw; i.e. yawing remains constant throughout the maneuver.

This is shown by considering $\mathbf{T}, \mathbf{B}$, and $\mathbf{N}$ over the various stages in the maneuver. After the first spiral period, the three vectors are unchanged. However, after traveling around the semicircle, we have the new vectors $\mathbf{T}_1, \mathbf{B}_1$, and $\mathbf{N}_1$. During the semicircle, the normal vector to the turn circle is given by

$$\mathbf{k}_1 = q\mathbf{B}_1 + r\mathbf{N}_1. \tag{42}$$

This is necessarily the same as the normal vector $\mathbf{k}_0$ at $t = 0$; if we let $\mathbf{B}_0 = \mathbf{B}(0), \mathbf{T}_0 = \mathbf{T}(0)$ and $\mathbf{N}_0 = \mathbf{N}(0)$, then

$$\mathbf{k}_0 = q\mathbf{B}_0 + r\mathbf{N}_0 = \mathbf{k}_1 = q\mathbf{B}_1 + r\mathbf{N}_1. \tag{43}$$

Since our aircraft traveled in a semicircle, we have

$$\mathbf{T}_1 = -\mathbf{T}_0. \tag{44}$$

Without loss of generality, assume $p > 0$. If $\mathbf{k} = p\mathbf{T}_0 + \mathbf{k}_0$ is the direction of the initial spiral, then we wish to show that the direction of the spiral on the third interval of the maneuver is $-\mathbf{k}$. Let the orientation vector of the latter spiral be $\mathbf{k}_L$. Let the roll rate of this spiral be $p_L = -p$. Then

$$\begin{aligned} \mathbf{k}_L &= p_L\mathbf{T}_1 + \mathbf{k}_1 \\ &= p\mathbf{T}_0 + \mathbf{k}_0 \\ &= \mathbf{k}. \end{aligned} \tag{45}$$

Since $p_L < 0$, the direction of the spiral is $-\mathbf{k}_L = -\mathbf{k}$ as desired.

7.3. *Advantage gained*

Let $A(p)$ be the advantage gained if we choose p for our advantage maneuver function. As our maneuver function has not changed, we still calculate advantage gained by subtracting the time taken for our maneuver from the duration of 2.5 revolutions. However, the period of a revolution is now shorter due to the yaw effect, so the calculation becomes

$$\begin{aligned} A(p) &= \frac{5\pi}{\sqrt{q^2 + r^2}} - \left(\frac{4\pi}{|\mathbf{k}|} + \frac{\pi}{\sqrt{q^2 + r^2}} \right) \\ &= 4\pi \left(\frac{1}{\sqrt{q^2 + r^2}} - \frac{1}{\sqrt{q^2 + r^2 + p^2}} \right) \\ &= 4\pi \left(1 - \sqrt{1 - \frac{p^2}{q^2 + r^2 + p^2}} \right). \end{aligned} \tag{46}$$

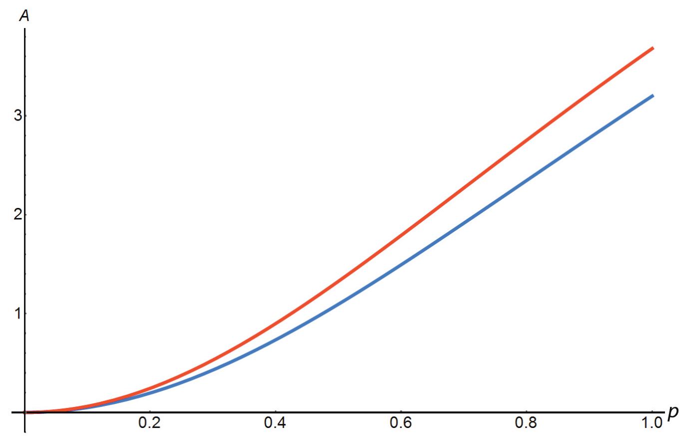

Fig. 8. Advantage gained with respect to the roll rate p. The red curve corresponds to advantage gained without yaw, i.e. $A(p) = 4\pi \left(1 - \sqrt{1 - \frac{p^2}{q^2+p^2}}\right)$, and the blue curve is with yaw; $A(p) = 4\pi \left(1 - \sqrt{1 - \frac{p^2}{q^2+r^2+p^2}}\right)$.

7.4. *Displacement*

In order to calculate displacement, we will work in the $\{\mathbf{T}, (q\mathbf{B} + r\mathbf{N})/\sqrt{q^2 + r^2}\}$ basis. Then $\mathbf{k} = \left(p, \sqrt{q^2 + r^2}\right)$. We then calculate $\mathbf{h}$ in the same manner as before and obtain

$$\mathbf{h} = \left(\frac{2\pi p^2}{(p^2 + q^2 + r^2)^{\frac{3}{2}}}, \frac{2\pi p\sqrt{q^2 + r^2}}{(p^2 + q^2 + r^2)^{\frac{3}{2}}} \right), \tag{47}$$

which has magnitude $\frac{2\pi p}{p^2+q^2+r^2}$.

7.5. *Connecting two parallel turn circles*

The procedure is analogous. Assume without loss of generality that the turn circles are horizontal; i.e. the vertical axis of P is the direction of $q\mathbf{B} + r\mathbf{N}$.

7.6. *Rotation of a non-parallel turn circle*

The procedure is analogous. We rotate $q\mathbf{B}(0) + r\mathbf{N}(0)$ around $\mathbf{k}$ instead. Replace $2\arctan\frac{p}{q}$ with $2\arctan\frac{p}{\sqrt{q^2+r^2}}$.

8. Derivation of the Kinematic Model

In this section, we state our assumptions and expound upon the reasoning that led to our kinematic model. We begin by considering a point with unit mass at rest that represents our aircraft. Considering motion in one dimension, we will describe our assumptions. The two forces that will be acting on our aircraft are thrust and drag. We will assume that thrust is an instantaneously applied force, and that it remains constant. For simplicity, let $F = 1$. Next, consider the drag equation

$$D = \frac{\rho C_D A}{2} u(t)^2 \tag{48}$$

where C_D is the drag coefficient, ρ is the air density, A is the reference area, and $u(t)$ is the velocity at time t. For simplicity, we will assume that the air density, the reference area and the drag coefficient are all constant, and that

$$\frac{\rho C_D A}{2} = 1 \tag{49}$$

so that the total drag is $D = u^2(t)$. Then

$$u'(t) = -u(t)^2 + 1. \tag{50}$$

This reduces to a separable ODE which has solution:

$$u(t) = (e^{2t} - 1)/(e^{2t} + 1). \tag{51}$$

Thus $u(t) \to 1$, so we can take $u(t) = 1$ to be the velocity of our aircraft after some initial time has passed. We will now consider motion in two dimensions, following from the assumption that our velocity stabilizes at 1. Then our velocity is given by the vector $\mathbf{v}(t)$ where $\mathbf{v}(0) = (1, 0)$.

We will now consider the application of lift. For simplicity, we will assume the lift force is $L = 1$. Initially, this will be in the direction $(0, 1)$. We then also assume for simplicity that the lift-induced drag doubles the drag coefficient C_D, so that the total drag is $D = 2$. Thrust is in the direction of the velocity, drag is in the opposite direction, and lift is perpendicular to the velocity, so the differential equation that describes the motion of our aircraft is given by

$$\begin{aligned}\mathbf{v}'(t) &= -2\|\mathbf{v}(t)\|\mathbf{v}(t) + \frac{1}{\|\mathbf{v}(t)\|}\mathbf{v}(t) \\ &\quad + \frac{1}{\|\mathbf{v}(t)\|}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{v}(t).\end{aligned} \tag{52}$$

We can then consider in polar coordinates $\mathbf{v}(t) = (v(t), \theta(t))$. Then $v(t)$ must satisfy

$$v'(t) = -v(t)^2 + 1. \tag{53}$$

This differential equation has solution

$$v(t) = \frac{(3 + 2\sqrt{2})e^{2\sqrt{2}t} + 1}{\sqrt{2}((3 + 2\sqrt{2})e^{2\sqrt{2}t} - 1)}. \tag{54}$$

If we apply the centripetal force formula $F = \frac{mv^2}{r}$ we find that

$$\begin{aligned} r(t) = v(t)^2 &\text{ and } \theta'(t) = \frac{v(t)}{r(t)} \\ &\implies \theta'(t) = \frac{1}{v(t)} \\ &\implies \theta(t) = \int_0^t \frac{1}{v(x)}dx \\ &= \sqrt{2}t + \ln\left(e^{-2\sqrt{2}t} + 2\sqrt{2} + 3\right) - \ln\left(2\sqrt{2} + 4\right). \end{aligned} \tag{55}$$

so that in Cartesian coordinates

$$\mathbf{v}(t) = (v(t)\cos(\theta(t)), v(t)\sin(\theta(t))) \tag{56}$$

which satisfies (52). As t increases, we have

$$v(t) \to (\frac{1}{\sqrt{2}}\cos(\sqrt{2}t), \frac{1}{\sqrt{2}}\sin(\sqrt{2}t)). \tag{57}$$

This shows that the velocity of our aircraft stabilizes in uniform circular motion so that if we integrate the velocity, the trajectory of our aircraft will also stabilize in uniform circular motion. This allows us to represent them using parameterized turn circles.

For the application of a continuous roll, we assume that the torque is applied instantly, and that there is no roll-induced drag. This results in the differential equations (14) and (15) which show that the trajectory is a spiral.

We have shown that our kinematic model is an accurate representation, provided that the variances in the true drag and lift equations are negligible, and that any roll-induced drag is negligible.

9. Conclusions

This notion of turn circles and thinking geometrically proves effective for the case of aircraft pitching at a constant rate, and that it is by moving out of the circle and obtaining an oblique angle relative to the other aircraft's circle that it is possible to increase relative angular velocity. This paper formulates an advantage maneuver utilizing this idea. This maneuver can be repeatedly applied to gain advantage in the given combat scenario. Additionally, this paper provides a procedure that transforms any given initial position and orientation of a fighter aircraft to that in which an advantage maneuver can be applied. We have also shown that the results extend similarly to the case where the aircraft are yawing at a constant rate. Future work includes the consideration of gravity and testing in a simulation with six degrees of freedom.

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