From Calculus to Machine Learning

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- Identify a space of "reasonable" models
- Construct a function which computes how well a given model fits the data
- Find the model(s) which maximize (or minimize) the function

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In this seminar we'll try to understand some of the theory behind all three steps.

The plan:

1. Optimization for functions of one variable
2. Linear algebra and PCA
3. Optimization for functions of several variables
4. Conditional probability and Bayesian statistics
5. Linear regression
6. Perceptrons
7. Back propagation and gradient descent

Optimizing quadratic functions of one variable

You wish to build a rectangular fence next to a river. You have 100m of fence to work with and you want to enclose as much area as possible. How do you do it?

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- Each possible fence is determined by its height x and its width y

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- Constraints: $x \geq 0$, $y \geq 0$, and $2x + y = 100$

What is the objective function?

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We want to maximize the area $A(x, y) = xy$

This is now just a math problem: maximize $A(x, y) = xy$ subject to the constraints $x \geq 0$, $y \geq 0$, and $2x + y = 100$

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The objective function is quadratic and the constraint is linear, so we can hope to solve it analytically.

Using the constraint, eliminate y to get:

$$A(x) = x(100 - 2x)$$

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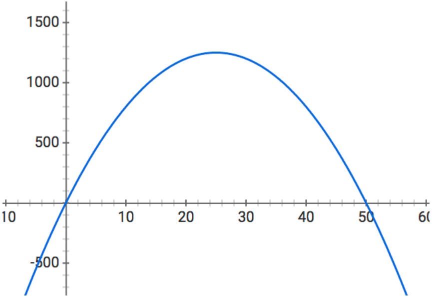

Algebraic magic:

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Since

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we get

$$A(x) \leq 1250$$

with equality if and only if $x = 25$

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Start with an expression of the form $x^2 + bx$.

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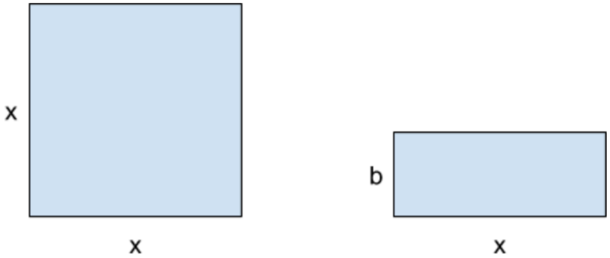

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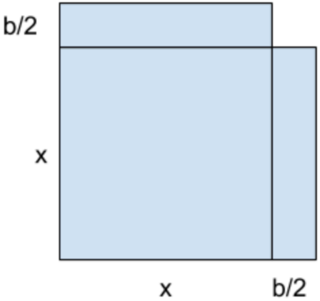

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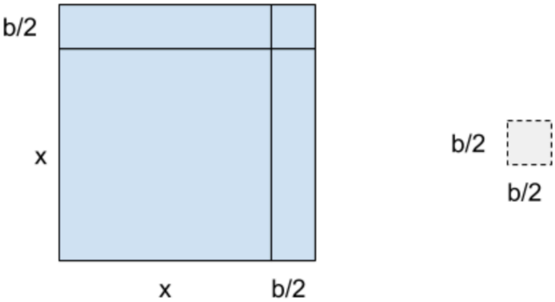

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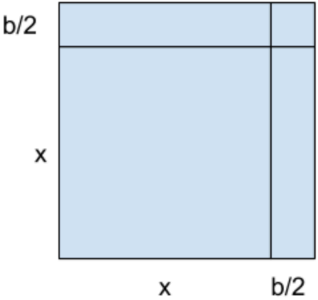

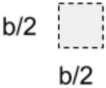

Conclusion:

$$x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4}$$

In the example above, it looks like this:

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$$\begin{aligned} A(x) &= x(100-2x) \\ &= -2x^2 + 100x \\ &= -2(x^2 - 50x) \\ &= -2((x-25)^2 - 625) \quad \text{magic!} \\ &= -2(x-25)^2 + 1250 \end{aligned}$$

You want to build an open box with a square base which holds $25m^3$ of water. How much material do you need?

Space of models:

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Each box is determined by the width x of the base and the height y

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Constraints: $x \geq 0$, $y \geq 0$, Volume $= x^2 y = 25$

Objective function:

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We want to minimize the surface area $S(x, y) = x^2 + 4xy$

Constrained optimization problem: minimize $S(x, y) = x^2 + 4xy$ subject to the constraints $x \geq 0$, $y \geq 0$, and $x^2 y = 25$.

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The objective function is quadratic, but the constraint $x^2 y = 25$ is cubic, so we expect this to be harder.

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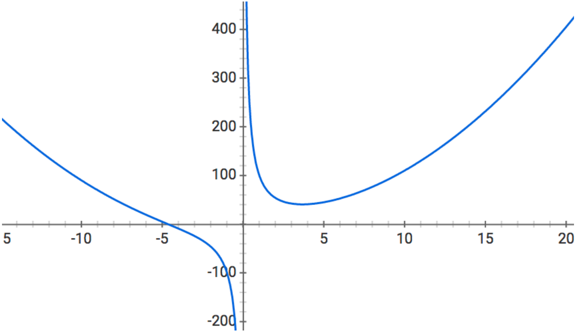

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Main idea: approximate a general function with a quadratic function.