Calculations for Transverse and Truncated ray transform pseudo differential operators.
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(* Use Alt 7 to make cells into comments *)

Tensor product

```
outer = Function[{t1, t2}, Outer[Times, t1, t2]]
```

```
General::spell1 : Possible spelling error: new
   symbol name "outer" is similar to existing symbol "Outer". More…
```

```
Function[{t1, t2}, Outer[Times, t1, t2]]
```

We want to make general symmetric rank two tensors as variables

AA = Table$[a_{\{i,j\}}, \{i, 1, 3\}, \{j, 1, 3\}]$

$\{\{a_{\{1,1\}}, a_{\{1,2\}}, a_{\{1,3\}}\}, \{a_{\{2,1\}}, a_{\{2,2\}}, a_{\{2,3\}}\}, \{a_{\{3,1\}}, a_{\{3,2\}}, a_{\{3,3\}}\}\}$

Now make it symmetric matrix by hand

A = $\{\{a_{\{1,1\}}, a_{\{1,2\}}, a_{\{1,3\}}\}, \{a_{\{1,2\}}, a_{\{2,2\}}, a_{\{2,3\}}\}, \{a_{\{1,3\}}, a_{\{2,3\}}, a_{\{3,3\}}\}\}$

$\{\{a_{\{1,1\}}, a_{\{1,2\}}, a_{\{1,3\}}\}, \{a_{\{1,2\}}, a_{\{2,2\}}, a_{\{2,3\}}\}, \{a_{\{1,3\}}, a_{\{2,3\}}, a_{\{3,3\}}\}\}$

```
A // MatrixForm
```

$$\begin{pmatrix} a_{\{1,1\}} & a_{\{1,2\}} & a_{\{1,3\}} \\ a_{\{1,2\}} & a_{\{2,2\}} & a_{\{2,3\}} \\ a_{\{1,3\}} & a_{\{2,3\}} & a_{\{3,3\}} \end{pmatrix}$$

Y will be the FT variable

Y = $\{y_1, y_2, y_3\}$

$\{y_1, y_2, y_3\}$

Some times we want to check what happens when we substitute for a specific choice of Y. This is used mainly when something doesnt work and it is usually clear what is wrong by making this substitution rather than looking at an enormous expression in ys

Y /. $\{y_1 \to 0, y_2 \text{->} 0, y_3 \to 1\}$

$\{0, 0, 1\}$

Functions for d and delta in the FT variable normalized by powers of Laplacian (as these commute thats ok)

δ^2

```
delta2 = Function[h, Y.h.Y / (Y.Y)]
```

$$\text{Function}\left[h, \frac{Y.h.Y}{Y.Y}\right]$$

```
delta2[A]
```

$$\frac{1}{y_1^2 + y_2^2 + y_3^2} (y_1 (a_{\{1,1\}} y_1 + a_{\{1,2\}} y_2 + a_{\{1,3\}} y_3) + y_2 (a_{\{1,2\}} y_1 + a_{\{2,2\}} y_2 + a_{\{2,3\}} y_3) + y_3 (a_{\{1,3\}} y_1 + a_{\{2,3\}} y_2 + a_{\{3,3\}} y_3))$$

dδ

```
ddelta = Function[h, ( outer[Y, Y].h + h.outer[Y, Y]) / (2 (Y.Y)) // Simplify]
```

$$\text{Function}\left[h,\ \text{Simplify}\left[\frac{\text{outer}[Y,\ Y].h+h.\text{outer}[Y,\ Y]}{2\ Y.Y}\right]\right]$$

```
ddelta[A] // MatrixForm
```

$$\begin{pmatrix} \frac{y_1\ (a_{\{1,1\}}\ y_1+a_{\{1,2\}}\ y_2+a_{\{1,3\}}\ y_3)}{y_1^2+y_2^2+y_3^2} & \frac{a_{\{1,1\}}\ y_1\ y_2+a_{\{2,2\}}\ y_1\ y_2+a_{\{1,2\}}\ (y_1^2+y_2^2)+a_{\{2,3\}}\ y_1\ y_3+a_{\{1,3\}}\ y_2\ y_3}{2\ (y_1^2+y_2^2+y_3^2)} & \frac{a_{\{2,3\}}\ y_1\ y_2+}{} \\ \frac{a_{\{1,1\}}\ y_1\ y_2+a_{\{2,2\}}\ y_1\ y_2+a_{\{1,2\}}\ (y_1^2+y_2^2)+a_{\{2,3\}}\ y_1\ y_3+a_{\{1,3\}}\ y_2\ y_3}{2\ (y_1^2+y_2^2+y_3^2)} & \frac{y_2\ (a_{\{1,2\}}\ y_1+a_{\{2,2\}}\ y_2+a_{\{2,3\}}\ y_3)}{y_1^2+y_2^2+y_3^2} & \frac{a_{\{1,3\}}\ y_1\ y_2+}{} \\ \frac{a_{\{2,3\}}\ y_1\ y_2+(a_{\{1,1\}}\ y_1+a_{\{3,3\}}\ y_1+a_{\{1,2\}}\ y_2)\ y_3+a_{\{1,3\}}\ (y_1^2+y_3^2)}{2\ (y_1^2+y_2^2+y_3^2)} & \frac{a_{\{1,3\}}\ y_1\ y_2+(a_{\{1,2\}}\ y_1+(a_{\{2,2\}}+a_{\{3,3\}})\ y_2)\ y_3+a_{\{2,3\}}\ (y_2^2+y_3^2)}{2\ (y_1^2+y_2^2+y_3^2)} & \end{pmatrix}$$

$d^2\ \delta^2$

```
d2delat2 = Function[h , delta2[h] outer[Y, Y] / (Y.Y) // Simplify]
```

$$\text{Function}\left[h,\ \text{Simplify}\left[\frac{\text{delta2}[h]\ \text{outer}[Y,\ Y]}{Y.Y}\right]\right]$$

```
d2delta2 = Function[h, Simplify[delta2[h] outer[Y, Y] / (Y.Y)]]
```

```
General::spell1 : Possible spelling error: new
   symbol name "d2delta2" is similar to existing symbol "d2delat2". More…
```

$$\text{Function}\left[h,\ \text{Simplify}\left[\frac{\text{delta2}[h]\ \text{outer}[Y,\ Y]}{Y.Y}\right]\right]$$

```
d2delta2[A] // MatrixForm
```

$$\begin{pmatrix} \frac{y_1^2\ (a_{\{1,1\}}\ y_1^2+2\ a_{\{1,2\}}\ y_1\ y_2+a_{\{2,2\}}\ y_2^2+2\ a_{\{1,3\}}\ y_1\ y_3+2\ a_{\{2,3\}}\ y_2\ y_3+a_{\{3,3\}}\ y_3^2)}{(y_1^2+y_2^2+y_3^2)^2} & \frac{y_1\ y_2\ (a_{\{1,1\}}\ y_1^2+2\ a_{\{1,2\}}\ y_1\ y_2+a_{\{2,2\}}\ y_2^2+2\ a_{\{1,3\}}\ y_1\ y_3+2\ a_{\{2,3\}}\ y_2}{(y_1^2+y_2^2+y_3^2)^2} \\ \frac{y_1\ y_2\ (a_{\{1,1\}}\ y_1^2+2\ a_{\{1,2\}}\ y_1\ y_2+a_{\{2,2\}}\ y_2^2+2\ a_{\{1,3\}}\ y_1\ y_3+2\ a_{\{2,3\}}\ y_2\ y_3+a_{\{3,3\}}\ y_3^2)}{(y_1^2+y_2^2+y_3^2)^2} & \frac{y_2^2\ (a_{\{1,1\}}\ y_1^2+2\ a_{\{1,2\}}\ y_1\ y_2+a_{\{2,2\}}\ y_2^2+2\ a_{\{1,3\}}\ y_1\ y_3+2\ a_{\{2,3\}}\ y_2\ y}{(y_1^2+y_2^2+y_3^2)^2} \\ \frac{y_1\ y_3\ (a_{\{1,1\}}\ y_1^2+2\ a_{\{1,2\}}\ y_1\ y_2+a_{\{2,2\}}\ y_2^2+2\ a_{\{1,3\}}\ y_1\ y_3+2\ a_{\{2,3\}}\ y_2\ y_3+a_{\{3,3\}}\ y_3^2)}{(y_1^2+y_2^2+y_3^2)^2} & \frac{y_2\ y_3\ (a_{\{1,1\}}\ y_1^2+2\ a_{\{1,2\}}\ y_1\ y_2+a_{\{2,2\}}\ y_2^2+2\ a_{\{1,3\}}\ y_1\ y_3+2\ a_{\{2,3\}}\ y_2}{(y_1^2+y_2^2+y_3^2)^2} \end{pmatrix}$$

Check some identities

```
Simplify[ddelta[d2delta2[A]] - d2delta2[A]] // Simplify // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

```
Simplify[ ddelta[ddelta[A]] - ddelta[A] / 2 - d2delta2[A] / 2] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

```
ddelta[ddelta[A]] - (1 / 2) ( ddelta[A] + d2delta2[A]) // Simplify
```

```
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

```
d2delta2[d2delta2[A]] - d2delta2[A] // Simplify
```

```
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

```
ddelta[ddelta[A]] - (1 / 2) ddelta[A] - (1 / 2) d2delta2[A] // Simplify
```

```
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

```
d2delat2[d2delta2[A]] - d2delta2[A] // Simplify
```

```
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

```
ddelta[eye delta2[A]] - d2delta2[A]
```

$$\{\{\frac{1}{2\ (y_1^2+y_2^2+y_3^2)}\ (\{\{y_1^2,\ y_1\,y_2,\ y_1\,y_3\},\ \{y_1\,y_2,\ y_2^2,\ y_2\,y_3\},\ \{y_1\,y_3,\ y_2\,y_3,\ y_3^2\}\}.\frac{1}{y_1^2+y_2^2+y_3^2}$$
$$\text{eye}\ (a_{\{1,1\}}\,y_1^2+2\,a_{\{1,2\}}\,y_1\,y_2+a_{\{2,2\}}\,y_2^2+2\,a_{\{1,3\}}\,y_1\,y_3+2\,a_{\{2,3\}}\,y_2\,y_3+a_{\{3,3\}}\,y_3^2)\ +$$
$$\frac{1}{y_1^2+y_2^2+y_3^2}\text{eye}\ (a_{\{1,1\}}\,y_1^2+2\,a_{\{1,2\}}\,y_1\,y_2+a_{\{2,2\}}\,y_2^2+2\,a_{\{1,3\}}\,y_1\,y_3+2\,a_{\{2,3\}}\,y_2\,y_3+a_{\{3,3\}}\,y_3^2)\,.$$
$$\{\{y_1^2,\ y_1\,y_2,\ y_1\,y_3\},\ \{y_1\,y_2,\ y_2^2,\ y_2\,y_3\},\ \{y_1\,y_3,\ y_2\,y_3,\ y_3^2\}\})\ -\ \frac{1}{(y_1^2+y_2^2+y_3^2)^2}$$
$$y_1^2\ (a_{\{1,1\}}\,y_1^2+2\,a_{\{1,2\}}\,y_1\,y_2+a_{\{2,2\}}\,y_2^2+2\,a_{\{1,3\}}\,y_1\,y_3+2\,a_{\{2,3\}}\,y_2\,y_3+a_{\{3,3\}}\,y_3^2),$$

$$\frac{1}{2\ (y_1^2+y_2^2+y_3^2)}\ (\{\{y_1^2,\ y_1\,y_2,\ y_1\,y_3\},\ \{y_1\,y_2,\ y_2^2,\ y_2\,y_3\},\ \{y_1\,y_3,\ y_2\,y_3,\ y_3^2\}\}.\frac{1}{y_1^2+y_2^2+y_3^2}$$
$$\text{eye}\ (a_{\{1,1\}}\,y_1^2+2\,a_{\{1,2\}}\,y_1\,y_2+a_{\{2,2\}}\,y_2^2+2\,a_{\{1,3\}}\,y_1\,y_3+2\,a_{\{2,3\}}\,y_2\,y_3+a_{\{3,3\}}\,y_3^2)\ +$$
$$\frac{1}{y_1^2+y_2^2+y_3^2}\text{eye}\ (a_{\{1,1\}}\,y_1^2+2\,a_{\{1,2\}}\,y_1\,y_2+a_{\{2,2\}}\,y_2^2+2\,a_{\{1,3\}}\,y_1\,y_3+2\,a_{\{2,3\}}\,y_2\,y_3+a_{\{3,3\}}\,y_3^2)\,.$$
$$\{\{y_1^2,\ y_1\,y_2,\ y_1\,y_3\},\ \{y_1\,y_2,\ y_2^2,\ y_2\,y_3\},\ \{y_1\,y_3,\ y_2\,y_3,\ y_3^2\}\})\ -\ \frac{1}{(y_1^2+y_2^2+y_3^2)^2}$$
$$y_1\,y_2\ (a_{\{1,1\}}\,y_1^2+2\,a_{\{1,2\}}\,y_1\,y_2+a_{\{2,2\}}\,y_2^2+2\,a_{\{1,3\}}\,y_1\,y_3+2\,a_{\{2,3\}}\,y_2\,y_3+a_{\{3,3\}}\,y_3^2),$$

$$\frac{1}{2\ (y_1^2+y_2^2+y_3^2)}\ (\{\{y_1^2,\ y_1\,y_2,\ y_1\,y_3\},\ \{y_1\,y_2,\ y_2^2,\ y_2\,y_3\},\ \{y_1\,y_3,\ y_2\,y_3,\ y_3^2\}\}.\frac{1}{y_1^2+y_2^2+y_3^2}$$
$$\text{eye}\ (a_{\{1,1\}}\,y_1^2+2\,a_{\{1,2\}}\,y_1\,y_2+a_{\{2,2\}}\,y_2^2+2\,a_{\{1,3\}}\,y_1\,y_3+2\,a_{\{2,3\}}\,y_2\,y_3+a_{\{3,3\}}\,y_3^2)\ +$$
$$\frac{1}{y_1^2+y_2^2+y_3^2}\text{eye}\ (a_{\{1,1\}}\,y_1^2+2\,a_{\{1,2\}}\,y_1\,y_2+a_{\{2,2\}}\,y_2^2+2\,a_{\{1,3\}}\,y_1\,y_3+2\,a_{\{2,3\}}\,y_2\,y_3+a_{\{3,3\}}\,y_3^2)\,.$$
$$\{\{y_1^2,\ y_1\,y_2,\ y_1\,y_3\},\ \{y_1\,y_2,\ y_2^2,\ y_2\,y_3\},\ \{y_1\,y_3,\ y_2\,y_3,\ y_3^2\}\})\ -\ \frac{1}{(y_1^2+y_2^2+y_3^2)^2}$$
$$y_1\,y_3\ (a_{\{1,1\}}\,y_1^2+2\,a_{\{1,2\}}\,y_1\,y_2+a_{\{2,2\}}\,y_2^2+2\,a_{\{1,3\}}\,y_1\,y_3+2\,a_{\{2,3\}}\,y_2\,y_3+a_{\{3,3\}}\,y_3^2)\},$$

$$\{\frac{1}{2\ (y_1^2+y_2^2+y_3^2)}\ (\{\{y_1^2,\ y_1\,y_2,\ y_1\,y_3\},\ \{y_1\,y_2,\ y_2^2,\ y_2\,y_3\},\ \{y_1\,y_3,\ y_2\,y_3,\ y_3^2\}\}.\frac{1}{y_1^2+y_2^2+y_3^2}$$
$$\text{eye}\ (a_{\{1,1\}}\,y_1^2+2\,a_{\{1,2\}}\,y_1\,y_2+a_{\{2,2\}}\,y_2^2+2\,a_{\{1,3\}}\,y_1\,y_3+2\,a_{\{2,3\}}\,y_2\,y_3+a_{\{3,3\}}\,y_3^2)\ +$$
$$\frac{1}{y_1^2+y_2^2+y_3^2}\text{eye}\ (a_{\{1,1\}}\,y_1^2+2\,a_{\{1,2\}}\,y_1\,y_2+a_{\{2,2\}}\,y_2^2+2\,a_{\{1,3\}}\,y_1\,y_3+2\,a_{\{2,3\}}\,y_2\,y_3+a_{\{3,3\}}\,y_3^2)\,.$$
$$\{\{y_1^2,\ y_1\,y_2,\ y_1\,y_3\},\ \{y_1\,y_2,\ y_2^2,\ y_2\,y_3\},\ \{y_1\,y_3,\ y_2\,y_3,\ y_3^2\}\})\ -\ \frac{1}{(y_1^2+y_2^2+y_3^2)^2}$$
$$y_1\,y_2\ (a_{\{1,1\}}\,y_1^2+2\,a_{\{1,2\}}\,y_1\,y_2+a_{\{2,2\}}\,y_2^2+2\,a_{\{1,3\}}\,y_1\,y_3+2\,a_{\{2,3\}}\,y_2\,y_3+a_{\{3,3\}}\,y_3^2),$$

$$\frac{1}{2\ (y_1^2+y_2^2+y_3^2)}\ (\{\{y_1^2,\ y_1\,y_2,\ y_1\,y_3\},\ \{y_1\,y_2,\ y_2^2,\ y_2\,y_3\},\ \{y_1\,y_3,\ y_2\,y_3,\ y_3^2\}\}.\frac{1}{y_1^2+y_2^2+y_3^2}$$
$$\text{eye}\ (a_{\{1,1\}}\,y_1^2+2\,a_{\{1,2\}}\,y_1\,y_2+a_{\{2,2\}}\,y_2^2+2\,a_{\{1,3\}}\,y_1\,y_3+2\,a_{\{2,3\}}\,y_2\,y_3+a_{\{3,3\}}\,y_3^2)\ +$$
$$\frac{1}{y_1^2+y_2^2+y_3^2}\text{eye}\ (a_{\{1,1\}}\,y_1^2+2\,a_{\{1,2\}}\,y_1\,y_2+a_{\{2,2\}}\,y_2^2+2\,a_{\{1,3\}}\,y_1\,y_3+2\,a_{\{2,3\}}\,y_2\,y_3+a_{\{3,3\}}\,y_3^2)\,.$$
$$\{\{y_1^2,\ y_1\,y_2,\ y_1\,y_3\},\ \{y_1\,y_2,\ y_2^2,\ y_2\,y_3\},\ \{y_1\,y_3,\ y_2\,y_3,\ y_3^2\}\})\ -\ \frac{1}{(y_1^2+y_2^2+y_3^2)^2}$$
$$y_2^2\ (a_{\{1,1\}}\,y_1^2+2\,a_{\{1,2\}}\,y_1\,y_2+a_{\{2,2\}}\,y_2^2+2\,a_{\{1,3\}}\,y_1\,y_3+2\,a_{\{2,3\}}\,y_2\,y_3+a_{\{3,3\}}\,y_3^2),$$

$$\frac{1}{2\ (y_1^2+y_2^2+y_3^2)}\ (\{\{y_1^2,\ y_1\,y_2,\ y_1\,y_3\},\ \{y_1\,y_2,\ y_2^2,\ y_2\,y_3\},\ \{y_1\,y_3,\ y_2\,y_3,\ y_3^2\}\}.\frac{1}{y_1^2+y_2^2+y_3^2}$$
$$\text{eye}\ (a_{\{1,1\}}\,y_1^2+2\,a_{\{1,2\}}\,y_1\,y_2+a_{\{2,2\}}\,y_2^2+2\,a_{\{1,3\}}\,y_1\,y_3+2\,a_{\{2,3\}}\,y_2\,y_3+a_{\{3,3\}}\,y_3^2)\ +$$

$$\frac{1}{y_1^2+y_2^2+y_3^2}\text{eye}\left(a_{\{1,1\}}\,y_1^2+2\,a_{\{1,2\}}\,y_1\,y_2+a_{\{2,2\}}\,y_2^2+2\,a_{\{1,3\}}\,y_1\,y_3+2\,a_{\{2,3\}}\,y_2\,y_3+a_{\{3,3\}}\,y_3^2\right).$$
$$\left.\{\{y_1^2,\ y_1\,y_2,\ y_1\,y_3\},\ \{y_1\,y_2,\ y_2^2,\ y_2\,y_3\},\ \{y_1\,y_3,\ y_2\,y_3,\ y_3^2\}\}\right)-\frac{1}{\left(y_1^2+y_2^2+y_3^2\right)^2}$$
$$y_2\,y_3\left(a_{\{1,1\}}\,y_1^2+2\,a_{\{1,2\}}\,y_1\,y_2+a_{\{2,2\}}\,y_2^2+2\,a_{\{1,3\}}\,y_1\,y_3+2\,a_{\{2,3\}}\,y_2\,y_3+a_{\{3,3\}}\,y_3^2\right)\Big\},$$
$$\Big\{\frac{1}{2\left(y_1^2+y_2^2+y_3^2\right)}\left(\{\{y_1^2,\ y_1\,y_2,\ y_1\,y_3\},\ \{y_1\,y_2,\ y_2^2,\ y_2\,y_3\},\ \{y_1\,y_3,\ y_2\,y_3,\ y_3^2\}\}.\frac{1}{y_1^2+y_2^2+y_3^2}\right.$$
$$\text{eye}\left(a_{\{1,1\}}\,y_1^2+2\,a_{\{1,2\}}\,y_1\,y_2+a_{\{2,2\}}\,y_2^2+2\,a_{\{1,3\}}\,y_1\,y_3+2\,a_{\{2,3\}}\,y_2\,y_3+a_{\{3,3\}}\,y_3^2\right)+$$
$$\frac{1}{y_1^2+y_2^2+y_3^2}\text{eye}\left(a_{\{1,1\}}\,y_1^2+2\,a_{\{1,2\}}\,y_1\,y_2+a_{\{2,2\}}\,y_2^2+2\,a_{\{1,3\}}\,y_1\,y_3+2\,a_{\{2,3\}}\,y_2\,y_3+a_{\{3,3\}}\,y_3^2\right).$$
$$\left.\{\{y_1^2,\ y_1\,y_2,\ y_1\,y_3\},\ \{y_1\,y_2,\ y_2^2,\ y_2\,y_3\},\ \{y_1\,y_3,\ y_2\,y_3,\ y_3^2\}\}\right)-\frac{1}{\left(y_1^2+y_2^2+y_3^2\right)^2}$$
$$y_1\,y_3\left(a_{\{1,1\}}\,y_1^2+2\,a_{\{1,2\}}\,y_1\,y_2+a_{\{2,2\}}\,y_2^2+2\,a_{\{1,3\}}\,y_1\,y_3+2\,a_{\{2,3\}}\,y_2\,y_3+a_{\{3,3\}}\,y_3^2\right),$$
$$\frac{1}{2\left(y_1^2+y_2^2+y_3^2\right)}\left(\{\{y_1^2,\ y_1\,y_2,\ y_1\,y_3\},\ \{y_1\,y_2,\ y_2^2,\ y_2\,y_3\},\ \{y_1\,y_3,\ y_2\,y_3,\ y_3^2\}\}.\frac{1}{y_1^2+y_2^2+y_3^2}\right.$$
$$\text{eye}\left(a_{\{1,1\}}\,y_1^2+2\,a_{\{1,2\}}\,y_1\,y_2+a_{\{2,2\}}\,y_2^2+2\,a_{\{1,3\}}\,y_1\,y_3+2\,a_{\{2,3\}}\,y_2\,y_3+a_{\{3,3\}}\,y_3^2\right)+$$
$$\frac{1}{y_1^2+y_2^2+y_3^2}\text{eye}\left(a_{\{1,1\}}\,y_1^2+2\,a_{\{1,2\}}\,y_1\,y_2+a_{\{2,2\}}\,y_2^2+2\,a_{\{1,3\}}\,y_1\,y_3+2\,a_{\{2,3\}}\,y_2\,y_3+a_{\{3,3\}}\,y_3^2\right).$$
$$\left.\{\{y_1^2,\ y_1\,y_2,\ y_1\,y_3\},\ \{y_1\,y_2,\ y_2^2,\ y_2\,y_3\},\ \{y_1\,y_3,\ y_2\,y_3,\ y_3^2\}\}\right)-\frac{1}{\left(y_1^2+y_2^2+y_3^2\right)^2}$$
$$y_2\,y_3\left(a_{\{1,1\}}\,y_1^2+2\,a_{\{1,2\}}\,y_1\,y_2+a_{\{2,2\}}\,y_2^2+2\,a_{\{1,3\}}\,y_1\,y_3+2\,a_{\{2,3\}}\,y_2\,y_3+a_{\{3,3\}}\,y_3^2\right),$$
$$\frac{1}{2\left(y_1^2+y_2^2+y_3^2\right)}\left(\{\{y_1^2,\ y_1\,y_2,\ y_1\,y_3\},\ \{y_1\,y_2,\ y_2^2,\ y_2\,y_3\},\ \{y_1\,y_3,\ y_2\,y_3,\ y_3^2\}\}.\frac{1}{y_1^2+y_2^2+y_3^2}\right.$$
$$\text{eye}\left(a_{\{1,1\}}\,y_1^2+2\,a_{\{1,2\}}\,y_1\,y_2+a_{\{2,2\}}\,y_2^2+2\,a_{\{1,3\}}\,y_1\,y_3+2\,a_{\{2,3\}}\,y_2\,y_3+a_{\{3,3\}}\,y_3^2\right)+$$
$$\frac{1}{y_1^2+y_2^2+y_3^2}\text{eye}\left(a_{\{1,1\}}\,y_1^2+2\,a_{\{1,2\}}\,y_1\,y_2+a_{\{2,2\}}\,y_2^2+2\,a_{\{1,3\}}\,y_1\,y_3+2\,a_{\{2,3\}}\,y_2\,y_3+a_{\{3,3\}}\,y_3^2\right).$$
$$\left.\{\{y_1^2,\ y_1\,y_2,\ y_1\,y_3\},\ \{y_1\,y_2,\ y_2^2,\ y_2\,y_3\},\ \{y_1\,y_3,\ y_2\,y_3,\ y_3^2\}\}\right)-\frac{1}{\left(y_1^2+y_2^2+y_3^2\right)^2}$$
$$y_3^2\left(a_{\{1,1\}}\,y_1^2+2\,a_{\{1,2\}}\,y_1\,y_2+a_{\{2,2\}}\,y_2^2+2\,a_{\{1,3\}}\,y_1\,y_3+2\,a_{\{2,3\}}\,y_2\,y_3+a_{\{3,3\}}\,y_3^2\right)\Big\}\Big\}$$

```
d2delta2[eye delta2[A]] - d2delta2[A]
```

$$\Big\{\Big\{\frac{1}{\left(y_1^2+y_2^2+y_3^2\right)^2}\{y_1,\ y_2,\ y_3\}.1\Big/\left(y_1^2+y_2^2+y_3^2\right)\text{eye}$$
$$\left(a_{\{1,1\}}\,y_1^2+2\,a_{\{1,2\}}\,y_1\,y_2+a_{\{2,2\}}\,y_2^2+2\,a_{\{1,3\}}\,y_1\,y_3+2\,a_{\{2,3\}}\,y_2\,y_3+a_{\{3,3\}}\,y_3^2\right).\{y_1,\ y_2,\ y_3\}\,y_1^2-$$
$$\frac{1}{\left(y_1^2+y_2^2+y_3^2\right)^2}y_1^2\left(a_{\{1,1\}}\,y_1^2+2\,a_{\{1,2\}}\,y_1\,y_2+a_{\{2,2\}}\,y_2^2+2\,a_{\{1,3\}}\,y_1\,y_3+2\,a_{\{2,3\}}\,y_2\,y_3+a_{\{3,3\}}\,y_3^2\right),$$
$$\frac{1}{\left(y_1^2+y_2^2+y_3^2\right)^2}\{y_1,\ y_2,\ y_3\}.1\Big/\left(y_1^2+y_2^2+y_3^2\right)\text{eye}\left(a_{\{1,1\}}\,y_1^2+2\,a_{\{1,2\}}\,y_1\,y_2+a_{\{2,2\}}\,y_2^2+\right.$$
$$\left.2\,a_{\{1,3\}}\,y_1\,y_3+2\,a_{\{2,3\}}\,y_2\,y_3+a_{\{3,3\}}\,y_3^2\right).\{y_1,\ y_2,\ y_3\}\,y_1\,y_2-\frac{1}{\left(y_1^2+y_2^2+y_3^2\right)^2}$$
$$y_1\,y_2\left(a_{\{1,1\}}\,y_1^2+2\,a_{\{1,2\}}\,y_1\,y_2+a_{\{2,2\}}\,y_2^2+2\,a_{\{1,3\}}\,y_1\,y_3+2\,a_{\{2,3\}}\,y_2\,y_3+a_{\{3,3\}}\,y_3^2\right),$$
$$\frac{1}{\left(y_1^2+y_2^2+y_3^2\right)^2}\{y_1,\ y_2,\ y_3\}.1\Big/\left(y_1^2+y_2^2+y_3^2\right)\text{eye}\left(a_{\{1,1\}}\,y_1^2+2\,a_{\{1,2\}}\,y_1\,y_2+a_{\{2,2\}}\,y_2^2+\right.$$

$$2\, a_{\{1,3\}}\, y_1\, y_3 + 2\, a_{\{2,3\}}\, y_2\, y_3 + a_{\{3,3\}}\, y_3^2\big) . \{y_1, y_2, y_3\}\, y_1\, y_3 - \frac{1}{\left(y_1^2 + y_2^2 + y_3^2\right)^2}$$

$$y_1\, y_3\, \left(a_{\{1,1\}}\, y_1^2 + 2\, a_{\{1,2\}}\, y_1\, y_2 + a_{\{2,2\}}\, y_2^2 + 2\, a_{\{1,3\}}\, y_1\, y_3 + 2\, a_{\{2,3\}}\, y_2\, y_3 + a_{\{3,3\}}\, y_3^2\right)\Big\},$$

$$\Big\{\frac{1}{\left(y_1^2 + y_2^2 + y_3^2\right)^2} \{y_1, y_2, y_3\} . 1 \Big/ \left(y_1^2 + y_2^2 + y_3^2\right) \text{eye}\, \big(a_{\{1,1\}}\, y_1^2 + 2\, a_{\{1,2\}}\, y_1\, y_2 + a_{\{2,2\}}\, y_2^2 +$$

$$2\, a_{\{1,3\}}\, y_1\, y_3 + 2\, a_{\{2,3\}}\, y_2\, y_3 + a_{\{3,3\}}\, y_3^2\big) . \{y_1, y_2, y_3\}\, y_1\, y_2 - \frac{1}{\left(y_1^2 + y_2^2 + y_3^2\right)^2}$$

$$y_1\, y_2\, \left(a_{\{1,1\}}\, y_1^2 + 2\, a_{\{1,2\}}\, y_1\, y_2 + a_{\{2,2\}}\, y_2^2 + 2\, a_{\{1,3\}}\, y_1\, y_3 + 2\, a_{\{2,3\}}\, y_2\, y_3 + a_{\{3,3\}}\, y_3^2\right),$$

$$\frac{1}{\left(y_1^2 + y_2^2 + y_3^2\right)^2} \{y_1, y_2, y_3\} . 1 \Big/ \left(y_1^2 + y_2^2 + y_3^2\right) \text{eye}$$

$$\left(a_{\{1,1\}}\, y_1^2 + 2\, a_{\{1,2\}}\, y_1\, y_2 + a_{\{2,2\}}\, y_2^2 + 2\, a_{\{1,3\}}\, y_1\, y_3 + 2\, a_{\{2,3\}}\, y_2\, y_3 + a_{\{3,3\}}\, y_3^2\right) . \{y_1, y_2, y_3\}\, y_2^2 -$$

$$\frac{1}{\left(y_1^2 + y_2^2 + y_3^2\right)^2} y_2^2\, \left(a_{\{1,1\}}\, y_1^2 + 2\, a_{\{1,2\}}\, y_1\, y_2 + a_{\{2,2\}}\, y_2^2 + 2\, a_{\{1,3\}}\, y_1\, y_3 + 2\, a_{\{2,3\}}\, y_2\, y_3 + a_{\{3,3\}}\, y_3^2\right),$$

$$\frac{1}{\left(y_1^2 + y_2^2 + y_3^2\right)^2} \{y_1, y_2, y_3\} . 1 \Big/ \left(y_1^2 + y_2^2 + y_3^2\right) \text{eye}\, \big(a_{\{1,1\}}\, y_1^2 + 2\, a_{\{1,2\}}\, y_1\, y_2 + a_{\{2,2\}}\, y_2^2 +$$

$$2\, a_{\{1,3\}}\, y_1\, y_3 + 2\, a_{\{2,3\}}\, y_2\, y_3 + a_{\{3,3\}}\, y_3^2\big) . \{y_1, y_2, y_3\}\, y_2\, y_3 - \frac{1}{\left(y_1^2 + y_2^2 + y_3^2\right)^2}$$

$$y_2\, y_3\, \left(a_{\{1,1\}}\, y_1^2 + 2\, a_{\{1,2\}}\, y_1\, y_2 + a_{\{2,2\}}\, y_2^2 + 2\, a_{\{1,3\}}\, y_1\, y_3 + 2\, a_{\{2,3\}}\, y_2\, y_3 + a_{\{3,3\}}\, y_3^2\right)\Big\},$$

$$\Big\{\frac{1}{\left(y_1^2 + y_2^2 + y_3^2\right)^2} \{y_1, y_2, y_3\} . 1 \Big/ \left(y_1^2 + y_2^2 + y_3^2\right) \text{eye}\, \big(a_{\{1,1\}}\, y_1^2 + 2\, a_{\{1,2\}}\, y_1\, y_2 + a_{\{2,2\}}\, y_2^2 +$$

$$2\, a_{\{1,3\}}\, y_1\, y_3 + 2\, a_{\{2,3\}}\, y_2\, y_3 + a_{\{3,3\}}\, y_3^2\big) . \{y_1, y_2, y_3\}\, y_1\, y_3 - \frac{1}{\left(y_1^2 + y_2^2 + y_3^2\right)^2}$$

$$y_1\, y_3\, \left(a_{\{1,1\}}\, y_1^2 + 2\, a_{\{1,2\}}\, y_1\, y_2 + a_{\{2,2\}}\, y_2^2 + 2\, a_{\{1,3\}}\, y_1\, y_3 + 2\, a_{\{2,3\}}\, y_2\, y_3 + a_{\{3,3\}}\, y_3^2\right),$$

$$\frac{1}{\left(y_1^2 + y_2^2 + y_3^2\right)^2} \{y_1, y_2, y_3\} . 1 \Big/ \left(y_1^2 + y_2^2 + y_3^2\right) \text{eye}\, \big(a_{\{1,1\}}\, y_1^2 + 2\, a_{\{1,2\}}\, y_1\, y_2 + a_{\{2,2\}}\, y_2^2 +$$

$$2\, a_{\{1,3\}}\, y_1\, y_3 + 2\, a_{\{2,3\}}\, y_2\, y_3 + a_{\{3,3\}}\, y_3^2\big) . \{y_1, y_2, y_3\}\, y_2\, y_3 - \frac{1}{\left(y_1^2 + y_2^2 + y_3^2\right)^2}$$

$$y_2\, y_3\, \left(a_{\{1,1\}}\, y_1^2 + 2\, a_{\{1,2\}}\, y_1\, y_2 + a_{\{2,2\}}\, y_2^2 + 2\, a_{\{1,3\}}\, y_1\, y_3 + 2\, a_{\{2,3\}}\, y_2\, y_3 + a_{\{3,3\}}\, y_3^2\right),$$

$$\frac{1}{\left(y_1^2 + y_2^2 + y_3^2\right)^2} \{y_1, y_2, y_3\} . 1 \Big/ \left(y_1^2 + y_2^2 + y_3^2\right) \text{eye}$$

$$\left(a_{\{1,1\}}\, y_1^2 + 2\, a_{\{1,2\}}\, y_1\, y_2 + a_{\{2,2\}}\, y_2^2 + 2\, a_{\{1,3\}}\, y_1\, y_3 + 2\, a_{\{2,3\}}\, y_2\, y_3 + a_{\{3,3\}}\, y_3^2\right) . \{y_1, y_2, y_3\}\, y_3^2 -$$

$$\frac{1}{\left(y_1^2 + y_2^2 + y_3^2\right)^2} y_3^2\, \left(a_{\{1,1\}}\, y_1^2 + 2\, a_{\{1,2\}}\, y_1\, y_2 + a_{\{2,2\}}\, y_2^2 + 2\, a_{\{1,3\}}\, y_1\, y_3 + 2\, a_{\{2,3\}}\, y_2\, y_3 + a_{\{3,3\}}\, y_3^2\right)\Big\}\Big\}$$

```
delta2[ddelta[A]] - delta2[A] // Simplify
```

0

```
delta2[d2delta2[A]] - delta2[A] // Simplify
```

0

```
ddelta[eye delta2[A]] - d2delta2[A] // Simplify // MatrixForm
```

```
( ({{y1^2, y1 y2, y1 y3}, {y1 y2, y2^2, y2 y3}, {y1 y3, y2 y3, y3^2}} . eye (a{1,1} y1^2 + 2 a{1,2} y1 y2 + a{2,2} y2^2 + 2 a{1,3} y1 y3 + 2 a{2,3} y2 y3 + a{3,3} y3^2)/(y1^2 + y2^2 + y3^2) + eye (a{1,1} y1^2 + 2 a{1,2} y1 y2 + a{2,
  ({{y1^2, y1 y2, y1 y3}, {y1 y2, y2^2, y2 y3}, {y1 y3, y2 y3, y3^2}} . eye (a{1,1} y1^2 + 2 a{1,2} y1 y2 + a{2,2} y2^2 + 2 a{1,3} y1 y3 + 2 a{2,3} y2 y3 + a{3,3} y3^2)/(y1^2 + y2^2 + y3^2) + eye (a{1,1} y1^2 + 2 a{1,2} y1 y2 + a{2,2}
  ({{y1^2, y1 y2, y1 y3}, {y1 y2, y2^2, y2 y3}, {y1 y3, y2 y3, y3^2}} . eye (a{1,1} y1^2 + 2 a{1,2} y1 y2 + a{2,2} y2^2 + 2 a{1,3} y1 y3 + 2 a{2,3} y2 y3 + a{3,3} y3^2)/(y1^2 + y2^2 + y3^2) + eye (a{1,1} y1^2 + 2 a{1,2} y1 y2 + a{2,2}
```

Now checking that I can do 6.6.10 and 6.6.11 with MMA. First 6.6.10 define operators. This is my Π as an operator, Sharafutdinov's ε

```
Proj = Function[a, a - outer[Y, Y].a / (Y.Y)]
```

$$\text{Function}\left[a,\ a-\frac{\text{outer}[Y,Y].a}{Y.Y}\right]$$

And Π as a matrix

```
Projm = Proj[IdentityMatrix[3]]
```

```
General::spell1 :
 Possible spelling error: new symbol name "Projm" is similar to existing symbol "Proj". More…
```

$$\left\{\left\{1-\frac{y_1^2}{y_1^2+y_2^2+y_3^2},\ -\frac{y_1\,y_2}{y_1^2+y_2^2+y_3^2},\ -\frac{y_1\,y_3}{y_1^2+y_2^2+y_3^2}\right\},\ \left\{-\frac{y_1\,y_2}{y_1^2+y_2^2+y_3^2},\ 1-\frac{y_2^2}{y_1^2+y_2^2+y_3^2},\ -\frac{y_2\,y_3}{y_1^2+y_2^2+y_3^2}\right\},\ \left\{-\frac{y_1\,y_3}{y_1^2+y_2^2+y_3^2},\ -\frac{y_2\,y_3}{y_1^2+y_2^2+y_3^2},\ 1-\frac{y_3^2}{y_1^2+y_2^2+y_3^2}\right\}\right\}$$

This is my tilde Π or Sharafutdinov's $\varepsilon 2$.

```
Proj2 = Function[a, (2 / 3) (Projm.a.Projm) + (1 / 3) Tr[Projm.a] Projm]
```

$$\text{Function}\left[a,\ \frac{2\,\text{Projm.a.Projm}}{3}+\frac{1}{3}\,\text{Tr}[\text{Projm.a}]\ \text{Projm}\right]$$

The set of permutations on three symbols

```
perms = Permutations[{i, j, k, l}]
```

```
{{i, j, k, l}, {i, j, l, k}, {i, k, j, l}, {i, k, l, j}, {i, l, j, k}, {i, l, k, j},
 {j, i, k, l}, {j, i, l, k}, {j, k, i, l}, {j, k, l, i}, {j, l, i, k}, {j, l, k, i},
 {k, i, j, l}, {k, i, l, j}, {k, j, i, l}, {k, j, l, i}, {k, l, i, j}, {k, l, j, i},
 {l, i, j, k}, {l, i, k, j}, {l, j, i, k}, {l, j, k, i}, {l, k, i, j}, {l, k, j, i}}
```

Here is the offical definition of epsilon2 using symmetry
ie (1/24) $\sum_\pi \sum_i \sum_j \Pi_{\pi(i)\,\pi(j)} \Pi_{\pi(k)\,\pi(l)}\, A_{\mathrm{ij}}$

```
Proj2test = (1 / 24)
   Table[Sum[Sum[Sum[ Projm[[perms[[m, 1]], perms[[m, 2]] ]] Projm[[perms[[m, 3]] ,
         perms[[m, 4]] ]] A[[i, j]], {i, 3}], {j, 3}], {m, 24}], {k, 3}, {l, 3}];
```

```
Proj2test - Proj2[A] // Simplify // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Also notice the formula

```
Proj2[A] - ( (2 / 3) A + (-2 / 3) (outer[Y, Y].A + A.outer[Y, Y]) / (Y.Y) +
      (1 / 3) Tr[A] IdentityMatrix[3] + (1) (Y.A.Y) outer[Y, Y] +
      (-1 / 3) (Y.A.Y) IdentityMatrix[3] / (Y.Y) + (-1 / 3) Tr[A] outer[Y, Y] / (Y.Y)) /.
   {y1 → 0, y2 → 0, y3 → 1} // Simplify // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

```
eye = IdentityMatrix[3]; YoY = outer[Y, Y] / (Y.Y);
```

```
Proj2[A] - ( (2 / 3) A - (2 / 3) (YoY.A + A.YoY) + (Y.A.Y) YoY / (Y.Y) + (1 / 3) (Tr[A]) eye -
     (1 / 3) Y.A.Y / (Y.Y) eye - (1 / 3) (Tr[A]) YoY) // Simplify // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We have now checked the formula for curly pi a.k.a epsilon2.

Now for 6.6.10

```
D2i = Function[a, a - (1 / 2) ( a.Projm + Projm.a) +
   (3 / 16) Proj2[a] + (1 / 4) Tr[Projm.a] IdentityMatrix[3]]
```

$$\text{Function}\left[a,\ a - \frac{1}{2}\,(a.\text{Projm} + \text{Projm}.a) + \frac{3\,\text{Proj2}[a]}{16} + \frac{1}{4}\,\text{Tr}[\text{Projm}.a]\ \text{IdentityMatrix}[3]\right]$$

```
D2iY = Function[a, (1 / 8) a + (3 / 8) (outer[Y, Y] .a + a .outer[Y, Y]) / (Y.Y) +
   (3 / 16) (Y.a.Y) outer[Y, Y] / ((Y.Y) ^2) + (-5 / 16) (Y.a.Y) IdentityMatrix[3] / (Y.Y)]
```

```
General::spell1 :
 Possible spelling error: new symbol name "D2iY" is similar to existing symbol "D2i". More…
```

$$\text{Function}\left[a,\ \frac{a}{8} + \frac{3\,(\text{outer}[Y, Y].a + a.\text{outer}[Y, Y])}{8\,Y.Y} + \frac{3\,Y.a.Y\ \text{outer}[Y, Y]}{16\,(Y.Y)^2} - \frac{5\,Y.a.Y\ \text{IdentityMatrix}[3]}{16\,Y.Y}\right]$$

```
(D2i[Atf] - D2iY[Atf]) /. {y1 → 0, y2 → 0, y3 → 1} // Simplify // MatrixForm
```

$$\begin{pmatrix} \frac{1}{16}\,(14\,\text{Atf} - 6\,\text{Atf}.\{\{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 1\}\} - 8\,\text{Atf}.\{\{1, 0, 0\}, \{0, 1, 0\}, \{0, 0, 0\}\} - 6 \\ \frac{1}{8}\,(7\,\text{Atf} - 3\,\text{Atf}.\{\{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 1\}\} - 4\,\text{Atf}. \\ \frac{1}{8}\,(7\,\text{Atf} - 3\,\text{Atf}.\{\{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 1\}\} - 4\,\text{Atf}. \end{pmatrix}$$

And here is 6.6.11

```
D2 = Function[a, 2 (4 a - 3 ( a.outer[Y, Y] / (Y.Y) + outer[Y, Y].a / (Y.Y)) +
     (Y.a.Y) outer[Y, Y] / (Y.Y) ^2 + (5 / 3) (Y.a.Y) IdentityMatrix[3] / (Y.Y))]
```

$$\text{Function}\left[a,\ 2\left(4\,a - 3\left(\frac{a.\text{outer}[Y, Y]}{Y.Y} + \frac{\text{outer}[Y, Y].a}{Y.Y}\right) + \frac{Y.a.Y\ \text{outer}[Y, Y]}{(Y.Y)^2} + \frac{5\,Y.a.Y\ \text{IdentityMatrix}[3]}{3\,Y.Y}\right)\right]$$

```
D2[D2i[A]] /. {y1 → 0, y2 → 0, y3 → 1} // Simplify // MatrixForm
```

$$\begin{pmatrix} \frac{1}{3}\,(13\,a_{\{1,1\}} + 10\,(a_{\{2,2\}} + a_{\{3,3\}})) & a_{\{1,2\}} & a_{\{1,3\}} \\ a_{\{1,2\}} & \frac{1}{3}\,(10\,a_{\{1,1\}} + 13\,a_{\{2,2\}} + 10\,a_{\{3,3\}}) & a_{\{2,3\}} \\ a_{\{1,3\}} & a_{\{2,3\}} & \frac{1}{3}\,(a_{\{1,1\}} + a_{\{2,2\}} + 4\,a_{\{3,3\}}) \end{pmatrix}$$

Requires a trace free matrix

```
Atf = {{a{1,1}, a{1,2}, a{1,3}}, {a{1,2}, a{2,2}, a{2,3}}, {a{1,3}, a{2,3}, -a{1,1} - a{2,2}}}
```

$\{\{a_{\{1,1\}}, a_{\{1,2\}}, a_{\{1,3\}}\}, \{a_{\{1,2\}}, a_{\{2,2\}}, a_{\{2,3\}}\}, \{a_{\{1,3\}}, a_{\{2,3\}}, -a_{\{1,1\}} - a_{\{2,2\}}\}\}$

```
Tr[Atf]
```

0

```
D2i[D2[Atf]] /. {y1 → 0, y2 → 0, y3 → 1} // Simplify // MatrixForm
```

$$\begin{pmatrix} a_{\{1,1\}} & a_{\{1,2\}} & a_{\{1,3\}} \\ a_{\{1,2\}} & a_{\{2,2\}} & a_{\{2,3\}} \\ a_{\{1,3\}} & a_{\{2,3\}} & -a_{\{1,1\}} - a_{\{2,2\}} \end{pmatrix}$$

```
D2i[D2[Atf]] // Simplify // MatrixForm
```

$$\begin{pmatrix} a_{\{1,1\}} & a_{\{1,2\}} & a_{\{1,3\}} \\ a_{\{1,2\}} & a_{\{2,2\}} & a_{\{2,3\}} \\ a_{\{1,3\}} & a_{\{2,3\}} & -a_{\{1,1\}} - a_{\{2,2\}} \end{pmatrix}$$

```
D2[D2i[Atf]] // Simplify // MatrixForm
```

$$\begin{pmatrix} a_{\{1,1\}} & a_{\{1,2\}} & a_{\{1,3\}} \\ a_{\{1,2\}} & a_{\{2,2\}} & a_{\{2,3\}} \\ a_{\{1,3\}} & a_{\{2,3\}} & -a_{\{1,1\}} - a_{\{2,2\}} \end{pmatrix}$$

```
maketracefree = Function[a, a - (1 / 3) Tr[a] IdentityMatrix[3]]
```

$$\text{Function}\left[a,\ a - \frac{1}{3}\,\text{Tr}[a]\ \text{IdentityMatrix}[3]\right]$$

```
maketracefree[Atf]
```

$\{\{a_{\{1,1\}}, a_{\{1,2\}}, a_{\{1,3\}}\}, \{a_{\{1,2\}}, a_{\{2,2\}}, a_{\{2,3\}}\}, \{a_{\{1,3\}}, a_{\{2,3\}}, -a_{\{1,1\}} - a_{\{2,2\}}\}\}$

```
D2[D2i[Atf]] // Simplify // MatrixForm
```

$$\begin{pmatrix} a_{\{1,1\}} & a_{\{1,2\}} & a_{\{1,3\}} \\ a_{\{1,2\}} & a_{\{2,2\}} & a_{\{2,3\}} \\ a_{\{1,3\}} & a_{\{2,3\}} & -a_{\{1,1\}} - a_{\{2,2\}} \end{pmatrix}$$

```
Tr[D2[D2i[Atf]]] // Simplify
```

0

So now checking 6.6.11 is a solution of 6.6.10 by direct substitution

```
D2i[D2[Atf]] // Simplify // MatrixForm
```

$$\begin{pmatrix} a_{\{1,1\}} & a_{\{1,2\}} & a_{\{1,3\}} \\ a_{\{1,2\}} & a_{\{2,2\}} & a_{\{2,3\}} \\ a_{\{1,3\}} & a_{\{2,3\}} & -a_{\{1,1\}} - a_{\{2,2\}} \end{pmatrix}$$

```
Tr[D2i[A]] // Simplify
```

$a_{\{1,1\}} + a_{\{2,2\}} + a_{\{3,3\}}$

```
Tr[D2i[Atf]] // Simplify
```

0

```
Tr[D2[Atf]] // Simplify
```

0

```
Tr[D2[A]] // Simplify
```

$8\,(a_{\{1,1\}} + a_{\{2,2\}} + a_{\{3,3\}})$

Now testing the formula in Thm 6 6 2 for equivalence with (6.6.1)

```
D2d =
 Function[a, 2 (4 a - 6 ddelta[a] + d2delta2[a] + (5 / 3) IdentityMatrix[3] delta2[a])]
```

$$\text{Function}\left[a,\ 2\left(4\,a - 6\,\text{ddelta}[a] + \text{d2delta2}[a] + \frac{5}{3}\ \text{IdentityMatrix}[3]\ \text{delta2}[a]\right)\right]$$

```
D2d[Atf] - D2[Atf] // Simplify // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

And that is ok as well, giving us confidence in our d and delta opporators

Now for the TRT

Constructing a formula for D1 inverse (like 6.6.10)

```
D1i = Function[a, a - (1 / 2) (Projm.a + a.Projm) + (3 / 8) Proj2[a]]
```

$$\text{Function}\left[a,\ a - \frac{1}{2}\ (\text{Projm.a} + \text{a.Projm}) + \frac{3\ \text{Proj2}[a]}{8}\right]$$

Now test with just Pi

```
D1i1 = Function[a,
  a - (1 / 2) (Projm.a + a.Projm) + (1 / 4) Projm.a.Projm + (1 / 8) Tr[Projm.a] Projm]
```

$$\text{Function}\left[a,\ a - \frac{1}{2}\ (\text{Projm.a} + \text{a.Projm}) + \frac{\text{Projm.a.Projm}}{4} + \frac{1}{8}\ \text{Tr}[\text{Projm.a}]\ \text{Projm}\right]$$

```
D1i[A] - D1i1[A] // Simplify // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

```
D1i2 = Function[a, (1 / 4) a + (1 / 4) (outer[Y, Y].a / (Y.Y) + a.outer[Y, Y] / (Y.Y)) +
   (3 / 8) (Y.a.Y) outer[Y, Y] / (Y.Y) ^2 + (1 / 8) Tr[a]
    (IdentityMatrix[3] - outer[Y, Y] / (Y.Y)) - (1 / 8) (Y.a.Y) IdentityMatrix[3] / (Y.Y)]
```

$$\text{Function}\left[a,\ \frac{a}{4} + \frac{1}{4}\left(\frac{\text{outer}[Y, Y].a}{Y.Y} + \frac{a.\text{outer}[Y, Y]}{Y.Y}\right) + \frac{3\ Y.a.Y\ \text{outer}[Y, Y]}{8\ (Y.Y)^2} + \frac{1}{8}\ \text{Tr}[a]\left(\text{IdentityMatrix}[3] - \frac{\text{outer}[Y, Y]}{Y.Y}\right) - \frac{Y.a.Y\ \text{IdentityMatrix}[3]}{8\ Y.Y}\right]$$

```
D1i[A] - D1i2[A] // Simplify // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

```
D1i[A] - D1i2[A] /. {y1 → 0, y2 -> 0, y3 → 1} // Simplify // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

```
D1test = Function[{a, k1, k2, k3, k4, k5},
  4 a - k4 (outer[Y, Y].a / (Y.Y) + a.outer[Y, Y] / (Y.Y)) +
   k1 (Y.a.Y) outer[Y, Y] / (Y.Y) ^2 + k2 Tr[a] (IdentityMatrix[3] - outer[Y, Y] / (Y.Y)) +
   k3 (Y.a.Y) IdentityMatrix[3] / (Y.Y) + k5 (Y.a.Y) outer[Y, Y] / (Y.Y) ^2]
```

$$\text{Function}\left[\{a, k1, k2, k3, k4, k5\},\ 4\,a - k4\left(\frac{\text{outer}[Y, Y].a}{Y.Y} + \frac{a.\text{outer}[Y, Y]}{Y.Y}\right) + \frac{k1\ Y.a.Y\ \text{outer}[Y, Y]}{(Y.Y)^2} + k2\ \text{Tr}[a]\left(\text{IdentityMatrix}[3] - \frac{\text{outer}[Y, Y]}{Y.Y}\right) + \frac{k3\ Y.a.Y\ \text{IdentityMatrix}[3]}{Y.Y} + \frac{k5\ Y.a.Y\ \text{outer}[Y, Y]}{(Y.Y)^2}\right]$$

```
tst = D1i[D1test[A, k1, k2, k3, k4, k5]] - A /. {y1 → 0, y2 -> 0, y3 → 1} // Simplify //
   MatrixForm;
tst
```

$$\begin{pmatrix} \frac{1}{2}\left((1+k2)\, a_{\{1,1\}} + (1+k2)\, a_{\{2,2\}} + (k2+k3)\, a_{\{3,3\}}\right) & 0 \\ 0 & \frac{1}{2}\left((1+k2)\, a_{\{1,1\}} + (1+k2)\, a_{\{2,2\}} + (k2+k3)\, a_{\{3} \\ -\frac{1}{2}(-2+k4)\, a_{\{1,3\}} & -\frac{1}{2}(-2+k4)\, a_{\{2,3\}} \end{pmatrix}$$

```
tst /. {k2 → -1, k3 → 1, k1 → 4, k4 → 2, k5 → -4}
```

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So Mathematica helped here to find D1

```
D1 = Function[a, 4 a - 2 (outer[Y, Y].a / (Y.Y) + a.outer[Y, Y] / (Y.Y)) -
   Tr[a] (IdentityMatrix[3] - outer[Y, Y] / (Y.Y)) + (Y.a.Y) IdentityMatrix[3] / (Y.Y) ]
```

$$\text{Function}\left[a,\ 4\,a - 2\left(\frac{\text{outer}[Y, Y].a}{Y.Y} + \frac{a.\text{outer}[Y, Y]}{Y.Y}\right) - \text{Tr}[a]\left(\text{IdentityMatrix}[3] - \frac{\text{outer}[Y, Y]}{Y.Y}\right) + \frac{Y.a.Y\ \text{IdentityMatrix}[3]}{Y.Y}\right]$$

```
D1i[D1[A]] // Simplify // MatrixForm
```

$$\begin{pmatrix} a_{\{1,1\}} & a_{\{1,2\}} & a_{\{1,3\}} \\ a_{\{1,2\}} & a_{\{2,2\}} & a_{\{2,3\}} \\ a_{\{1,3\}} & a_{\{2,3\}} & a_{\{3,3\}} \end{pmatrix}$$

```
D1[D1i[A]] // Simplify // MatrixForm
```

$$\begin{pmatrix} a_{\{1,1\}} & a_{\{1,2\}} & a_{\{1,3\}} \\ a_{\{1,2\}} & a_{\{2,2\}} & a_{\{2,3\}} \\ a_{\{1,3\}} & a_{\{2,3\}} & a_{\{3,3\}} \end{pmatrix}$$

So to me that looks like a left and right inverse!

```
dd = Function[a, outer[Y, Y] a / (Y.Y)]
```

$$\text{Function}\left[a,\ \frac{\text{outer}[Y, Y]\ a}{Y.Y}\right]$$

```
D1d = Function[a,
  4 a - 4 ddelta[a] - Tr[a] IdentityMatrix[3] + dd[Tr[a]] + delta2[a] IdentityMatrix[3]]
```

```
Function[a,
 4 a - 4 ddelta[a] - Tr[a] IdentityMatrix[3] + dd[Tr[a]] + delta2[a] IdentityMatrix[3]]
```

```
D1i[D1d[A]] // Simplify // MatrixForm
```

$$\begin{pmatrix} a_{\{1,1\}} & a_{\{1,2\}} & a_{\{1,3\}} \\ a_{\{1,2\}} & a_{\{2,2\}} & a_{\{2,3\}} \\ a_{\{1,3\}} & a_{\{2,3\}} & a_{\{3,3\}} \end{pmatrix}$$

```
D1id = Function[a, (1 / 4) a + (1 / 2) ddelta[a]
   + (3 / 8) d2delta2[a] + (1 / 8) Tr[a] IdentityMatrix[3] -
   (1 / 8) dd[Tr[a]] - (1 / 8) delta2[a] IdentityMatrix[3]]
```

```
General::spell : Possible spelling error: new
   symbol name "D1id" is similar to existing symbols {D1d, D1i}. More…
```

$$\text{Function}\left[a,\ \frac{a}{4} + \frac{\text{ddelta}[a]}{2} + \frac{3\ \text{d2delta2}[a]}{8} + \frac{1}{8}\text{Tr}[a]\ \text{IdentityMatrix}[3] - \frac{1}{8}\text{dd}[\text{Tr}[a]] - \frac{1}{8}\text{delta2}[a]\ \text{IdentityMatrix}[3]\right]$$

```
D1id[D1[A]] /. {y1 → 0, y2 -> 0, y3 → 1} // Simplify // MatrixForm
```

$$\begin{pmatrix} a_{\{1,1\}} & a_{\{1,2\}} & a_{\{1,3\}} \\ a_{\{1,2\}} & a_{\{2,2\}} & a_{\{2,3\}} \\ a_{\{1,3\}} & a_{\{2,3\}} & a_{\{3,3\}} \end{pmatrix}$$

```
D1[D1id[A]] // Simplify // MatrixForm
```

$$\begin{pmatrix} a_{\{1,1\}} & a_{\{1,2\}} & a_{\{1,3\}} \\ a_{\{1,2\}} & a_{\{2,2\}} & a_{\{2,3\}} \\ a_{\{1,3\}} & a_{\{2,3\}} & a_{\{3,3\}} \end{pmatrix}$$

```
Tr[D1i[A]] /. {y1 → 0, y2 -> 0, y3 → 1} // Simplify
```

$$\frac{1}{2}\left(a_{\{1,1\}} + a_{\{2,2\}} + 2\, a_{\{3,3\}}\right)$$

```
Tr[D2[Atf]] /. {y1 → 0, y2 -> 0, y3 → 1}
```

$$2\left(4\, a_{\{1,1\}} + \frac{5}{3}\left(-a_{\{1,1\}} - a_{\{2,2\}}\right)\right) + 2\left(\frac{5}{3}\left(-a_{\{1,1\}} - a_{\{2,2\}}\right) + 4\, a_{\{2,2\}}\right) + 2\left(-a_{\{1,1\}} - a_{\{2,2\}} + \frac{1}{3}\left(a_{\{1,1\}} + a_{\{2,2\}}\right)\right)$$

```
Tr[D1i[A]] - ((1 / 2) Tr[A] + (1 / 2) Y.A.Y / (Y.Y)) /. {y1 → 0, y2 -> 0, y3 → 1} // Simplify
```

0

```
Tr[D1i[A]] - ((1 / 2) Tr[A] + (1 / 2) Y.A.Y / (Y.Y)) // Simplify
```

0

Trace free version of D1 inverse

```
D1itrf = Function[a, (1 / 4) a + (1 / 4) (outer[Y, Y].a / (Y.Y) + a.outer[Y, Y] / (Y.Y)) +
    (3 / 8) (Y.a.Y) outer[Y, Y] / (Y.Y) ^2 + (-1 / 24) Tr[a] IdentityMatrix[3] -
    (1 / 8) Tr[a] outer[Y, Y] / (Y.Y) - (7 / 24) (Y.a.Y) IdentityMatrix[3] / (Y.Y)]
```

$$\text{Function}\left[a,\ \frac{a}{4} + \frac{1}{4}\left(\frac{\text{outer}[Y, Y].a}{Y.Y} + \frac{a.\text{outer}[Y, Y]}{Y.Y}\right) + \frac{3\, Y.a.Y\, \text{outer}[Y, Y]}{8\, (Y.Y)^2} - \frac{1}{24}\,\text{Tr}[a]\,\text{IdentityMatrix}[3] - \frac{\text{Tr}[a]\,\text{outer}[Y, Y]}{8\, Y.Y} - \frac{7\, Y.a.Y\,\text{IdentityMatrix}[3]}{24\, Y.Y}\right]$$

```
Tr[D1itrf[A]] // Simplify
```

0

Now to calculate the composite of D2 with the trace free D1inverse

```
d2 = Function[a, a outer[Y, Y] / (Y.Y)]
```

$$\text{Function}\left[a,\ \frac{a\,\text{outer}[Y, Y]}{Y.Y}\right]$$

```
D2D1ttst =
 Function[a, (2) a + (-8 / 9) IdentityMatrix[3] Tr[a] + (-2) ddelta[a] + (2 / 3) d2[Tr[a]]
    IdentityMatrix[3] + (4 / 9) IdentityMatrix[3] delta2[a] + (2 / 3) d2delta2[a]]
```

$$\text{Function}\left[a,\ 2\,a - \frac{8}{9}\,\text{IdentityMatrix}[3]\,\text{Tr}[a] - 2\,\text{ddelta}[a] + \frac{2}{3}\,\text{d2}[\text{Tr}[a]]\,\text{IdentityMatrix}[3] + \frac{4}{9}\,\text{IdentityMatrix}[3]\,\text{delta2}[a] + \frac{2\,\text{d2delta2}[a]}{3}\right]$$

```
(D2[D1itrf[A]] - D2D1ttst[A]) // Simplify // MatrixForm
```

$$\begin{pmatrix} 0 & \frac{2\left(a_{\{1,1\}}+a_{\{2,2\}}+a_{\{3,3\}}\right) y_1 y_2}{3\left(y_1^2+y_2^2+y_3^2\right)} & \frac{2\left(a_{\{1,1\}}+a_{\{2,2\}}+a_{\{3,3\}}\right) y_1 y_3}{3\left(y_1^2+y_2^2+y_3^2\right)} \\ \frac{2\left(a_{\{1,1\}}+a_{\{2,2\}}+a_{\{3,3\}}\right) y_1 y_2}{3\left(y_1^2+y_2^2+y_3^2\right)} & 0 & \frac{2\left(a_{\{1,1\}}+a_{\{2,2\}}+a_{\{3,3\}}\right) y_2 y_3}{3\left(y_1^2+y_2^2+y_3^2\right)} \\ \frac{2\left(a_{\{1,1\}}+a_{\{2,2\}}+a_{\{3,3\}}\right) y_1 y_3}{3\left(y_1^2+y_2^2+y_3^2\right)} & \frac{2\left(a_{\{1,1\}}+a_{\{2,2\}}+a_{\{3,3\}}\right) y_2 y_3}{3\left(y_1^2+y_2^2+y_3^2\right)} & 0 \end{pmatrix}$$

```
(D2[D1itrf[A]] - D2D1ttst[A]) /. {k1 -> k3 → -2 / 3, k5 → 4 / 9, k4 → 2 / 3, k6 → 2 / 3} //
  Simplify // MatrixForm
```

$$\begin{pmatrix} 0 & \frac{2\left(a_{\{1,1\}}+a_{\{2,2\}}+a_{\{3,3\}}\right) y_1 y_2}{3\left(y_1^2+y_2^2+y_3^2\right)} & \frac{2\left(a_{\{1,1\}}+a_{\{2,2\}}+a_{\{3,3\}}\right) y_1 y_3}{3\left(y_1^2+y_2^2+y_3^2\right)} \\ \frac{2\left(a_{\{1,1\}}+a_{\{2,2\}}+a_{\{3,3\}}\right) y_1 y_2}{3\left(y_1^2+y_2^2+y_3^2\right)} & 0 & \frac{2\left(a_{\{1,1\}}+a_{\{2,2\}}+a_{\{3,3\}}\right) y_2 y_3}{3\left(y_1^2+y_2^2+y_3^2\right)} \\ \frac{2\left(a_{\{1,1\}}+a_{\{2,2\}}+a_{\{3,3\}}\right) y_1 y_3}{3\left(y_1^2+y_2^2+y_3^2\right)} & \frac{2\left(a_{\{1,1\}}+a_{\{2,2\}}+a_{\{3,3\}}\right) y_2 y_3}{3\left(y_1^2+y_2^2+y_3^2\right)} & 0 \end{pmatrix}$$

```
(D2[D1itrf[A]] - D2D1ttst[A]) /. {k1 → 2, k2 → 8 / 9, k3 → -2} // Simplify // MatrixForm
```

$$\begin{pmatrix} 0 & \frac{2\left(a_{\{1,1\}}+a_{\{2,2\}}+a_{\{3,3\}}\right) y_1 y_2}{3\left(y_1^2+y_2^2+y_3^2\right)} & \frac{2\left(a_{\{1,1\}}+a_{\{2,2\}}+a_{\{3,3\}}\right) y_1 y_3}{3\left(y_1^2+y_2^2+y_3^2\right)} \\ \frac{2\left(a_{\{1,1\}}+a_{\{2,2\}}+a_{\{3,3\}}\right) y_1 y_2}{3\left(y_1^2+y_2^2+y_3^2\right)} & 0 & \frac{2\left(a_{\{1,1\}}+a_{\{2,2\}}+a_{\{3,3\}}\right) y_2 y_3}{3\left(y_1^2+y_2^2+y_3^2\right)} \\ \frac{2\left(a_{\{1,1\}}+a_{\{2,2\}}+a_{\{3,3\}}\right) y_1 y_3}{3\left(y_1^2+y_2^2+y_3^2\right)} & \frac{2\left(a_{\{1,1\}}+a_{\{2,2\}}+a_{\{3,3\}}\right) y_2 y_3}{3\left(y_1^2+y_2^2+y_3^2\right)} & 0 \end{pmatrix}$$

```
(D2[D1itrf[A]] - D2D1ttst[A]) /. {k1 → 2, k3 → -2, k5 → 4 / 9, k6 → 2 / 3} /.
   {y₁ → 1, y₂ → 0, y₃ → 1} // Simplify // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & \frac{1}{3}\left(a_{\{1,1\}}+a_{\{2,2\}}+a_{\{3,3\}}\right) \\ 0 & 0 & 0 \\ \frac{1}{3}\left(a_{\{1,1\}}+a_{\{2,2\}}+a_{\{3,3\}}\right) & 0 & 0 \end{pmatrix}$$

```
(D2[D1itrf[A]] - D2D1ttst[A]) /.
   {k1 → 2, k2 → -8 / 9, k3 → 2, k4 → 2 / 3, k5 → 4 / 9, k6 → 4 / 3} // Simplify // MatrixForm
```

$$\begin{pmatrix} 0 & \frac{2\left(a_{\{1,1\}}+a_{\{2,2\}}+a_{\{3,3\}}\right) y_1 y_2}{3\left(y_1^2+y_2^2+y_3^2\right)} & \frac{2\left(a_{\{1,1\}}+a_{\{2,2\}}+a_{\{3,3\}}\right) y_1 y_3}{3\left(y_1^2+y_2^2+y_3^2\right)} \\ \frac{2\left(a_{\{1,1\}}+a_{\{2,2\}}+a_{\{3,3\}}\right) y_1 y_2}{3\left(y_1^2+y_2^2+y_3^2\right)} & 0 & \frac{2\left(a_{\{1,1\}}+a_{\{2,2\}}+a_{\{3,3\}}\right) y_2 y_3}{3\left(y_1^2+y_2^2+y_3^2\right)} \\ \frac{2\left(a_{\{1,1\}}+a_{\{2,2\}}+a_{\{3,3\}}\right) y_1 y_3}{3\left(y_1^2+y_2^2+y_3^2\right)} & \frac{2\left(a_{\{1,1\}}+a_{\{2,2\}}+a_{\{3,3\}}\right) y_2 y_3}{3\left(y_1^2+y_2^2+y_3^2\right)} & 0 \end{pmatrix}$$

```
D2D1ttst = Function[a, (k1) a + k2 IdentityMatrix[3] Tr[a] + (k3) ddelta[a] +
   (k4) d2[Tr[a]] IdentityMatrix[3] + k5 IdentityMatrix[3] delta2[a] + k6 d2delta2[a] ]
```

```
Function[a, k1 a + k2 IdentityMatrix[3] Tr[a] + k3 ddelta[a] +
  k4 d2[Tr[a]] IdentityMatrix[3] + k5 IdentityMatrix[3] delta2[a] + k6 d2delta2[a]]
```

```
D2D1ttst =
 Function[a, (2) a + (-8 / 9) IdentityMatrix[3] Tr[a] + (-2) ddelta[a] + (2 / 3) d2[Tr[a]]
    IdentityMatrix[3] + (4 / 9) IdentityMatrix[3] delta2[a] + (2 / 3) d2delta2[a]]
```

$$\text{Function}\Big[a,\ 2\,a - \frac{8}{9}\ \text{IdentityMatrix}[3]\ \text{Tr}[a] - 2\ \text{ddelta}[a] + \frac{2}{3}\ \text{d2}[\text{Tr}[a]]\ \text{IdentityMatrix}[3] + \frac{4}{9}\ \text{IdentityMatrix}[3]\ \text{delta2}[a] + \frac{2\ \text{d2delta2}[a]}{3}\Big]$$

```
D2D1ttst1 = Function[a, (k1) a + (k3) (outer[Y, Y].a / (Y.Y) + a.outer[Y, Y] / (Y.Y)) / 2 +
   (k6) (Y.a.Y) outer[Y, Y] / (Y.Y) ^2 + (k2) Tr[a] IdentityMatrix[3] +
   (k4) Tr[a] outer[Y, Y] / (Y.Y) + (k5) (Y.a.Y) IdentityMatrix[3] / (Y.Y)]
```

$$\text{Function}\Big[a,\ \text{k1}\ a + \frac{1}{2}\ \text{k3}\left(\frac{\text{outer}[Y,Y].a}{Y.Y} + \frac{a.\text{outer}[Y,Y]}{Y.Y}\right) + \frac{\text{k6}\ Y.a.Y\ \text{outer}[Y,Y]}{(Y.Y)^2} + \text{k2}\ \text{Tr}[a]\ \text{IdentityMatrix}[3] + \frac{\text{k4}\ \text{Tr}[a]\ \text{outer}[Y,Y]}{Y.Y} + \frac{\text{k5}\ Y.a.Y\ \text{IdentityMatrix}[3]}{Y.Y}\Big]$$

```
(D2[D1itrf[A]] - D2D1ttst1[A]) /.
   {k1 → 2, k2 → -8 / 9, k3 → -2, k4 → 2 / 3, k5 → 4 / 9, k6 → 2 / 3} // Simplify // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

D2D1it = Function[a, (2) a + (-2) (outer[Y, Y].a / (Y.Y) + a.outer[Y, Y] / (Y.Y)) / 2 + (2 / 3) (Y.a.Y) outer[Y, Y] / (Y.Y) ^2 + (-8 / 9) Tr[a] IdentityMatrix[3] + (2 / 3) Tr[a] outer[Y, Y] / (Y.Y) + (4 / 9) (Y.a.Y) IdentityMatrix[3] / (Y.Y)]

$$\text{Function}\left[a,\ 2\,a-\frac{2}{2}\left(\frac{\text{outer}[Y,Y].a}{Y.Y}+\frac{a.\text{outer}[Y,Y]}{Y.Y}\right)+\frac{2\,Y.a.Y\,\text{outer}[Y,Y]}{3\,(Y.Y)^2}-\frac{8}{9}\,\text{Tr}[a]\,\text{IdentityMatrix}[3]+\frac{2\,\text{Tr}[a]\,\text{outer}[Y,Y]}{3\,Y.Y}+\frac{4\,Y.a.Y\,\text{IdentityMatrix}[3]}{9\,Y.Y}\right]$$

(D2[D1itrf[A]] - D2D1it[A]) // Simplify // MatrixForm

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Now we want to see if D2D1i du = du if tr du=0

u = {u1, u2, u3}

{u1, u2, u3}

Y

$\{y_1, y_2, y_3\}$

du = (outer[Y, u] + outer[u, Y]) / 2 / (Y.Y)

$$\left\{\left\{\frac{u1\,y_1}{y_1^2+y_2^2+y_3^2},\frac{u2\,y_1+u1\,y_2}{2\left(y_1^2+y_2^2+y_3^2\right)},\frac{u3\,y_1+u1\,y_3}{2\left(y_1^2+y_2^2+y_3^2\right)}\right\},\left\{\frac{u2\,y_1+u1\,y_2}{2\left(y_1^2+y_2^2+y_3^2\right)},\frac{u2\,y_2}{y_1^2+y_2^2+y_3^2},\frac{u3\,y_2+u2\,y_3}{2\left(y_1^2+y_2^2+y_3^2\right)}\right\},\left\{\frac{u3\,y_1+u1\,y_3}{2\left(y_1^2+y_2^2+y_3^2\right)},\frac{u3\,y_2+u2\,y_3}{2\left(y_1^2+y_2^2+y_3^2\right)},\frac{u3\,y_3}{y_1^2+y_2^2+y_3^2}\right\}\right\}$$

du // MatrixForm

$$\begin{pmatrix} \frac{u1\,y_1}{y_1^2+y_2^2+y_3^2} & \frac{u2\,y_1+u1\,y_2}{2\left(y_1^2+y_2^2+y_3^2\right)} & \frac{u3\,y_1+u1\,y_3}{2\left(y_1^2+y_2^2+y_3^2\right)} \\ \frac{u2\,y_1+u1\,y_2}{2\left(y_1^2+y_2^2+y_3^2\right)} & \frac{u2\,y_2}{y_1^2+y_2^2+y_3^2} & \frac{u3\,y_2+u2\,y_3}{2\left(y_1^2+y_2^2+y_3^2\right)} \\ \frac{u3\,y_1+u1\,y_3}{2\left(y_1^2+y_2^2+y_3^2\right)} & \frac{u3\,y_2+u2\,y_3}{2\left(y_1^2+y_2^2+y_3^2\right)} & \frac{u3\,y_3}{y_1^2+y_2^2+y_3^2} \end{pmatrix}$$

dut = Simplify[du - (1 / 3) Tr[du] IdentityMatrix[3]]

$$\left\{\left\{-\frac{-2\,u1\,y_1+u2\,y_2+u3\,y_3}{3\left(y_1^2+y_2^2+y_3^2\right)},\frac{u2\,y_1+u1\,y_2}{2\left(y_1^2+y_2^2+y_3^2\right)},\frac{u3\,y_1+u1\,y_3}{2\left(y_1^2+y_2^2+y_3^2\right)}\right\},\left\{\frac{u2\,y_1+u1\,y_2}{2\left(y_1^2+y_2^2+y_3^2\right)},-\frac{u1\,y_1-2\,u2\,y_2+u3\,y_3}{3\left(y_1^2+y_2^2+y_3^2\right)},\frac{u3\,y_2+u2\,y_3}{2\left(y_1^2+y_2^2+y_3^2\right)}\right\},\left\{\frac{u3\,y_1+u1\,y_3}{2\left(y_1^2+y_2^2+y_3^2\right)},\frac{u3\,y_2+u2\,y_3}{2\left(y_1^2+y_2^2+y_3^2\right)},-\frac{u1\,y_1+u2\,y_2-2\,u3\,y_3}{3\left(y_1^2+y_2^2+y_3^2\right)}\right\}\right\}$$

D2D1ttst1[dut] - dut /. {y_1 → 0, y_2 -> 0, y_3 → 1} // Simplify // MatrixForm

$$\begin{pmatrix} -\frac{1}{3}\,(-1+k1-2\,k5)\,u3 & 0 & \frac{1}{4}\,(-2+2\,k1+k3)\,u1 \\ 0 & -\frac{1}{3}\,(-1+k1-2\,k5)\,u3 & \frac{1}{4}\,(-2+2\,k1+k3)\,u2 \\ \frac{1}{4}\,(-2+2\,k1+k3)\,u1 & \frac{1}{4}\,(-2+2\,k1+k3)\,u2 & \frac{2}{3}\,(-1+k1+k3+k5+k6)\,u3 \end{pmatrix}$$

```
D1i[D2[dut]] - dut // Simplify // MatrixForm
```

$$\begin{pmatrix} \frac{(u1\,y_1+u2\,y_2+u3\,y_3)\left(2\,y_1^2+y_2^2+y_3^2\right)}{9\left(y_1^2+y_2^2+y_3^2\right)^2} & \frac{y_1\,y_2\,(u1\,y_1+u2\,y_2+u3\,y_3)}{9\left(y_1^2+y_2^2+y_3^2\right)^2} & \frac{y_1\,y_3\,(u1\,y_1+u2\,y_2+u3\,y_3)}{9\left(y_1^2+y_2^2+y_3^2\right)^2} \\ \frac{y_1\,y_2\,(u1\,y_1+u2\,y_2+u3\,y_3)}{9\left(y_1^2+y_2^2+y_3^2\right)^2} & \frac{(u1\,y_1+u2\,y_2+u3\,y_3)\left(y_1^2+2\,y_2^2+y_3^2\right)}{9\left(y_1^2+y_2^2+y_3^2\right)^2} & \frac{y_2\,y_3\,(u1\,y_1+u2\,y_2+u3\,y_3)}{9\left(y_1^2+y_2^2+y_3^2\right)^2} \\ \frac{y_1\,y_3\,(u1\,y_1+u2\,y_2+u3\,y_3)}{9\left(y_1^2+y_2^2+y_3^2\right)^2} & \frac{y_2\,y_3\,(u1\,y_1+u2\,y_2+u3\,y_3)}{9\left(y_1^2+y_2^2+y_3^2\right)^2} & \frac{(u1\,y_1+u2\,y_2+u3\,y_3)\left(y_1^2+y_2^2+2\,y_3^2\right)}{9\left(y_1^2+y_2^2+y_3^2\right)^2} \end{pmatrix}$$