

This document outlines ideas for a project regarding MOTSs in anti de-Sitter (AdS) spacetimes:

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1. INTRODUCTION

Penrose diagram of pure AdS spacetime can be found in Hawking-Ellis pg. 132.

2. ADS SPACETIMES IN PAINLEVÉ-GULLSTRAND COORDINATES

The usual Painlevé-Gullstrand coordinates read

$$(2.1) \quad ds^2 = -f(r)dt^2 + 2\sqrt{1-f(r)}dtdr + dr^2 + r^2d\Omega^2$$

where for a Schwarzschild-AdS black hole would have the metric function

$$(2.2) \quad f(r) = 1 - \frac{2M}{r} + \frac{r^2}{\ell^2}$$

The MOTSodesic equations are given in general form in 2111.0937

$$(2.3) \quad T^i D_i T^j = \kappa N^j$$

Where

$$\begin{aligned} T^i &= \dot{r}\partial_r + \dot{\theta}\partial_\theta \\ N^i &= r\sqrt{p(r)} \left[\frac{\dot{\theta}}{p(r)}\partial_r - \frac{\dot{r}}{r^2}\partial_\theta \right] \\ \hat{\phi}^i &= \frac{1}{r\sin\theta}\partial_r \\ p(r)\dot{r}^2 + r^2\dot{\theta}^2 &= 1 \\ \kappa &= k_u - N_j \hat{\phi}^i D_i \hat{\phi}^j \end{aligned}$$

Explicitly, the MOTSodesic equations are

$$\ddot{r} = -\frac{p'\dot{r}^2 - 2r\dot{\theta}^2}{2p} + \frac{r\dot{\theta}\kappa}{\sqrt{p}}$$

$$\ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r} - \frac{\sqrt{p}\dot{r}\kappa}{r}$$

$$\kappa = -\frac{1}{r\sqrt{p}} \left[p\dot{r} \cot \theta - r\dot{\theta} \right] + \frac{1}{2r\sqrt{p(1-pf)}} \left[rp^2\dot{r}^2 f' + r\dot{r}^2 p' - 2(r^2\dot{\theta}^2 + 1)(1-pf) \right]$$

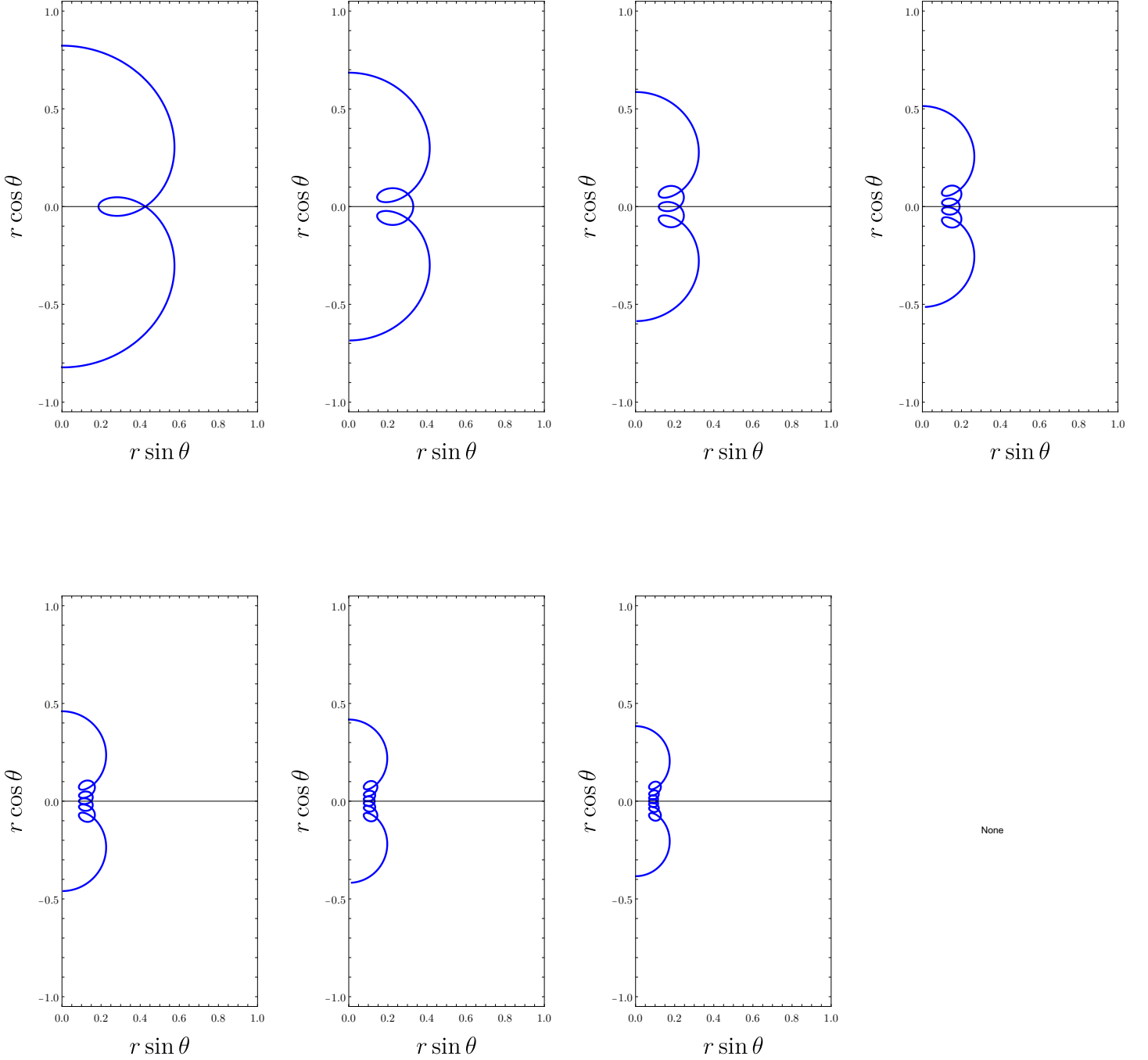


Figure 1. Uses parameters $M = 1$ and $\Lambda = 1$.

There seems to be an infinitely many self-intersecting MOTSs within the Schwarzschild-AdS black hole. This is consistent with Dr. Robie's conjecture that the number of self-intersections correlate to the functional minimum of the metric function in spherically symmetric cases. Since there is no minimum for $f(r) = 1 - \frac{2M}{r} + \Lambda r^2$ (or that it is $\lim_{r \rightarrow 0} f(r) = -\infty$), then it makes sense for it to have a very large number of self-intersecting MOTSs.

2.1. Hawking-Page Transition. Looking at the 3+1 Schwarzschild-AdS in terms of its event/apparent horizon radius r_h (arXiv:2205.09938, between (1) and (2)):

$$(2.4) \quad \left(1 - \frac{r_h}{r}\right) \left(\frac{r^2 + rr_h + r_h^2}{\ell^2} + 1\right)$$

$$(2.5) \quad M = \frac{r_h}{2} \left(\frac{r_h^2}{\ell^2} + 1\right),$$

we can calculate the Hawking temperature T_H as a function of M , using the time-like killing vector $\ell^\alpha = \frac{\partial}{\partial t}$: For $\ell^\alpha = \frac{\partial}{\partial t}$ of the Euclidean sector:

$$(2.6) \quad \begin{aligned} \ell^2 &= f(r) \\ \nabla_\alpha \ell^2 &= f'(r) dr \\ g^{\alpha\beta} \nabla_\alpha \ell^2 \nabla_\beta \ell^2 &= g^{rr} (f'(r))^2 \\ &= f(r) (f'(r))^2 \\ \frac{g^{\alpha\beta} \nabla_\alpha \ell^2 \nabla_\beta \ell^2}{4\ell^2} &= \frac{1}{4} (f'(r))^2 \\ f'(2M) &= \frac{3r_h^2 + \ell^2}{r_h \ell^2} \\ \frac{g^{\alpha\beta} \nabla_\alpha \ell^2 \nabla_\beta \ell^2}{4\ell^2} &= \frac{1}{4} \left(\frac{3r_h^2 + \ell^2}{r_h \ell^2}\right)^2 = \kappa^2 \\ \boxed{T_H = \frac{\kappa}{2\pi} = \frac{1}{4\pi} \frac{3r_h^2 + \ell^2}{r_h \ell^2}} \end{aligned}$$

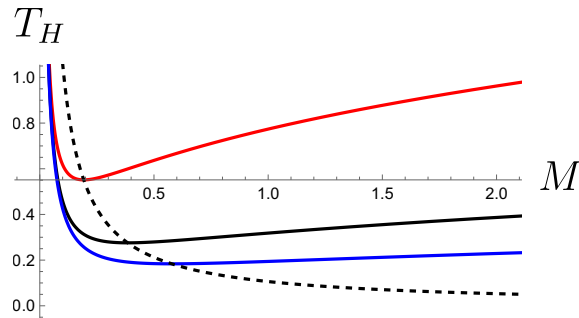


Figure 2. The red, black, then blue curves are for $\ell = 0.5, 1.0, 1.5$, respectively. The dashed black curve is the critical point for varying ℓ .

By solving $\frac{\partial T}{\partial M} = \frac{\partial T}{\partial r_h} / \frac{\partial M}{\partial r_h} = 0$, the critical point occurs at $r_{h,critical} = \ell/\sqrt{3}$. A nice choice would be to fix $\ell = \sqrt{3}$ actually. For this choice, $r_{h,critical} = 1$ and $M_{critical} = 2/3 \approx 0.66$.

Immediately and naively, it does not seem like the self-intersecting MOTS behaviour changes compared to Figure 1

3. ADS SPACETIMES IN KRUSKAL-SZEKERES EXTENSIONS

It seems that someone has already given the Kruskal-Szekeres extension for AdS_5 : arXiv:gr-qc/0005115. In 5-dimensions, the metric function's roots is in its best form:

$$(3.1) \quad ds^2 = -\frac{r_H^2}{1+4\mu} \left(1 + \frac{r_0^2}{r^2}\right) (r_H + r)^2 \exp\left(-\frac{2r_0}{r_H} \arctan\left(\frac{r}{r_0}\right)\right) dU dV + r^2 d\Omega_3^2.$$

The unit 3-sphere is $d\Omega_3^2 = d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2$ for $\theta \in (0, \pi/2)$, $\phi, \psi \in (0, 2\pi)$. and the other parameters are defined as such:

$$(3.2) \quad \begin{aligned} r_H &= \sqrt{\frac{1}{2} \left(\sqrt{1+4\mu} - 1 \right)} \\ r_0 &= \sqrt{\frac{1}{2} \left(\sqrt{1+4\mu} + 1 \right)} \\ T_H &= \frac{\sqrt{1+4\mu}}{2\pi r_H} \\ r_* &= \frac{1}{\sqrt{1+4\mu}} \left(r_0 \arctan\left(\frac{r}{r_0}\right) + \frac{1}{2} r_H \ln\left(\frac{r-r_H}{r+r_H}\right) \right) \\ u &= t - r_* \\ v &= t + r_* \\ U &= -\exp(-2\pi T_H u) \\ V &= \exp(2\pi T_H v) \end{aligned}$$

In 4+1 dimensions, $F(r) = 1 - \frac{\mu}{r^2} + r^2$ has four roots at $\pm r_H$ and $\pm ir_0$.

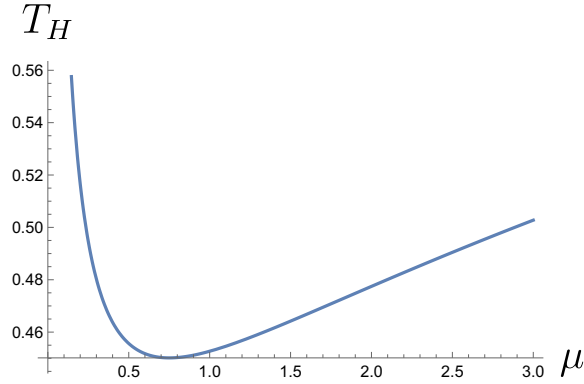


Figure 3. The Hawking-Page transition happens at $\mu_{critical} = 3/4$ (the critical point).

We can do some math:

$$(3.3) \quad \begin{aligned} U \cdot V &= -\exp(2\pi T_H(v - u)) = -\exp(4\pi T_H r_*) = -\exp\left(\frac{2\pi T_H \left(2r_0 \arctan\left(\frac{r}{r_0}\right) + r_H \ln\left(\frac{r-r_H}{r+r_H}\right)\right)}{\sqrt{4\mu+1}}\right) \\ U &= T - X \\ V &= T + X \\ -T^2 + X^2 &= \exp\left(\frac{2\pi T_H \left(2r_0 \arctan\left(\frac{r}{r_0}\right) + r_H \ln\left(\frac{r-r_H}{r+r_H}\right)\right)}{\sqrt{4\mu+1}}\right) = \left(\frac{r-r_H}{r+r_H}\right) \exp\left(\frac{2r_0}{r_H} \arctan\left(\frac{r}{r_0}\right)\right) = G(r). \end{aligned}$$

In the Schwarzschild case ($r_0 \rightarrow 0$), we do not expect this to go back to the Schwarzschild case (remember there is a denominator of r^2 now.)

There is no obvious function for r that we know of (analogous to the Lambert-W function), but we have the above form and can still use that implicitly. Even from this we can draw Penrose-Carter diagrams and Kruskal-Szekeres diagrams.

3.1. Conformal diagrams. The Kruskal and Penrose diagrams are spoken in exactly the languages of these coordinates (for the Kruskal diagrams plot $(x = X, y = T)$, for Penrose diagrams plot and rotate $(x = \tilde{V} = \arctan(V), y = \tilde{U} = \arctan(U))$ by 45 degrees.)

3.1.1. Penrose Diagram. Incredibly, letting the computer plot the V and U in terms of Schwarzschild-like coordinates $t \in \mathbb{R}, r_H < r < \infty$, we can get the timelike surface at large r and null surface at $r \rightarrow r_H$.

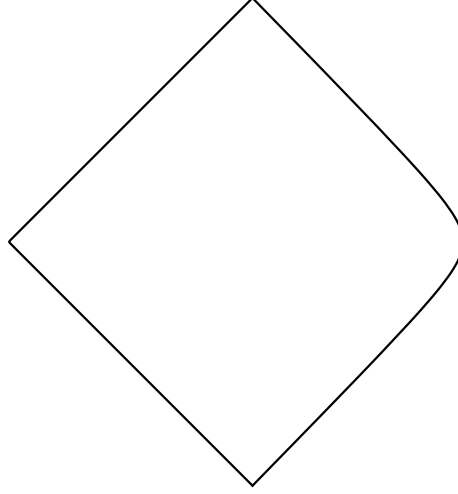


Figure 4. $r \rightarrow r_H$ future and past null surface on the left, $r \rightarrow \infty$ future and past timelike surface on the right. Numerically obtained by plotting \tilde{U} and \tilde{V} .

3.2. Derivatives via implicit function theorem. In fact, the implicit function theorem can be applied for setting

$$(3.4) \quad F(T, X, r(T, X)) = -T^2 + X^2 - G(r(T, X)) = 0 ,$$

where

$$(3.5) \quad G(r(T, X)) = \exp \left(\frac{2\pi T_H \left(2r_0 \arctan\left(\frac{r}{r_0}\right) + r_H \ln\left(\frac{r-r_H}{r+r_H}\right) \right)}{\sqrt{4\mu + 1}} \right)$$

The theorem states (from this source): “If $F : \mathbb{R}^3 \rightarrow \mathbb{R}$ and a point $(T_0, X_0, r_0) \in \mathbb{R}^3$ so that $F(T_0, X_0, r_0) = c$. If $\frac{\partial F}{\partial r} \neq 0$, then there is a neighborhood so that whenever (T, X) is sufficiently close to (T_0, X_0) there is a unique $r = r(T, X)$ that is such that $F(T, X, r) = c$.” In this case, $c = 0$.

So, later, the MOTSodesic equations will need

$$(3.6) \quad r_X = \frac{\partial r}{\partial X} = -\frac{F_X}{F_r} = -\frac{2X}{G'(r)}$$

$$(3.7) \quad r_T = \frac{\partial r}{\partial T} = -\frac{F_T}{F_r} = \frac{2T}{G'(r)}$$

Before we move on, it would be fun to simplify $G(r)$ a little:

$$\begin{aligned}
 G(r) &= \exp \left(\frac{2\pi T_H \left(2r_0 \arctan\left(\frac{r}{r_0}\right) + r_H \ln\left(\frac{r-r_H}{r+r_H}\right) \right)}{\sqrt{4\mu+1}} \right) \\
 &= e^{\frac{4\pi r_0 T_H \arctan\left(\frac{r}{r_0}\right)}{\sqrt{4\mu+1}}} \cdot \left(\frac{r-r_H}{r+r_H} \right)^{\frac{2\pi r_H T_H}{\sqrt{4\mu+1}}}
 \end{aligned}$$

Haha! $G(r=r_H) = 0$, but $G'(r=r_H) = e^{\frac{4\pi r_0 T_H \arctan\left(\frac{r_H}{r_0}\right)}{\sqrt{4\mu+1}}} / 2r_H \neq 0$!

3.3. MOTSodesics in Schwarzschild-AdS₅ Kruskal-type coordinates.

$$(3.8) \quad ds^2 = \frac{r_H^2}{1+4\mu} \left(1 + \frac{r_0^2}{r^2}\right) (r_H + r)^2 \exp\left(-\frac{2r_0}{r_H} \arctan\left(\frac{r}{r_0}\right)\right) (-dT^2 + dX^2) + r^2 d\Omega_3^2 ,$$

where r_H and r_0 are roots of

$$1 - \frac{\mu}{r^2} + r^2 = 0$$

The lapse N is defined by

$$(3.9) \quad N^2 := \frac{r_H^2}{1+4\mu} \left(1 + \frac{r_0^2}{r^2}\right) (r_H + r)^2 \exp\left(-\frac{2r_0}{r_H} \arctan\left(\frac{r}{r_0}\right)\right) .$$

The coordinates $T, X \in \mathbb{R}$ relate to $r = r(T, X)$ by

$$(3.10) \quad -T^2 + X^2 = \left(\frac{r - r_H}{r + r_H}\right) \exp\left(\frac{2r_0}{r_H} \arctan\left(\frac{r}{r_0}\right)\right) = G(r) .$$

$$(3.11) \quad G'(r) = \frac{2(r_0^2 + r_H^2)}{r_H} \frac{r^2}{(r^2 + r_0^2)(r^2 + r_H^2)} \exp\left(\frac{2r_0}{r_H} \arctan\left(\frac{r}{r_0}\right)\right)$$

We can (almost identically) derive the MOTSodesic equations from the Marginally Outer trapped tori paper and the n -dimension rotating paper [link to tori, link to rotating]

Σ_T surfaces of constant T has the induced metric

$$(3.12) \quad h_{ij} dx^i dx^j = N^2 dX^2 + r^2 d\Omega_{(3)}^2 .$$

The normal one-form to this foliation:

$$(3.13) \quad u_\alpha dx^\alpha = -N dT$$

The quarter-plane orbit space with curve parameter s , that is $X, \theta = P(s), \Theta(s)$ (consistent with paper).

$$(3.14) \quad N^2 dX^2 + r^2 d\theta^2 = (N^2 \dot{P}^2 + r^2 \dot{\Theta}^2) ds^2$$

This will be our arc-length parameterization by the way,

$$(3.15) \quad N^2 \dot{P}^2 + r^2 \dot{\Theta}^2 \equiv 1$$

(Just like equation 20 in Kruskal paper)

Let indices α, β run over $\{T, X, \theta, \phi, \psi\}$, i, j over $\{X, \theta, \phi, \psi\}$, a, b over $\{X, \theta\}$, and A, B over $\{\phi, \psi\}$

$$(3.16) \quad d\Omega_3^2 = d\theta^2 + \sin^2 \theta d\phi^2 + \cos^2 \theta d\psi^2 .$$

$$(3.17) \quad \underline{h}_{ab} dx^a dx^b = N^2 dX^2 + r^2 d\theta^2$$

$$(3.18) \quad \underline{h}_{AB} d\phi^A d\phi^B = r^2 \sin^2 \theta d\phi^2 + r^2 \cos^2 \theta d\psi^2$$

$$(3.19) \quad \underline{h}^{AB} \frac{\partial}{\partial \phi^A} \frac{\partial}{\partial \phi^B} = \frac{1}{r^2 \sin^2 \theta} \left(\frac{\partial}{\partial \phi}\right)^2 + \frac{1}{r^2 \cos^2 \theta} \left(\frac{\partial}{\partial \psi}\right)^2$$

$$(3.20) \quad \underline{K}_{ab} = K_{ij} e_a^i e_b^j , \quad \underline{K}_{AB} = K_{ij} e_A^i e_B^j .$$

$$(3.21) \quad \begin{aligned} \hat{T}^a \frac{\partial}{\partial x^a} &= \dot{P} \frac{\partial}{\partial X} + \dot{\Theta} \frac{\partial}{\partial \theta} \\ \hat{N}^a \frac{\partial}{\partial x^a} &= \frac{r \dot{\Theta}}{N} \frac{\partial}{\partial X} + \frac{-N \dot{P}}{r} \frac{\partial}{\partial \theta} \\ \hat{N}_a dx^a &= Nr (\Theta'(s) dX - P'(s) d\theta) \end{aligned}$$

$$(3.22) \quad \hat{T}^b \underline{\nabla}_b \hat{T}^a = \kappa_{\text{MOTS}} \hat{N}^a$$

$$(3.23) \quad \kappa_{\text{MOTS}} = \mathcal{K} + \mathcal{K}_{\hat{N}} + \mathcal{K}_{\hat{T}\hat{T}}$$

$$(3.24) \quad \begin{aligned} \mathcal{K} &:= \underline{h}^{AB} \underline{K}_{AB} \\ \mathcal{K}_{\hat{N}} &:= \hat{N}^a (\underline{D}_a \ln \sqrt{\underline{h}}) \\ \mathcal{K}_{\hat{T}\hat{T}} &:= \underline{K}_{ab} \hat{T}^a \hat{T}^b \end{aligned}$$

where \underline{D} lives on the orbit quarter-plane \underline{h}_{ab} .

$$(3.25) \quad \begin{aligned} \mathcal{K} &:= \underline{h}^{\phi\phi} \underline{K}_{\phi\phi} + \underline{h}^{\psi\psi} \underline{K}_{\psi\psi} , \\ \boxed{\mathcal{K} = \frac{2r_T}{Nr}} . \\ \mathcal{K}_{\hat{N}} &:= \hat{N}^a (\underline{D}_a \ln \sqrt{\underline{h}_{\phi\phi} \underline{h}_{\psi\psi}}) , \\ \boxed{\mathcal{K}_{\hat{N}} = \frac{2\dot{\Theta} r_X}{N} - \frac{N(\cot \Theta - \tan \Theta) \dot{P}}{r}} . \\ \mathcal{K}_{\hat{T}\hat{T}} &:= \underline{K}_{ab} \hat{T}^a \hat{T}^b \\ \boxed{\mathcal{K}_{\hat{T}\hat{T}} = N_T \dot{P}^2 + \frac{r}{N} r_T \dot{\Theta}^2} \end{aligned}$$

$$(3.26) \quad \boxed{\begin{aligned} \ddot{P} &= - \left(\frac{N_X}{N} \right) \dot{P}^2 + \left(\frac{r r_X}{N^2} \right) \dot{\Theta}^2 + \left(\frac{r \kappa_{\text{MOTS}}}{N} \right) \dot{\Theta} . \\ \ddot{\Theta} &= - \left(\frac{2r_X}{r} \right) \dot{P} \dot{\Theta} - \left(\frac{N \kappa_{\text{MOTS}}}{r} \right) \dot{P} . \end{aligned}}$$

Check equations for $r = r_H$,

$$(3.27) \quad \begin{aligned} G(r = r_H) &= 0 . \\ G'(r = r_H) &= \frac{1}{2r_H} \exp \left(\frac{2r_0}{r_H} \arctan \left(\frac{r_H}{r_0} \right) \right) \\ r_X \Big|_{r=r_H} &= - \frac{2X}{G'(r_H)} = - r_T \Big|_{r=r_H} = -4T r_H \exp \left(- \frac{2r_0}{r_H} \arctan \left(\frac{r_H}{r_0} \right) \right) \\ N^2(r_H) &= \frac{4r_H^2 (r_H^2 + r_0^2)}{(1 + 2r_H^2)^2} \exp \left(- \frac{2r_0}{r_H} \arctan \left(\frac{r_H}{r_0} \right) \right) \\ N(r_H) &= \frac{2r_H \sqrt{r_H^2 + r_0^2}}{1 + 2r_H^2} \exp \left(- \frac{r_0}{r_H} \arctan \left(\frac{r_H}{r_0} \right) \right) \\ N'(r_H) &= \frac{(r_H^2 - 3r_0^2)}{(1 + 2r_H^2) \sqrt{r_H^2 + r_0^2}} \exp \left(- \frac{r_0}{r_H} \arctan \left(\frac{r_H}{r_0} \right) \right) \\ P &= X = T \quad ; \quad \dot{P} = 0 \quad ; \quad \ddot{P} = 0 \\ \Theta &= \frac{s}{r_H} \quad ; \quad \dot{\Theta} = \frac{1}{r_H} \quad ; \quad \ddot{\Theta} = 0 \end{aligned}$$

Need to show $\left(\frac{rr_X}{N^2}\right)\dot{\Theta}^2 + \left(\frac{r\kappa_{\text{MOTS}}}{N}\right)\dot{\Theta} = 0$.

$$\begin{aligned}
 \left(\frac{rr_X}{N^2}\right)\dot{\Theta}^2 &= \frac{r_H(-4Tr_H)(1+2r_H^2)^2}{4r_H^2(r_H^2+r_0^2)} \frac{1}{r_H^2} \\
 &= \frac{-T(1+2r_H^2)^2}{r_H^2(r_H^2+r_0^2)} \\
 \left(\frac{r\kappa_{\text{MOTS}}}{N}\right)\dot{\Theta} &= \frac{2r_T}{N^2r} + \frac{2\dot{\Theta}r_X}{N^2} + \frac{r}{N^2}r_T\dot{\Theta}^2 \\
 (3.28) \quad &= \frac{8Tr_H \exp\left(-\frac{2r_0}{r_H} \arctan\left(\frac{r_H}{r_0}\right)\right)}{\frac{4r_H^2(r_H^2+r_0^2)}{(1+2r_H^2)^2} \exp\left(-\frac{2r_0}{r_H} \arctan\left(\frac{r_H}{r_0}\right)\right) r_H} + \frac{-8\frac{1}{r_H}Tr_H \exp\left(-\frac{2r_0}{r_H} \arctan\left(\frac{r_H}{r_0}\right)\right)}{\frac{4r_H^2(r_H^2+r_0^2)}{(1+2r_H^2)^2} \exp\left(-\frac{2r_0}{r_H} \arctan\left(\frac{r_H}{r_0}\right)\right)} \\
 &\quad + \frac{4Tr_H \exp\left(-\frac{2r_0}{r_H} \arctan\left(\frac{r_H}{r_0}\right)\right)}{\frac{4r_H^2(r_H^2+r_0^2)}{(1+2r_H^2)^2} \exp\left(-\frac{2r_0}{r_H} \arctan\left(\frac{r_H}{r_0}\right)\right) r_H} \\
 &= \frac{T(1+2r_H^2)^2}{r_H^2(r_H^2+r_0^2)}
 \end{aligned}$$

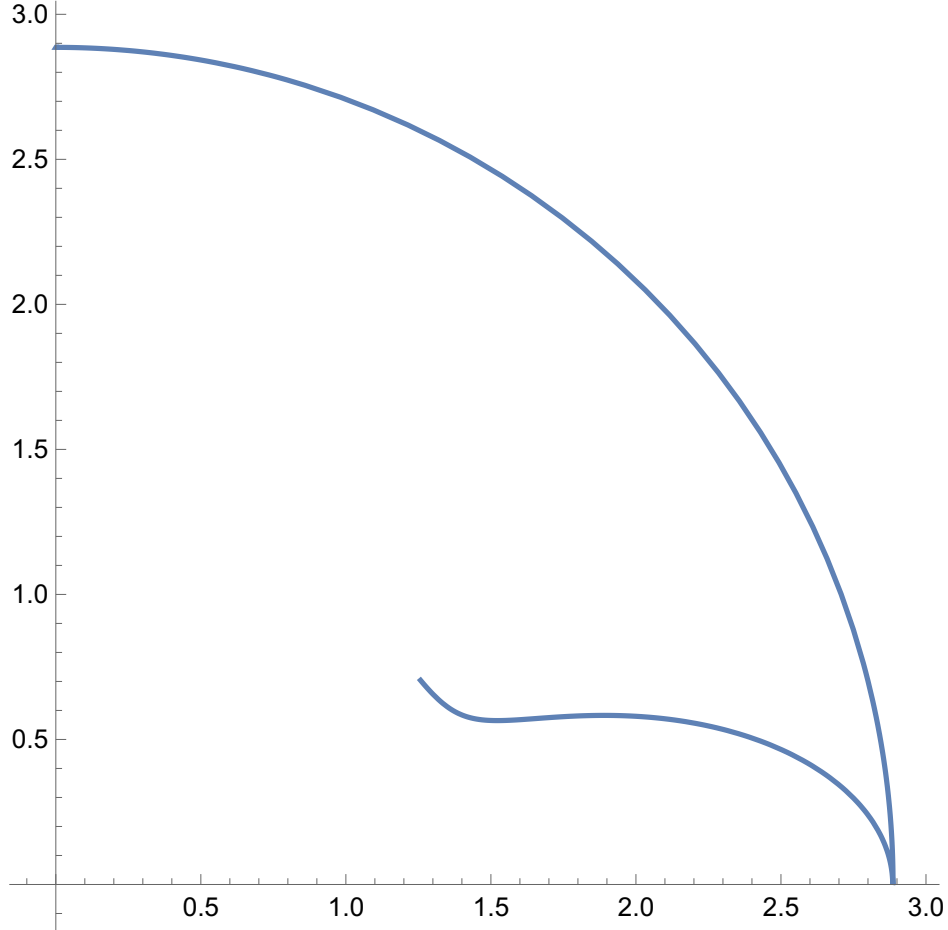


Figure 5. The first MOTSs made (the $r = r_H = 1$ MOTS). Axes are $e^X\{\sin \theta, \cos \theta\}$

3.4. Properties of the inverse function $G^{-1}(X, T)$ (analogous to Lambert W function). We consider the relationship between the coordinates X, T and r, t , given by

$$(3.29) \quad -T^2 + X^2 = \left(\frac{r - r_H}{r + r_H} \right) \exp \left(\frac{2r_0}{r_H} \arctan \left(\frac{r}{r_0} \right) \right) = G(r) .$$

In the asymptotically flat Schwarzschild spacetime, the relationship is

$$(3.30) \quad -T^2 + X^2 = -\left(1 - \frac{r}{2M}\right) \exp(r/2M) ,$$

which is inverted to give $r = r(X, T) \propto W(-T^2 + X^2)$. The case in Schwarzschild-AdS is clearly going to be more involved.

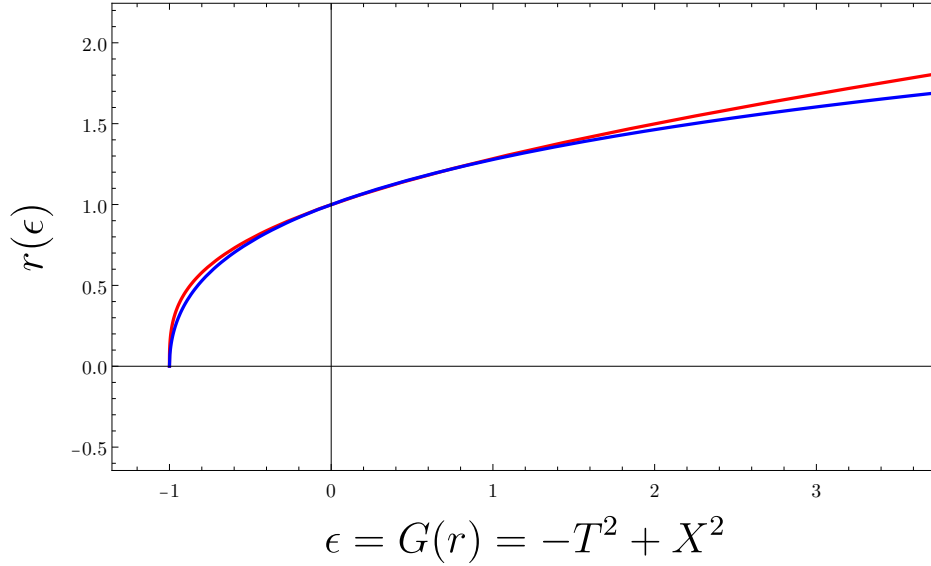


Figure 6. r as a function of $\epsilon = -T^2 + X^2$ in the Schwarzschild case (blue) and Schwarzschild-AdS case (red).

What one could do is a chosen few series expansions:

$$\begin{aligned} \frac{r - r_H}{r + r_H} &\sim -1 + \frac{2r}{r_H} - \frac{2r^2}{r_H^2} + \frac{2r^3}{r_H^3} + \mathcal{O}(r^4) , \\ \arctan \left(\frac{r}{r_0} \right) &\sim \frac{r}{r_0} - \frac{r^3}{3r_0^3} + \frac{r^5}{r_0^5} + \mathcal{O}(r^7) . \end{aligned}$$

(note these series do not converge for $r > 0 \in \mathbb{R}$, but are asymptotic for $r/r_H \rightarrow 0$, $r/r_0 \rightarrow 0$.)

Then $G(r)$ looks like:

$$\begin{aligned} G(r) &= \left(\frac{r - r_H}{r + r_H} \right) \exp \left(\frac{2r_0}{r_H} \arctan \left(\frac{r}{r_0} \right) \right) \\ &\sim \left(-1 + \frac{2r}{r_H} + \mathcal{O}(r^2) \right) \exp \left(\frac{2r_0}{r_H} \left(\frac{r}{r_0} + \mathcal{O}(r^3) \right) \right) \\ &= \left(-1 + \frac{2r}{r_H} \right) \exp \left(\frac{2r}{r_H} \right) (\exp(\mathcal{O}(r^3))) + \mathcal{O}(r^2) \end{aligned}$$

In the limit $r/r_H \rightarrow 0$ (and since r_0 is parameterized via μ dependent on the parameter r_H , we get $r/r_0 \rightarrow 0$ for free), $G(r) \sim G_{Schwarzschild}(r)$ and the function behaves similar to the Lambert- W function.

From the plot, we it would not be surprising if the same analysis for $r \rightarrow r_H$ (at $\epsilon \rightarrow 0$) shows a similar result.

3.5. Derivation of the Maximal extension of Schwarzschild-AdS. Consider the d th-dimensional Schwarzschild-AdS spacetime ($n = d - 3$)

$$(3.31) \quad ds^2 = -f_n(r)dt^2 + \frac{dr^2}{f_n(r)} + r^2 d\Omega_{n+1}^2 \quad ; \quad f_n = 1 - \frac{\mu}{r^n} + \frac{r^2}{\ell^2} .$$

3.6. $d=5$. We had found literature on a maximal extension for the $(4+1)$ -dimensional Schwarzschild-AdS spacetime to be

$$(3.32) \quad ds^2 = \frac{r_H^2}{1 + 4\mu} \left(1 + \frac{r_0^2}{r^2} \right) (r_H + r)^2 \exp \left(-\frac{2r_0}{r_H} \arctan \left(\frac{r}{r_0} \right) \right) (-dT^2 + dX^2) + r^2 d\Omega_3^2 .$$

Starting from the Schwarzschild-like radial coordinates,

$$(3.33) \quad ds^2 = -f_5(r)dt^2 + \frac{dr^2}{f_5(r)} + r^2 d\Omega_3^2 \quad ; \quad f = 1 - \frac{\mu}{r^2} + \frac{r^2}{\ell^2} ,$$

The metric function can be brought to factored form with

$$(3.34) \quad \mu = \frac{r_H^2 r_0^2}{r_0^2 - r_H^2} \quad ; \quad l^2 = r_0^2 - r_H^2 ,$$

$$(3.35) \quad f_5(r) = \frac{(r^2 - r_H^2)(r^2 + r_0^2)}{r^2(r_0^2 - r_H^2)} .$$

This is constructed such that the roots are at $r = r_H$ and $r = ir_0$. In Planck units ($\ell = 1$), the factor of $r_0^2 - r_H^2$ is unity by definition. We would like to know that ℓ is not imaginary, that is $r_0 > r_H$, so explicitly:

$$(3.36) \quad ir_0 = \pm \sqrt{\frac{-l^2}{2}} \sqrt{1 + \sqrt{l^2 + 4\mu}} \quad ; \quad r_H = \pm \sqrt{\frac{l^2}{2}} \sqrt{-1 + \sqrt{l^2 + 4\mu}}$$

It is helpful to note the surface gravity

$$(3.37) \quad \kappa_i = \frac{1}{2} f'(r_i) = \frac{r_i^4 + r_0^2 r_H^2}{r_i^3 (r_0^2 - r_H^2)}$$

We first find the tortoise coordinates such that the metric takes the form

$$ds^2 = f(r) (-dt^2 + dr^{*2}) + d\Omega_3^2 .$$

This is done by setting

$$dr^2 = f^2 dr^{*2}$$

Asserting that $\text{Sign}(dr) = \text{Sign}(dr^*)$ (so that we may get rid of the squares without worries regarding the plus-minus signs), we have

$$dr^* = \frac{dr}{f}$$

$$r^* = \int \frac{dr}{f} = \frac{(r_0^2 - r_H^2) \left(r_0 \tan^{-1} \left(\frac{r}{r_0} \right) - r_H \tanh^{-1} \left(\frac{r}{r_H} \right) \right)}{r_0^2 + r_H^2} + \text{const.} .$$

Just as a note: $\tanh^{-1}(\pm 1)$ and $\tan^{-1}(\pm i)$ diverges.

Now introduce null coordinates:

$$u \equiv t - r^* \quad ; \quad v \equiv t + r^* .$$

$$du = dt - \frac{dr}{f} \quad ; \quad dv = dt + \frac{dr}{f} ,$$

$$du = dt - dr^* \quad ; \quad dv = dt + dr^* ,$$

$$dr = -\frac{f}{2}(du - dv) \quad ; \quad dt = \frac{1}{2}(du + dv) ,$$

$$ds^2 = -f du dv + r^2 d\Omega_3^2 .$$

Just to make ourselves feel better, remember these are truly null coordinates and we can recover the Eddington-Finkelstein coordinates (ingoing & outgoing) with $dt \rightarrow du + \frac{dr}{f}$ and $dt \rightarrow dv - \frac{dr}{f}$:

$$ds^2 = -du^2 - 2dudr + r^2 d\Omega_3^2 \quad ds^2 = -dv^2 + 2dvdr + r^2 d\Omega_3^2$$

Now remember we want to rid ourselves of the coordinate misbehaviour at $r = r_H$. What we could do is use a \tanh^{-1} to \ln relation:

$$\tanh^{-1}(z) = \frac{1}{2} \ln\left(\frac{1+z}{1-z}\right),$$

which makes our tortoise coordinate look like:

$$(3.38) \quad r^* = \frac{r_0^2 - r_H^2}{r_0^2 + r_H^2} \left(r_0 \arctan\left(\frac{r}{r_0}\right) - \frac{r_H}{2} \ln\left(\frac{1 + \frac{r}{r_H}}{1 - \frac{r}{r_H}}\right) \right) = \frac{1}{2}(-u + v).$$

This can actually be written in terms of the surface gravity $\kappa_{k\pm} = \frac{1}{2}f'(r_{k\pm})$ (to make consistent with Harvey's notes and Poisson's text):

$$(3.39) \quad r^* = \frac{1}{\kappa_{0+}} \arctan\left(\frac{r}{r_0}\right) + \frac{1}{2\kappa_{H-}} \ln(r_H + r) + \frac{1}{2\kappa_{H+}} \ln(r_H - r) = \frac{1}{2}(-u + v).$$

Now, even though none of the texts explicitly say this, they typically then define

$$(3.40) \quad U_{k\pm} \equiv -\exp(-\kappa_{k\pm}u) \quad ; \quad V_{k\pm} \equiv \exp(\kappa_{k\pm}v),$$

seemingly picking out the coordinate singularity to single out with the choice of $\kappa_{k\pm}$. So let's check this out and pick the only real positive root that matters here at $r = r_H$ (that is, pick κ_{H+}).

$$(3.41) \quad U = -\exp(-\kappa_{H+}u) \quad ; \quad V = \exp(\kappa_{H+}v),$$

$$(3.42) \quad dU = \kappa_{H+} \exp(-\kappa_{H+}u) du \quad dV = \kappa_{H+} \exp(\kappa_{H+}v) dv,$$

$$(3.43) \quad dU dV = \kappa_{H+}^2 \exp(\kappa_{H+}(-u + v)) du dv,$$

$$(3.44) \quad = \kappa_{H+}^2 \exp\left(\frac{2\kappa_{H+}}{\kappa_{0+}} \arctan\left(\frac{r}{r_0}\right) + \frac{\kappa_{H+}}{\kappa_{H-}} \ln(r_H + r) + \ln(r_H - r)\right) du dv,$$

$$(3.45) \quad \text{note that } \frac{\kappa_{H+}}{\kappa_{H-}} = -1$$

$$(3.46) \quad = \kappa_{H+}^2 \left(\frac{r_H - r}{r_H + r}\right) \exp\left(\frac{2\kappa_{H+}}{\kappa_{0+}} \arctan\left(\frac{r}{r_0}\right)\right) du dv,$$

$$(3.47) \quad du dv = \kappa_{H+}^{-2} \left(\frac{r_H + r}{r_H - r}\right) \exp\left(-\frac{2r_0}{r_H} \arctan\left(\frac{r}{r_0}\right)\right) dU dV.$$

Substituting this into our line element for null coordinates

$$ds^2 = -f du dv + r^2 d\Omega_3^2,$$

oh, we can see that the vanishing factor of $r - r_H$ in f will be suppressed with the appearance of $r_H - r$ in the denominator having chosen the κ_{H+} choice.

$$ds^2 = -\frac{(r^2 - r_H^2)(r^2 + r_0^2)}{r^2(r_0^2 - r_H^2)} \left(\frac{r_H^2(r_0^2 - r_H^2)^2}{(r_0^2 + r_H^2)^2}\right) \left(\frac{r_H + r}{r_H - r}\right) \exp\left(-\frac{2r_0}{r_H} \arctan\left(\frac{r}{r_0}\right)\right) dU dV + r^2 d\Omega_3^2,$$

$$\boxed{ds^2 = r_H^2 \left(\frac{r_0^2 - r_H^2}{(r_0^2 + r_H^2)^2}\right) \left(1 + \frac{r_0^2}{r^2}\right) (r + r_H)^2 \exp\left(-\frac{2r_0}{r_H} \arctan\left(\frac{r}{r_0}\right)\right) dU dV + r^2 d\Omega_3^2}.$$

This exactly matches the form of the metric we have been using so far if

$$(3.48) \quad \left(\frac{r_0^2 - r_H^2}{(r_0^2 + r_H^2)^2}\right) = \frac{1}{1 + 4\mu}.$$

Haha! Sure enough, we find that

$$(3.49) \quad \left(\frac{r_0^2 - r_H^2}{(r_0^2 + r_H^2)^2} \right) = \frac{1}{l^2 + 4\mu} ,$$

and we are happy with the dimensions the parameters imply. In fact, we would like the boxed form of the metric more as it purely depends on the roots r_H and r_0 with no implicit mentioning of μ or l ...

It is very easy to update the MOTSodesic equations as they are written in terms of $N = \sqrt{N^2}$, so we can instead (of (3.9)) set

$$(3.50) \quad N^2 := r_H^2 \left(\frac{r_0^2 - r_H^2}{(r_0^2 + r_H^2)^2} \right) \left(1 + \frac{r_0^2}{r^2} \right) (r + r_H)^2 \exp \left(-\frac{2r_0}{r_H} \arctan \left(\frac{r}{r_0} \right) \right) .$$

3.7. **d=4.** The Schwarzschild-AdS spacetime has typical coordinates that look like

$$(3.51) \quad ds^2 = -f_4 dt^2 + \frac{dr^2}{f_4} + r^2 d\Omega_2^2 ,$$

where

$$(3.52) \quad f_4 = 1 - \frac{\mu}{r} + \frac{r^2}{l^2} .$$

We rewrite this function in factored form:

$$(3.53) \quad \frac{(r - r_H)(r - r_1)(r - r_2)}{l^2 r} ,$$

where (with explicit dependencies that involve cube roots and other nonsense)

$$(3.54) \quad r_H + r_1 + r_2 = 0 , \quad r_H r_1 r_2 = l^2 \mu , \quad r_1 r_2 + r_1 r_H + r_2 r_H = l^2 .$$

Of the three roots, r_H is the only real positive root whereas $r_1 = r_0$, $r_2 = r_0^*$ are complex conjugate pairs. From the above relations, this means $r_H = -2 \operatorname{Re}\{r_0\}$ and $|r_0|^2 - \frac{1}{2} r_H^2 = l^2$, which is somewhat analogous to the Sch-AdS₅ case. The surface gravity is

$$(3.55) \quad \kappa_k = \frac{2r_k^3 + l^2 \mu}{2l^2 r_k^2} = \frac{2r_k^3 + r_H r_1 r_2}{2(r_1 r_2 + r_1 r_H + r_2 r_H) r_k^2}$$

$$(3.56) \quad \kappa_H = \frac{(r_H - r_1)(r_H - r_2)}{2l^2 r_H} , \quad \kappa_1 = \frac{(r_1 - r_2)(r_1 - r_H)}{2l^2 r_1} , \quad \kappa_2 = \frac{(r_2 - r_1)(r_2 - r_H)}{2l^2 r_2} .$$

$$(3.57) \quad r^* = \int \frac{dr}{f} = \frac{1}{2\kappa_H} \ln(r - r_H) + \frac{1}{2\kappa_1} \ln(r - r_1) + \frac{1}{2\kappa_2} \ln(r - r_2)$$

Following the epithany in the previous part, we can simply find exponentiated null coordinates that is regular at $r = r_H$.

$$(3.58) \quad dU dV = \kappa_H^2 \exp(\kappa_H(-u + v)) du dv ,$$

$$(3.59) \quad = \kappa_H^2 \exp(2\kappa_H r^*) du dv ,$$

$$(3.60) \quad = \kappa_H^2 \exp\left(\ln(r - r_H) + \frac{\kappa_H}{\kappa_1} \ln(r - r_1) + \frac{\kappa_H}{\kappa_2} \ln(r - r_2)\right) du dv ,$$

$$(3.61) \quad = \kappa_H^2 (r - r_H)(r - r_1)^{\frac{\kappa_H}{\kappa_1}} (r - r_2)^{\frac{\kappa_H}{\kappa_2}} du dv ,$$

$$(3.62) \quad du dv = \kappa_H^{-2} (r - r_H)^{-1} (r - r_1)^{-\frac{\kappa_H}{\kappa_1}} (r - r_2)^{-\frac{\kappa_H}{\kappa_2}} dU dV .$$

Then the null coordinates

$$ds^2 = -f du dv + r^2 d\Omega_2^2$$

becomes

$$(3.63) \quad ds^2 = -\frac{\kappa_H^{-2}}{l^2 r} (r - r_1)^{1 - \frac{\kappa_H}{\kappa_1}} (r - r_2)^{1 - \frac{\kappa_H}{\kappa_2}} dU dV + r^2 d\Omega_2^2 ,$$

or

$$(3.64) \quad \boxed{ds^2 = -\frac{4r_H^2}{r} \frac{(r - r_1)^{1 - \frac{\kappa_H}{\kappa_1}} (r - r_2)^{1 - \frac{\kappa_H}{\kappa_2}}}{(r_H - r_1)(r_H - r_2)} dU dV + r^2 d\Omega_2^2} ,$$

The relationship between $-T^2 + X^2 = U \cdot V = G(r)$ is:

$$G(r) = U \cdot V = -\exp(2\kappa_H r^*) = -\exp\left(\ln(r - r_H) + \frac{\kappa_H}{\kappa_1} \ln(r - r_1) + \frac{\kappa_H}{\kappa_2} \ln(r - r_2)\right) ,$$

$$-T^2 + X^2 = -(r - r_H)(r - r_1)^{\frac{\kappa_H}{\kappa_1}} (r - r_2)^{\frac{\kappa_H}{\kappa_2}} .$$

A note on the dimensions: looking at equation (3.61), $([\text{length}])^{\frac{\kappa_H}{\kappa_1}} \cdot ([\text{length}])^{\frac{\kappa_H}{\kappa_2}}$ has dimensions $[\frac{1}{\text{length}}]$.

*** We will want to avoid explicit mentionings of r_1 or r_2 in the metric, since the numerics will use any excuse to entertain the idea of a complex numerical approximation. We know that $r_1 = r_2^*$ (where the $*$ denotes a complex conjugation, perhaps it is better for us to use \bar{r}_0 as to not conflict with the tortoise coordinate r^*), so we will rewrite $r_1 = r_0 = r_{0R} + ir_{0I}$. Explicitly (in terms of r_H):

$$r_{0R} = \frac{\sqrt[3]{2} \left(\sqrt{81 (r_H^3 + r_H)^2 + 12} - 9 (r_H^3 + r_H) \right)^{2/3} - 2\sqrt[3]{3}}{2 \cdot 6^{2/3} \sqrt[3]{\sqrt{81 (r_H^3 + r_H)^2 + 12} - 9 (r_H^3 + r_H)}},$$

$$r_{0I} = \frac{1}{2} \sqrt{\frac{\left(\sqrt{81 (r_H^3 + r_H)^2 + 12} - 9 (r_H^3 + r_H) \right)^{2/3}}{2^{2/3} \sqrt[3]{3}} + \frac{2^{2/3} \sqrt[3]{3}}{\left(\sqrt{81 (r_H^3 + r_H)^2 + 12} - 9 (r_H^3 + r_H) \right)^{2/3}}} + 2.$$

(having set the Planck units $\ell = 1$ [length] has messed up some unit understandings...)

The portion of the metric that contains $r_1 = r_0$ and $r_2 = r_0^*$ (which follows that $\kappa_1 = \kappa_0$ and $\kappa_2 = \kappa_0^*$ looking at (3.56)) terms can then be analyzed such that they are written in purely real terminology:

$$(3.65) \quad \frac{(r - r_1)^{1 - \frac{\kappa_H}{\kappa_1}} (r - r_2)^{1 - \frac{\kappa_H}{\kappa_2}}}{(r_H - r_1)(r_H - r_2)} = \frac{(r^2 - 2r_{0R}r + |r_0|^2) \left| (r - r_0)^{-\frac{\kappa_H}{\kappa_0}} \right|}{r_H^2 - 2r_{0R}r_H + |r_0|^2},$$

where everything is now real, but we would also love to write down explicitly $\left| (r - r_0)^{-\frac{\kappa_H}{\kappa_0}} \right|$ in terms of its components. κ_0 is complex (with nonzero real and imaginary parts) making the term a non-natural-number complex exponential...

I wonder if Hari has any idea how to approach this... ↑

Perhaps if we did a change of base

$$(3.66) \quad (r - r_0)^{-\frac{\kappa_H}{\kappa_0}} = \exp\left(-\frac{\kappa_H}{\kappa_0} \ln(r - r_0)\right),$$

then the modulus could be more easily seen...

Let $-\frac{\kappa_H}{\kappa_0} = \kappa = \kappa_R + i\kappa_I$ and $r - r_0 = \rho = \rho_R + I\rho_I$.

$$(3.67) \quad \exp[(\kappa_R + i\kappa_I) \ln(\rho_R + I\rho_I)] = \exp[(\kappa_R + i\kappa_I)(\ln|\rho| + i\arg(\rho))],$$

$$(3.68) \quad = \exp[\kappa_R \ln|\rho| - \kappa_I \arg(\rho) + i(\kappa_R \arg(\rho) + \kappa_I \ln|\rho|)],$$

$$(3.69) \quad |\exp(\dots)| = \exp(\kappa_R \ln|\rho| - \kappa_I \arg(\rho)).$$

Fascinatingly, the $\arg(\rho)$ bit will give us an $\arctan\left(\frac{-r_{0I}}{r - r_{0R}}\right)$, which bears resemblance to the Schw-AdS₅ case as it also has an $\exp(\arctan)$ factor (but it never had to deal with complex roots, only strictly real or strictly imaginary ones...).

3.8. MOTSodesics in Schw-AdS₄ in Kruskal type coordinates.

This is hilariously simple as the equations have been tried and true from our Kruskal paper 2312.00769 (eqns 14 through 27)

4. AdS SPACETIMES IN FURTHER-GENERALIZED-PAINLEVÉ-GULLSTRAND COORDINATES

Martel-Poisson had already ‘generalized’ the Painlevé-Gullstrand coordinates in [arXiv:gr-qc/0001069] for a *constant* p parameter which is interpreted as related to the “initial velocity” of the infalling timelike observer at $r \rightarrow \infty$ via $v_\infty = \sqrt{1-p}$ (in units of $c = 1$). In this interpretation, $p \rightarrow 1$ means the observer is initially at rest at infinity before falling in (Painlevé-Gullstrand coordinates) and $p \rightarrow 0$ is when the observer is already (terminally) null (Eddington-Finkelstein coordinates). In our paper [arXiv:2111.09373 [gr-qc]], Ivan and Robie had shown that you can Further generalize this setting $p = p(r)$, allowing for the coordinates of an accelerated time-like observer.

The further-generalized-Painlevé-Gullstrand coordinates reads

$$(4.1) \quad ds^2 = -f(r)dt^2 + 2\sqrt{1-p(r)f(r)}dtdr + p(r)dr^2 + r^2d\Omega^2 .$$

The Schwarzschild-AdS metric function is

$$(4.2) \quad f(r) = 1 - \frac{2M}{r} + \frac{r^2}{\ell^2} \quad \text{or} \quad = \frac{r\ell^2 - 2M\ell^2 + r^3}{r\ell^2} .$$

This metric function goes $f(r \rightarrow 0) \rightarrow -\infty$ and $f(r \rightarrow \infty) \rightarrow +\infty$. However, the crossterm become imaginary if $p(r)f(r) > 1$. Thus, we may use $p(r)$ to *suppress* the metric function.

5. EXTREMAL SLICINGS OF THESE SPACETIMES AND MOTSS IN THEM (ROBIE'S NOTES)