This document outlines ideas for a project regarding MOTSs in anti de-Sitter (AdS) spacetimes:

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1. Introduction

Penrose diagram of pure AdS spacetime can be found in Hawking-Ellis pg. 132.

2. AdS spacetimes in Painlevé-Gullstrand coordinates

The usual Painlevé–Gullstrand coordinates read

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + 2\sqrt{1-f(r)}\mathrm{d}t\mathrm{d}r + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \tag{2.1}$$

where for a Schwarschild-AdS black hole would have the metric function

$$f(r) = 1 - \frac{2M}{r} + \frac{r^2}{\ell^2} \tag{2.2}$$

The MOTSodesic equations are given in general form in 2111.0937

$$T^i D_i T^j = \kappa N^j \tag{2.3}$$

Where

$$\begin{aligned}
T^i &= \dot{r}\partial_r + \dot{\theta}\partial_\theta \\
N^i &= r\sqrt{p(r)}\left[\frac{\dot{\theta}}{p(r)}\partial_r - \frac{\dot{r}}{r^2}\partial_\theta\right] \\
\hat{\phi}^i &= \frac{1}{r\sin\theta}\partial_r \\
p(r)\dot{r}^2 + r^2\dot{\theta}^2 &= 1 \\
\kappa &= k_u - N_j\hat{\phi}^i D_i\hat{\phi}^j
\end{aligned}$$

Explicitly, the MOTSodesic equations are

$$
\begin{aligned}
\ddot{r} &= -\frac{p'\dot{r}^2 - 2r\dot{\theta}^2}{2p} + \frac{r\dot{\theta}\kappa}{\sqrt{p}} \\
\ddot{\theta} &= -\frac{2\dot{r}\dot{\theta}}{r} - \frac{\sqrt{p}\dot{r}\kappa}{r} \\
\kappa &= -\frac{1}{r\sqrt{p}}\left[p\dot{r}\cot\theta - r\dot{\theta}\right] + \frac{1}{2r\sqrt{p(1-pf)}}\left[rp^2\dot{r}^2 f' + r\dot{r}^2 p' - 2(r^2\dot{\theta}^2+1)(1-pf)\right]
\end{aligned}
$$

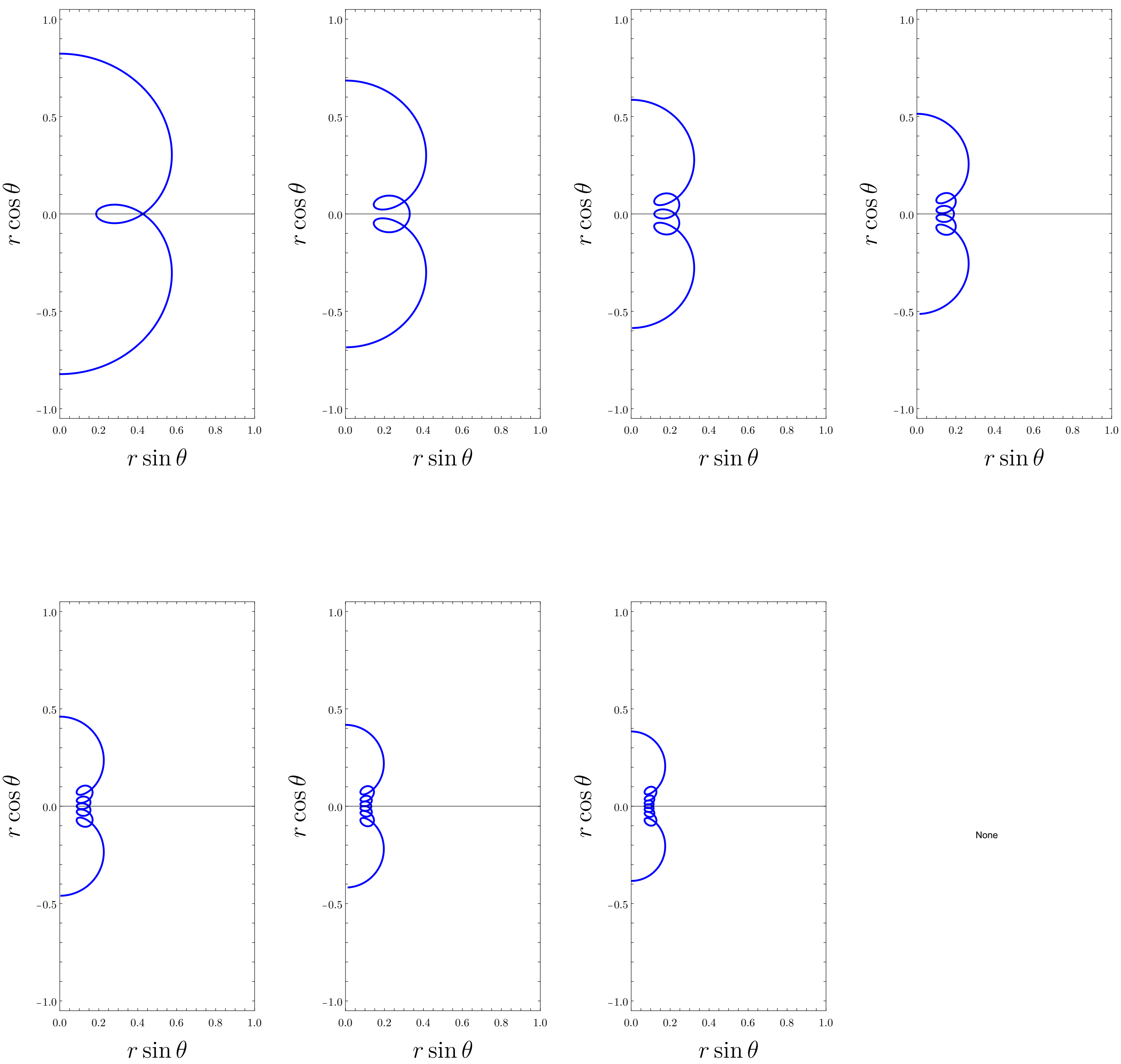

Figure 1. Uses parameters $M = 1$ and $\Lambda = 1$.

There seems to be an infinitely many self-intersecting MOTSs within the Schwarzschild-AdS black hole. This is consistent with Dr. Robie's conjecture that the number of self-intersections correlate to the functional minimum of the metric function in spherically symmetric cases. Since there is no minimum for $f(r) = 1 - \frac{2M}{r} + \Lambda r^2$ (or that it is $\lim_{r\to 0} f(r) = -\infty$), then it makes sense for it to have a very large number of self-intersecting MOTSs.

2.1. **Hawking-Page Transition.** Looking at the 3+1 Schwarzschild-AdS in terms of its event/apparent horizon radius r_h (arXiv:2205.09938, between (1) and (2)):

$$\left(1 - \frac{r_h}{r}\right)\left(\frac{r^2 + r r_h + {r_h}^2}{\ell^2} + 1\right) \tag{2.4}$$

$$M = \frac{r_h}{2}\left(\frac{{r_h}^2}{\ell^2} + 1\right) \ , \tag{2.5}$$

we can calculate the Hawking temperature T_H as a function of M, using the time-like killing vector $\ell^\alpha = \frac{\partial}{\partial t}$: For $\ell^\alpha = \frac{\partial}{\partial t}$ of the Euclidean sector:

$$\begin{aligned}
\ell^2 &= f(r) \\
\nabla_\alpha \ell^2 &= f'(r)\mathrm{d}r \\
g^{\alpha\beta}\nabla_\alpha \ell^2 \nabla_\beta \ell^2 &= g^{rr}(f'(r))^2 \\
&= f(r)(f'(r))^2 \\
\frac{g^{\alpha\beta}\nabla_\alpha \ell^2 \nabla_\beta \ell^2}{4\ell^2} &= \frac{1}{4}(f'(r))^2 \\
f'(2M) &= \frac{3{r_h}^2 + \ell^2}{r_h \ell^2} \\
\frac{g^{\alpha\beta}\nabla_\alpha \ell^2 \nabla_\beta \ell^2}{4\ell^2} &= \frac{1}{4}\left(\frac{3{r_h}^2 + \ell^2}{r_h \ell^2}\right)^2 = \kappa^2 \\
&\boxed{T_H = \frac{\kappa}{2\pi} = \frac{1}{4\pi}\frac{3{r_h}^2 + \ell^2}{r_h \ell^2}}
\end{aligned} \tag{2.6}$$

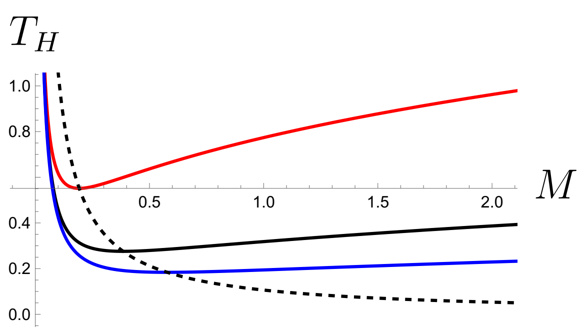

Figure 2. The red, black, then blue curves are for $\ell = 0.5,\ 1.0,\ 1.5$, respectively. The dashed black curve is the critical point for varying ℓ.

By solving $\frac{\partial T}{\partial M} = \frac{\partial T}{\partial r_h} / \frac{\partial M}{\partial r_h} = 0$, the critical point occurs at $r_{h,critical} = \ell/\sqrt{3}$. A nice choice would be to fix $\ell = \sqrt{3}$ actually. For this choice, $r_{h,critical} = 1$ and $M_{critcal} = 2/3 \approx 0.66$.

Immediately and naively, it does not seem like the self-intersecting MOTS behaviour changes compared to Figure 1

3. AdS spacetimes in Kruskal-Szekeres extensions

It seems that someone has already given the Kruskal-Szekeres extension for AdS_5: arXiv:gr-qc/0005115. In 5-dimensions, the metric function's roots is in its best form:

$$\mathrm{d}s^2 = -\frac{r_H^2}{1+4\mu}\left(1+\frac{r_0^2}{r^2}\right)(r_H+r)^2\exp\left(-\frac{2r_0}{r_H}\arctan\left(\frac{r}{r_0}\right)\right)\mathrm{d}U\mathrm{d}V + r^2\mathrm{d}\Omega_3^2\ . \tag{3.1}$$

The unit 3-sphere is $\mathrm{d}\Omega_3^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2 + \cos^2\theta\mathrm{d}\psi$ for $\theta \in (0,\pi/2)$, $\phi,\psi \in (0,2\pi)$. and the other parameters are defined as such:

$$\begin{aligned}
r_H &= \sqrt{\frac{1}{2}\left(\sqrt{1+4\mu}-1\right)}\\
r_0 &= \sqrt{\frac{1}{2}\left(\sqrt{1+4\mu}+1\right)}\\
T_H &= \frac{\sqrt{1+4\mu}}{2\pi r_H}\\
r_* &= \frac{1}{\sqrt{1+4\mu}}\left(r_0\arctan\left(\frac{r}{r_0}\right)+\frac{1}{2}r_H\ln\left(\frac{r-r_H}{r+r_H}\right)\right)\\
u &= t-r_*\\
v &= t+r_*\\
U &= -\exp(-2\pi T_H u)\\
V &= \exp(2\pi T_H v)
\end{aligned} \tag{3.2}$$

In 4+1 dimensions, $F(r) = 1-\frac{\mu}{r^2}+r^2$ has four roots at $\pm r_H$ and $\pm i r_0$.

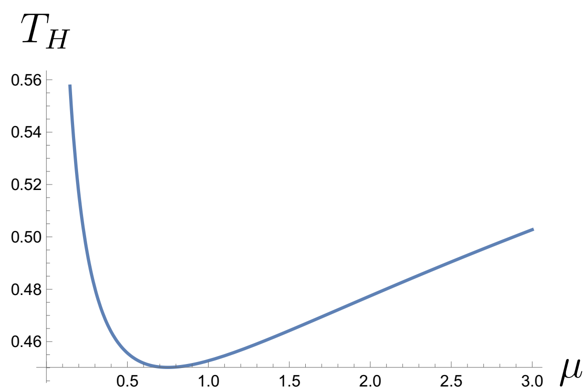

Figure 3. The Hawking-Page transition happens at $\mu_{critical} = 3/4$ (the critical point).

We can do some math:

$$U\cdot V = -\exp(2\pi T_H(v-u)) = -\exp(4\pi T_H r_*) = -\exp\left(\frac{2\pi T_H\left(2r_0\arctan\left(\frac{r}{r_0}\right)+r_H\ln\left(\frac{r-r_H}{r+r_H}-1\right)\right)}{\sqrt{4\mu+1}}\right) \tag{3.3}$$

$$\begin{aligned}
U &= T-X\\
V &= T+X
\end{aligned}$$

$$-T^2+X^2 = \exp\left(\frac{2\pi T_H\left(2r_0\arctan\left(\frac{r}{r_0}\right)+r_H\ln\left(\frac{r-r_H}{r+r_H}\right)\right)}{\sqrt{4\mu+1}}\right) = \left(\frac{r-r_H}{r+r_H}\right)\exp\left(\frac{2r_0}{r_H}\arctan\left(\frac{r}{r_0}\right)\right) = G(r)\ .$$

In the Schwarzschild case ($r_0 \to 0$), we do not expect this to go back to the Schwarzschild case (remember there is a denominator of r^2 now.)

There is no obvious function for r that we know of (analogous to the Lambert-W function), but we have the above form and can still use that implicitly. Even from this we can draw Penrose-Carter diagrams and Kruskal-Szekeres diagrams.

3.1. **Conformal diagrams.** The Kruskal and Penrose diagrams are spoken in exactly the languages of these coordinates (for the Kruskal diagrams plot $(x = X, y = T)$, for Penrose diagrams plot and rotate $(x = \tilde{V} = \arctan(V), y = \tilde{U} = \arctan(U))$ by 45 degrees.)

3.1.1. *Penrose Diagram.* Incredibly, letting the computer plot the V and U in terms of Schwarzschild-like coordinates $t \in \mathbb{R}, rH < r < \infty$, we can get the timelike surface at large r and null surface at $r \to r_H$.

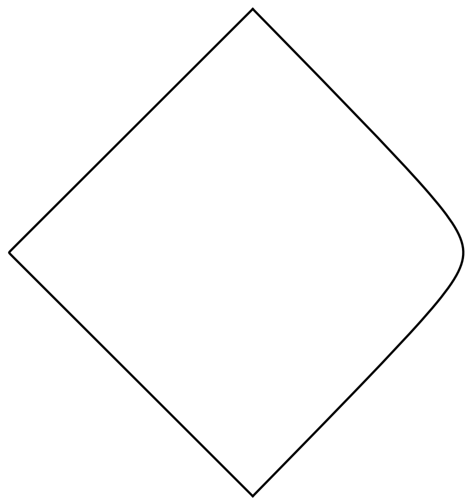

Figure 4. $r \to r_H$ future and past null surface on the left, $r \to \infty$ future and past timelike surface on the right. Numerically obtained by plotting $\tilde{U}$ and $\tilde{V}$.

3.2. **Derivatives via implicit function theorem.** In fact, the implicit function theorem can be applied for setting

$$F(T, X, r(T, X)) = -T^2 + X^2 - G(r(T, X)) = 0 \ , \tag{3.4}$$

where

$$G(r(T, X)) = \exp\left(\frac{2\pi T_H \left(2r_0 \arctan\left(\frac{r}{r_0}\right) + r_H \ln\left(\frac{r-r_H}{r+r_H}\right)\right)}{\sqrt{4\mu + 1}}\right) \tag{3.5}$$

The theorem states (from this source): "If $F : \mathbb{R}^3 \to \mathbb{R}$ and a point $(T_0, X_0, r_0) \in \mathbb{R}^3$ so that $F(T_0, X_0, r_0) = c$. If $\frac{\partial F}{\partial r} \neq 0$, then there is a neighborhood so that whenever (T, X) is sufficiently close to (T_0, X_0) there is a unique $r = r(T, X)$ that is such that $F(T, X, r) = c$." In this case, $c = 0$.

So, later, the MOTSodesic equations will need

$$r_X = \frac{\partial r}{\partial X} = -\frac{F_X}{F_r} = -\frac{2X}{G'(r)} \tag{3.6}$$

$$r_T = \frac{\partial r}{\partial T} = -\frac{F_T}{F_r} = \frac{2T}{G'(r)} \tag{3.7}$$

Before we move on, it would be fun to simplify $G(r)$ a little:

$$G(r) = \exp\left(\frac{2\pi T_H\left(2r_0 \arctan\left(\frac{r}{r_0}\right) + r_H \ln\left(\frac{r-r_H}{r+r_H}\right)\right)}{\sqrt{4\mu+1}}\right)$$
$$= e^{\frac{4\pi r_0 T_H \arctan\left(\frac{r}{r_0}\right)}{\sqrt{4\mu+1}}} \cdot \left(\frac{r-r_H}{r+r_H}\right)^{\frac{2\pi r_H T_H}{\sqrt{4\mu+1}}}$$

Haha! $G(r = r_H) = 0$, but $G'(r = r_H) = e^{\frac{4\pi r_0 T_H \arctan\left(\frac{r_H}{r_0}\right)}{\sqrt{4\mu+1}}}/2r_H \neq 0$!

3.3. **MOTSodesics in Schwarzschild-AdS$_5$ Kruskal-type coordinates.**

$$\mathrm{d}s^2 = \frac{r_H^2}{1+4\mu}\left(1+\frac{r_0^2}{r^2}\right)(r_H+r)^2\exp\left(-\frac{2r_0}{r_H}\arctan\left(\frac{r}{r_0}\right)\right)(-\mathrm{d}T^2+\mathrm{d}X^2)+r^2\mathrm{d}\Omega_3^2\ , \tag{3.8}$$

where r_H and r_0 are roots of

$$1-\frac{\mu}{r^2}+r^2=0$$

The lapse N is defined by

$$N^2 := \frac{r_H^2}{1+4\mu}\left(1+\frac{r_0^2}{r^2}\right)(r_H+r)^2\exp\left(-\frac{2r_0}{r_H}\arctan\left(\frac{r}{r_0}\right)\right)\ . \tag{3.9}$$

The coordinates $T, X\in\mathbb{R}$ relate to $r=r(T,X)$ by

$$-T^2+X^2=\left(\frac{r-r_H}{r+r_H}\right)\exp\left(\frac{2r_0}{r_H}\arctan\left(\frac{r}{r_0}\right)\right)=G(r)\ . \tag{3.10}$$

$$G'(r)=\frac{2({r_0}^2+r_H^2)}{r_H}\frac{r^2}{(r^2+{r_0}^2)(r^2+r_H^2)}\exp\left(\frac{2r_0}{r_H}\arctan\left(\frac{r}{r_0}\right)\right) \tag{3.11}$$

We can (almost identically) derive the MOTSodesic equations from the Marginally Outer trapped tori paper and the n-dimension rotating paper [link to tori, link to rotating]

Σ_T surfaces of constant T has the induced metric

$$h_{ij}\mathrm{d}x^i\mathrm{d}x^j = N^2\mathrm{d}X^2+r^2\mathrm{d}\Omega_{(3)}^2\ . \tag{3.12}$$

The normal one-form to this foliation:

$$u_\alpha \mathrm{d}x^\alpha = -N\mathrm{d}T \tag{3.13}$$

The quarter-plane orbit space with curve parameter s, that is $X,\theta = P(s),\Theta(s)$ (consistent with paper).

$$N^2\mathrm{d}X^2+r^2\mathrm{d}\theta^2=(N^2\dot{P}^2+r^2\dot{\Theta}^2)\mathrm{d}s^2 \tag{3.14}$$

This will be our arc-length parameterization by the way,

$$N^2\dot{P}^2+r^2\dot{\Theta}^2\equiv 1 \tag{3.15}$$

(Just like equation 20 in Kruskal paper)

Let indices α,β run over $\{T,X,\theta,\phi,\psi\}$, i,j over $\{X,\theta,\phi,\psi\}$, a,b over $\{X,\theta\}$, and A,B over $\{\phi,\psi\}$

$$\mathrm{d}\Omega_3^2=\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2+\cos^2\theta\mathrm{d}\psi^2\ . \tag{3.16}$$

$$\underline{h}_{ab}\mathrm{d}x^a\mathrm{d}x^b=N^2\mathrm{d}X^2+r^2\mathrm{d}\theta^2 \tag{3.17}$$

$$\underline{h}_{AB}\mathrm{d}\phi^A\mathrm{d}\phi^B=r^2\sin^2\theta\mathrm{d}\phi^2+r^2\cos^2\theta\mathrm{d}\psi^2 \tag{3.18}$$

$$\underline{h}^{AB}\frac{\partial}{\partial\phi^A}\frac{\partial}{\partial\phi^B}=\frac{1}{r^2\sin^2\theta}\left(\frac{\partial}{\partial\phi}\right)^2+\frac{1}{r^2\cos^2\theta}\left(\frac{\partial}{\partial\psi}\right)^2 \tag{3.19}$$

$$\underline{K}_{ab}=K_{ij}e^i_ae^j_b\ ,\quad \underline{K}_{AB}=K_{ij}e^i_Ae^j_B\ . \tag{3.20}$$

$$\begin{aligned}\hat{T}^a\frac{\partial}{\partial x^a}&=\dot{P}\frac{\partial}{\partial X}+\dot{\Theta}\frac{\partial}{\partial\theta}\\ \hat{N}^a\frac{\partial}{\partial x^a}&=\frac{r\dot{\Theta}}{N}\frac{\partial}{\partial X}+\frac{-N\dot{P}}{r}\frac{\partial}{\partial\theta}\\ \hat{N}_a\mathrm{d}x^a&=Nr\left(\Theta'(s)\mathrm{d}X-P'(s)\mathrm{d}\theta\right)\end{aligned} \tag{3.21}$$

$$\hat{T}^b\underline{\nabla}_b\hat{T}^a=\kappa_{\mathrm{MOTS}}\hat{N}^a \tag{3.22}$$

$$\begin{aligned} \kappa_{\mathrm{MOTS}} &= \mathcal{K} + \mathcal{K}_{\hat{N}} + \mathcal{K}_{\hat{T}\hat{T}} \\ \mathcal{K} &:= \underline{h}^{AB}\underline{K}_{AB} \\ \mathcal{K}_{\hat{N}} &:= \hat{N}^a(\underline{D}_a \ln\sqrt{\underline{h}}) \\ \mathcal{K}_{\hat{T}\hat{T}} &:= \underline{K}_{ab}\hat{T}\hat{T} \end{aligned} \tag{3.23, 3.24}$$

where $\underline{D}$ lives on the orbit quarter-plane $\underline{h}_{ab}$.

$$\begin{aligned} \mathcal{K} &:= \underline{h}^{\phi\phi}\underline{K}_{\phi\phi} + \underline{h}^{\psi\psi}\underline{K}_{\psi\psi} \ , \\ & \boxed{\mathcal{K} = \frac{2r_T}{Nr}} \ . \\ \mathcal{K}_{\hat{N}} &:= \hat{N}^a(\underline{D}_a \ln\sqrt{\underline{h}_{\phi\phi}\underline{h}_{\psi\psi}}) \ , \\ & \boxed{\mathcal{K}_{\hat{N}} = \frac{2\dot{\Theta}r_X}{N} - \frac{N(\cot\Theta - \tan\Theta)\dot{P}}{r}} \ . \\ \mathcal{K}_{\hat{T}\hat{T}} &:= \underline{K}_{ab}\hat{T}^a\hat{T}^b \\ & \boxed{\mathcal{K}_{\hat{T}\hat{T}} = N_T\dot{P}^2 + \frac{r\ r_T}{N}\dot{\Theta}^2} \end{aligned} \tag{3.25}$$

$$\boxed{\begin{aligned} \ddot{P} &= -\left(\frac{N_X}{N}\right)\dot{P}^2 + \left(\frac{rr_X}{N^2}\right)\dot{\Theta}^2 + \left(\frac{r\kappa_{\mathrm{MOTS}}}{N}\right)\dot{\Theta} \ . \\ \ddot{\Theta} &= -\left(\frac{2r_X}{r}\right)\dot{P}\dot{\Theta} - \left(\frac{N\kappa_{\mathrm{MOTS}}}{r}\right)\dot{P} \ . \end{aligned}} \tag{3.26}$$

Check equations for $r = r_H$,

$$\begin{aligned} G(r=r_H) &= 0 \ . \\ G'(r=r_H) &= \frac{1}{2r_H}\exp\left(\frac{2r_0}{r_H}\arctan\left(\frac{r_H}{r_0}\right)\right) \\ r_X\Big|_{r=r_H} &= -\frac{2X}{G'(r_H)} = -\ r_T\Big|_{r=r_H} = -4Tr_H\exp\left(-\frac{2r_0}{r_H}\arctan\left(\frac{r_H}{r_0}\right)\right) \\ N^2(r_H) &= \frac{4r_H^2(r_H^2+{r_0}^2)}{(1+2r_H^2)^2}\exp\left(-\frac{2r_0}{r_H}\arctan\left(\frac{r_H}{r_0}\right)\right) \\ N(r_H) &= \frac{2r_H\sqrt{r_H^2+{r_0}^2}}{1+2r_H^2}\exp\left(-\frac{r_0}{r_H}\arctan\left(\frac{r_H}{r_0}\right)\right) \\ N'(r_H) &= \frac{(r_H^2-3{r_0}^2)}{(1+2r_H^2)\sqrt{r_H^2+{r_0}^2}}\exp\left(-\frac{r_0}{r_H}\arctan\left(\frac{r_H}{r_0}\right)\right) \\ & P = X = T \quad ; \quad \dot{P} = 0 \quad ; \quad \ddot{P} = 0 \\ & \Theta = \frac{s}{r_H} \quad ; \quad \dot{\Theta} = \frac{1}{r_H} \quad ; \quad \ddot{\Theta} = 0 \end{aligned} \tag{3.27}$$

Need to show $\left(\frac{r r_X}{N^2}\right)\dot{\Theta}^2 + \left(\frac{r\kappa_{\text{MOTS}}}{N}\right)\dot{\Theta} = 0$.

$$\left(\frac{r r_X}{N^2}\right)\dot{\Theta}^2 = \frac{r_H(-4Tr_H)(1+2r_H^2)^2}{4r_H^2(r_H^2+r_0{}^2)}\frac{1}{r_H^2}$$
$$= \frac{-T(1+2r_H^2)^2}{r_H^2(r_H^2+r_0{}^2)}$$
$$\left(\frac{r\kappa_{\text{MOTS}}}{N}\right)\dot{\Theta} = \frac{2r_T}{N^2 r} + \frac{2\dot{\Theta} r_X}{N^2} + \frac{r\ r_T}{N^2}\dot{\Theta}^2$$

$$= \frac{8Tr_H \exp\left(-\frac{2r_0}{r_H}\arctan\left(\frac{r_H}{r_0}\right)\right)}{\frac{4r_H{}^2(r_H{}^2+r_0{}^2)}{(1+2r_H{}^2)^2}\exp\left(-\frac{2r_0}{r_H}\arctan\left(\frac{r_H}{r_0}\right)\right)r_H} + \frac{-8\frac{1}{r_H}Tr_H \exp\left(-\frac{2r_0}{r_H}\arctan\left(\frac{r_H}{r_0}\right)\right)}{\frac{4r_H{}^2(r_H{}^2+r_0{}^2)}{(1+2r_H{}^2)^2}\exp\left(-\frac{2r_0}{r_H}\arctan\left(\frac{r_H}{r_0}\right)\right)} \tag{3.28}$$
$$+ \frac{4Tr_H \exp\left(-\frac{2r_0}{r_H}\arctan\left(\frac{r_H}{r_0}\right)\right)}{\frac{4r_H{}^2(r_H{}^2+r_0{}^2)}{(1+2r_H{}^2)^2}\exp\left(-\frac{2r_0}{r_H}\arctan\left(\frac{r_H}{r_0}\right)\right)r_H}$$
$$= \frac{T(1+2r_H^2)^2}{r_H^2(r_H^2+r_0{}^2)}$$

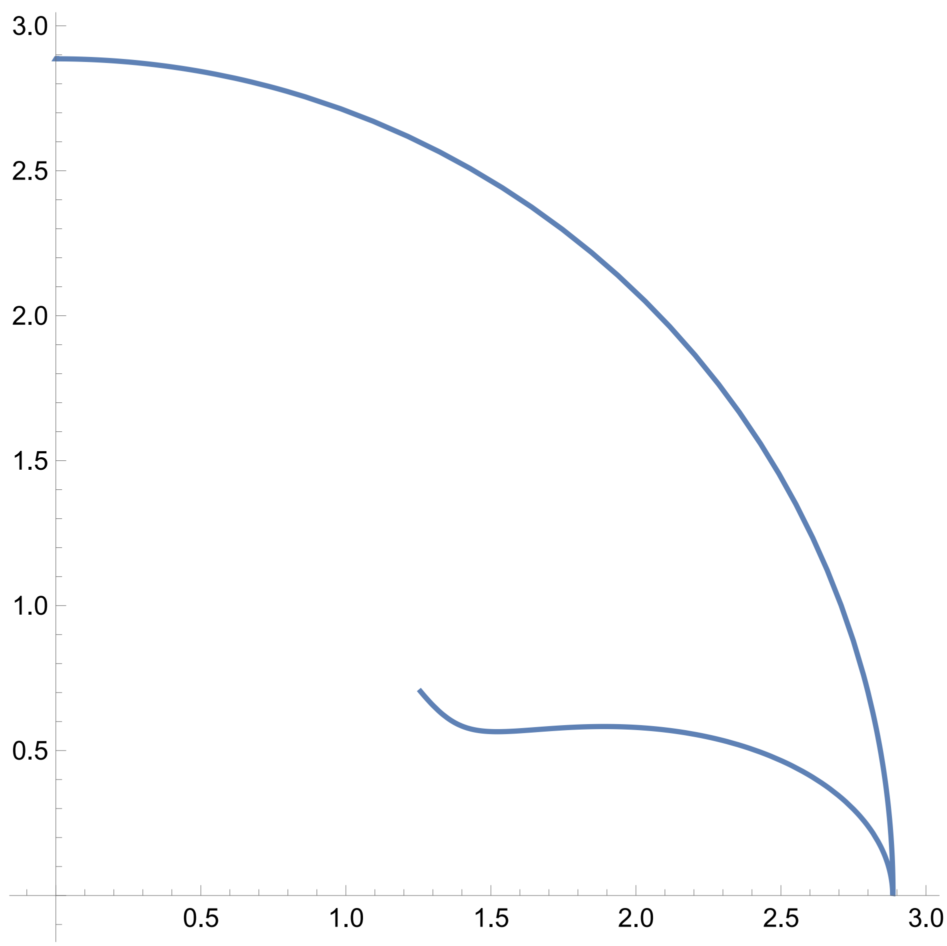

Figure 5. The first MOTSs made (the $r = rH = 1$ MOTS). Axes are $e^X\{\sin\theta, \cos\theta\}$

3.4. **Properties of the inverse function $G^{-1}(X,T)$ (analogous to Lambert W function).** We consider the relationship between the coordinates X, T and r, t, given by

$$-T^2 + X^2 = \left(\frac{r - r_H}{r + r_H}\right) \exp\left(\frac{2r_0}{r_H} \arctan\left(\frac{r}{r_0}\right)\right) = G(r) \ . \tag{3.29}$$

In the asymptotically flat Schwarzschild spacetime, the relationship is

$$-T^2 + X^2 = -(1 - \frac{r}{2M}) \exp(r/2M) \ , \tag{3.30}$$

which is inverted to give $r = r(X,T) \propto W(-T^2 + X^2)$. The case in Schwarschild-AdS is clearly going to be more involved.

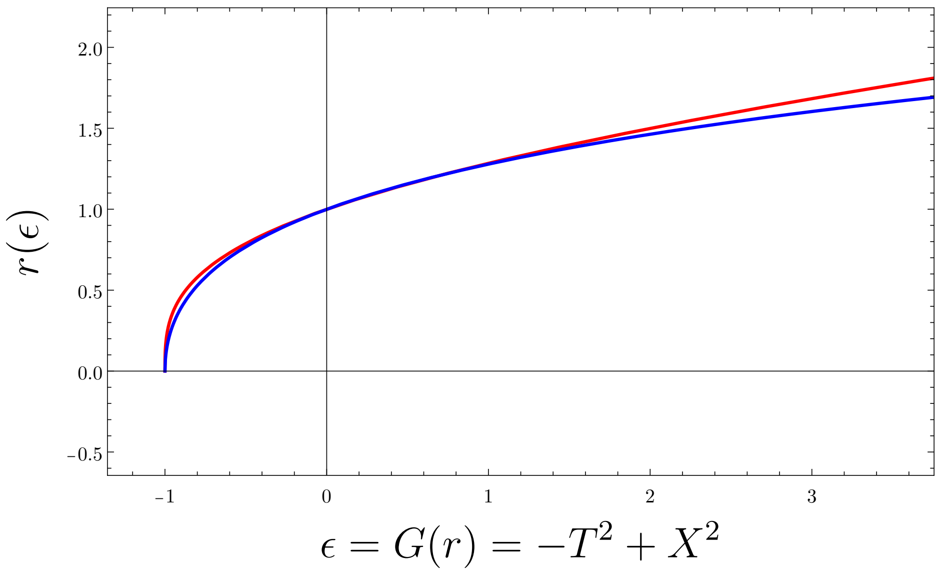

Figure 6. r as a function of $\epsilon = -T^2 + X^2$ in the Schwarzschild case (blue) and Schwarzschild-AdS case (red).

What one could do is a chosen few series expansions:

$$\frac{r - r_H}{r + r_H} \sim -1 + \frac{2r}{r_H} - \frac{2r^2}{r_H^2} + \frac{2r^3}{r_H^3} + \mathcal{O}(r^4) \ ,$$

$$\arctan\left(\frac{r}{r_0}\right) \sim \frac{r}{r_0} - \frac{r^3}{3{r_0}^3} + \frac{r^5}{{r_0}^5} + \mathcal{O}(r^7) \ .$$

(note these series do not converge for $r > 0 \in \mathbb{R}$, but are asymptotic for $r/r_H \to 0,\ r/r_0 \to 0$.) Then $G(r)$ looks like:

$$\begin{aligned} G(r) &= \left(\frac{r - r_H}{r + r_H}\right) \exp\left(\frac{2r_0}{r_H} \arctan\left(\frac{r}{r_0}\right)\right) \\ &\sim \left(-1 + \frac{2r}{r_H} + \mathcal{O}(r^2)\right) \exp\left(\frac{2r_0}{r_H} (\frac{r}{r_0} + \mathcal{O}(r^3))\right) \\ &= \left(-1 + \frac{2r}{r_H}\right) \exp\left(\frac{2r}{r_H}\right) \left(\exp(\mathcal{O}(r^3))\right) + \mathcal{O}(r^2) \end{aligned}$$

In the limit $r/r_H \to 0$ (and since r_0 is parameterized via μ dependent on the parameter r_H, we get $r/r_0 \to 0$ for free), $G(r) \sim G_{Scwharzschild}(r)$ and the function behaves similar to the Lambert-W function.

From the plot, we it would not be surprising if the same analysis for $r \to r_H$ (at $\epsilon \to 0$) shows a similar result.

3.5. **Derivation of the Maximal extension of Schwarzschild-AdS.** Consider the dth-dimensional Schwarszschild-AdS spacetime $(n = d-3)$

$$\mathrm{d}s^2 = -f_n(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f_n(r)} + r^2\mathrm{d}\Omega_{n+1}^2 \qquad ; \qquad f_n = 1 - \frac{\mu}{r^n} + \frac{r^2}{\ell^2} \ . \tag{3.31}$$

3.6. **d=5.** We had found literature on a maximal extension for the $(4+1)$-dimensional Schwarzschild-AdS spacetime to be

$$\mathrm{d}s^2 = \frac{r_H^2}{1+4\mu}\left(1+\frac{r_0^2}{r^2}\right)(r_H+r)^2\exp\left(-\frac{2r_0}{r_H}\arctan\left(\frac{r}{r_0}\right)\right)(-\mathrm{d}T^2+\mathrm{d}X^2) + r^2\mathrm{d}\Omega_3^2 \ . \tag{3.32}$$

Starting from the Schwarzschild-like radial coordinates,

$$\mathrm{d}s^2 = -f_5(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f_5(r)} + r^2\mathrm{d}\Omega_3^2 \qquad ; \qquad f = 1 - \frac{\mu}{r^2} + \frac{r^2}{\ell^2} \ , \tag{3.33}$$

The metric function can be brought to factored form with

$$\mu = \frac{{r_H}^2{r_0}^2}{{r_0}^2 - r_H^2} \quad ; \quad l^2 = {r_0}^2 - r_H^2 \ , \tag{3.34}$$

$$f_5(r) = \frac{(r^2 - r_H^2)(r^2 + {r_0}^2)}{r^2({r_0}^2 - r_H^2)} \ . \tag{3.35}$$

This is constructed such that the roots are at $r = r_H$ and $r = ir_0$. In Planck units ($\ell = 1$), the factor of ${r_0}^2 - r_H^2$ is unity by definition. We would like to know that ℓ is not imaginary, that is $r_0 > r_H$, so explicitly:

$$ir_0 = \pm\sqrt{\frac{-l^2}{2}}\sqrt{1+\sqrt{l^2+4\mu}} \quad ; \quad r_H = \pm\sqrt{\frac{l^2}{2}}\sqrt{-1+\sqrt{l^2+4\mu}} \tag{3.36}$$

It is helpful to note the surface gravity

$$\kappa_i = \frac{1}{2}f'(r_i) = \frac{{r_i}^4 + {r_0}^2 r_H^2}{{r_i}^3({r_0}^2 - r_H^2)} \tag{3.37}$$

We first find the tortoise coordinates such that the metric takes the form

$$\mathrm{d}s^2 = f(r)\left(-\mathrm{d}t^2 + \mathrm{d}r^{*2}\right) + \mathrm{d}{\Omega_3}^2 \ .$$

This is done by setting

$$\mathrm{d}r^2 = f^2\mathrm{d}r^{*2}$$

Asserting that $\mathrm{Sign}(\mathrm{d}r) = \mathrm{Sign}(\mathrm{d}r^*)$ (so that we may get rid of the squares without worries regarding the plus-minus signs), we have

$$\mathrm{d}r^* = \frac{\mathrm{d}r}{f}$$

$$r^* = \int\frac{\mathrm{d}r}{f} = \frac{(r_0^2 - r_H^2)\left(r_0\tan^{-1}\left(\frac{r}{r_0}\right) - r_H\tanh^{-1}\left(\frac{r}{r_H}\right)\right)}{r_0^2 + r_H^2} + \text{const.} \ .$$

Just as a note: $\tanh^{-1}(\pm 1)$ and $\tan^{-1}(\pm i)$ diverges.
Now introduce null coordinates:

$$u \equiv t - r^* \quad ; \quad v \equiv t + r^* \ .$$

$$\mathrm{d}u = \mathrm{d}t - \frac{\mathrm{d}r}{f} \quad ; \quad \mathrm{d}v = \mathrm{d}t + \frac{\mathrm{d}r}{f} \ ,$$

$$\mathrm{d}u = \mathrm{d}t - \mathrm{d}r^* \quad ; \quad \mathrm{d}v = \mathrm{d}t + \mathrm{d}r^* \ ,$$

$$\mathrm{d}r = -\frac{f}{2}(\mathrm{d}u - \mathrm{d}v) \quad ; \quad \mathrm{d}t = \frac{1}{2}(\mathrm{d}u + \mathrm{d}v) \ ,$$

$$\mathrm{d}s^2 = -f\mathrm{d}u\mathrm{d}v + r^2\mathrm{d}{\Omega_3}^2 \ .$$

Just to make ourselves feel better, remember these are truly null coordinates and we can recover the Eddington-Finkelstein coordinates (ingoing & outgoing) with $\mathrm{d}t \to \mathrm{d}u + \frac{\mathrm{d}r}{f}$ and $\mathrm{d}t \to \mathrm{d}v - \frac{\mathrm{d}r}{f}$:

$$\mathrm{d}s^2 = -\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + r^2\mathrm{d}{\Omega_3}^2 \qquad \mathrm{d}s^2 = -\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2\mathrm{d}{\Omega_3}^2$$

Now remember we want to rid ourselves of the coordinate misbehaviour at $r = r_H$. What we could do is use a $\tanh^{-1}$ to ln relation:

$$\tanh^{-1}(z) = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) ,$$

which makes our tortoise coordinate look like:

$$r^* = \frac{{r_0}^2 - r_H^2}{{r_0}^2 + r_H^2}\left(r_0 \arctan\left(\frac{r}{r_0}\right) - \frac{r_H}{2}\ln\left(\frac{1+\frac{r}{r_H}}{1-\frac{r}{r_H}}\right)\right) = \frac{1}{2}(-u+v) . \tag{3.38}$$

This can actually be written in terms of the surface gravity $\kappa_{k\pm} = \frac{1}{2}f'(r_{k\pm})$ (to make consistent with Harvey's notes and Poisson's text):

$$r^* = \frac{1}{\kappa_{0+}}\arctan\left(\frac{r}{r_0}\right) + \frac{1}{2\kappa_{H-}}\ln(r_H + r) + \frac{1}{2\kappa_{H+}}\ln(r_H - r) = \frac{1}{2}(-u+v) . \tag{3.39}$$

Now, even though none of the texts explicitly say this, they typically then define

$$U_{k\pm} \equiv -\exp(-\kappa_{k\pm}u) \quad ; \quad V_{k\pm} \equiv \exp(\kappa_{k\pm}v) , \tag{3.40}$$

seemingly picking out the coordinate singularity to single out with the choice of $\kappa_{k\pm}$. So let's check this out and pick the only real positive root that matters here at $r = r_H$ (that is, pick κ_{H+}).

$$U = -\exp(-\kappa_{H+}u) \quad ; \quad V = \exp(\kappa_{H+}v) , \tag{3.41}$$

$$\mathrm{d}U = \kappa_{H+}\exp(-\kappa_{H+}u)\mathrm{d}u \qquad \mathrm{d}V = \kappa_{H+}\exp(\kappa_{H+}v)\mathrm{d}v , \tag{3.42}$$

$$\mathrm{d}U\mathrm{d}V = \kappa_{H+}^2\exp(\kappa_{H+}(-u+v))\mathrm{d}u\mathrm{d}v , \tag{3.43}$$

$$= \kappa_{H+}^2\exp\left(\frac{2\kappa_{H+}}{\kappa_{0+}}\arctan\left(\frac{r}{r_0}\right) + \frac{\kappa_{H+}}{\kappa_{H-}}\ln(r_H + r) + \ln(r_H - r)\right)\mathrm{d}u\mathrm{d}v , \tag{3.44}$$

$$\text{note that } \frac{\kappa_{H+}}{\kappa_{H-}} = -1 \tag{3.45}$$

$$= \kappa_{H+}^2\left(\frac{r_H - r}{r_H + r}\right)\exp\left(\frac{2\kappa_{H+}}{\kappa_{0+}}\arctan\left(\frac{r}{r_0}\right)\right)\mathrm{d}u\mathrm{d}v , \tag{3.46}$$

$$\mathrm{d}u\mathrm{d}v = \kappa_{H+}^{-2}\left(\frac{r_H + r}{r_H - r}\right)\exp\left(-\frac{2r_0}{r_H}\arctan\left(\frac{r}{r_0}\right)\right)\mathrm{d}U\mathrm{d}V . \tag{3.47}$$

Substituting this into our line element for null coordinates

$$\mathrm{d}s^2 = -f\mathrm{d}u\mathrm{d}v + r^2\mathrm{d}{\Omega_3}^2 ,$$

oh, we can see that the vanishing factor of $r - r_H$ in f will be suppressed with the appearance of $r_H - r$ in the denominator having chosen the κ_{H+} choice.

$$\mathrm{d}s^2 = -\frac{(r^2 - r_H^2)(r^2 + {r_0}^2)}{r^2({r_0}^2 - r_H^2)}\left(\frac{r_H^2({r_0}^2 - r_H^2)^2}{({r_0}^2 + r_H^2)^2}\right)\left(\frac{r_H + r}{r_H - r}\right)\exp\left(-\frac{2r_0}{r_H}\arctan\left(\frac{r}{r_0}\right)\right)\mathrm{d}U\mathrm{d}V + r^2\mathrm{d}{\Omega_3}^2 ,$$

$$\boxed{\mathrm{d}s^2 = r_H^2\left(\frac{{r_0}^2 - r_H^2}{({r_0}^2 + r_H^2)^2}\right)\left(1 + \frac{{r_0}^2}{r^2}\right)(r + r_H)^2\exp\left(-\frac{2r_0}{r_H}\arctan\left(\frac{r}{r_0}\right)\right)\mathrm{d}U\mathrm{d}V + r^2\mathrm{d}{\Omega_3}^2} .$$

This exactly matches the form of the metric we have been using so far if

$$\left(\frac{{r_0}^2 - r_H^2}{({r_0}^2 + r_H^2)^2}\right) = \frac{1}{1+4\mu} . \tag{3.48}$$

Haha! Sure enough, we find that

$$\left(\frac{{r_0}^2 - r_H^2}{({r_0}^2 + r_H^2)^2}\right) = \frac{1}{l^2 + 4\mu} \ , \tag{3.49}$$

and we are happy with the dimensions the parameters imply. In fact, we would like the boxed form of the metric more as it purely depends on the roots r_H and r_0 with no implicit mentioning of μ or l...

It is very easy to update the MOTSodesic equations as they are written in terms of $N = \sqrt{N^2}$, so we can instead (of (3.9)) set

$$N^2 := r_H^2 \left(\frac{{r_0}^2 - r_H^2}{({r_0}^2 + r_H^2)^2}\right)\left(1 + \frac{{r_0}^2}{r^2}\right)(r + r_H)^2 \exp\left(-\frac{2r_0}{r_H}\arctan\left(\frac{r}{r_0}\right)\right) \ . \tag{3.50}$$

3.7. **d=4.** The Schwarzschild-AdS spacetime has typical coordinates that look like

$$\mathrm{d}s^2 = -f_4\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f_4} + r^2\mathrm{d}{\Omega_2}^2 \ , \tag{3.51}$$

where

$$f_4 = 1 - \frac{\mu}{r} + \frac{r^2}{l^2} \ . \tag{3.52}$$

We rewrite this function in factored form:

$$\frac{(r-r_H)(r-r_1)(r-r_2)}{l^2 r} \ , \tag{3.53}$$

where (with explicit dependencies that involve cube roots and other nonsense)

$$r_H + r_1 + r_2 = 0 \ , \quad r_H r_1 r_2 = l^2\mu \ , \quad r_1 r_2 + r_1 r_H + r_2 r_H = l^2 \ . \tag{3.54}$$

Of the three roots, r_H is the only real positive root whereas $r_1 = r_0,\ r_2 = r_0^*$ are complex conjugate pairs. From the above relations, this means $r_H = -2\,\mathrm{Re}\{r_0\}$ and $|r_0|^2 - \frac{1}{2}r_H^2 = \ell^2$, which is somewhat analogous to the Sch-AdS$_5$ case. The surface gravity is

$$\kappa_k = \frac{2{r_k}^3 + l^2\mu}{2l^2{r_k}^2} \quad = \frac{2{r_k}^3 + r_H r_1 r_2}{2(r_1 r_2 + r_1 r_H + r_2 r_H){r_k}^2} \tag{3.55}$$

$$\kappa_H = \frac{(r_H - r_1)(r_H - r_2)}{2l^2 r_H} \ , \quad \kappa_1 = \frac{(r_1 - r_2)(r_1 - r_H)}{2l^2 r_1} \ , \quad \kappa_2 = \frac{(r_2 - r_1)(r_2 - r_H)}{2l^2 r_2} \ . \tag{3.56}$$

$$r^* = \int \frac{\mathrm{d}r}{f} = \frac{1}{2\kappa_H}\ln(r - r_H) + \frac{1}{2\kappa_1}\ln(r - r_1) + \frac{1}{2\kappa_2}\ln(r - r_2) \tag{3.57}$$

Following the epithany in the previous part, we can simply find exponentiated null coordinates that is regular at $r = r_H$.

$$\mathrm{d}U\mathrm{d}V = \kappa_H^2 \exp(\kappa_H(-u+v))\mathrm{d}u\mathrm{d}v \ , \tag{3.58}$$

$$= \kappa_H^2 \exp(2\kappa_H r^*)\mathrm{d}u\mathrm{d}v \ , \tag{3.59}$$

$$= \kappa_H^2 \exp\left(\ln(r - r_H) + \frac{\kappa_H}{\kappa_1}\ln(r - r_1) + \frac{\kappa_H}{\kappa_2}\ln(r - r_2)\right)\mathrm{d}u\mathrm{d}v \ , \tag{3.60}$$

$$= \kappa_H^2 (r - r_H)(r - r_1)^{\frac{\kappa_H}{\kappa_1}}(r - r_2)^{\frac{\kappa_H}{\kappa_2}}\mathrm{d}u\mathrm{d}v \ , \tag{3.61}$$

$$\mathrm{d}u\mathrm{d}v = {\kappa_H}^{-2}(r - r_H)^{-1}(r - r_1)^{-\frac{\kappa_H}{\kappa_1}}(r - r_2)^{-\frac{\kappa_H}{\kappa_2}}\mathrm{d}U\mathrm{d}V \ . \tag{3.62}$$

Then the null coordinates

$$\mathrm{d}s^2 = -f\mathrm{d}u\mathrm{d}v + r^2\mathrm{d}{\Omega_2}^2$$

becomes

$$\mathrm{d}s^2 = -\frac{{\kappa_H}^{-2}}{l^2 r}(r - r_1)^{1-\frac{\kappa_H}{\kappa_1}}(r - r_2)^{1-\frac{\kappa_H}{\kappa_2}}\mathrm{d}U\mathrm{d}V + r^2\mathrm{d}{\Omega_2}^2 \ , \tag{3.63}$$

or

$$\boxed{\mathrm{d}s^2 = -\frac{4r_H^2}{r}\frac{(r - r_1)^{1-\frac{\kappa_H}{\kappa_1}}(r - r_2)^{1-\frac{\kappa_H}{\kappa_2}}}{(r_H - r_1)(r_H - r_2)}\mathrm{d}U\mathrm{d}V + r^2\mathrm{d}{\Omega_2}^2} \ , \tag{3.64}$$

The relationship between $-T^2 + X^2 = U \cdot V = G(r)$ is:

$$G(r) = U \cdot V = -\exp(2\kappa_H r^*) = -\exp\left(\ln(r - r_H) + \frac{\kappa_H}{\kappa_1}\ln(r - r_1) + \frac{\kappa_H}{\kappa_2}\ln(r - r_2)\right) \ ,$$

$$-T^2 + X^2 = -(r - r_H)(r - r_1)^{\frac{\kappa_H}{\kappa_1}}(r - r_2)^{\frac{\kappa_H}{\kappa_2}} \ .$$

A note on the dimensions: looking at equation (3.61), $([\text{length}])^{\frac{\kappa_H}{\kappa_1}} \cdot ([\text{length}])^{\frac{\kappa_H}{\kappa_2}}$ has dimensions $[\frac{1}{\text{length}}]$. *** We will want to avoid explicit mentionings of r_1 or r_2 in the metric, since the numerics will use any excuse to entertain the idea of a complex numerical approximation. We know that $r_1 = {r_2}^*$ (where the * denotes a complex conjugation, perhaps it is better for us to use $\bar{r}_0$ as to not conflict with the tortoise coordinate r^*.), so we will rewrite $r_1 = r_0 = r_{0R} + i r_{0I}$. Explicitly (in terms of r_H and in Planck units $\ell = 1$):

$$r_{0R} = \frac{\sqrt[3]{2}\left(\sqrt{81\left(r_H^3+r_H\right)^2+12}-9\left(r_H^3+r_H\right)\right)^{2/3}-2\sqrt[3]{3}}{2\; 6^{2/3}\sqrt[3]{\sqrt{81\left(r_H^3+r_H\right)^2+12}-9\left(r_H^3+r_H\right)}} \,,$$

$$r_{0I} - \frac{1}{2}\sqrt{\frac{\left(\sqrt{81\left(r_H^3+r_H\right)^2+12}-9\left(r_H^3+r_H\right)\right)^{2/3}}{2^{2/3}\sqrt[3]{3}}+\frac{2^{2/3}\sqrt[3]{3}}{\left(\sqrt{81\left(r_H^3+r_H\right)^2+12}-9\left(r_H^3+r_H\right)\right)^{2/3}}+2} \,.$$

(having set the $\ell = 1$ [length] has messed up some unit understandings...)

The portion of the metric that contains $r_1 = r_0$ and $r_2 = {r_0}^*$ (which follows that $\kappa_1 = \kappa_0$ and $\kappa_2 = {\kappa_0}^*$ looking at (3.56)) terms can then be analyzed such that they are written in purely real terminology:

$$\frac{(r-r_1)^{1-\frac{\kappa_H}{\kappa_1}}(r-r_2)^{1-\frac{\kappa_H}{\kappa_2}}}{(r_H-r_1)(r_H-r_2)} = \frac{(r^2-2r_{0R}r+|r_0|^2)\left|(r-r_0)^{-\frac{\kappa_H}{\kappa_0}}\right|^2}{r_H^2-2r_{0R}r_H+|r_0|^2} \,, \tag{3.65}$$

where everything is now real, but we would also love to write down explicitly $\left|(r-r_0)^{-\frac{\kappa_H}{\kappa_0}}\right|$ in terms of its components. κ_0 is complex (with nonzero real and imaginary parts) making the term a non-natural-number complex exponential...

I wonder if Hari has an idea of how to approach this ↑ but see attempt below...

Perhaps if we did a change of base

$$(r-r_0)^{-\frac{\kappa_H}{\kappa_0}} = \exp\left(-\frac{\kappa_H}{\kappa_0}\ln(r-r_0)\right) \,, \tag{3.66}$$

then the modulus could be more easily seen...

Let $-\frac{\kappa_H}{\kappa_0} = \kappa = \kappa_R + i\kappa_I$ and $r - r_0 = \rho = \rho_R + I\rho_I$.

$$\exp[(\kappa_R+i\kappa_I)\ln(\rho_R+I\rho_I)] = \exp[(\kappa_R+i\kappa_I)(\ln|\rho|+i\arg(\rho))] \,, \tag{3.67}$$

$$= \exp[\kappa_R\ln|\rho| - \kappa_I\arg(\rho) + i(\kappa_R\arg(\rho)+\kappa_I\ln|\rho|)] \,, \tag{3.68}$$

$$\text{the modulus is then } |\exp(\ldots)| = \exp(\kappa_R\ln|\rho| - \kappa_I\arg(\rho)) \,. \tag{3.69}$$

Fascinatingly, the $\arg(\rho)$ bit will give us an $\arctan\left(\frac{-r_{0I}}{r-r_{0R}}\right)$, which bears resemblance to the Schw-AdS$_5$ case as it also has an $\exp(\arctan(r/r_0))$ factor (but it never had to deal with complex roots, only strictly real or strictly imaginary ones...).

3.8. **MOTSodesics in Schw-AdS$_4$ in Kruskal type coordinates.**

This is hilariously simple as the equations have been tried and true from our Kruskal paper 2312.00769 (eqns 14 through 27)

4. AdS spacetimes in Further-Generalized-Painlevé-Gullstrand coordinates

Martel-Poisson had already 'generalized' the Painlevé-Gullstrand coordinates in [arXiv:gr-qc/0001069] for a *constant* p parameter which is interpreted as related to the "initial velocity" of the infalling timelike observer at $r \to \infty$ via $v_\infty = \sqrt{1-p}$ (in units of $c = 1$). In this interpretation, $p \to 1$ means the observer is initially at rest at infinity before falling in (Painlevé-Gullstrand coordinates) and $p \to 0$ is when the observer is already (terminally) null (Eddington-Finkelstein coordinates). In our paper [arXiv:2111.09373 [gr-qc]], Ivan and Robie had shown that you can Further generalize this setting $p = p(r)$, allowing for the coordinates of an accelerated time-like observer.

The further-generalized-Painlevé-Gullstrand coordinates reads

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + 2\sqrt{1 - p(r)f(r)}\mathrm{d}t\mathrm{d}r + p(r)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \ . \tag{4.1}$$

The Schwarzschild-AdS metric function is

$$f(r) = 1 - \frac{2M}{r} + \frac{r^2}{\ell^2} \qquad \text{or} \qquad = \frac{r\ell^2 - 2M\ell^2 + r^3}{r\ell^2} \ . \tag{4.2}$$

This metric function goes $f(r \to 0) \to -\infty$ and $f(r \to \infty) \to +\infty$. However, the crossterm become imaginary if $p(r)f(r) > 1$. Thus, we may use $p(r)$ to *suppress* the metric function.

5. Extremal slicings of these spacetimes and MOTSs in them (Robie's notes)