This document outlines ideas for a project regarding MOTSs in anti de-Sitter (AdS) spacetimes:

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Contents

1. Kruskal Extension of AdS_4

The Schwarzschild-AdS spacetime has typical coordinates that look like

$$\mathrm{d}s^2 = -f_4 \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f_4} + r^2 \mathrm{d}{\Omega_2}^2 \ , \tag{1.1}$$

where

$$f_4 = 1 - \frac{\mu}{r} + \frac{r^2}{l^2} \ . \tag{1.2}$$

We rewrite this function in factored form:

$$\frac{(r - r_H)(r - r_0)(r - \bar{r_0})}{l^2 r} \ , \tag{1.3}$$

where (with explicit dependencies that involve cube roots and other nonsense)

$$\begin{aligned} r_H &= \frac{\sqrt[3]{2}\left(9\mu\ell^2 + \sqrt{12\ell^6 + 81\mu^2\ell^4}\right)^{2/3} - 2\sqrt[3]{3}\ell^2}{6^{2/3}\sqrt[3]{9\mu\ell^2 + \sqrt{12\ell^6 + 81\mu^2\ell^4}}} \\ r_0 &= \frac{\left(1 + i\sqrt{3}\right)\ell^2}{2^{2/3}\sqrt[3]{3}\sqrt[3]{9\mu\ell^2 + \sqrt{3}\sqrt{4\ell^6 + 27\mu^2\ell^4}}} - \frac{\left(1 - i\sqrt{3}\right)\sqrt[3]{9\mu\ell^2 + \sqrt{3}\sqrt{4\ell^6 + 27\mu^2\ell^4}}}{2\sqrt[3]{2}\cdot 3^{2/3}} \\ \bar{r_0} &= \frac{\left(1 - i\sqrt{3}\right)\ell^2}{2^{2/3}\sqrt[3]{3}\sqrt[3]{9\mu\ell^2 + \sqrt{3}\sqrt{4\ell^6 + 27\mu^2\ell^4}}} - \frac{\left(1 + i\sqrt{3}\right)\sqrt[3]{9\mu\ell^2 + \sqrt{3}\sqrt{4\ell^6 + 27\mu^2\ell^4}}}{2\sqrt[3]{2}\cdot 3^{2/3}} \end{aligned} \tag{1.4}$$

$$r_H + r_0 + \bar{r_0} = 0 \ , \quad r_H r_0 \bar{r_0} = l^2\mu \ , \quad r_0\bar{r_0} + r_0 r_H + \bar{r_0} r_H = l^2 \ . \tag{1.5}$$

Of the three roots, r_H is the only real positive root whereas $r_0, \ \bar{r_0}$ are complex conjugate pairs. From the above relations, this gives a few nice relations

$$\begin{aligned} r_H &= -2\,\mathrm{Re}\{r_0\} \\ |r_0|^2 &= \frac{\ell^2\mu}{r_H} \\ |r_0|^2 &= -(r_1 + r_2)r_H + \ell^2 = -2\,\mathrm{Re}\{r_0\}r_H + \ell^2 = r_H^2 + \ell^2 \end{aligned}$$

The second and last relations give an expression for μ:

$$\mu = r_H + \frac{r_H^3}{\ell^2} \ .$$

The surface gravity is

$$\kappa_k = \frac{2{r_k}^3 + l^2\mu}{2l^2{r_k}^2} \quad = \frac{2{r_k}^3 + r_H r_0 \bar{r_0}}{2(r_0\bar{r_0} + r_0 r_H + \bar{r_0} r_H){r_k}^2} \tag{1.6}$$

$$\kappa_H = \frac{(r_H - r_0)(r_H - \bar{r_0})}{2l^2 r_H}\ , \quad \kappa_0 = \frac{(r_0 - \bar{r_0})(r_0 - r_H)}{2l^2 r_0}\ , \quad \bar{\kappa_0} = \frac{(\bar{r_0} - r_0)(\bar{r_0} - r_H)}{2l^2 \bar{r_0}}\ . \tag{1.7}$$

$$r^* = \int \frac{\mathrm{d}r}{f} = \frac{1}{2\kappa_H}\ln(r - r_H) + \frac{1}{2\kappa_0}\ln(r - r_0) + \frac{1}{2\bar{\kappa_0}}\ln(r - \bar{r_0}) + C\ . \tag{1.8}$$

Now it turns out that we may allow for Mathematica to do some simplifications for us if we defined $r_0 = r_{0R} + i r_{0I}$:

$$r^* = \left(\frac{\ell^2\left(r_{0I}^2 + r_{0R}(r_{0R} - r_H)\right)}{r_{0I}\left(r_{0I}^2 + (r_{0R} - r_H)^2\right)}\right)\arctan\left(\frac{r - r_{0R}}{r_{0I}}\right) - \frac{r_H\ell^2\log\left(r^2 - 2rr_{0R} + r_{0I}^2 + r_{0R}^2\right)}{2\left(r_{0I}^2 + (r_{0R} - r_H)^2\right)} + \frac{r_H\ell^2\log(r - r_H)}{r_{0I}^2 + (r_{0R} - r_H)^2} + C\ . \tag{1.9}$$

The first two terms can be recovered from the $\frac{1}{2\kappa_0}\ln(r - r_0) + \frac{1}{2\bar{\kappa_0}}\ln(r - \bar{r_0})$ terms (done quickly below) whereas the final term $\frac{r_H\ell^2\log(r - r_H)}{r_{0I}^2 + (r_{0R} - r_H)^2}$ is precisely $\frac{1}{2\kappa_H}\ln(r - r_H)$. The integration constant C will be relevant later, but for now will be forgotten.

$$\frac{1}{2\kappa_0}\ln(r - r_0) + \frac{1}{2\bar{\kappa_0}}\ln(r - \bar{r_0}) = \ln[(r - r_0)^{\frac{1}{2\kappa_0}}(r - \bar{r_0})^{\frac{1}{2\bar{\kappa_0}}}]\ , \tag{1.10}$$

$$= \ln\left(\left|(r - r_0)^{\frac{1}{2\kappa_0}}\right|^2\right)\ , \tag{1.11}$$

$$= 2\ln\left(\left|\exp\left(\tfrac{1}{2\kappa_0}\ln(r - r_0)\right)\right|\right)\ , \tag{1.12}$$

(1.13)

Now we go to the null coordinates

$$u \equiv t - r^* \quad ; \quad v \equiv t + r^*\ .$$

$$U_{k\pm} \equiv -\exp(-\kappa_{k\pm}u) \equiv T - X \quad ; \quad V_{k\pm} \equiv \exp(\kappa_{k\pm}v) \equiv T + X\ , \tag{1.14}$$

$$\mathrm{d}U\mathrm{d}V = \kappa_H^2\exp(\kappa_H(-u + v))\mathrm{d}u\mathrm{d}v\ , \tag{1.15}$$

$$= \kappa_H^2\exp(2\kappa_H r^*)\mathrm{d}u\mathrm{d}v\ , \tag{1.16}$$

$$= \kappa_H^2\exp\Bigg[\log(r - r_H) + 2\kappa_H\left(\frac{\ell^2\left(r_{0I}^2 + r_{0R}(r_{0R} - r_H)\right)}{r_{0I}\left(r_{0I}^2 + (r_{0R} - r_H)^2\right)}\right)\arctan\left(\frac{r - r_{0R}}{r_{0I}}\right) \tag{1.17}$$

$$-\frac{1}{2}\log\left(r^2 - 2rr_{0R} + r_{0I}^2 + r_{0R}^2\right) + C\Bigg]\mathrm{d}u\mathrm{d}v\ , \tag{1.18}$$

$$= e^C\kappa_H^2(r - r_H)\left(r^2 - 2rr_{0R} + r_{0I}^2 + r_{0R}^2\right)^{-1/2}\exp\left(2\kappa_H\alpha\arctan\left(\frac{r - r_{0R}}{r_{0I}}\right)\right)\ , \tag{1.19}$$

$$\text{where } \alpha = \left(\frac{\ell^2\left(r_{0I}^2 + r_{0R}(r_{0R} - r_H)\right)}{r_{0I}\left(r_{0I}^2 + (r_{0R} - r_H)^2\right)}\right) \tag{1.20}$$

$$\text{and } \left(r^2 - 2rr_{0R} + r_{0I}^2 + r_{0R}^2\right)^{-1/2} = |r - r_0|^{-1} \tag{1.21}$$

Then the null coordinates

$$\mathrm{d}s^2 = -f\mathrm{d}u\mathrm{d}v + r^2\mathrm{d}{\Omega_2}^2$$

becomes

$$\boxed{\mathrm{d}s^2 = -\frac{\kappa_H^{-2}}{(e^C)\ell^2 r}\left((r - r_{0R})^2 + r_{0I}^2\right)^{3/2}\exp\left[-2\kappa_H\alpha\arctan\left(\frac{r - r_{0R}}{r_{0I}}\right)\right]\mathrm{d}U\mathrm{d}V + r^2\mathrm{d}\Omega^2}\ , \tag{1.22}$$

with the $r(T,X)$ relation $G(r)$ looking like

$$G(r) = -T^2 + X^2 = -U \cdot V = (e^C)(r - r_H)\left((r - r_{0R})^2 + r_{0I}^2\right)^{-1/2} \exp\left(2\kappa_H \alpha \arctan\left(\frac{r - r_{0R}}{r_{0I}}\right)\right). \tag{1.23}$$

We may find the normalization factor of e^C here asserting that $-T^2 + X^2 = -1$ when $r = 0$.

$$e^C = \frac{\sqrt{r_{0I}^2 + r_{0R}^2}\exp\left\{2\alpha\kappa_H \tan^{-1}\left(\frac{r_{0R}}{r_{0I}}\right)\right\}}{r_H} \tag{1.24}$$

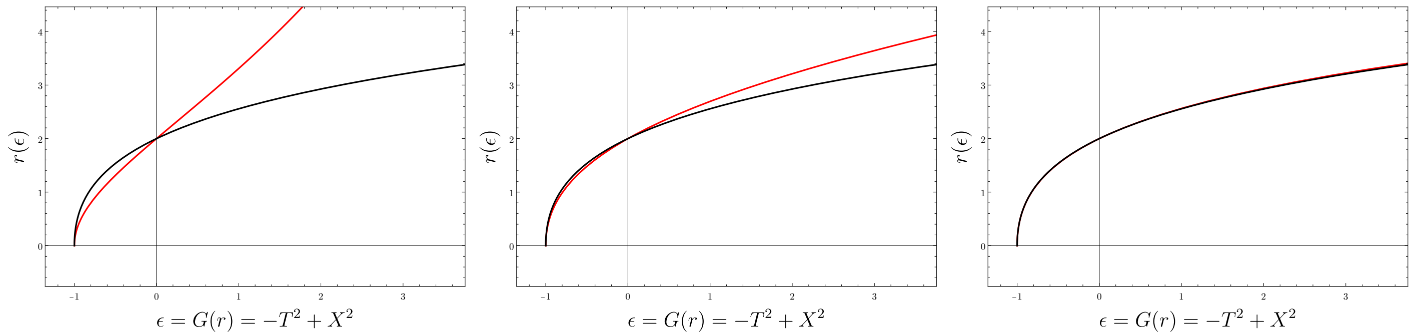

Figure 1. $\ell = 1$, $\ell = 5$, $\ell = 25$ in red from left (least Schwarzschild) to right (more Schwarzschild). The black curve is that of the Schwarzschild case $\ell \to \infty$.

Implicit function theorem relations:

$$G(r) = -T^2 + X^2 = -U \cdot V = (e^C)(r - r_H)\left((r - r_{0R})^2 + r_{0I}^2\right)^{-1/2} \exp\left(2\kappa_H \alpha \arctan\left(\frac{r - r_{0R}}{r_{0I}}\right)\right). \tag{1.25}$$

$$F(X,T,r) = T^2 - X^2 + G(r) = 0$$

$$r_X = \frac{\partial r}{\partial X} = \frac{-\frac{\partial F}{\partial X}}{\frac{\partial F}{\partial r}} = \frac{2X}{G'(r)} \tag{1.26}$$

$$r_T = \frac{\partial r}{\partial T} = \frac{-2T}{G'(r)} \tag{1.27}$$

$$r_{XX} = \frac{\partial^2 r}{\partial X^2} = -\frac{2G'(r) - 2XG''(r)r_X}{(G'(r))^2} \tag{1.28}$$

$$r_{TT} = \frac{\partial^2 r}{\partial T^2} = -\frac{-2G'(r) + 2TG''(r)r_T}{(G'(r))^2} \tag{1.29}$$

2. Setting Constants

From these relations, we generate the following relations to be fed to the program:

For arb. choice of ℓ & r_H

$$\boxed{\begin{aligned}
\mu &= r_H + r_H^3/\ell^2 \ , \\
r_{0R} &= -r_H/2 \ , \\
r_{0I} &= \pm\sqrt{\frac{\ell^2\mu}{r_H} - \frac{r_H^2}{4}} \ \text{(choose +)} \\
\alpha &= \frac{\ell^2\left(r_{0I}^2 + r_{0R}(r_{0R} - r_H)\right)}{r_{0I}\left(r_{0I}^2 + (r_{0R} - r_H)^2\right)} \ , \\
e^C &= \frac{\sqrt{r_{0I}^2 + r_{0R}^2}\exp\left\{2\alpha\kappa_H \tan^{-1}\left(\frac{r_{0R}}{r_{0I}}\right)\right\}}{r_H} \ , \\
\kappa_H &= \frac{(r_H - r_{0R})^2 + r_{0I}^2}{2r_H\ell^2}
\end{aligned}}$$