343 OPERATIONS RESEARCH

Duality Theory

05 November 2020

Second part of the course

Netiquette

- Meeting is recorded, slides and video will be shared
- Please mute your mic and turn off your camera
- For questions, please use Piazza rather than the teams chat
- To ask Q: “raise your hand”, wait to be called, “lower hand”

Organization

- Lecture: Mon 10.00 - 12.00 + Fri 10.00 - 11.00
- Exercises: Fri 11.00 - 12.00

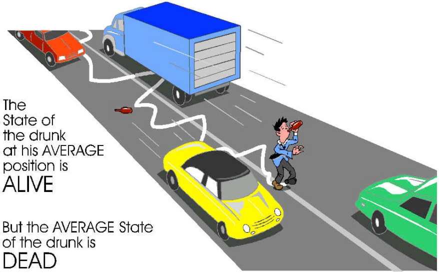
Content

- Duality
- Sensitivity
- Game theory
- Integer programming

Overview of duality

- Fundamental technique in optimization
- **In nutshell**: for every opt prblm, construct another opt prblm

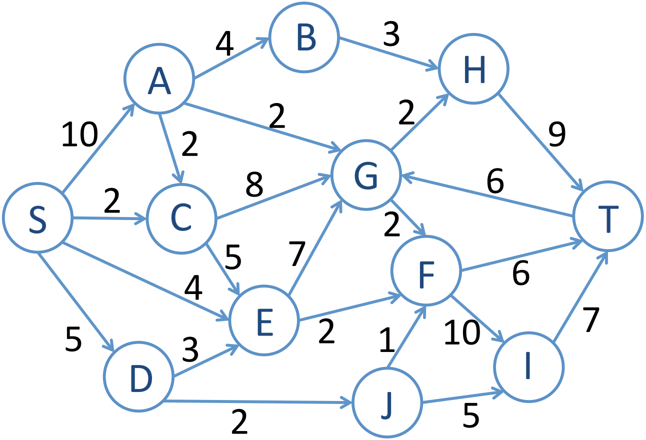
- Countless "applications":
  - Game theory
  - Making difficult optimization problems easier to solve
    - `https://arxiv.org/pdf/1910.13393.pdf`
    - `https://arxiv.org/pdf/1406.5429.pdf`
  - Optimization under uncertainty



This Lecture

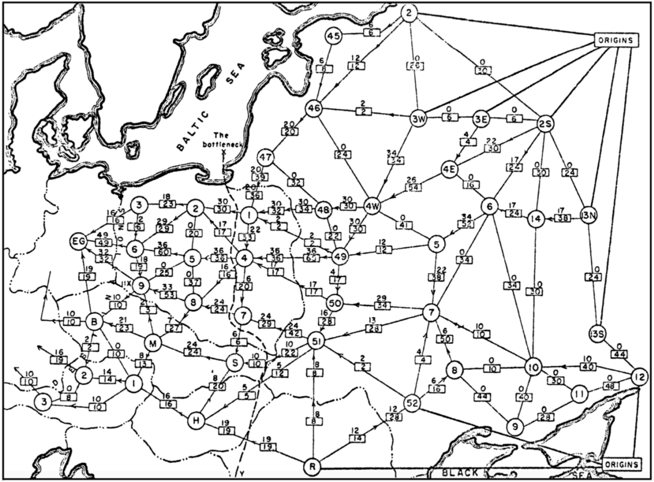
- Dual Problem
- Weak Duality
- Strong Duality
- Characterization of Duality

In “class” challenge



Sanity Check (x2).

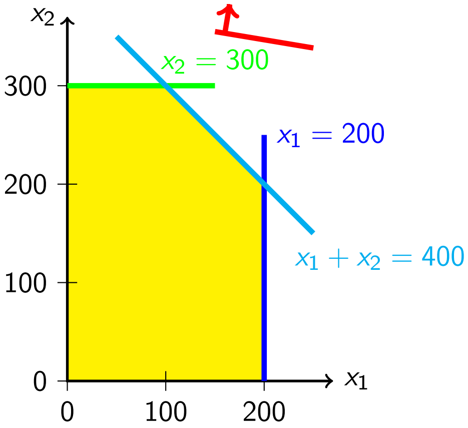
What about a more realistic graph?



- Harris & Ross (1955) developed this map (declassified 1999)
- On the map, the min-cut is called the *bottleneck*. There are 44 verticies, 105 edges, and the max-flow is 1.63×10^5.

Motivating Example

$$
\begin{array}{llllll}
\max\limits_{x_1,x_2} & z = & x_1 + 6x_2 & & & \\
\text{s.t.} & & x_1 & & \leq 200 & (1) \\
& & & x_2 & \leq 300 & (2) \\
& & x_1 + & x_2 & \leq 400 & (3) \\
& & x_1, x_2 \geq 0 & & &
\end{array}
$$



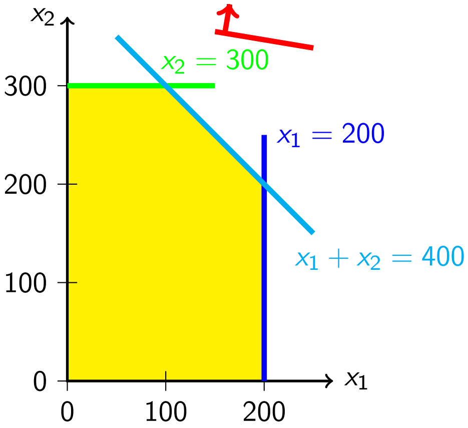
- Someone says optimum is $[x_1^*, x_2^*] = [100, 300]$, $z^* = 1900$.
- How can we check this claim? Consider combinations of the constraints to produce new *valid inequalities* that upper bound the objective function when evaluated on the feasible set.

$$(1) + 6(2) \Rightarrow x_1 + 6x_2 \leq 2000$$

Motivating Example

$$\max_{x_1,x_2} \; z = x_1 + 6x_2$$

$$\begin{aligned} \text{s.t.} \quad & x_1 & \le 200 \quad & (1) \\ & \quad\quad x_2 & \le 300 \quad & (2) \\ & x_1 + x_2 & \le 400 \quad & (3) \\ & x_1, x_2 \ge 0 & & \end{aligned}$$



- Valid inequality $(1) + 6(2) \Rightarrow x_1 + 6x_2 \le 2000$ implies that it is impossible for us to have $z^* > 2000$.
- Sanity Check. Can we bring down the bound any further? What values for each of the multipliers?

$$??(1) + ??(2) + ??(3) \Rightarrow x_1 + 6x_2 \le ??$$

Systemising the Motivating Example [1/2]

- Introduce one multiplier for (y_1, y_2, y_3) each constraint:

$$\begin{array}{rcll} x_1 & \leq 200 & (1) & y_1 \\ x_2 & \leq 300 & (2) & y_2 \\ x_1 + x_2 & \leq 400 & (3) & y_3 \end{array}$$

- We need $y_1, y_2, y_3 \geq 0$ to preserve the inequalities after multiplication.
- After we multiply and add $y_1(1) + y_2(2) + y_3(3)$, we obtain a new valid inequality of the form:

$$(y_1 + y_3)x_1 + (y_2 + y_3)x_2 \leq 200y_1 + 300y_2 + 400y_3$$

- We need the LHS (and hence the RHS) to upper bound the objective function $x_1 + 6x_2$. This can be achieved by enforcing:

$$y_1 + y_3 \geq 1$$
$$y_2 + y_3 \geq 6$$

These imply our desired upper bounds since $x_1, x_2 \geq 0$.

Systemising the Motivating Example [2/2]

- In summary, we have the following linear program for the best possible upper bound of the original problem:

$$
\begin{aligned}
\min_{y_1, y_2, y_3} \quad & 200y_1 + 300y_2 + 400y_3 \\
\text{s.t.} \quad & y_1 + y_3 \geq 1 \\
& y_2 + y_3 \geq 6 \\
& y_1, y_2, y_3 \geq 0
\end{aligned}
$$

  This new problem is called the dual LP!
- The optimal solution of the primal is $[x_1, x_2]^T = [100, 300]^T$ with optimal value 1900.
- The optimal solution of the dual is $[y_1, y_2, y_3]^T = [0, 5, 1]^T$ with optimal value 1900!

Comparing Primal & Dual Linear Programs

Primal LP

$$\begin{aligned}
\max_{x_1, x_2} \ z = \quad & x_1 + 6x_2 \\
\text{s.t.} \quad & x_1 \leq 200 \\
& x_2 \leq 300 \\
& x_1 + x_2 \leq 400 \\
& x_1, x_2 \geq 0
\end{aligned}$$

Dual LP

$$\begin{aligned}
\min_{y_1, y_2, y_3} \quad & 200y_1 + 300y_2 + 400y_3 \\
\text{s.t.} \quad & y_1 + y_3 \geq 1 \\
& y_2 + y_3 \geq 6 \\
& y_1, y_2, y_3 \geq 0
\end{aligned}$$

Sanity Check. Could we take the dual of the dual LP? What would we get?

Definition

- Primal Problem.

$$\max \left\{ c^T x : Ax \leq b, \ x \geq 0 \right\}, \tag{P}$$

where $c, x \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$.

- Dual Problem.

$$\min \left\{ b^T y : A^T y \geq c, \ y \geq 0 \right\}, \tag{D}$$

where c, A, b as in (P) and $y \in \mathbb{R}^m$.

- Definition is 'symmetric'. The dual of (D) is (P).
  - Follows from the transformation rules shown later.

Weak Duality

Theorem (Weak Duality).

Assume that the problems

$$\max \left\{c^T x : Ax \le b,\ x \ge 0\right\} \tag{P}$$

and

$$\min \left\{b^T y : A^T y \ge c,\ y \ge 0\right\}. \tag{D}$$

are both feasible. Let $x \in \mathbb{R}^n$ be feasible for (P) and $y \in \mathbb{R}^m$ be feasible for (D). Then

$$c^T x \le b^T y.$$

Weak Duality

Proof. (P) requires that

$$Ax \leq b \quad \Rightarrow \quad y^T Ax \leq y^T b$$

since $y \geq 0$. Similarly, (D) implies

$$(A^T y)^T \geq c^T \quad \Rightarrow \quad y^T Ax \geq c^T x$$

since $x \geq 0$ and $(A^T y)^T = y^T A$.
Then if both LPs are feasible

$$c^T x \leq y^T Ax \leq y^T b$$

The theorem follows after noting that $y^T b = b^T y$, since both vector multiplications give a scalar (1×1 matrix).

Weak Duality (draw)

Strong Duality

Theorem (Strong Duality). Assume that problem (P) is feasible with a bounded optimum. Let B be optimal basis for (P), together with optimal basic solution (x_B^*, x_N^*).

Then we have that:

(a) $y^* = (B^{-1})^T c_B$ is an optimal solution for (D).

(b) $c^T x^* = b^T y^*$, that is, the objective values coincide.

Recall the simplex tableau:

BV	z	x_B^T	x_N^T	RHS
z	1	0^T	$-r^T$	$c_B^T B^{-1} b$
x_B	0	I	$B^{-1} N$	$B^{-1} b$

Note: If (P) is unbounded, then (D) is infeasible and viceversa. Also, note that I can construct a similar theorem if (D) is feasible with a bounded optimum.

Strong Duality (draw)

Duality & Shadow Prices [Shadow prices covered in upcoming lecture]

- Because of strong duality we have that the optimal solution of the dual problem is $y^* = (B^{-1})^T c_B$.
- However, the shadow prices of the primal problem are given $\Pi = (B^{-1})^T c_B$
- Thus shadow prices can also be obtained by solving the dual problem.

Duality & Optimisation

- The simplex algorithm we have seen is often called the primal simplex algorithm.
  - Start from feasible solution (but suboptimal), then search for optimal feasible solution.
- The dual simplex algorithm is similar, but operates on the dual problem.
- Strong duality guarantees that the two algorithms return the same optimal solution.

Primal/Dual Possibilities

Again, we consider the following forms of the primal and dual:

$$(P)\quad \begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax \le b \\ & x \ge 0 \end{array} \qquad\qquad (D)\quad \begin{array}{ll} \min & b^T y \\ \text{s.t.} & A^T y \ge c \\ & y \ge 0 \end{array}$$

We know that an LP either: (i) has a finite optimum, (ii) is unbounded, or (iii) is infeasible. Here are the possibilities that we can have when we consider a primal/dual pair. Can you explain each entry of the table?

		Primal		
		Finite optimal	Unbounded	Infeasible
Dual	Finite optimal			
	Unbounded			
	Infeasible			

Indirect Way

Idea.

Bring problem to form of (P) or (D) and apply duality definition.

'Algorithm'.

1. Bring LP to the form of either (P) or (D).
   - Replace variables $x_i \in \mathbb{R}$ with $(x_i^+ - x_i^-)$ where $x_i^+, x_i^- \geq 0$.
   - Replace equality constraints with two inequality constraints.
   - Change constraint direction $(\leq, \geq)$ by multiplication with (-1) if necessary.
   - Change direction of objective function by multiplication with (-1) if necessary.

Indirect Way

2. Obtain dual according to definition.
   - If LP is in the form of (P), its dual is (D).
   - If LP is in the form of (D), its dual is (P).
3. Simplify dual problem. (Optional)
   - Replace variable pairs $y_i, y_j \geq 0$, $i \neq j$, that occur in all functions as $\alpha y_i - \alpha y_j$ by one variable $y_k \in \mathbb{R}$.
   - Replace matching inequality constraints by equality constraints.

Indirect Way: Example

Obtain the dual of

$$\max_{x_1, x_2} \; 2x_1 + x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &= 2 \\ 2x_1 - x_2 &\geq 3 \\ x_1 - x_2 &\leq 1, \end{aligned}$$

where $x_1 \geq 0$ and $x_2 \in \mathbb{R}$.

Indirect Way: Example

$$\max_{x_1, x_2} \; 2x_1 + x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &= 2 \\ 2x_1 - x_2 &\geq 3 \\ x_1 - x_2 &\leq 1, \end{aligned}$$

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'Algorithm'.

1. Bring LP to the form of either (P) or (D).
   - Replace variables $x_i \in \mathbb{R}$ with $(x_i^+ - x_i^-)$ where $x_i^+, x_i^- \geq 0$.

Indirect Way: Example

$$\max_{x_1, x_2} \; 2x_1 + x_2^+ - x_2^-$$

subject to

$$\begin{aligned} x_1 + x_2^+ - x_2^- &= 2 \\ 2x_1 - x_2^+ + x_2^- &\geq 3 \\ x_1 - x_2^+ + x_2^- &\leq 1, \end{aligned}$$

where $x_1, x_2^+, x_2^- \geq 0$.

'Algorithm'.

1. Bring LP to the form of either (P) or (D).
   - Replace equality constraints with two inequality constraints.

Indirect Way: Example

$$\max_{x_1, x_2} \; 2x_1 + x_2^+ - x_2^-$$

subject to

$$\begin{aligned} x_1 + x_2^+ - x_2^- &\leq 2 \\ x_1 + x_2^+ - x_2^- &\geq 2 \\ 2x_1 - x_2^+ + x_2^- &\geq 3 \\ x_1 - x_2^+ + x_2^- &\leq 1, \end{aligned}$$

where $x_1, x_2^+, x_2^- \geq 0$.

'Algorithm'.

1. Bring LP to the form of either (P) or (D).
   - Change constraint direction $(\leq, \geq)$ by multiplication with (-1) if necessary.

Indirect Way: Example

$$\max_{x_1, x_2} \; 2x_1 + x_2^+ - x_2^-$$

subject to

$$\begin{aligned} x_1 + x_2^+ - x_2^- &\leq 2 \\ -x_1 - x_2^+ + x_2^- &\leq -2 \\ -2x_1 + x_2^+ - x_2^- &\leq -3 \\ x_1 - x_2^+ + x_2^- &\leq 1, \end{aligned}$$

where $x_1, x_2^+, x_2^- \geq 0$.

'Algorithm'.

1. Bring LP to the form of either (P) or (D).
   - Change direction of objective function by multiplication with (-1) if necessary.

Indirect Way: Example

$$\max_{x_1, x_2} \; 2x_1 + x_2^+ - x_2^-$$

subject to

$$\begin{aligned} x_1 + x_2^+ - x_2^- &\le 2 \\ -x_1 - x_2^+ + x_2^- &\le -2 \\ -2x_1 + x_2^+ - x_2^- &\le -3 \\ x_1 - x_2^+ + x_2^- &\le 1, \end{aligned}$$

where $x_1, x_2^+, x_2^- \ge 0$.

'Algorithm'.

1. Obtain dual according to definition.
   - If LP is in the form of (P), its dual is (D).

Indirect Way: Example

Primal Problem:

$$\max_x \; c^T x \quad \text{with} \quad c = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \text{ and } x = \begin{pmatrix} x_1 \\ x_2^+ \\ x_2^- \end{pmatrix}$$

subject to

$$Ax \leq b \quad \text{with} \quad A = \begin{pmatrix} 1 & 1 & -1 \\ -1 & -1 & 1 \\ -2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 2 \\ -2 \\ -3 \\ 1 \end{pmatrix}$$

$$x \geq 0.$$

'Algorithm'.

1. Obtain dual according to definition.
   - If LP is in the form of (P), its dual is (D).

Indirect Way: Example

Dual Problem:

$$\min_y \; b^T y \quad \text{with} \quad b = \begin{pmatrix} 2 \\ -2 \\ -3 \\ 1 \end{pmatrix} \text{ and } y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

subject to

$$A^T y \geq c \quad \text{with} \quad A^T = \begin{pmatrix} 1 & -1 & -2 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix} \text{ and } c = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

$$y \geq 0.$$

'Algorithm'.

1. Obtain dual according to definition.
   - If LP is in the form of (P), its dual is (D).

Indirect Way: Example

Dual Problem:

$$\min_{y_1,y_2,y_3,y_4} \quad 2y_1 - 2y_2 - 3y_3 + y_4$$

subject to

$$\begin{aligned} y_1 - y_2 - 2y_3 + y_4 &\geq 2 \\ y_1 - y_2 + y_3 - y_4 &\geq 1 \\ -y_1 + y_2 - y_3 + y_4 &\geq -1, \end{aligned}$$

where $y_1, \ldots, y_4 \geq 0$.

'Algorithm'.

1. Simplify dual problem. (Optional)
   - Replace variable pairs $y_i, y_j \geq 0$, $i \neq j$, that occur in all functions as $\alpha y_i - \alpha y_j$ by one variable $y_k \in \mathbb{R}$.

Indirect Way: Example

$$\min_{y_1', y_3, y_4} \; 2y_1' - 3y_3 + y_4$$

subject to

$$\begin{aligned} y_1' - 2y_3 + y_4 &\geq 2 \\ y_1' + y_3 - y_4 &\geq 1 \\ -y_1' - y_3 + y_4 &\geq -1, \end{aligned}$$

where $y_1' \in \mathbb{R}$ and $y_3, y_4 \geq 0$.

'Algorithm'.

1. Simplify dual problem. (Optional)
   - Replace matching inequality constraints by equality constraints.

Indirect Way: Example

The simplified dual problem is:

$$\min_{y_1',y_3,y_4} \; 2y_1' - 3y_3 + y_4$$

subject to

$$\begin{aligned} y_1' - 2y_3 + y_4 &\geq 2 \\ y_1' + y_3 - y_4 &= 1, \end{aligned}$$

where $y_1' \in \mathbb{R}$ and $y_3, y_4 \geq 0$.

Direct Way

Transforming the initial LP to (P) and then obtain (D) can be tedious (so called Indirect way).

Direct way. Apply duality without detour via (P) or (D).

1. For every primal constraint, create one dual variable.
   For every primal variable, create one dual constraint.

2. Dual coefficient matrix is A^T.
   Former right-hand sides b become new costs.
   Former costs c become new right-hand sides.

Direct Way

3. If primal is max problem: Dual is min problem.
   - If i^{th} primal constraint is $[\geq, =, \leq]$, i^{th} dual variable becomes $[y_i \leq 0,\ y_i \in \mathbb{R},\ y_i \geq 0]$, respectively.
   - If j^{th} primal variable is $[x_j \geq 0,\ x_j \in \mathbb{R},\ x_j \leq 0]$, j^{th} dual constraint becomes $[\geq, =, \leq]$, respectively.

4. If primal is min problem: Dual is max problem.
   - If i^{th} primal constraint is $[\geq, =, \leq]$, i^{th} dual variable becomes $[y_i \geq 0,\ y_i \in \mathbb{R},\ y_i \leq 0]$, respectively.
   - If j^{th} primal variable is $[x_j \geq 0,\ x_j \in \mathbb{R},\ x_j \leq 0]$, j^{th} dual constraint becomes $[\leq, =, \geq]$, respectively.

Example: Direct Way

Same example as before: obtain the dual of

$$\max_{x_1, x_2} \; 2x_1 + x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &= 2 \\ 2x_1 - x_2 &\geq 3 \\ x_1 - x_2 &\leq 1, \end{aligned}$$

where $x_1 \geq 0$ and $x_2 \in \mathbb{R}$.

Example: Primal Problem

$$\max_{x_1, x_2} \; 2x_1 + x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &= 2 \\ 2x_1 - x_2 &\geq 3 \\ x_1 - x_2 &\leq 1, \end{aligned}$$

where $x_1 \geq 0$ and $x_2 \in \mathbb{R}$.

1. For every primal constraint, create one dual variable.
   For every primal variable, create one dual constraint.

Example: Dual Problem

Variables:

- y_1 variable for $x_1 + x_2 = 2$
- y_2 variable for $2x_1 - x_2 \geq 3$
- y_3 variable for $x_1 - x_2 \leq 1$

Since the primal problem has 2 variables, the dual will have 2 constraints one for x_1, another for x_2.

Example: Primal Problem

$$\max_{x_1,x_2} 2x_1 + x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &= 2 \\ 2x_1 - x_2 &\geq 3 \\ x_1 - x_2 &\leq 1, \end{aligned}$$

where $x_1 \geq 0$ and $x_2 \in \mathbb{R}$.

2. Dual coefficient matrix is A^T.
   Former right-hand sides b become new costs.
   Former costs c become new right-hand sides.

Example: Dual Problem

$$\underset{y_1,y_2,y_3}{?} \quad 2y_1 + 3y_2 + y_3$$

subject to

$$\begin{aligned} y_1 + 2y_2 + y_3 &\ ?\ 2 && [x_1] \\ y_1 - y_2 - y_3 &\ ?\ 1, && [x_2] \end{aligned}$$

where the domain of y_1, y_2, y_3 is not yet defined.

Example: Primal Problem

$$\max_{x_1, x_2} \; 2x_1 + x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &= 2 \\ 2x_1 - x_2 &\geq 3 \\ x_1 - x_2 &\leq 1, \end{aligned}$$

where $x_1 \geq 0$ and $x_2 \in \mathbb{R}$.

3. If primal is max problem: Dual is min problem.
   - If i^{th} primal constraint is $[\geq, =, \leq]$, i^{th} dual variable becomes $[y_i \leq 0,\ y_i \in \mathbb{R},\ y_i \geq 0]$, respectively.

Example: Dual Problem

$$\min_{y_1, y_2, y_3} \; 2y_1 + 3y_2 + y_3$$

subject to

$$y_1 + 2y_2 + y_3 \; ? \; 2$$
$$y_1 - y_2 - y_3 \; ? \; 1,$$

where $y_1 \in \mathbb{R}$, $y_2 \leq 0$ and $y_3 \geq 0$.

Example: Primal Problem

$$\max_{x_1, x_2} \; 2x_1 + x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &= 2 \\ 2x_1 - x_2 &\geq 3 \\ x_1 - x_2 &\leq 1, \end{aligned}$$

where $x_1 \geq 0$ and $x_2 \in \mathbb{R}$.

3. ▶ If j^{th} primal variable is $[x_j \geq 0,\ x_j \in \mathbb{R},\ x_j \leq 0]$, j^{th} dual constraint becomes $[\geq, =, \leq]$, respectively.

Example: Result

The resulting dual problem is:

$$\min_{y_1, y_2, y_3} \quad 2y_1 + 3y_2 + y_3$$

subject to

$$\begin{aligned} y_1 + 2y_2 + y_3 &\geq 2 \\ y_1 - y_2 - y_3 &= 1, \end{aligned}$$

where $y_1 \in \mathbb{R}$, $y_2 \leq 0$ and $y_3 \geq 0$.

Equivalence of Indirect and Direct Way

Indirect way led us to the problem:

$$\min_{y_1', y_3, y_4} \; 2y_1' - 3y_3 + y_4$$

subject to

$$y_1' - 2y_3 + y_4 \geq 2$$
$$y_1' + y_3 - y_4 = 1,$$

where $y_1' \in \mathbb{R}$ and $y_3, y_4 \geq 0$.

Equivalence of Indirect and Direct Way

Direct way

led us to the problem:

$$\min_{y_1, y_2, y_3} \; 2y_1 + 3y_2 + y_3$$

subject to

$$\begin{aligned} y_1 + 2y_2 + y_3 &\geq \; 2 \\ y_1 - y_2 - y_3 &= 1, \end{aligned}$$

where $y_1 \in \mathbb{R}$, $y_2 \leq 0$ and $y_3 \geq 0$.

Interpreting the Dual

Grocery store - Dual

Baker

Student - Primal

	Chocolate	Sugar	Cream	Price
Brownies	3 oz	2 oz	2 oz	50 p
Cheesecake	0 oz	4 oz	5 oz	80 p
Requirements	6 oz	10 oz	8 oz	

- A student's diet requires her to eat at least 6oz of chocolate, 10oz of sugar, and 8oz of cream.
- She will buy a snack from a bakery by choosing the least expensive combination of brownies x_1 and cheesecake x_2.

Diet Problem

Primal How much brownies x_1 and cheesecake x_2 to purchase?

$$\begin{aligned} \min_{x_1,x_2} \quad & 50x_1 + 80x_2 \\ \text{s.t.} \quad & 3x_1 \geq 6 \\ & 2x_1 + 4x_2 \geq 10 \\ & 2x_1 + 5x_2 \geq 8 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Dual The store maximises profit. They set prices on chocolate y_1, sugar y_2, and cream y_3 so that the ingredients are not more costly than the product.

$$\begin{aligned} \max_{y_1,y_2,y_3} \quad & 6y_1 + 10y_2 + 8y_3 \\ \text{s.t.} \quad & 3y_1 + 2y_2 + 2y_3 \leq 50 \\ & 4y_2 + 5y_3 \leq 80 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$