60016 OPERATIONS RESEARCH

Introduction to Operations Research

Autumn Term - 2020-21

Netiquette

- The meeting is recorded, slides and videos will be shared
- Please mute your mic and switch off your webcam
- For questions, we will use live Q&A and Piazza, rather than the Teams chat.
- To ask a question, click on the "Raise your hand" button
  - Wait to be called by the lecturer
  - At times, it may take a few slides
  - Do not forget to "Lower hand" afterwards
- If the lecturer connection drops, please wait for him to rejoin.
- If uncomfortable to use mic/speak up, please use Piazza, we will come back later.

Course Information

- Lecturers:
  - Giuliano Casale (g.casale@imperial.ac.uk)
  - Dario Paccagnan (d.paccagnan@imperial.ac.uk)
- Logistics for the first half:
  - Lectures: Mon 10:00-12:00
Fri 10:00-11:00
  - Tutorial: Fri 11:00-12:00
  - Changes of schedule will be announced in due course, if needed.
  - Tutorials will develop Q&A on tutorial sheets and exercises.
- Materials website:
  - Basic Linear Algebra refresher
  - Information for external students
- Other teaching aids:
  - Everything on Panopto
  - Course forum on Piazza
- One assessed coursework (CATE), please collaborate!

Some Books (Optional Readings)

- *F.Hillier & J.Lieberman:*
  Introduction to Operations Research.
- *H.Taha:*
  Operations Research.
- *D.G.Luenberger & Y.Ye:*
  Linear and Nonlinear Programming.
- *W. Winston:*
  Operations Research: Applications and Algorithms.

What is Operations Research?

- OR is a multidisciplinary branch of mathematics involving
  - mathematical modelling
  - mathematical optimisation
  - statistical analysis

  in order to find "good" solutions for complex decision problems.
- Typical objectives in OR are:
  - maximise profit
  - minimise cost
  - minimise risk
  - minimise completion time
  - maximise efficiency
  - etc.

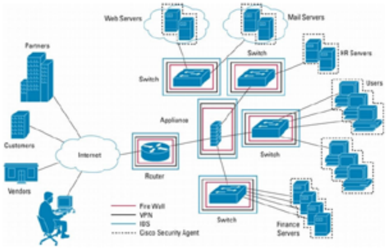
Scope of OR

OR techniques are ubiquitous in operations management, industrial engineering, economics and finance, ICT management, machine learning, health care, security, etc.

Designing the layout of a factory

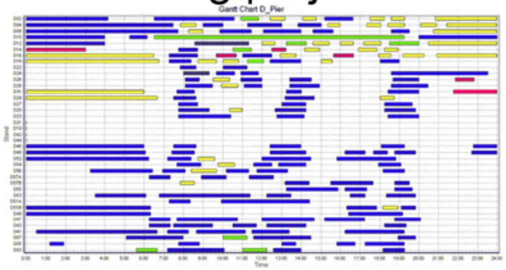
road traffic management

Constructing a telec-
ommunication network

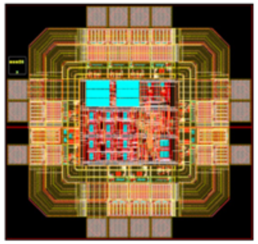


Determining the routes of city buses

scheduling project tasks



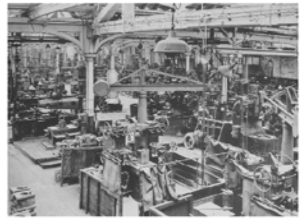
Designing the layout of a computer chip



Financial planning

History of OR

19th century:
Industrial Revolution



Efficiency of
production processes

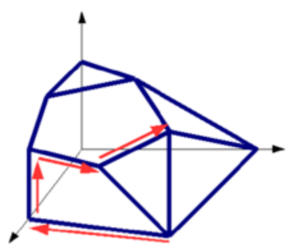
World War II:
Birth of Modern OR

Optimal design of
convoy system

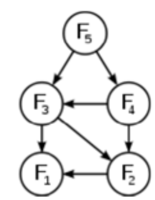
Where to add armour
in RAF bombers?

Allocate scarce resources to
various military operations
in an effective manner

1947
Simplex Algorithm



1953
Dynamic Programming



1970's
Personal Computers

Industry-size problems
can be solved efficiently

Phases of an OR Study

Hillier and Lieberman p. 8:

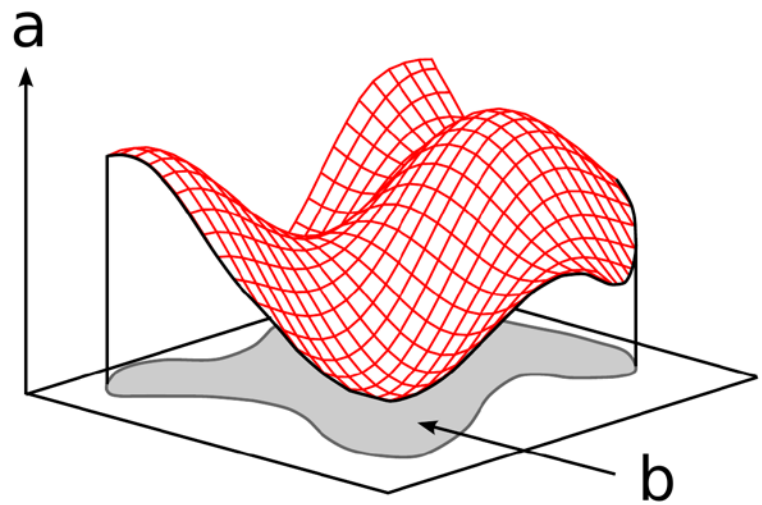
1. Define the problem of interest and gather relevant data.
2. Formulate a mathematical model to represent the problem.
3. Develop a computer-based procedure for deriving solutions to the problem from the model.
4. Test the model and refine as needed.
5. Prepare for the ongoing application of the model as prescribed by management.
6. Implement.

Course focuses on phases 2 and 3.

Mathematical Programming

OR solves mathematical programming models:

$$\begin{aligned} \underset{x}{\text{minimise}} \quad & z = f(x) \\ \text{subject to} \quad & x \in \mathcal{X}, \end{aligned}$$



where

- $x \in \mathbb{R}^n$ are the decision variables
- $f : \mathbb{R}^n \to \mathbb{R}$ is the objective function (e.g., cost)
- $\mathcal{X} \subseteq \mathbb{R}^n$ is the feasible set (set of admissible decisions)
- any vector x that minimises f is an optimal solution of the program and is denoted by x^*
- $z^* = f(x^*)$ is the optimal value achieved by x^*

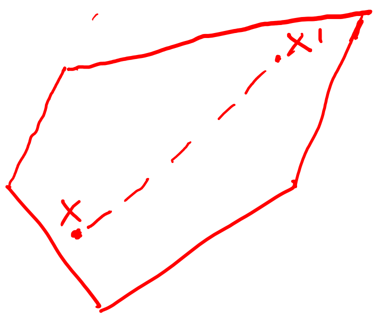
Topics of this Course

- Linear Programming
- Integer Programming
- Duality and sensitivity analysis
- Game Theory
- Modelling, skill mostly acquired through:
  - Exercises: verify your solutions with the GNU GLPK solver.
  - Case studies: in tutorials (one exam question about this).

Linear Programming

- A Linear program (LP) is a mathematical program that
  - optimises (maximises or minimises) a linear objective function
  - over a feasible set described by linear equality and/or inequality constraints.
- Optimal decision tool
- Widely adopted, many success stories (see Hillier & Lieberman)
- LPs are much simpler to cope with than non-linear programs
  - *CO477 - Computational Optimisation* deals with the theory of non-linear optimization.

Z
X
x₀
x₀'



x
x'

LP Running Case: Example 1

Resource Allocation Problem: A manufacturer produces A (acid) and C (caustic) and wants to decide a production plan.

Ingredients used for producing A and C are: X (e.g., a sulphate) and Y (e.g., sodium).

- Each ton of A requires: 2ton of X; 1ton of Y
- Each ton of C requires: 1ton of X ; 3ton of Y
- Supply of X limited to: 11ton/week
- Supply of Y limited to: 18ton/week
- A sells for: £1000/ton
- C sells for: £1000/ton
- Market research: max 4 tons of A/week can be sold.

$\boxed{\text{Maximize weekly value of sales of A and C.}}$

Example 1 (Modelling)

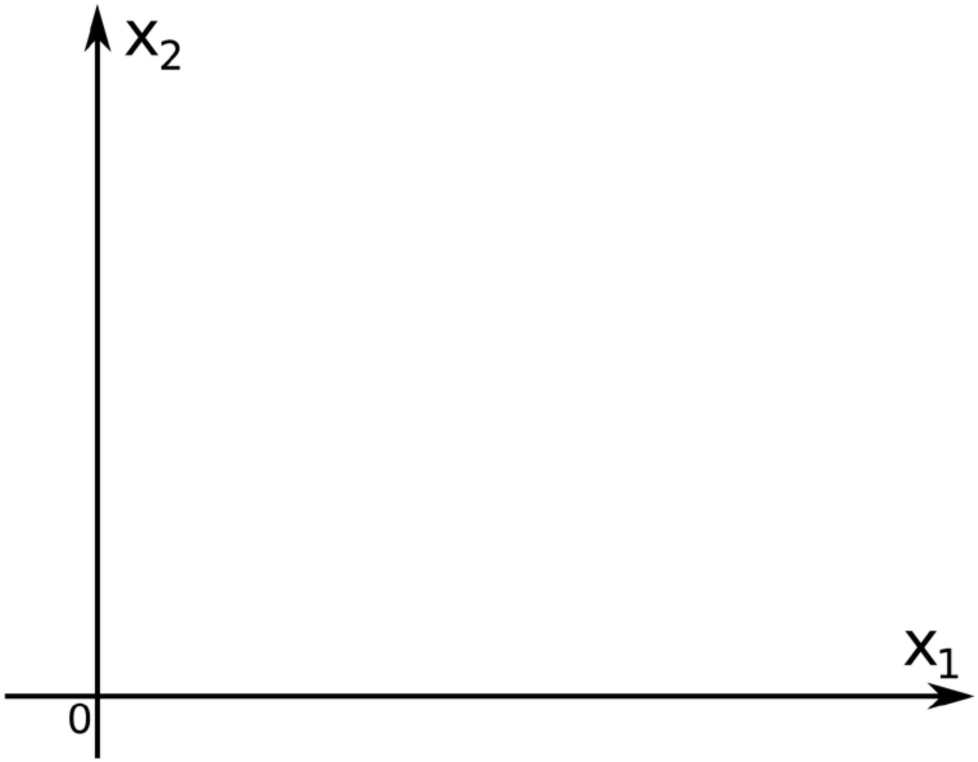
How much A and C to produce?

$\Rightarrow$ Formulate a mathematical programming model!

- Decision variables
  - x_1 = weekly production of A (in tons)
  - x_2 = weekly production of C (in tons)
- Objective function
  - $z = f(x_1, x_2)$ = weekly profit (in 1000 £)
- Feasible set
  - $\mathcal{X}$ = set of all implementable/admissible production plans $x = (x_1, x_2)$
  - e.g., $x = (27, 2)$ is not possible (not enough supply!)

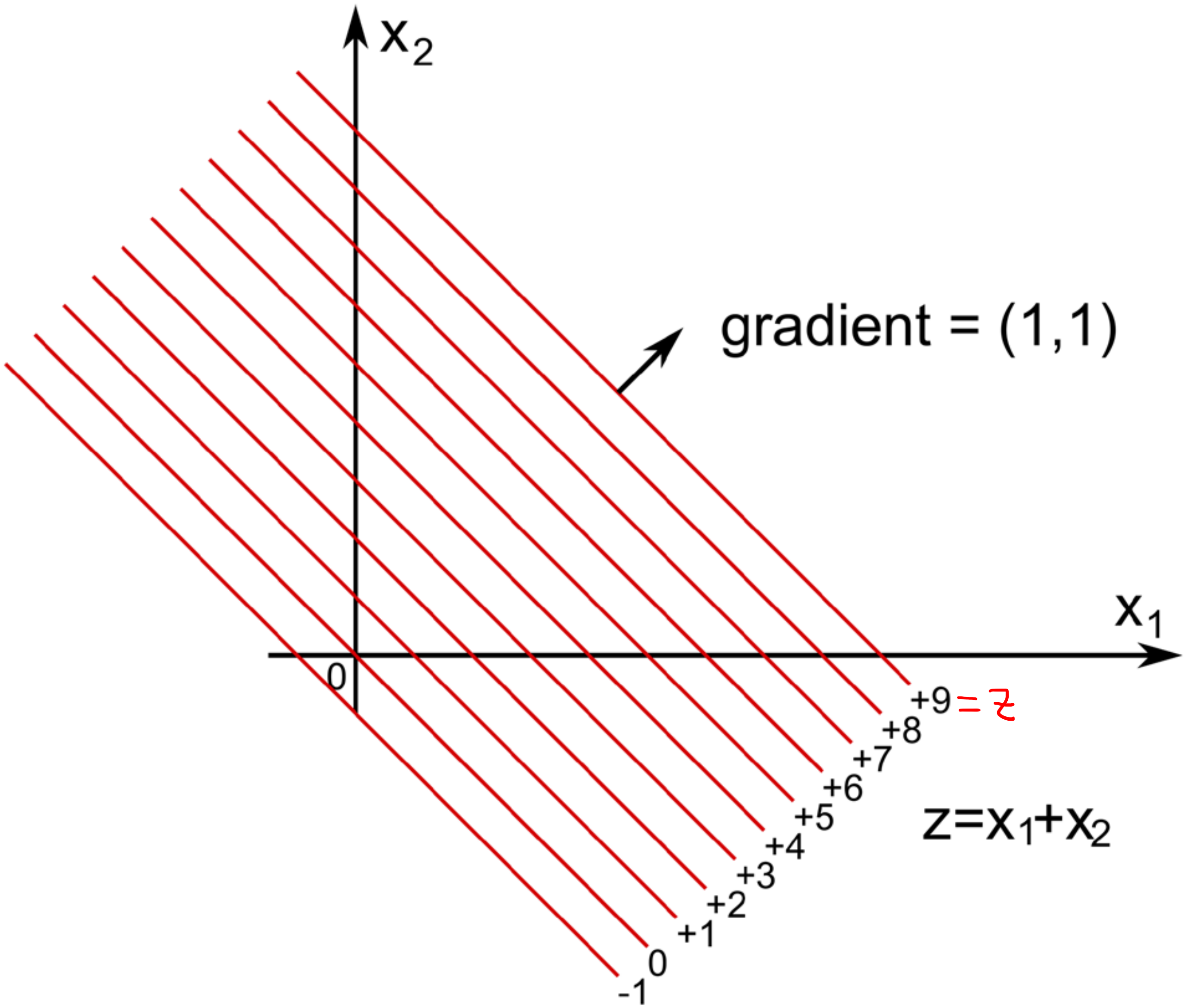
Example 1 (Decision Variables)

A production plan is representable as $x = (x_1, x_2)$



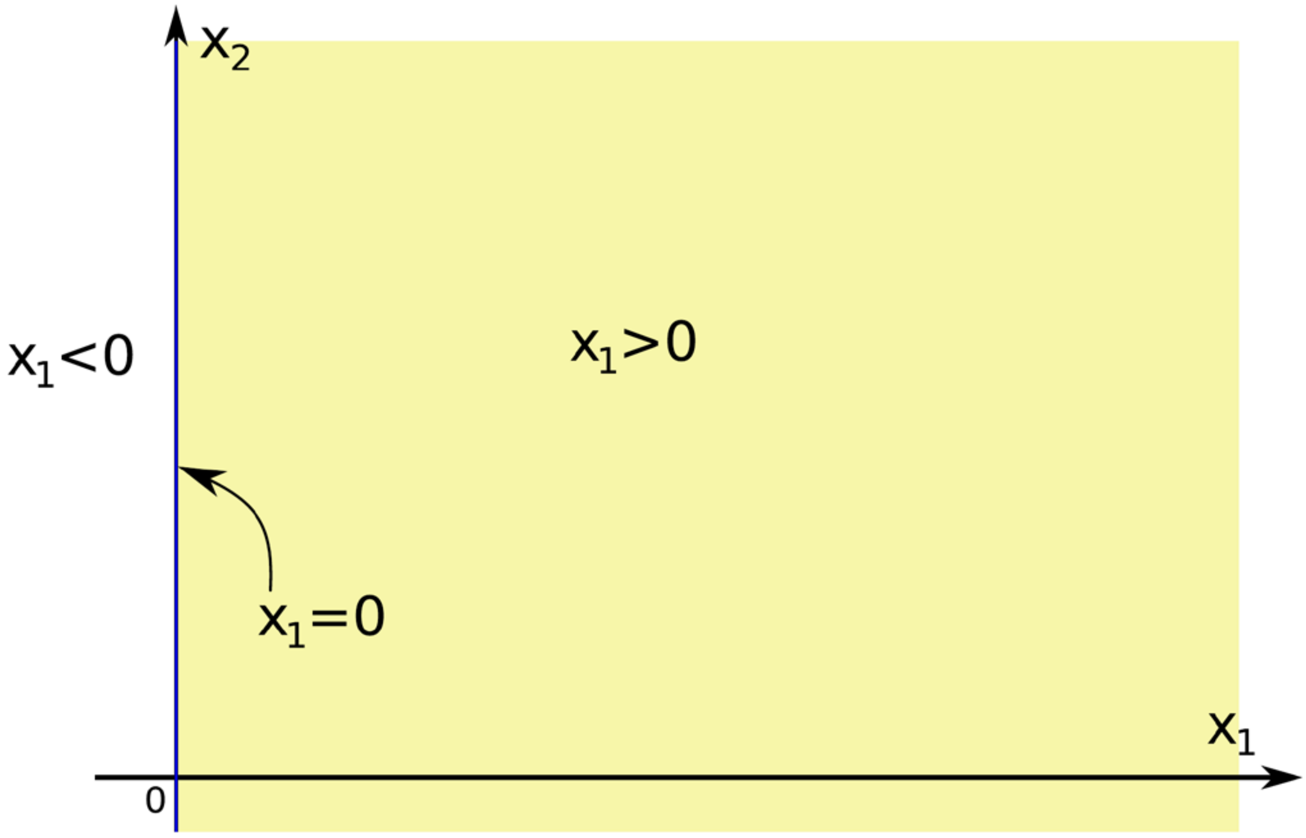
Example 1 (Objective Function)

Profit: $z = x_1 + x_2$ (in 1000 £)



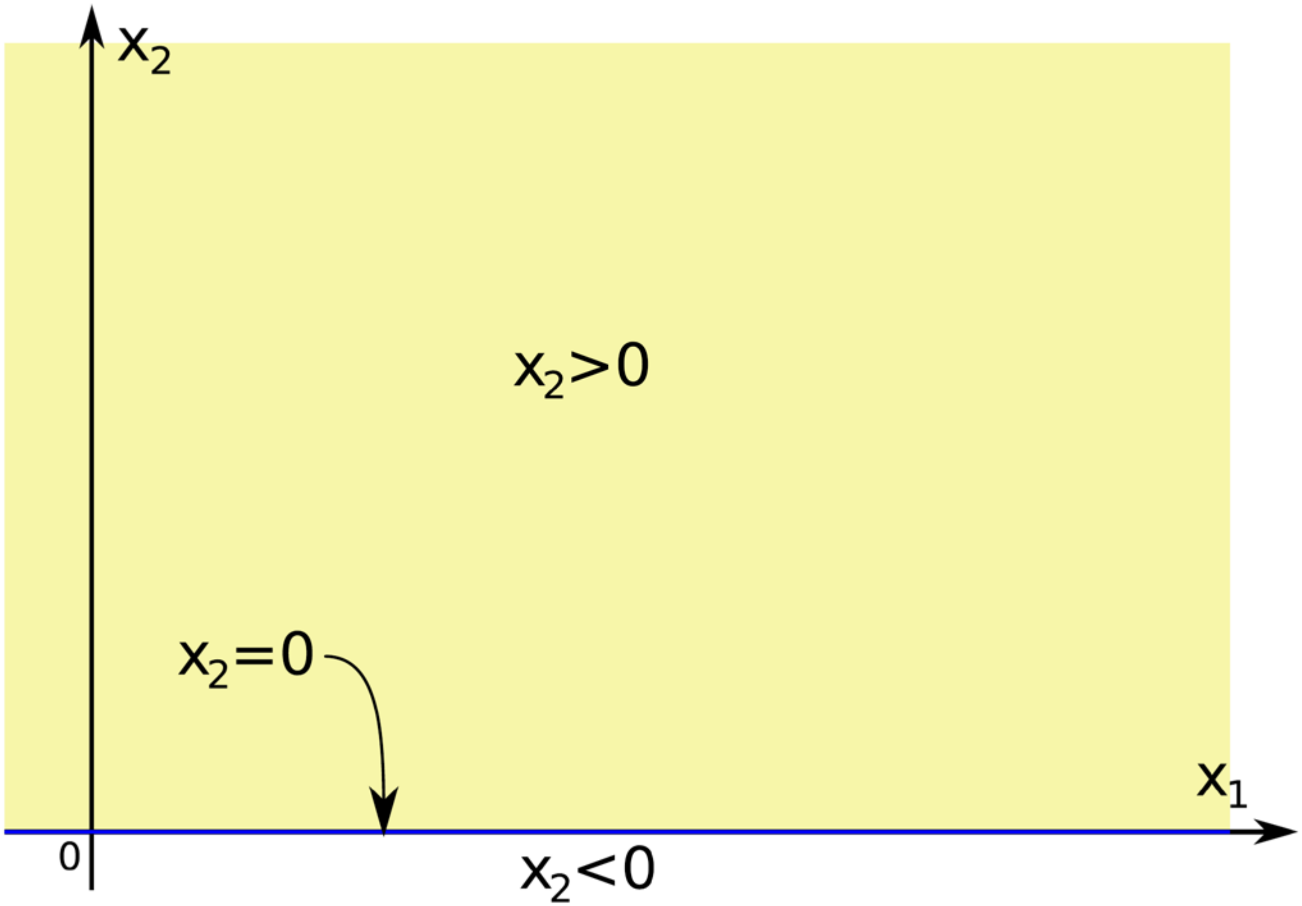
Example 1 (Feasible Set)

Amount of A produced is non-negative: $x_1 \geq 0$



Example 1 (Feasible Set)

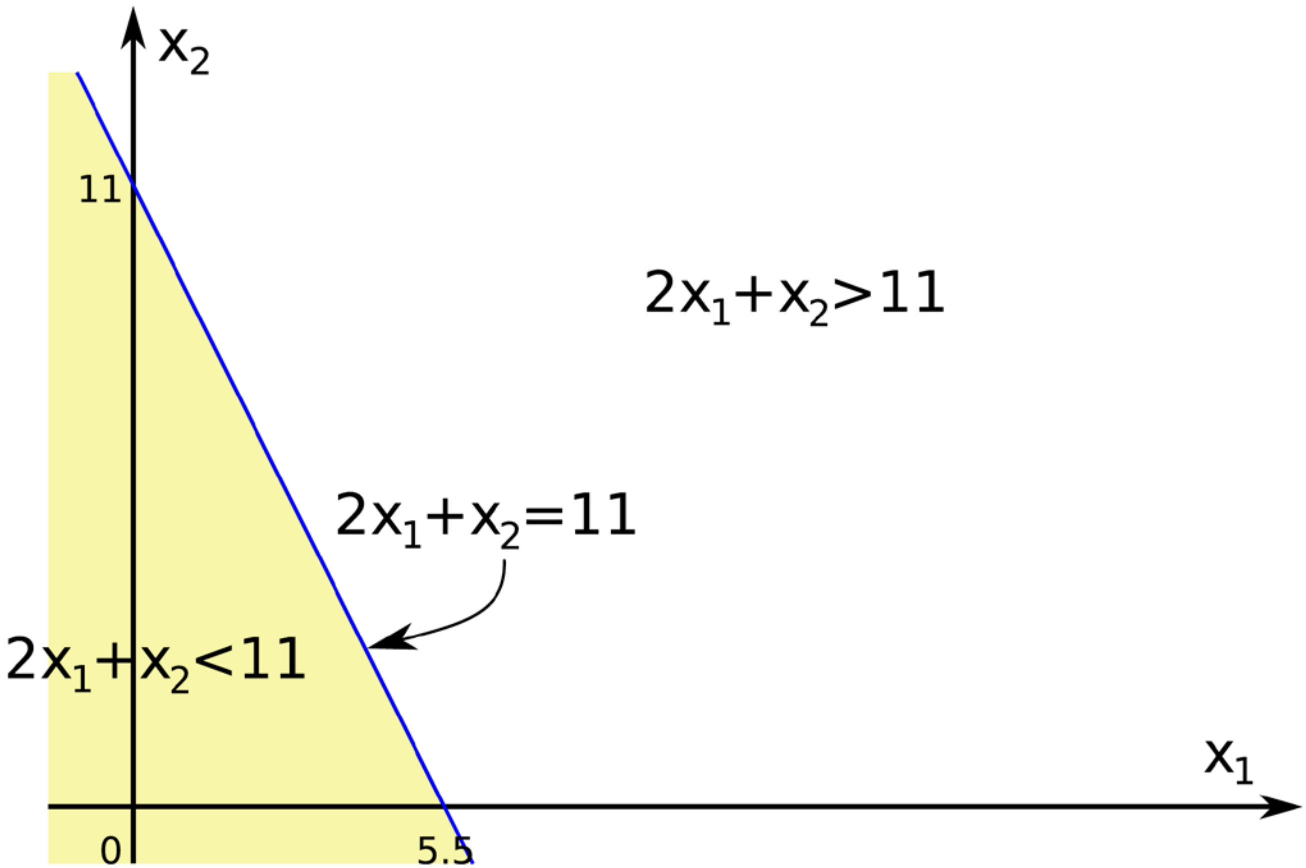
Amount of C produced is non-negative: $x_2 \geq 0$



Example 1 (Feasible Set)

x_1 tons of A & x_2 tons of C require $2x_1 + x_2$ ton of X

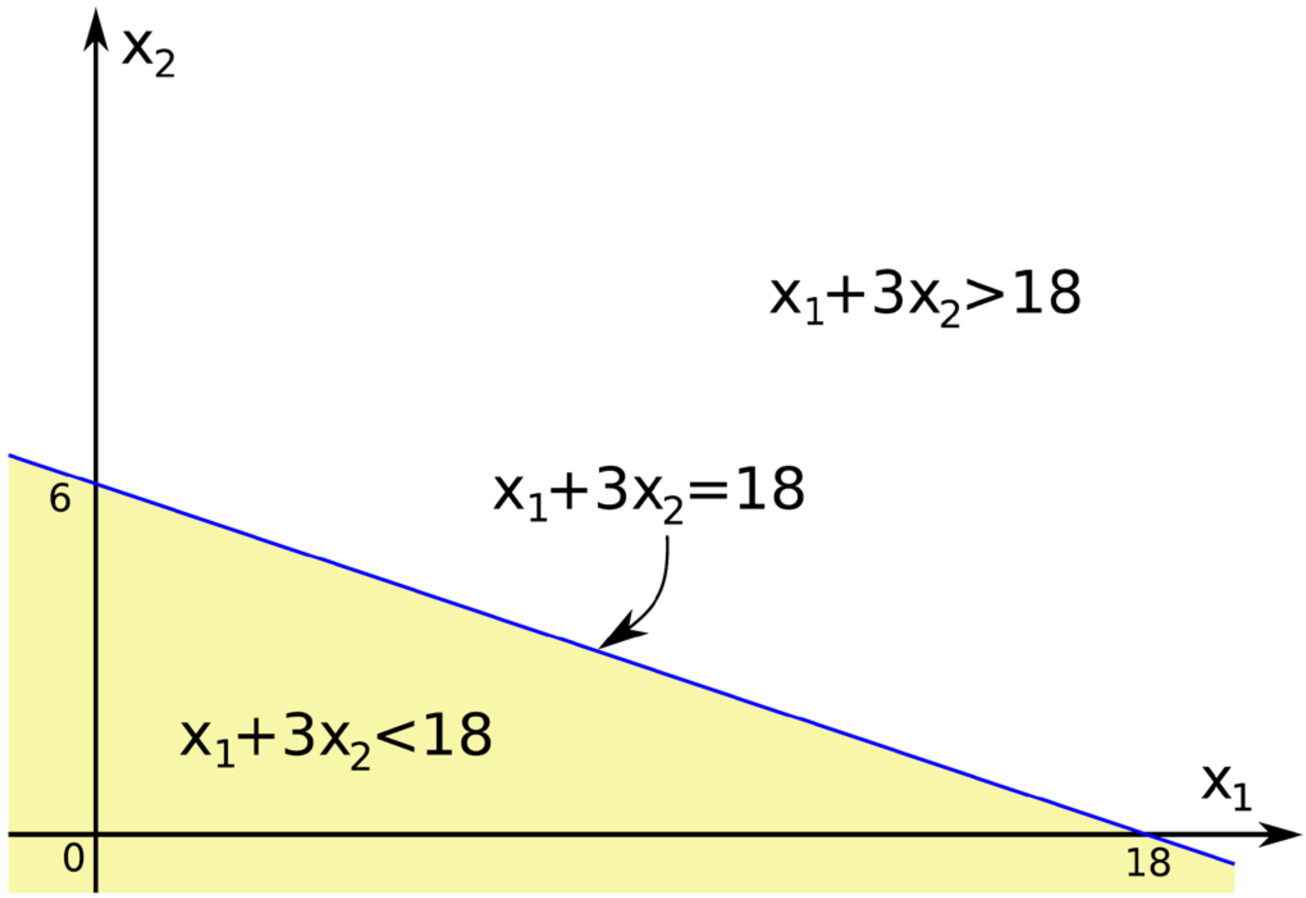
X is limited to 11ton/week: $2x_1 + x_2 \leq 11$



Example 1 (Feasible Set)

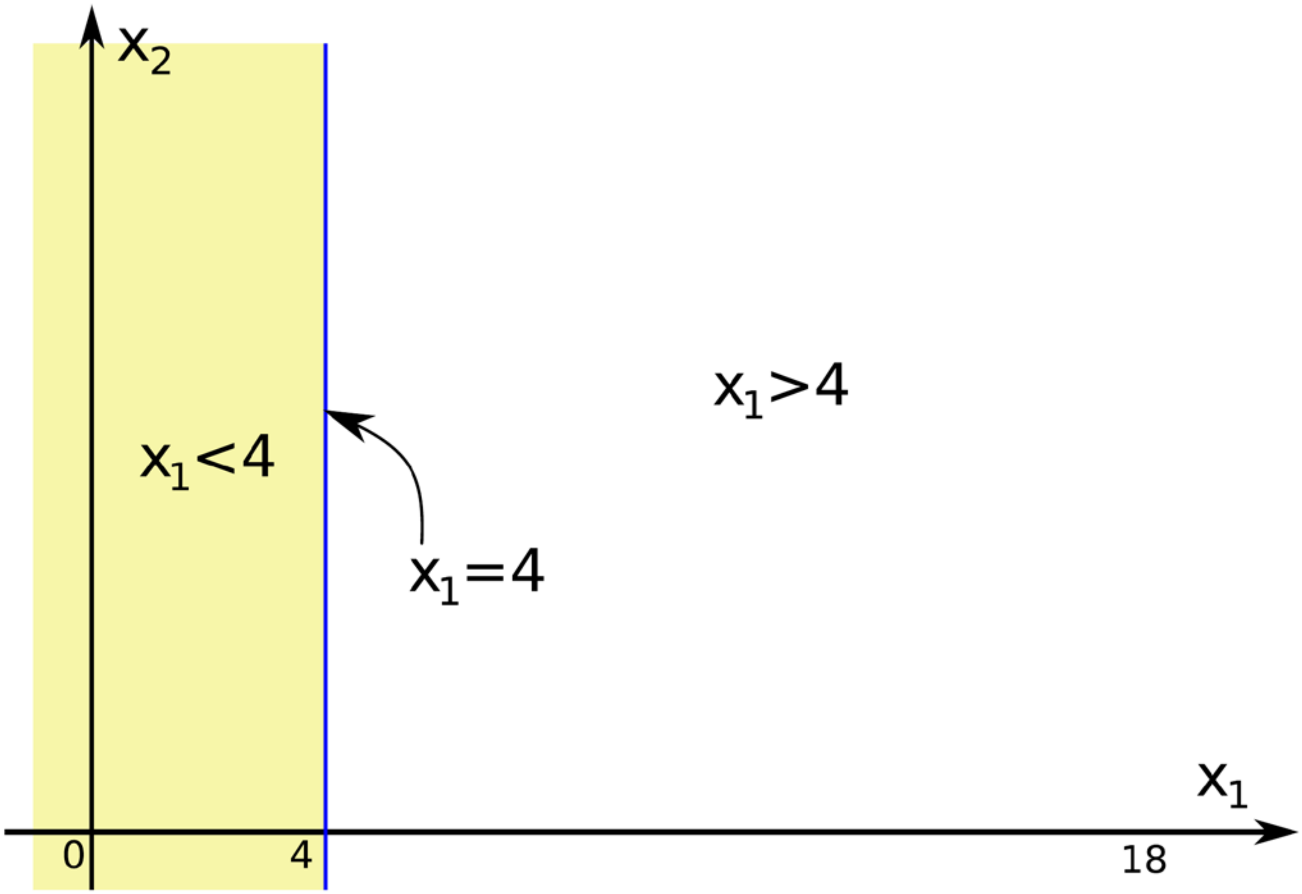
x_1 tons of A & x_2 tons of C require $x_1 + 3x_2$ ton of Y

Y is limited to 18ton/week: $x_1 + 3x_2 \leq 18$



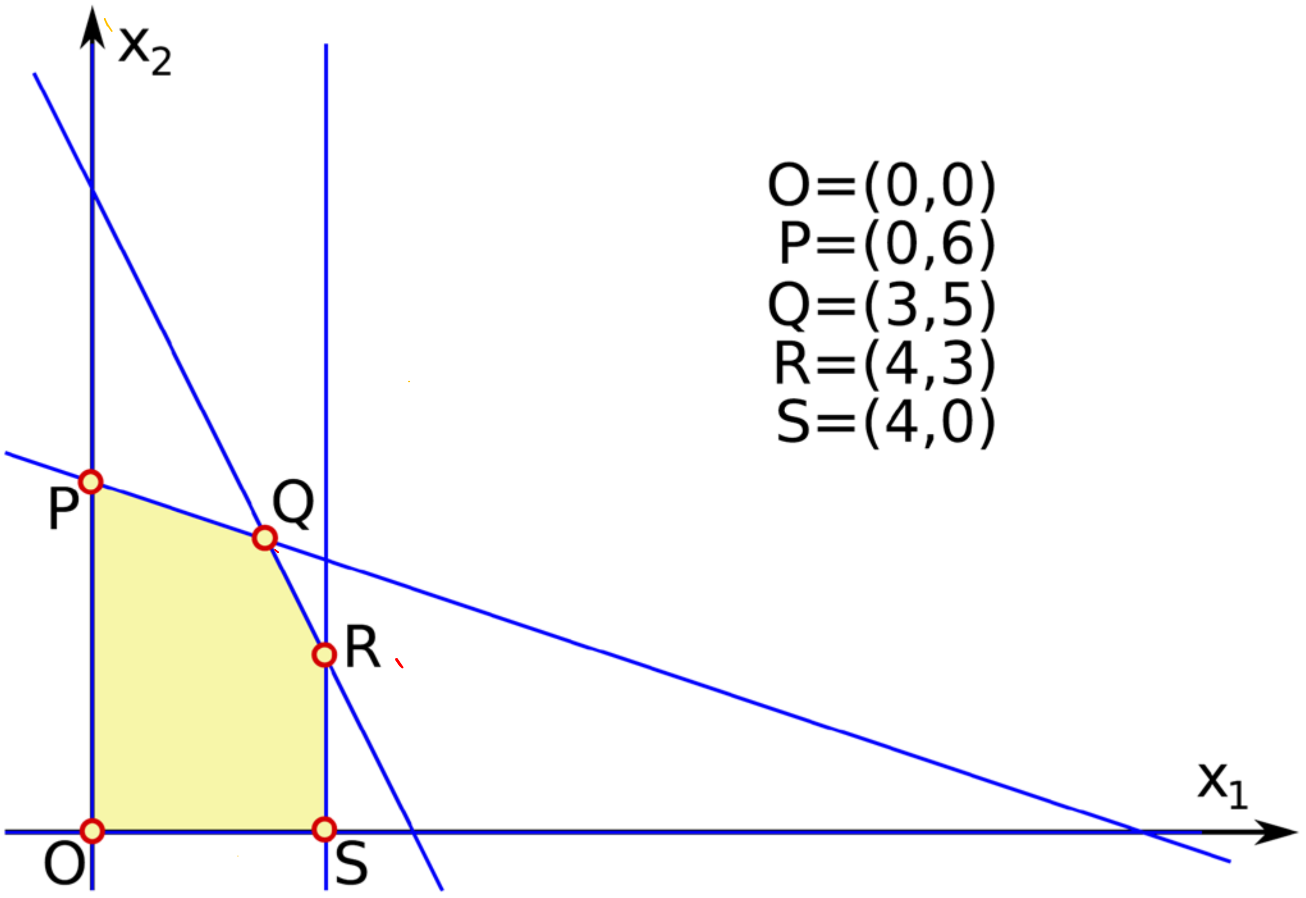
Example 1 (Feasible Set)

Cannot sell more than 4 tons of A/week: $x_1 \leq 4$



Example 1 (Feasible Set)

To obtain the overall feasible set,
intersect the feasible sets of all individual constraints



Example 1 (Feasible Set)

- The feasible set is a convex polygon
- The corner points O,P,Q,R,S of the feasible set are termed vertices
- Each vertex is given by the intersection of two blue lines
  - its coordinates can be computed by jointly solving the two linear equations defining the blue lines
- We obtain O=(0,0), P=(0,6), Q=(3,5), R=(4,3), S=(4,0)

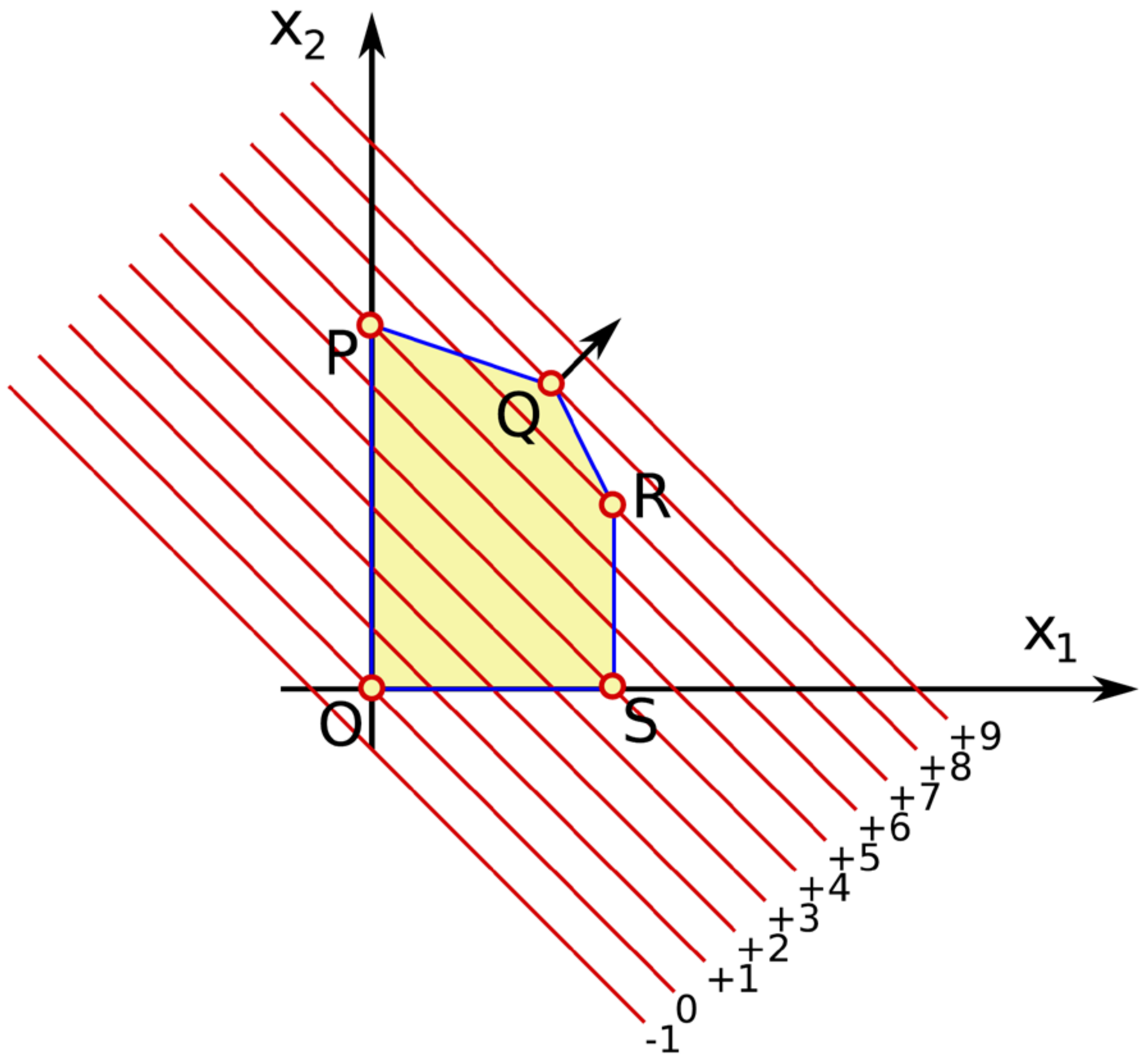
Example 1 (Summary)

The best production plan is obtained by solving the following mathematical problem:

$$
\begin{array}{lll}
\text{maximise} & z = x_1 + x_2 & : \text{ objective function} \\
\\
\text{subject to} & 2x_1 + x_2 \leq 11 & : \text{ constraint on availability of X} \\
& x_1 + 3x_2 \leq 18 & : \text{ constraint on availability of Y} \\
& x_1 \leq 4 & : \text{ constraint on demand of A} \\
& x_1, x_2 \geq 0 & : \text{ non-negativity constraints}
\end{array}
$$

This is a linear program.

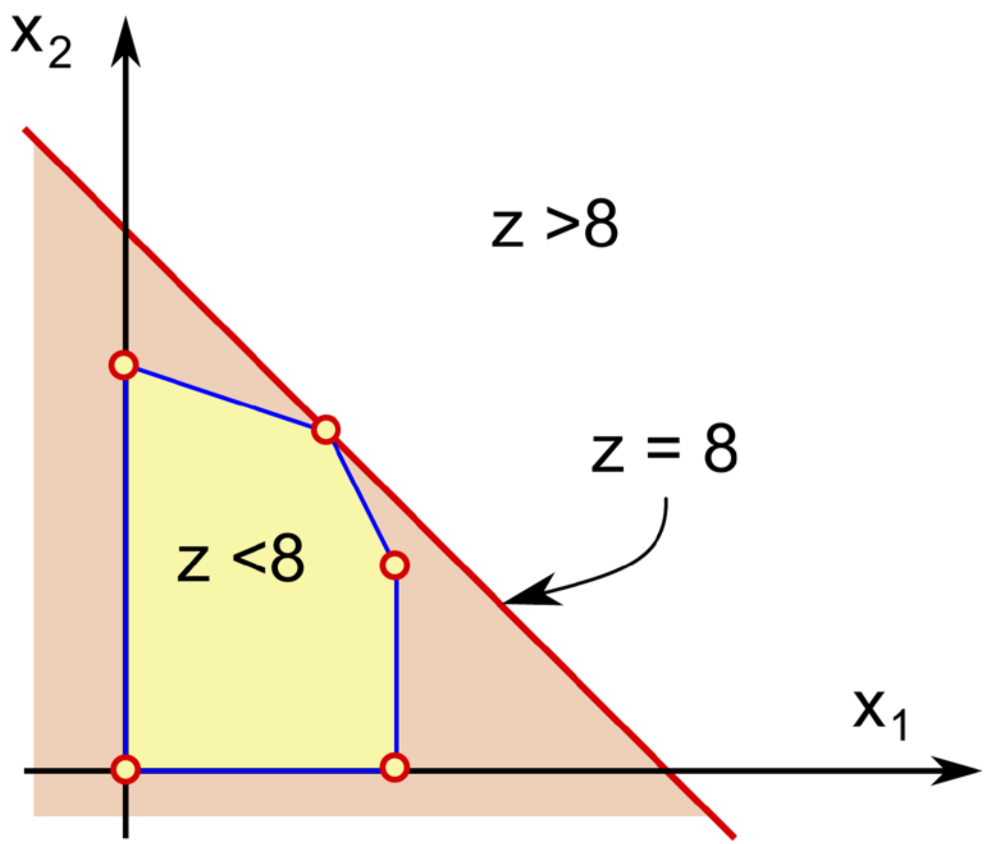
Example 1 (Graphical Solution)



Example 1 (Graphical Solution)

All feasible points satisfy $z \leq 8$

Q is the only feasible point (x_1, x_2) with $z = x_1 + x_2 = 8$



Linear Programming

- Assume feasible set $\mathcal{X}$ bounded and nonempty
- We can prove that LPs have an optimal vertex solution
  - LPs may be solved by inspecting all vertices, but ...
  - The number of vertices grows exponentially with the number of constraints and variables in the LP
- How to program a computer to efficiently solve LPs?
  - Simplex Algorithm finds an optimal vertex
  - Vertices inspected by the Simplex algorithm are often a small subset of the total
  - What made the Simplex algorithm famous is that it works well on most instances

Variants of Example 1

- Minimise $z = 3x_1 - x_2$ over feasible set of Example 1

  Examine the objective function at all vertices:

O=(0,0)	P=(0,6)	Q=(3,5)	R=(4,3)	S=(4,0)
0	-6	4	9	12

  $\Rightarrow$ P: $x_1 = 0$, $x_2 = 6$ is optimal.

- Maximise $z = 2x_1 + x_2$ over feasible set of Example 1:

  Any point on the line segment QR is optimal.

$\Rightarrow$ points other than vertices can be optimal, but there is at least one optimal vertex

Variants of Example 1

- Set a minimum weekly production goal of 7 tons of C

  We add a new constraint $x_1 \geq 7$. Then the feasible set becomes empty, because we previously imposed $x_1 \leq 4$

  $\Rightarrow$ the LP is infeasible

- Remove constraints on availability of X and Y

  Objective function can now grow to $+\infty$ on the feasible set. There is no maximum!

  $\Rightarrow$ the LP is unbounded