343 OPERATIONS RESEARCH

Basic Linear Algebra Refresher

Part I

Basic Linear Algebra

Matrix

Rectangular array of (real) numbers:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m \times n}$$

Example:

$$A = \begin{bmatrix} 1.0 & 5.5 & 6.3 \\ 3.1 & 2.4 & 8.9 \end{bmatrix} \in \mathbb{R}^{2 \times 3}$$

Matrix Equality

Two matrices A and B are *equal*, i.e.,

$$A = B$$

if and only if (iff):

$$a_{ij} = b_{ij}$$

$$\forall i,\ i = 1, \ldots, m$$
$$\forall j,\ j = 1, \ldots, n$$

Vector

Matrix with only *one* column:

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} \in \mathbb{R}^{m \times 1} = \mathbb{R}^m$$

Example:

$$v = \begin{bmatrix} 1.0 \\ 2.5 \end{bmatrix} \in \mathbb{R}^2$$

Scalar Product

The scalar product of two vectors v and u is:

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$v^T u = u^T v = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$$

Multiplication by a Scalar

For a $m \times n$ Matrix A:

$$3\,A = \begin{bmatrix} 3\,a_{11} & 3\,a_{12} & \dots & 3\,a_{1n} \\ 3\,a_{21} & 3\,a_{22} & \dots & 3\,a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 3\,a_{m1} & 3\,a_{m2} & \dots & 3\,a_{mn} \end{bmatrix}$$

Matrix Addition

For two $m \times n$ Matrices A and B:

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{bmatrix}$$

Matrix Transpose

For a $m \times n$ Matrix A, its transpose is

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix}$$

that is a $n \times m$ Matrix with:

$$(A^T)^T = A$$

Matrix Product

For a Matrix $A(m \times r)$ and a Matrix $B(r \times n)$, their product is the matrix

$$A\,B = C = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mn} \end{bmatrix}$$

that is a $m \times n$ Matrix with

$$c_{ij} = (\text{row } i \text{ of } A)^T (\text{column } j \text{ of } B).$$

Matrix Product

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$c_{11} = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 5$$

Matrix Product

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$c_{12} = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = 8$$

Matrix Product

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$c_{21} = \begin{bmatrix} 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 7$$

Matrix Product

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$c_{22} = \begin{bmatrix} 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = 11$$

Matrix Product

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$A\,B = C = \begin{bmatrix} 5 & 8 \\ 7 & 11 \end{bmatrix}$$

Properties: $C = A\ B$

Associative:

$$A\,(B\ C) = (A\ B)\ C$$

Distributive:

$$A\,(B + C) = (A\ B) + (A\ C)$$

$$(A + B)\ C = (A\ C) + (B\ C)$$

In general, matrix product is not commutative:

$$A\ B \neq B\ A$$

Part II

Matrices and Systems of Linear Equations

Linear Equations

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m
\end{array}$$

m Equations in n Variables

Matrix Notation

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$A\,x = b$$

Sometimes denoted:

$$\left[\begin{array}{c|c} A & b \end{array}\right]$$

Example

$$\begin{array}{rcrcl} x_1 & + & 2x_2 & = & 5 \\ 2x_1 & - & x_2 & = & 0 \end{array}$$

Matrices: $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

Example

$$\begin{array}{rcrcr} x_1 & + & 2x_2 & = & 5 \\ 2x_1 & - & x_2 & = & 0 \end{array}$$

$$\text{Compact: } \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 2 & -1 & 0 \end{array}\right]$$

$$\text{Solution: } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Gauss-Jordan Method

A System of Linear Equations

$$A\,x = b$$

may have:

- no solutions;
- a unique solution;
- infinitely many solutions.

Commonly solved by the Gauss-Jordan Method, which uses *Elementary Row Operations (ERO)* to progressively simplify the coefficient matrix A.

ERO 1

Multiplying any row by a non-zero scalar $c \neq 0$:

$$(\text{row } i \text{ of } A') = c\,(\text{row } i \text{ of } A)$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 6 \\ 0 & 1 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 9 & 15 & 18 \\ 0 & 1 & 2 & 3 \end{bmatrix} = A'$$

ERO 2

Multiplying any row by a non-zero scalar $c \neq 0$
and add it to another one $(i \neq j)$:

$$(\text{row } j \text{ of } A') = c\,(\text{row } i \text{ of } A) + (\text{row } j \text{ of } A)$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 6 \\ 0 & 1 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 6 \\ \color{red}{4} & \color{red}{13} & \color{red}{22} & \color{red}{27} \end{bmatrix} = A'$$

$$4 \begin{bmatrix} 1 & 3 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 13 & 22 & 27 \end{bmatrix}$$

ERO 3

Interchange any two rows i and j:

$$\begin{aligned} (\text{row } j \text{ of } A') &= (\text{row } i \text{ of } A) \\ (\text{row } i \text{ of } A') &= (\text{row } j \text{ of } A) \end{aligned}$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 6 \\ 0 & 1 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} \color{red}{0} & \color{red}{1} & \color{red}{2} & \color{red}{3} \\ 1 & 3 & 5 & 6 \\ \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} \end{bmatrix} = A'$$

Solving Equations

Use ERO1 and ERO2 to simplify Equations.

$$\begin{aligned} x_1 + x_2 &= 2 \\ 2x_1 + 4x_2 &= 7 \end{aligned}$$

Solving Equations

Use ERO1 and ERO2 to simplify Equations.

$$\begin{aligned} x_1 + x_2 &= 2 \\ 2x_1 + 4x_2 &= 7 \end{aligned} \qquad\qquad \begin{aligned} x_1 + x_2 &= 2 \\ 2x_2 &= 3 \end{aligned}$$

$$(\text{row 2 of } A') = -2\ (\text{row 1 of } A) + (\text{row 2 of A})$$

Solving Equations

Use ERO1 and ERO2 to simplify Equations.

$$\begin{aligned} x_1 + x_2 &= 2 \\ 2x_1 + 4x_2 &= 7 \end{aligned}$$

$$\begin{aligned} x_1 + x_2 &= 2 \\ 2x_2 &= 3 \end{aligned}$$

$$\begin{aligned} x_1 + x_2 &= 2 \\ x_2 &= \frac{3}{2} \end{aligned}$$

$$(\text{row 2 of } A') = \frac{1}{2}(\text{row 2 of } A)$$

Solving Equations

Use ERO1 and ERO2 to simplify Equations.

$$\begin{aligned} x_1 + x_2 &= 2 \\ 2x_1 + 4x_2 &= 7 \end{aligned} \qquad \begin{aligned} x_1 + x_2 &= 2 \\ 2x_2 &= 3 \end{aligned}$$

$$\begin{aligned} x_1 + x_2 &= 2 \\ x_2 &= \frac{3}{2} \end{aligned} \qquad \begin{aligned} x_1 &= \frac{1}{2} \\ x_2 &= \frac{3}{2} \end{aligned}$$

$$(\text{row } 1 \text{ of } A') = -1\,(\text{row } 2 \text{ of } A) + (\text{row } 1 \text{ of } A)$$

Solving Equations (cont)

In Compact Form:

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 4 & 7 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 2 & 3 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & \frac{3}{2} \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \end{array}\right]$$

Basic Principle

If

$$\left[\begin{array}{c|c} A' & b' \end{array}\right]$$

is obtained by ERO1, ER02, and ER03 from

$$\left[\begin{array}{c|c} A & b \end{array}\right]$$

Then:

$$\boxed{A'\,x = b'} \quad \text{and} \quad \boxed{A\,x = b}$$

are equivalent.

Example

$$
\begin{aligned}
2x_1 + 2x_2 + x_3 &= 9 \\
2x_1 - x_2 + 2x_3 &= 6 \\
x_1 - x_2 + 2x_3 &= 5
\end{aligned}
$$

Gauss-Jordan Method:
Solve by systematically applying the ERO

Example (cont)

$$\left[\begin{array}{c|c} A & b \end{array}\right] = \left[\begin{array}{ccc|c} 2 & 2 & 1 & 9 \\ 2 & -1 & 2 & 6 \\ 1 & -1 & 2 & 5 \end{array}\right]$$

Example (cont)

$$\left[\begin{array}{ccc|c} \color{red}{2} & 2 & 1 & 9 \\ 2 & -1 & 2 & 6 \\ 1 & -1 & 2 & 5 \end{array}\right]$$

ERO 1

$$(\text{row } 1) = \frac{1}{2}(\text{row } 1)$$

Example (cont)

$$\left[\begin{array}{ccc|c} 1 & 1 & \frac{1}{2} & \frac{9}{2} \\ 2 & -1 & 2 & 6 \\ 1 & -1 & 2 & 5 \end{array}\right]$$

Example (cont)

$$\left[\begin{array}{ccc|c} 1 & 1 & \frac{1}{2} & \frac{9}{2} \\ \color{red}{2} & -1 & 2 & 6 \\ 1 & -1 & 2 & 5 \end{array}\right]$$

ERO 2

$$(\text{row } 2) = -2\,(\text{row } 1) + (\text{row } 2)$$

Example (cont)

$$\left[\begin{array}{ccc|c} 1 & 1 & \frac{1}{2} & \frac{9}{2} \\ \color{red}{0} & \color{blue}{-3} & \color{blue}{1} & \color{blue}{-3} \\ 1 & -1 & 2 & 5 \end{array}\right]$$

Example (cont)

$$\left[\begin{array}{ccc|c} 1 & 1 & \frac{1}{2} & \frac{9}{2} \\ 0 & -3 & 1 & -3 \\ \color{red}{1} & -1 & 2 & 5 \end{array}\right]$$

ERO 2

$$(\text{row } 3) = -1\,(\text{row } 1) + (\text{row } 3)$$

Example (cont)

$$\left[\begin{array}{ccc|c} 1 & 1 & \frac{1}{2} & \frac{9}{2} \\ 0 & -3 & 1 & -3 \\ \color{red}{0} & \color{blue}{-2} & \color{blue}{\frac{3}{2}} & \color{blue}{\frac{1}{2}} \end{array}\right]$$

Example (cont)

$$\left[\begin{array}{ccc|c} 1 & 1 & \frac{1}{2} & \frac{9}{2} \\ 0 & \color{red}{-3} & 1 & -3 \\ 0 & -2 & \frac{3}{2} & \frac{1}{2} \end{array}\right]$$

ERO 1

$$(\text{row } 2) = -\frac{1}{3}\,(\text{row } 2)$$

Example (cont)

$$\left[\begin{array}{ccc|c} 1 & 1 & \frac{1}{2} & \frac{9}{2} \\ 0 & {\color{red}1} & {\color{blue}-\frac{1}{3}} & {\color{blue}1} \\ 0 & -2 & \frac{3}{2} & \frac{1}{2} \end{array}\right]$$

Example (cont)

$$\left[\begin{array}{ccc|c} 1 & \color{red}{1} & \frac{1}{2} & \frac{9}{2} \\ 0 & 1 & -\frac{1}{3} & 1 \\ 0 & -2 & \frac{3}{2} & \frac{1}{2} \end{array}\right]$$

ERO 2

$$(\text{row } 1) = -1\,(\text{row } 2) + (\text{row } 1)$$

Example (cont)

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{5}{6} & \frac{7}{2} \\ 0 & 1 & -\frac{1}{3} & 1 \\ 0 & -2 & \frac{3}{2} & \frac{1}{2} \end{array}\right]$$

Example (cont)

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{5}{6} & \frac{7}{2} \\ 0 & 1 & -\frac{1}{3} & 1 \\ 0 & -2 & \frac{3}{2} & \frac{1}{2} \end{array}\right]$$

ERO 2

$$(\text{row } 3) = 2\,(\text{row } 2) + (\text{row } 3)$$

Example (cont)

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{5}{6} & \frac{7}{2} \\ 0 & 1 & -\frac{1}{3} & 1 \\ 0 & {\color{red}0} & {\color{blue}\frac{5}{6}} & {\color{blue}\frac{5}{2}} \end{array}\right]$$

Example (cont)

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{5}{6} & \frac{7}{2} \\ 0 & 1 & -\frac{1}{3} & 1 \\ 0 & 0 & \frac{5}{6} & \frac{5}{2} \end{array}\right]$$

ERO 1

$$(\text{row } 3) = \frac{6}{5}(\text{row } 3)$$

Example (cont)

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{5}{6} & \frac{7}{2} \\ 0 & 1 & -\frac{1}{3} & 1 \\ 0 & 0 & \color{red}{1} & \color{blue}{3} \end{array}\right]$$

Example (cont)

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{5}{6} & \frac{7}{2} \\ 0 & 1 & -\frac{1}{3} & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

ERO 2

$$(\text{row } 1) = -\frac{5}{6}\,(\text{row } 3) + (\text{row } 1)$$

Example (cont)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & -\frac{1}{3} & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Example (cont)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & -\frac{1}{3} & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

ERO 2

$$(\text{row } 2) = \frac{1}{3}(\text{row } 3) + (\text{row } 2)$$

Example (cont)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & \color{red}{0} & \color{blue}{2} \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Example (cont)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Example (cont)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

The solution is now straightforward:
$x_1 = 1$, $x_2 = 2$, and $x_3 = 3$

Remark (ER03)

ERO3 can be used to re-arrange rows such that ERO1 and ERO2 can be applied systematically.

$$\begin{array}{rcrcrcr} & & 2x_2 & + & x_3 & = & 6 \\ x_1 & + & x_2 & - & x_3 & = & 2 \\ 2x_1 & + & x_2 & + & x_3 & = & 4 \end{array}$$

$$\left[\begin{array}{ccc|c} 0 & 2 & 1 & 6 \\ 1 & 1 & -1 & 2 \\ 2 & 1 & 1 & 4 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 2 & 1 & 6 \\ 2 & 1 & 1 & 4 \end{array}\right]$$

Part III

Basic Variables and Solutions

Basic Variables

For any system of linear equations

$$A\,x = b$$

a variable x_j with:

- one coefficient a_{ij} equal to 1, and
- all other coefficients in column j equal to 0

is called a *Basic Variable*.

All other variables are called *Non-Basic Variables*.

Solutions I

Condition I:

$$A' \, x = b' \text{ has at least one row of the form}$$

$$\left[\begin{array}{cccc|c} 0 & 0 & \dots & 0 & c \end{array}\right]$$

$$\text{with } c \neq 0.$$

$$\text{Condition I} \implies A \, x = b \text{ has no solution.}$$

$$\text{e.g.} \quad \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right] \quad \text{has no solution!}$$

Solutions II

Suppose Condition I does not hold and the set of non-basic variables is empty, NBV$= \emptyset$. Then $A' \, x = b'$ and hence $A \, x = b$ has a unique solution.

$$\text{e.g.} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

has a unique solution:

$$x_1 = 1, \; x_2 = 2, \; x_3 = 3$$

Solutions III

Suppose Condition I does not hold but the set of non-basic variables is non-empty, NBV$\neq \emptyset$.
Then $A' \, x = b'$ and $A \, x = b$ have an infinite number of solutions.
In order to obtain these solutions, set each NBV to an arbitrary value. Then solve for the BVs in terms of the NBVs.

Example

$$
\begin{array}{ccccccccc}
x_1 & & & + & x_4 & + & x_5 & = & 3 \\
& x_2 & & + & 2x_4 & & & = & 2 \\
& & x_3 & & & + & x_5 & = & 1
\end{array}
$$

Example

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 1 & 3 \\ 0 & 1 & 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Rank = 3 < M(5) => No Solution

Example

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 1 & 3 \\ 0 & 1 & 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

Condition I does not apply, and:

$$BV = \{x_1, x_2, x_3\} \;\text{ and }\; NBV = \{x_4, x_5\}$$

Set: $x_4 = a$ and $x_5 = b$, for any value of a and b we get:

$$\begin{aligned} x_1 &= 3 - a - b \\ x_2 &= 2 - 2a \\ x_3 &= 1 - b \end{aligned}$$

Gauss-Jordan: Summary

Does $[\ A \mid b\ ]$ have a row $[\ 0\ 0\ \ldots\ 0 \mid c\ ]$ with $c \neq 0$?

- YES: $\nexists$ solution
- NO: Find BV and NBV. Is NBV$= \emptyset$?
  - YES: $\exists!$ solution
  - NO: $\exists$ solutions (infinite)

Part IV

Linear Combinations

Linear Combinations

Let V be a set of (column) vectors,

$$V = \{v_1, v_2, v_3, \ldots, v_k\}$$

all of the same dimension.
A *linear combination* of vectors in V is any vector of the form:

$$v = c_1 v_1 + c_2 v_2 + \ldots + c_k v_k.$$

with $c_1, c_2, \ldots, c_k \in \mathbb{R}$.

Example

$$V = \left\{ v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$

$$2v_1 - v_2 = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$v_1 + 3v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

$$0v_1 + 3v_2 = 3\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

Linear Independence I

Let V be a set of (column) vectors in $\mathbb{R}^m$

$$V = \{v_1, v_2, v_3, \ldots, v_k\}.$$

and the null vector

$$N = \left[\begin{array}{cccc} 0 & 0 & \ldots & 0 \end{array}\right]^T.$$

In order to determine if V is *linearly independent* (l.i.), we try to find a linear combinations of vectors in V which results in N.

Clearly the *trivial linear combination* always works:

$$0v_1 + 0v_2 + \ldots + 0v_k = N.$$

Linear Independence II

- A set of vectors V is *linearly independent* iff the only linear combination which gives the null vector is the trivial linear combination.
- Otherwise the vectors in V are said to be *linearly dependent*.
- Any V containing N is linearly dependent (for $c \neq 0$)

$$\text{e.g. } c \begin{bmatrix} 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = N.$$

Examples

The vectors $e_1 = [1 \;\; 0]^T$ and $e_2 = [0 \;\; 1]^T$ are l.i.:

$$c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = N \quad \Rightarrow \qquad c_1 = 0 = c_2.$$

The vectors $e_1 = [1 \;\; 2]^T$ and $e_2 = [2 \;\; 4]^T$ are l.d.:

$$2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - 1 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = N.$$

Rank of a Matrix

Take a $m \times n$ matrix A. Consider its column vectors:

$$C = \{c_1, c_2, \ldots, c_n\}.$$

The rank of A is the maximal number of linear independent vectors in C.

$$\text{e.g. } rank\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = 0 \quad \text{e.g.} rank\left(\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\right) = 1$$

$$\text{e.g. } rank\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 2$$

Inverse of a Matrix

Let A be an $m \times m$ square matrix.
The $m \times m$ *identity matrix* is given by:

$$I = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

A matrix A^{-1} for which the following holds:

$$A\,A^{-1} = A^{-1}\,A = I$$

is called the *inverse* matrix of A.

Inverse of a Matrix II

- The inverse of a matrix may not exist!
- An invertible matrix A is called *non-singular*.
- The inverse of an $m \times m$ matrix does not exist if $rank(A) < m$.

Matrix inversion can be used to solve linear systems:

$$\begin{aligned} A\,x &= b \\ A^{-1}\,A\,x &= A^{-1}\,b \\ I\,x &= A^{-1}\,b \\ x &= A^{-1}\,b \end{aligned}$$

The Gauss-Jordan process indirectly computes A^{-1}.