Math Models of OR: Degeneracy

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Outline

1. Basic variables equal to zero

2. Solving a degenerate linear optimization problem

3. Cycling

Standard form

Recall our standard form linear optimization problem,

$$\begin{array}{lrcl} \min_{x \in \mathbb{R}^n} & c^T x & & \\ \text{subject to} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

where c and x are n-vectors, b is an m-vector, and A is an $m \times n$ matrix. A basic feasible solution has m basic variables, each one corresponding to a column of the identity matrix in a canonical form representation of the LOP.

Note that the rank of A must equal m if a basic feasible solution exists, since we must be able to perform a sequence of elementary row operations leading to a canonical form tableau.

Variables equal to zero

The nonbasic variables are all equal to zero. Just because a variable is basic, it doesn't have to be strictly positive.

Definition

A basic feasible solution is **degenerate** *if at least one of the basic variables is equal to zero. A standard form linear optimization problem is* **degenerate** *if at least one of its basic feasible solutions is degenerate.*

If every basic variable is strictly positive in a basic feasible solution then the BFS is **nondegenerate***. A linear optimization problem is* **nondegenerate** *if each of its basic feasible solutions is nondegenerate.*

A degenerate example

$$\begin{array}{lrcrcrcl} \min_{x\in\mathbb{R}^3} & & - & 3x_2 & & & & \\ \text{subject to} & & & x_2 & + & x_3 & = & 4 \\ & x_1 & + & 2x_2 & & & = & 8 \\ & & & x_j \geq 0, & & j = 1,2,3 & & \end{array}$$

Nondegenerate BFS:
$x = (8, 0, 4)$.

$$\begin{array}{c} \text{ratio} \\ \\ \frac{4}{1} \\ \frac{8}{2} \end{array} \quad \begin{array}{c|ccc} & x_1 & \overset{\downarrow}{x_2} & x_3 \\ \hline 0 & 0 & -2 & 0 \\ \hline 4 & 0 & \color{red}{1} & 1 \\ 8 & 1 & 2 & 0 \end{array} \quad \xrightarrow{R_0 + 2R_1,\ R_2 - 2R_1} \quad \begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ \hline 0 & 0 & \color{blue}{0} & 2 \\ \hline 4 & 0 & \color{red}{1} & 1 \\ 0 & 1 & \color{blue}{0} & -2 \end{array}$$

Since there was a tie in the minimum ratio when bringing x_2 into the basis, we arbitrarily decided to pivot on the first row, so x_3 became nonbasic. The resulting BFS has $x_1 = 0$ and basic, so this LOP is degenerate.

A degenerate example

$$\begin{array}{lrcrcrcr} \min_{x\in\mathbb{R}^3} & & - & 3x_2 & & & & \\ \text{subject to} & & & x_2 & + & x_3 & = & 4 \\ & x_1 & + & 2x_2 & & & = & 8 \\ & & & \multicolumn{5}{l}{x_j \geq 0, \quad j = 1,2,3} \end{array}$$

Nondegenerate BFS: $x = (8, 0, 4)$.

$$\begin{array}{c} \\ \text{ratio} \\ \\ \frac{4}{1} \\ \frac{8}{2} \end{array} \quad \begin{array}{c|ccc} & & \downarrow & \\ & x_1 & x_2 & x_3 \\ \hline 0 & 0 & -2 & 0 \\ \hline 4 & 0 & \color{red}{1} & 1 \\ 8 & 1 & 2 & 0 \\ \hline \end{array} \quad \xrightarrow{R_0 + 2R_1,\ R_2 - 2R_1} \quad \begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ \hline 0 & 0 & \color{blue}{0} & 2 \\ \hline 4 & 0 & \color{red}{1} & 1 \\ 0 & 1 & \color{blue}{0} & -2 \\ \hline \end{array}$$

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A degenerate example

$$\begin{array}{lrcrcrcr} \min_{x\in\mathbb{R}^3} & & - & 3x_2 & & & & \\ \text{subject to} & & & x_2 & + & x_3 & = & 4 \\ & x_1 & + & 2x_2 & & & = & 8 \\ & & & x_j \geq 0, & & j = 1,2,3 & & \end{array}$$

Nondegenerate BFS: $x = (8, 0, 4)$.

$$\begin{array}{c|c|ccc|} \text{ratio} & & x_1 & \stackrel{\downarrow}{x_2} & x_3 \\ \hline & 0 & 0 & -2 & 0 \\ \hline \frac{4}{1} & 4 & 0 & \color{red}{1} & 1 \\ \frac{8}{2} & 8 & 1 & 2 & 0 \\ \hline \end{array} \quad \xrightarrow{R_0 + 2R_1,\ R_2 - 2R_1} \quad \begin{array}{|c|ccc|} & x_1 & x_2 & x_3 \\ \hline 0 & 0 & \color{blue}{0} & 2 \\ \hline 4 & 0 & \color{red}{1} & 1 \\ 0 & 1 & \color{blue}{0} & -2 \\ \hline \end{array}$$

Since there was a tie in the minimum ratio when bringing x_2 into the basis, we arbitrarily decided to pivot on the first row, so x_3 became nonbasic. The resulting BFS has $x_1 = 0$ and basic, so this LOP is degenerate.

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2 Solving a degenerate linear optimization problem

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An example

Consider the linear optimization problem

$$\begin{array}{lrcrcl} \min_{x \in \mathbb{R}^2} & -x_1 & - & 2x_2 & & \\ \text{subject to} & x_1 & + & 3x_2 & \leq & 6 \\ & x_1 & & & \leq & 3 \\ & & & x_2 & \leq & 2 \\ & & & x_1,\, x_2 & \geq & 0 \end{array}$$

We add slack variables to get standard form:

$$\begin{array}{lrcrcrcrcrcl} \min_{x \in \mathbb{R}^2} & -x_1 & - & 2x_2 & & & & & & & & \\ \text{subject to} & x_1 & + & 3x_2 & + & x_3 & & & & & = & 6 \\ & x_1 & & & & & + & x_4 & & & = & 3 \\ & & & x_2 & & & & & + & x_5 & = & 2 \\ & & & & & & & & & & & \end{array}$$

$$x_j \geq 0, \quad j = 1, \ldots, 5$$

Canonical form

Since this representation is in canonical form, we can proceed directly with the simplex algorithm, starting from the initial BFS with nonbasic variables $x_1 = x_2 = 0$ and basic variables $x_3 = 6$, $x_4 = 3$, $x_5 = 2$.

$$
\begin{array}{c|c|ccccc|}
 & & & \downarrow & & & \\
\text{ratio} & & x_1 & x_2 & x_3 & x_4 & x_5 \\
\cline{2-7}
 & 0 & -1 & -2 & 0 & 0 & 0 \\
\cline{2-7}
\frac{6}{3} & 6 & 1 & 3 & 1 & 0 & 0 \\
- & 3 & 1 & 0 & 0 & 1 & 0 \\
\frac{2}{1} & 2 & 0 & \color{red}{1} & 0 & 0 & 1 \\
\cline{2-7}
\end{array}
$$

Pivot

$$
\begin{array}{c|c|ccccc|}
 & \multicolumn{1}{c}{} & & \downarrow & & & \\
\text{ratio} & \multicolumn{1}{c}{} & x_1 & x_2 & x_3 & x_4 & x_5 \\
\cline{2-7}
 & 0 & -1 & -2 & 0 & 0 & 0 \\
\cline{2-7}
\frac{6}{3} & 6 & 1 & 3 & 1 & 0 & 0 \\
- & 3 & 1 & 0 & 0 & 1 & 0 \\
\frac{2}{1} & 2 & 0 & \color{red}{1} & 0 & 0 & 1 \\
\cline{2-7}
\end{array}
$$

$$
\xrightarrow{R_0 + 2R_3,\ R_1 - 3R_3}
\begin{array}{|c|ccccc|}
\multicolumn{1}{c}{} & x_1 & x_2 & x_3 & x_4 & x_5 \\
\hline
4 & -1 & \color{blue}{0} & 0 & 0 & 2 \\
\hline
0 & 1 & \color{blue}{0} & 1 & 0 & -3 \\
3 & 1 & \color{blue}{0} & 0 & 1 & 0 \\
2 & 0 & \color{red}{1} & 0 & 0 & 1 \\
\hline
\end{array}
$$

There was a tie in the minimum ratio test when x_2 enters the basis, which we arbitrarily broke in favor of the final row, so x_5 became nonbasic.

The pivot row for the degenerate pivot

$$
\begin{array}{c|c|ccccc|}
 & & \downarrow & & & & \\
\text{ratio} & & x_1 & x_2 & x_3 & x_4 & x_5 \\
\hline
 & 4 & -1 & 0 & 0 & 0 & 2 \\
\hline
\frac{0}{1} & 0 & \color{red}{1} & 0 & 1 & 0 & -3 \\
\frac{3}{1} & 3 & 1 & 0 & 0 & 1 & 0 \\
- & 2 & 0 & 1 & 0 & 0 & 1 \\
\hline
\end{array}
$$

The updated BFS has nonbasic variables $x_1 = x_5 = 0$ and basic variables $x_3 = 0$, $x_4 = 3$, $x_2 = 2$, a degenerate BFS with value -4.

The pivot row corresponds to the constraint

$$
\begin{array}{cccc}
x_1 & +\, x_3 & -\, 3x_5 & = \quad 0 \\
\text{nonbasic,} & \text{basic,} & \text{nonbasic,} & \\
\text{entering basis} & \text{leaving basis} & \text{stays nonbasic} &
\end{array}
$$

Since we keep $x_5 = 0$, we're forced to keep both $x_1 = x_3 = 0$.

Second pivot

$$
\begin{array}{c|c|ccccc|}
 & & \downarrow & & & & \\
\text{ratio} & & x_1 & x_2 & x_3 & x_4 & x_5 \\
\hline
 & 4 & -1 & 0 & 0 & 0 & 2 \\
\hline
\frac{0}{1} & 0 & \color{red}{1} & 0 & 1 & 0 & -3 \\
\frac{3}{1} & 3 & 1 & 0 & 0 & 1 & 0 \\
- & 2 & 0 & 1 & 0 & 0 & 1 \\
\hline
\end{array}
$$

On the second pivot, x_1 enters the basis and x_3 leaves the basis.

$$
\xrightarrow{R_0 + R_1,\ R_2 - R_1}
\quad
\begin{array}{|c|ccccc|}
 & x_1 & x_2 & x_3 & x_4 & x_5 \\
\hline
4 & \color{blue}{0} & 0 & 1 & 0 & -1 \\
\hline
0 & \color{red}{1} & 0 & 1 & 0 & -3 \\
3 & \color{blue}{0} & 0 & -1 & 1 & 3 \\
2 & \color{blue}{0} & 1 & 0 & 0 & 1 \\
\hline
\end{array}
$$

The updated BFS has nonbasic variables $x_3 = x_5 = 0$ and basic variables $\color{magenta}{x_1 = 0}$, $x_4 = 3$, $x_2 = 2$.

Point is the same, basic sequence has changed

		x_1	x_2	x_3	x_4	x_5
	4	0	0	1	0	-1
$-$	0	1	0	1	0	-3
$\frac{3}{3}$	3	0	0	-1	1	3
$\frac{2}{1}$	2	0	1	0	0	1

The updated BFS has nonbasic variables $x_3 = x_5 = 0$ and basic variables $x_1 = 0$, $x_4 = 3$, $x_2 = 2$.

Notice that the point x has not changed, although different variables are now designated as basic variables.

Previously, we had $x_3 = 0$ as a basic variable, and now we have $x_1 = 0$ as a basic variable.

Third pivot

ratio		x_1	x_2	x_3	x_4	x_5 ↓
	4	0	0	1	0	-1
$-$	0	1	0	1	0	-3
$\frac{3}{3}$	3	0	0	-1	1	3
$\frac{2}{1}$	2	0	1	0	0	1

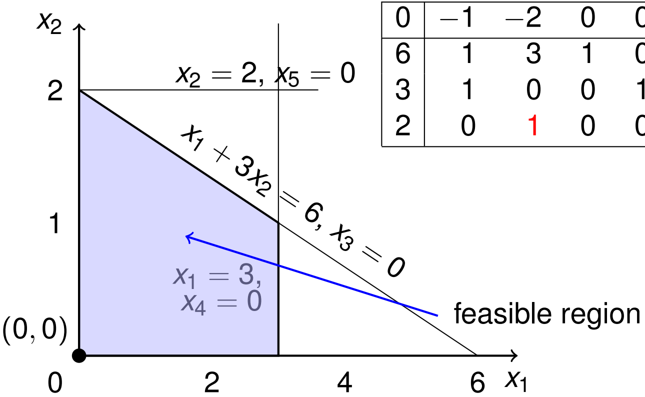
x_5 enters the basis, replacing x_4.

$$\xrightarrow{\frac{1}{3}R_2 \text{ then } R_0 + R_2,\ R_1 + 3R_2,\ R_3 - R_2}$$

	x_1	x_2	x_3	x_4	x_5
5	0	0	$\frac{2}{3}$	$\frac{1}{3}$	0
3	1	0	0	1	0
1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	1
1	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	0

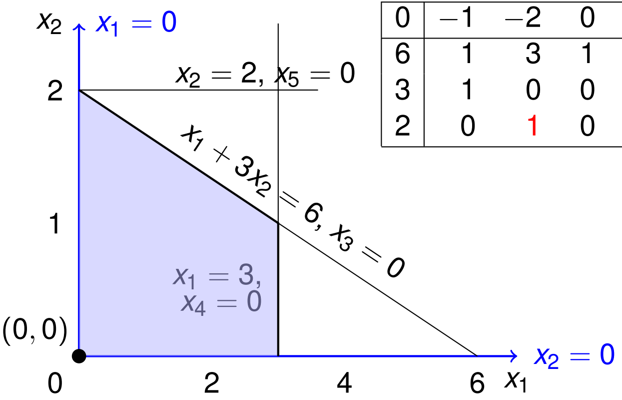
The tableau is in optimal form. The optimal solution has nonbasic variables $x_3 = x_4 = 0$ and basic variables $x_1 = 3$, $x_5 = 1$, $x_2 = 1$, with value -5.

We can graph this LOP in $(x_1, x_2)-$space.



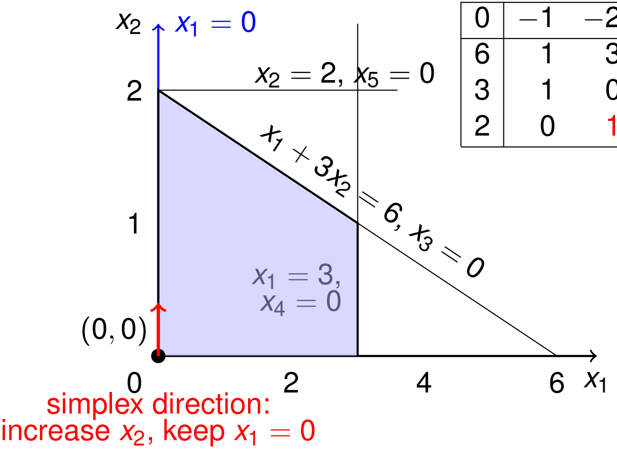
	x_1	x_2	x_3	x_4	x_5
0	-1	-2	0	0	0
6	1	3	1	0	0
3	1	0	0	1	0
2	0	1	0	0	1

We can graph this LOP in $(x_1, x_2)-$space.



	x_1	x_2	x_3	x_4	x_5
0	-1	-2	0	0	0
6	1	3	1	0	0
3	1	0	0	1	0
2	0	1	0	0	1

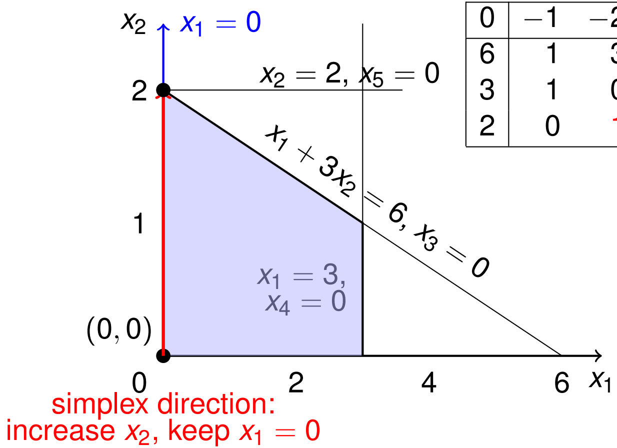
We can graph this LOP in $(x_1, x_2)-$space.



	x_1	x_2	x_3	x_4	x_5
0	-1	-2	0	0	0
6	1	3	1	0	0
3	1	0	0	1	0
2	0	1	0	0	1

As we increase x_2, we drive x_3 and x_5 towards 0.

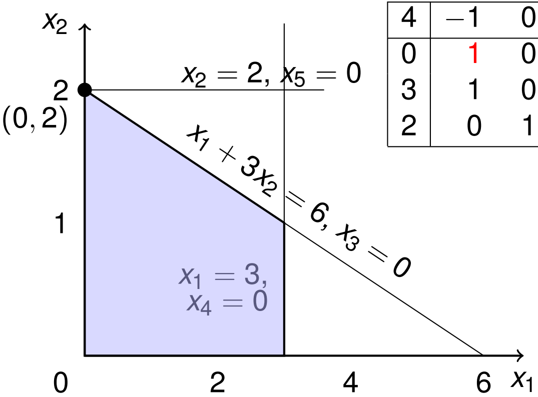
We can graph this LOP in $(x_1, x_2)-$space.



	x_1	x_2	x_3	x_4	x_5
0	-1	-2	0	0	0
6	1	3	1	0	0
3	1	0	0	1	0
2	0	1	0	0	1

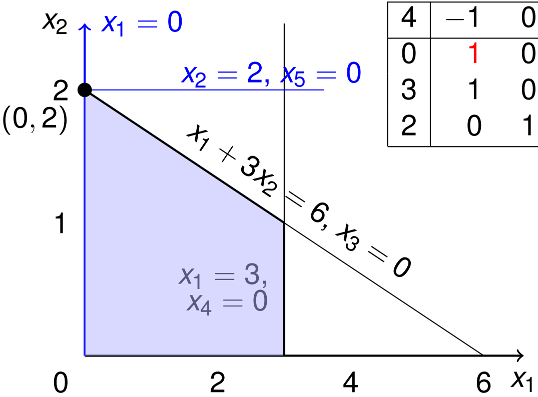
As we increase x_2, we drive x_3 and x_5 towards 0. x_3 and x_5 reach 0 simultaneously, giving $x = (0, 2, 0, 3, 0)$.

The second pivot



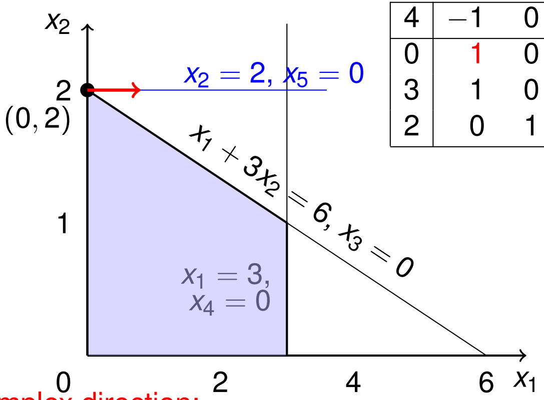
	x_1	x_2	x_3	x_4	x_5
4	-1	0	0	0	2
0	1	0	1	0	-3
3	1	0	0	1	0
2	0	1	0	0	1

The second pivot



	x_1	x_2	x_3	x_4	x_5
4	-1	0	0	0	2
0	1	0	1	0	-3
3	1	0	0	1	0
2	0	1	0	0	1

The second pivot

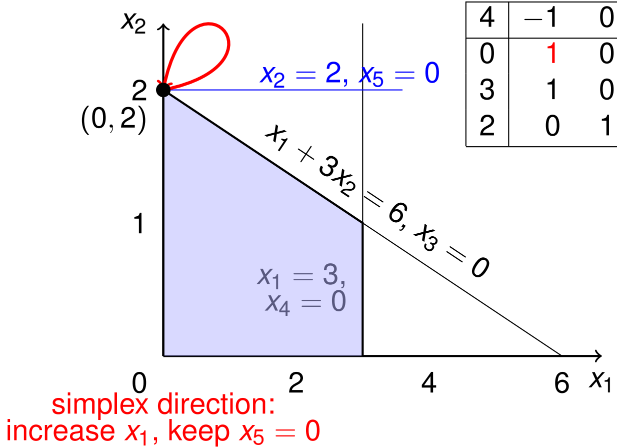


	x_1	x_2	x_3	x_4	x_5
4	-1	0	0	0	2
0	1	0	1	0	-3
3	1	0	0	1	0
2	0	1	0	0	1

simplex direction:
increase x_1, keep $x_5 = 0$

As we increase x_1, we drive x_3 towards 0.

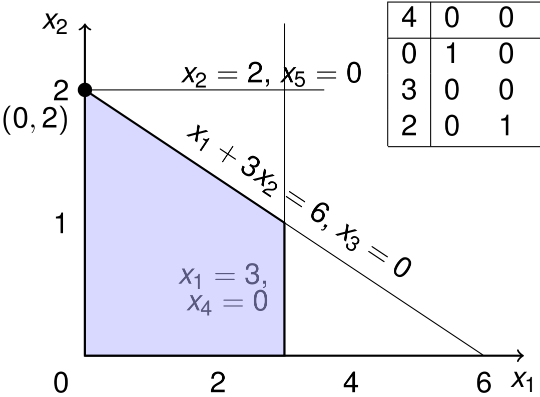
The second pivot



	x_1	x_2	x_3	x_4	x_5
4	-1	0	0	0	2
0	1	0	1	0	-3
3	1	0	0	1	0
2	0	1	0	0	1

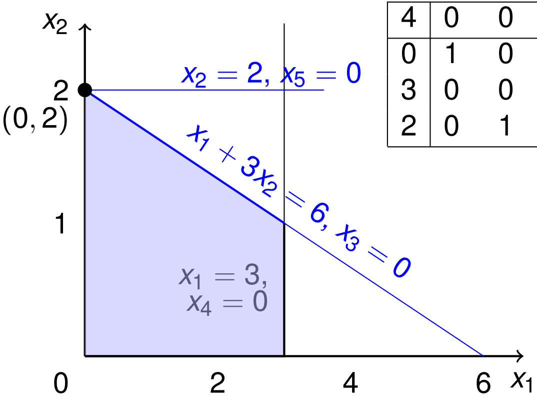
As we increase x_1, we drive x_3 towards 0. But x_3 is already zero. Stay at $x = (0, 2, 0, 3, 0)$.

The third pivot



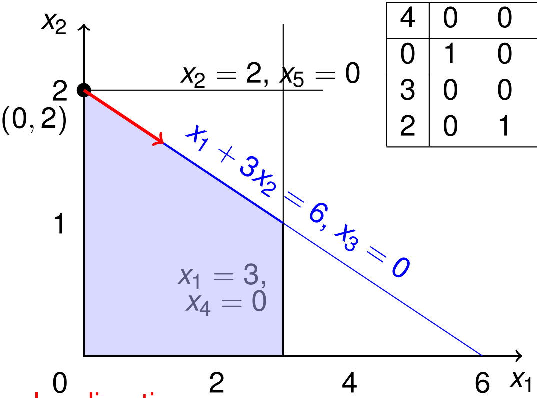
	x_1	x_2	x_3	x_4	x_5
4	0	0	1	0	−1
0	1	0	1	0	−3
3	0	0	−1	1	3
2	0	1	0	0	1

The third pivot



	x_1	x_2	x_3	x_4	x_5
4	0	0	1	0	-1
0	1	0	1	0	-3
3	0	0	-1	1	3
2	0	1	0	0	1

The third pivot

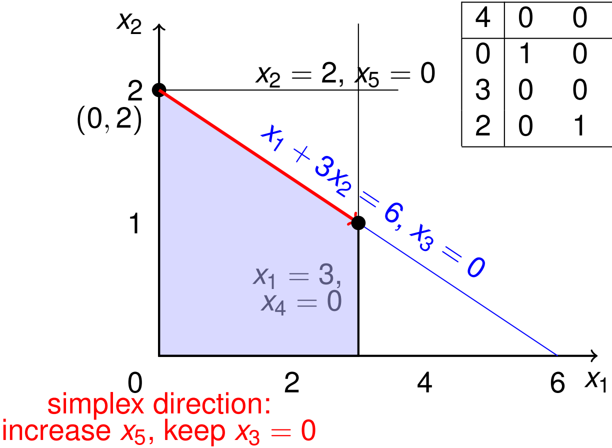


	x_1	x_2	x_3	x_4	x_5
4	0	0	1	0	-1
0	1	0	1	0	-3
3	0	0	-1	1	3
2	0	1	0	0	1

simplex direction:
increase x_5, keep $x_3 = 0$

As we increase x_5, we drive x_2 and x_4 towards 0.

The third pivot

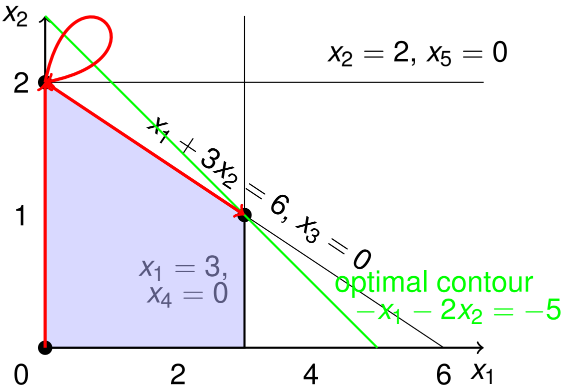


	x_1	x_2	x_3	x_4	x_5
4	0	0	1	0	-1
0	1	0	1	0	-3
3	0	0	-1	1	3
2	0	1	0	0	1

As we increase x_5, we drive x_2 and x_4 towards 0. x_4 reaches zero first, giving $x = (3, 1, 0, 0, 1)$.

All the iterations

Note that the degenerate point is “overdetermined”: we have three lines passing through the point, corresponding to the active constraints $x_1 = 0$, $x_2 = 2$, and $x_1 + 3x_2 = 6$. Equivalently, $x_1 = x_3 = x_5 = 0$.



An alternative pivot sequence

Return now to the initial pivot. We arbitrarily broke the tie in the minimum ratio in favor of x_5 leaving the basis.

What happens if we make the other choice: x_3 leaves the basis?

$$
\begin{array}{c|c|ccccc|}
\text{ratio} & & x_1 & \overset{\downarrow}{x_2} & x_3 & x_4 & x_5 \\
\hline
 & 0 & -1 & -2 & 0 & 0 & 0 \\
\hline
\frac{6}{3} & 6 & 1 & \color{red}{3} & 1 & 0 & 0 \\
- & 3 & 1 & 0 & 0 & 1 & 0 \\
\frac{2}{1} & 2 & 0 & 1 & 0 & 0 & 1 \\
\hline
\end{array}
$$

$$
\xrightarrow{\frac{1}{3}R_1 \text{ then } R_0 + 2R_1,\ R_3 - R_1}
\quad
\begin{array}{|c|ccccc|}
 & x_1 & x_2 & x_3 & x_4 & x_5 \\
\hline
4 & -\frac{1}{3} & \color{blue}{0} & \frac{2}{3} & 0 & 0 \\
\hline
2 & \frac{1}{3} & \color{red}{1} & \frac{1}{3} & 0 & 0 \\
3 & 1 & \color{blue}{0} & 0 & 1 & 0 \\
0 & -\frac{1}{3} & \color{blue}{0} & -\frac{1}{3} & 0 & 1 \\
\hline
\end{array}
$$

Second iterate

	x_1	x_2	x_3	x_4	x_5
4	$-\frac{1}{3}$	0	$\frac{2}{3}$	0	0
2	$\frac{1}{3}$	1	$\frac{1}{3}$	0	0
3	1	0	0	1	0
0	$-\frac{1}{3}$	0	$-\frac{1}{3}$	0	1

We get the same point as before, $x = (0, 2, 0, 3, 0)$, but now the basic sequence is $S = (2, 4, 5)$.

Earlier, it was $S = (3, 4, 2)$, so now x_5 is basic, whereas before x_3 was basic.

Second pivot

On the second pivot, the variable x_1 enters the basis, and by the minimum ratio test x_4 becomes nonbasic.

$$
\begin{array}{c}
\text{ratio} \\
\\
\frac{2}{1/3} \\
\frac{3}{1} \\
-
\end{array}
\quad
\begin{array}{c|ccccc}
 & \downarrow & & & & \\
 & x_1 & x_2 & x_3 & x_4 & x_5 \\
\hline
4 & -\frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\
\hline
2 & \frac{1}{3} & 1 & \frac{1}{3} & 0 & 0 \\
3 & 1 & 0 & 0 & 1 & 0 \\
0 & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & 1 \\
\hline
\end{array}
$$

x_1 replaces x_4 in the basis.

$$
\xrightarrow{R_0 + \frac{1}{3}R_2,\ R_1 - \frac{1}{3}R_2,\ R_3 + \frac{1}{3}R_2}
\quad
\begin{array}{c|ccccc}
 & x_1 & x_2 & x_3 & x_4 & x_5 \\
\hline
5 & 0 & 0 & \frac{2}{3} & \frac{1}{3} & 0 \\
\hline
1 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & 0 \\
3 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 1 \\
\hline
\end{array}
$$

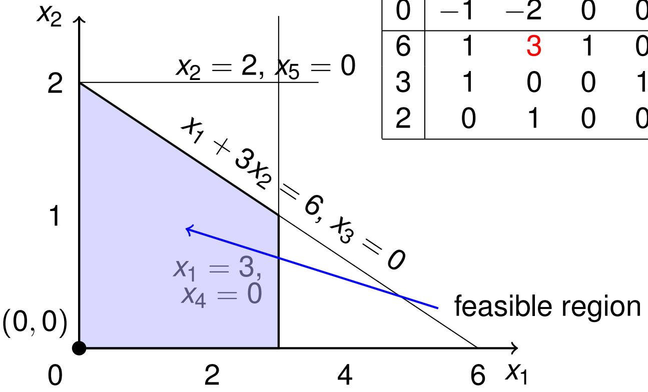
Optimal

$$
\begin{array}{c|ccccc|}
 & x_1 & x_2 & x_3 & x_4 & x_5 \\
\hline
5 & \color{blue}{0} & 0 & \frac{2}{3} & \frac{1}{3} & 0 \\
\hline
1 & \color{blue}{0} & 1 & \frac{1}{3} & -\frac{1}{3} & 0 \\
3 & \color{red}{1} & 0 & 0 & 1 & 0 \\
1 & \color{blue}{0} & 0 & -\frac{1}{3} & \frac{1}{3} & 1 \\
\hline
\end{array}
$$

This tableau is in optimal form, returning the same solution as before.

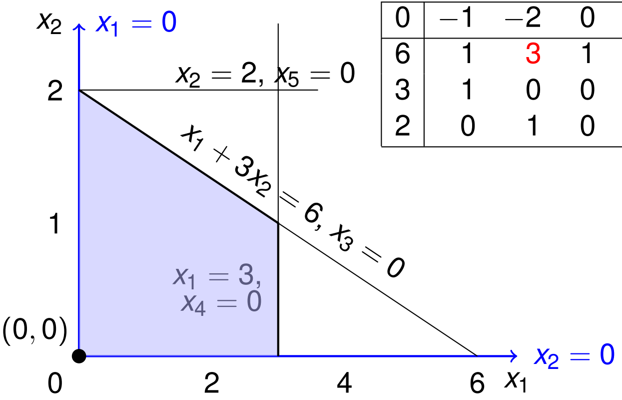
With the different choice of variable to leave the basis on the first pivot, we ended up only requiring 2 pivots instead of three.

Graphing the progress of the algorithm



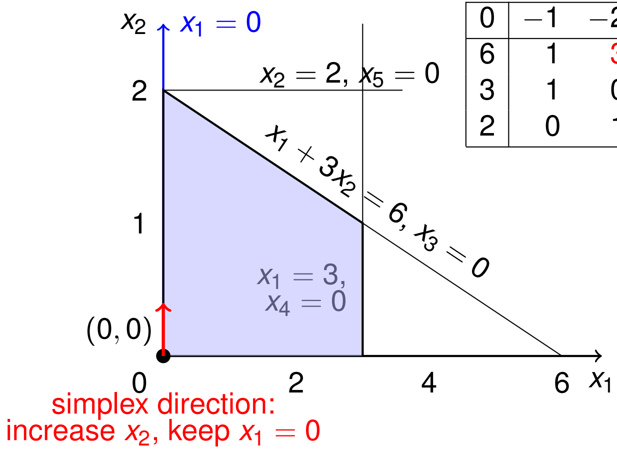
	x_1	x_2	x_3	x_4	x_5
0	-1	-2	0	0	0
6	1	3	1	0	0
3	1	0	0	1	0
2	0	1	0	0	1

Graphing the progress of the algorithm



	x_1	x_2	x_3	x_4	x_5
0	-1	-2	0	0	0
6	1	3	1	0	0
3	1	0	0	1	0
2	0	1	0	0	1

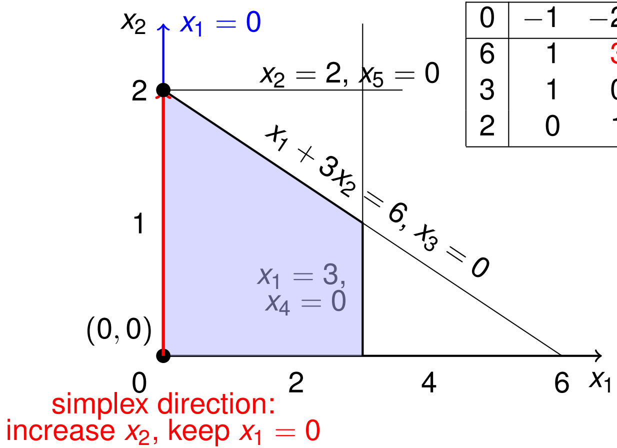
Graphing the progress of the algorithm



	x_1	x_2	x_3	x_4	x_5
0	-1	-2	0	0	0
6	1	3	1	0	0
3	1	0	0	1	0
2	0	1	0	0	1

As we increase x_2, we drive x_3 and x_5 towards 0.

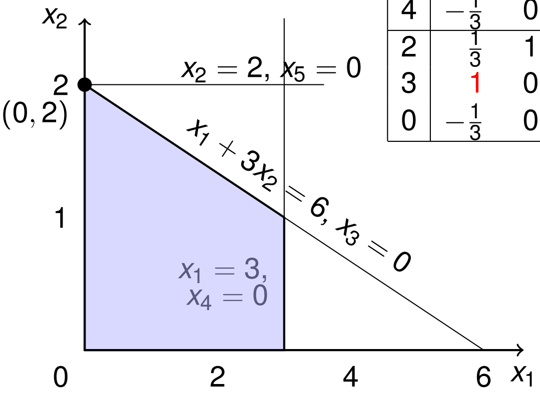
Graphing the progress of the algorithm



	x_1	x_2	x_3	x_4	x_5
0	-1	-2	0	0	0
6	1	3	1	0	0
3	1	0	0	1	0
2	0	1	0	0	1

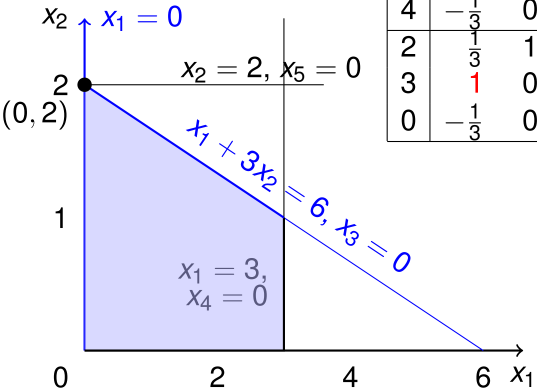
As we increase x_2, we drive x_3 and x_5 towards 0. x_3 and x_5 reach 0 simultaneously, giving $x = (0, 2, 0, 3, 0)$.

The second pivot



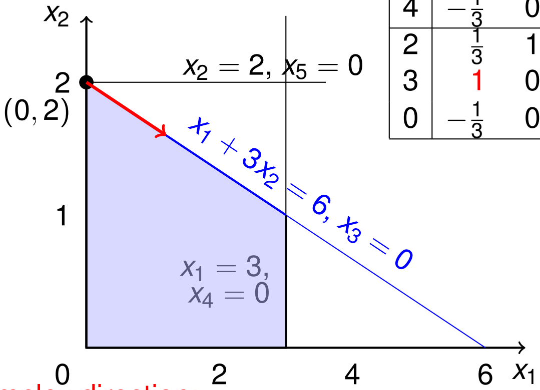
	x_1	x_2	x_3	x_4	x_5
4	$-\frac{1}{3}$	0	$\frac{2}{3}$	0	0
2	$\frac{1}{3}$	1	$\frac{1}{3}$	0	0
3	1	0	0	1	0
0	$-\frac{1}{3}$	0	$-\frac{1}{3}$	0	1

The second pivot



	x_1	x_2	x_3	x_4	x_5
4	$-\frac{1}{3}$	0	$\frac{2}{3}$	0	0
2	$\frac{1}{3}$	1	$\frac{1}{3}$	0	0
3	1	0	0	1	0
0	$-\frac{1}{3}$	0	$-\frac{1}{3}$	0	1

The second pivot

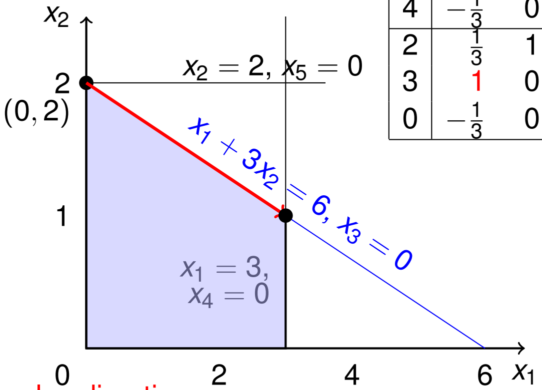


	x_1	x_2	x_3	x_4	x_5
4	$-\frac{1}{3}$	0	$\frac{2}{3}$	0	0
2	$\frac{1}{3}$	1	$\frac{1}{3}$	0	0
3	1	0	0	1	0
0	$-\frac{1}{3}$	0	$-\frac{1}{3}$	0	1

simplex direction:
increase x_1, keep $x_3 = 0$

As we increase x_1, we drive x_2 and x_4 towards 0.

The second pivot

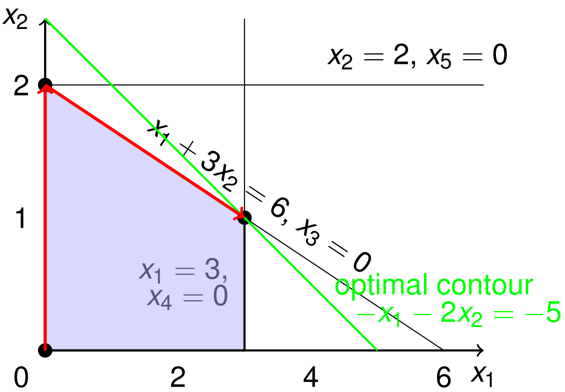


	x_1	x_2	x_3	x_4	x_5
4	$-\frac{1}{3}$	0	$\frac{2}{3}$	0	0
2	$\frac{1}{3}$	1	$\frac{1}{3}$	0	0
3	1	0	0	1	0
0	$-\frac{1}{3}$	0	$-\frac{1}{3}$	0	1

simplex direction:
increase x_1, keep $x_3 = 0$

As we increase x_1, we drive x_2 and x_4 towards 0. x_3 reaches zero first, giving $x = (3, 1, 0, 0, 1)$.

Both iterations



Outline

1 Basic variables equal to zero

2 Solving a degenerate linear optimization problem

3 Cycling

Cycling

It is possible that degeneracy can lead to **cycling**, where we perform a sequence of pivots and end up back where we started.

At each iteration, the point x is not changed, although the basic sequence does change.

For an example, see page 52 in the text by Ecker and Kupferschmid.

Bland's Rule

Cycling can be prevented by modifying the rule for choosing the entering and leaving variables.

For example, we can use Bland's least index rule:

1. *Pivot column:* choose the nonbasic variable with negative cost c_j that has the **least index** j. For example, if the only two nonbasic variables with negative cost have $c_3 = -4$ and $c_9 = -15$, we choose x_3 to enter the basis, because $3 < 9$.
2. *Pivot row:* If there is a tie in the minimum ratio test, choose the variable to leave the basis to be the one with **least index** among those that are involved in the tie. For example, if the minimum ratio test results in a tie between x_5 with a ratio of $8/2$ and x_8 with a ratio of $6/3$, choose x_5 to leave the basis.

Using Bland's rule on the cycling example results in moving away from the degenerate BFS.

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Using Bland's rule on the cycling example results in moving away from the degenerate BFS.

Anti-cycling rules in practice

Anti-cycling pivot rules do not prioritize deceasing the objective function, so they generally perform poorly in practice, and are only used in implementations when cycling is detected.

This is despite the observation that it seems that **most real-world linear optimization problems are degenerate.**