Tutorial 3 - 60016 Operations Research

Simplex Algorithm

Exercise 1. A plant manufactures three types of vehicles: automobiles, trucks and vans, on which the company makes a profit of £ 4000, £ 6000, and £ 3000, respectively, per vehicle.

The plant has three main departments: parts, assembly and finishing operating 120, 100, and 80 hours, respectively, each two-week period. It takes 50, 40, and 30 hours, respectively, to manufacture the parts for automobiles, trucks and vans. Assembly takes 40, 30, and 20 hours, respectively, for an automobile, truck and van. Finishing takes 20, 40, and 10 hours, respectively, for an automobile, truck and van.

How many of each type should the company manufacture in each two-week period to maximise its profits? Formulate a linear program to answer this question and solve it using the Simplex Algorithm.

Solution

x_1	number of trucks manufactured
x_2	number of automobiles manufactured
x_3	number of vans manufactured

$$\max\ y = 6x_1 + 4x_2 + 3x_3$$

subject to:

$$\begin{array}{rcrcrcr} 4x_1 & + & 5x_2 & + & 3x_3 & \leq & 12 \\ 3x_1 & + & 4x_2 & + & 2x_3 & \leq & 10 \\ 4x_1 & + & 2x_2 & + & x_3 & \leq & 8 \end{array}$$

and

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

In standard form, we get the following LP problem:

$$\min\ z = -6x_1 - 4x_2 - 3x_3$$

subject to:

$$\begin{array}{rcrcrcrcrcrcrcr} 4x_1 & + & 5x_2 & + & 3x_3 & + & x_4 & & & & & = & 12 \\ 3x_1 & + & 4x_2 & + & 2x_3 & & & + & x_5 & & & = & 10 \\ 4x_1 & + & 2x_2 & + & x_3 & & & & & + & x_6 & = & 8 \end{array}$$

and

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0, x_6 \geq 0.$$

BV	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	6	4	3	0	0	0	0
x_4	4	5	3	1	0	0	12
x_5	3	4	2	0	1	0	10
x_6	$\boxed{4}$	2	1	0	0	1	8
z	0	1	$\frac{3}{2}$	0	0	$-\frac{3}{2}$	-12
x_4	0	3	$\boxed{2}$	1	0	-1	4
x_5	0	$\frac{5}{2}$	$\frac{5}{4}$	0	1	$-\frac{3}{4}$	4
x_1	1	$\frac{1}{2}$	$\frac{1}{4}$	0	0	$\frac{1}{4}$	2
z	0	$-\frac{5}{4}$	0	$-\frac{3}{4}$	0	$-\frac{3}{4}$	-15
x_3	0	$\frac{3}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	2
x_5	0	$\frac{5}{8}$	0	$-\frac{5}{8}$	1	$-\frac{1}{8}$	$\frac{3}{2}$
x_1	1	$\frac{1}{8}$	0	$-\frac{1}{8}$	0	$\frac{3}{8}$	$\frac{3}{2}$

The optimal BFS to the (original max) problem is thus given by:

$$(z, x_1, x_2, x_3, x_4, x_5, x_6) = \left(15, \frac{3}{2}, 0, 2, 0, \frac{3}{2}, 0\right)$$

i.e. the maximal profit is £ 15,000.

Exercise 2. Consider the following LP problem:

$$\max\ y = 2x_1 + 3x_2$$

subject to:

$$\begin{array}{rcrcr} x_1 & + & x_2 & \leq & 4 \\ -x_1 & + & 2x_2 & \geq & -1 \end{array}$$

and

$$x_1 \geq 0, x_2 \geq 0.$$

Solve this problem using the Simplex Algorithm.

Solution

$$\min\ z = -2x_1 - 3x_2$$

subject to

$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & + & x_3 & & & = & 4 \\ x_1 & - & 2x_2 & & & + & x_4 & = & 1 \end{array}$$

and

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0.$$

BV	x_1	x_2	x_3	x_4	RHS
z	2	3	0	0	0
x_3	1	$\boxed{1}$	1	0	4
x_4	1	-2	0	1	1
z	-1	0	-3	0	-12
x_2	1	1	1	0	4
x_4	3	0	2	1	9

The optimal BFS (for the standard min problem) is thus given as:

$$(z, x_1, x_2, x_3, x_4) = (-12, 0, 4, 0, 9).$$

Exercise 3. Solve the following LP problem using the simplex algorithm:

$$\max\ y = 2x_1 - 3x_2 + x_3$$

subject to:

$$\begin{array}{rcrcrcl} 3x_1 & + & 6x_2 & + & x_3 & \leq & 6 \\ 4x_1 & + & 2x_2 & + & x_3 & \leq & 4 \\ x_1 & - & x_2 & + & x_3 & \leq & 3 \end{array}$$

and

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

Solution

$$\min\ z = -2x_1 + 3x_2 - x_3$$

subject to:

$$\begin{array}{rcrcrcrcrcrcl} 3x_1 & + & 6x_2 & + & x_3 & + & x_4 & & & & & = & 6 \\ 4x_1 & + & 2x_2 & + & x_3 & & & + & x_5 & & & = & 4 \\ x_1 & - & x_2 & + & x_3 & & & & & + & x_6 & = & 3 \end{array}$$

and

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0, x_6 \geq 0.$$

BV	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	2	-3	1	0	0	0	0
x_4	3	6	1	1	0	0	6
x_5	$\boxed{4}$	2	1	0	1	0	4
x_6	1	-1	1	0	0	1	3
z	0	-4	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	-2
x_4	0	$\frac{9}{2}$	$\frac{1}{4}$	1	$-\frac{3}{4}$	0	3
x_1	1	$\frac{1}{2}$	$\frac{1}{4}$	0	$\frac{1}{4}$	0	1
x_6	0	$-\frac{3}{2}$	$\boxed{\frac{3}{4}}$	0	$-\frac{1}{4}$	1	2
z	0	-3	0	0	$-\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{10}{3}$
x_4	0	5	0	1	$-\frac{2}{3}$	$-\frac{1}{3}$	$\frac{7}{3}$
x_1	1	1	0	0	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$
x_3	0	-2	1	0	$-\frac{1}{3}$	$\frac{4}{3}$	$\frac{8}{3}$

The optimum BS is thus given by:

$$(x_1, x_2, x_3, x_4, x_5, x_6) = \left(\frac{1}{3}, 0, \frac{8}{3}, \frac{7}{3}, 0, 0\right).$$

Exercise 4. Consider the linear programming problem:

$$\max y = \ 10x_1 - 57x_2 - 9x_3 - 24x_4$$

subject to

$$\begin{array}{rcrcrcrcr} 0.5x_1 & - & 5.5x_2 & - & 2.5x_3 & + & 9x_4 & \leq & 0 \\ 0.5x_1 & - & 1.5x_2 & - & 0.5x_3 & + & x_4 & \leq & 0 \\ -x_1 & & & & & & & \geq & -1 \end{array}$$

and

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0,$$

1. Write the problem in standard form.
2. Write the simplex tableau for an initial basis composed by all slack variables.
3. Starting from the all-slack basis, the simplex algorithm with standard pivoting rules arrives after 4 iterations to the following intermediate tableau:

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z	-20	-9			10.50	-70.50		0
x_3	-2	4	1		0.50	-4.50		0
x_4	-0.50	0.50		1	0.25	-1.25		0
x_7	1						1	1

   Continue the simplex algorithm and show that the algorithm cycles.

4. Starting from the same intermediate tableau used at the previous question, solve the problem using the Simplex algorithm with Bland's rule.
5. Does index set $I = \{2, 3, 4\}$ define a valid basis for this LP?

Solution 1. In standard form, we get the following LP problem:

$$\min z = \ -10x_1 + 57x_2 + 9x_3 + 24x_4$$

subject to:

$$\begin{array}{rcrcrcrcrcrcrcr} 0.5x_1 & - & 5.5x_2 & - & 2.5x_3 & + & 9x_4 & + & x_5 & & & & & = & 0 \\ 0.5x_1 & - & 1.5x_2 & - & 0.5x_3 & + & x_4 & & & + & x_6 & & & = & 0 \\ x_1 & & & & & & & & & & & + & x_7 & = & 1 \end{array}$$

and

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0, x_6 \geq 0, x_7 \geq 0.$$

2.

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z	10	-57	-9	-24	0	0	0	0
x_5	$\boxed{0.5}$	-5.5	-2.5	9	1	0	0	0
x_6	0.5	-1.5	-0.5	1	0	1	0	0
x_7	1	0	0	0	0	0	1	1

3. Solving with the Simplex Algorithm using standard pivoting rules we get:

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z	10	-57	-9	-24	0	0	0	0
x_5	$\boxed{0.5}$	-5.5	-2.5	9	1	0	0	0
x_6	0.5	-1.5	-0.5	1	0	1	0	0
x_7	1	0	0	0	0	0	1	1
z		53	41	-204	-20			0
x_1	1	-11	-5	18	2			0
x_6		$\boxed{4}$	2	-8	-1	1		0
x_7		11	5	-18	-2		1	1
z			14.50	-98	-6.75	-13.25		0
x_1	1		$\boxed{0.50}$	-4	-0.75	2.75		0
x_2		1	0.50	-2	-0.25	0.25		0
x_7			-0.50	4	0.75	-2.75	1	1
z	-29			18	15	-93		0
x_3	2		1	-8	-1.50	5.50		0
x_2	-1	1		$\boxed{2}$	0.50	-2.50		0
x_7	1						1	1
z	-20	-9			10.50	-70.50		0
x_3	-2	4	1		$\boxed{0.50}$	-4.50		0
x_4	-0.50	0.50		1	0.25	-1.25		0
x_7	1						1	1
z	22	-93	-21			24		0
x_5	-4	8	2		1	-9		0
x_4	0.50	-1.50	-0.50	1		$\boxed{1}$		0
x_7	1						1	1
z	10	-57	-9	-24				0
x_5	0.50	-5.50	-2.50	9	1			0
x_6	0.50	-1.50	-0.50	1		1		0
x_7	1						1	1

and since the last tableau returns to the initial basis of all slacks, the algorithm is cycling.

4. Solving with the Simplex Algorithm using Bland's rule, the tableaus are identical until we reach the index set $I = \{5, 4, 7\}$, where we choose as pivot column x_1 instead of x_6, since it is the column with smallest index such that

$\beta_q > 0$. Thus, we have:

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z	22	-93	-21			24		0
x_5	-4	8	2		1	-9		0
x_4	$\boxed{0.50}$	-1.50	-0.50	1		1		0
x_7	1						1	1
z		-27	1	-44		-20		0
x_5		-4	-2	8	1	-1		0
x_1	1	-3	-1	2		2		0
x_7		3	$\boxed{1}$	-2		-2	1	1
z		-30		-42		-18	-1	-1
x_5		2		4	1	-5	2	2
x_1	1						1	1
x_3		3	1	-2		-2	1	1

The optimal BFS to the (original max) problem is thus given by:

$$(y^*, x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*, x_7^*) = (1, 1, 0, 1, 0, 2, 0, 0)$$

since $y^* = -z^*$.

5. For the index set $I = \{2, 3, 4\}$, we would have a basis matrix

$$B = \begin{bmatrix} -5.5 & -2.5 & 9 \\ -1.5 & -0.5 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

which is singular. Since this is a square matrix, the columns associated to the indexes in I are linearly dependent. Therefore, I does not define a valid basis.