Tutorial 1 - 60016 Operations Research

Linear Programming

Exercise 1. *Consider the following linear programming (LP) problem:*

$$
\begin{array}{ll}
\textit{maximise} & y = 2x_1 + x_2 \\
\textit{subject to} & x_1 - 4x_2 \leq 1 \\
& -x_1 - 5x_2 \leq -3 \\
& x_1, x_2 \geq 0
\end{array}
$$

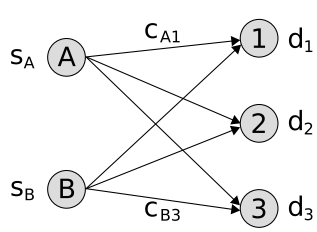
Write the LP in standard form.

Solution

$$
\begin{array}{ll}
\text{minimise} & z = -2x_1 - x_2 \\
\text{subject to} & x_1 - 4x_2 + s_1 = 1 \\
& x_1 + 5x_2 - s_2 = 3 \\
& x_1, x_2, s_1, s_2 \geq 0
\end{array}
$$

In the second constraint we first multipled both sides by -1 to ensure that the right-hand coefficient is positive and then introduced the excess variable s_2 to obtain an equality. Once the optimal value z^* is found, we will set $y^* = -z^*$.

Exercise 2 (Transportation and Trans-shipment Problems)**.** *In this problem*[1]*, an IT company produces laptops at two factories A and B. In factory A, s_A laptops are produced per year, whereas the output of factory B is s_B laptops/year. The company owns the three stores 1, 2 and 3. At store 1, d_1 laptops are sold every year. The corresponding numbers for stores 2 and 3 are d_2 and d_3, respectively. The costs of shipping one laptop from factory $i \in \{A, B\}$ to store $j \in \{1, 2, 3\}$ is c_{ij} pounds. Assume that $s_A + s_B \geq d_1 + d_2 + d_3$, i.e., the demand of all stores can be satisfied.*



1. *Assume that $(s_A, s_B) = (3, 3)$ and $(d_1, d_2, d_3) = (2, 2, 2)$. Furthermore, supposed that matrix (c_{ij}) is as follows:*

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix},$$

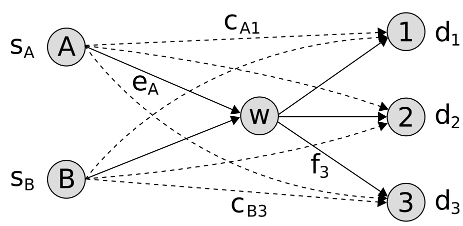
where the first row corresponds to factory A, the second row to factory B, and the columns correspond to the different stores. The costs c_{B2}, for example, are one pound per laptop.

How should the laptops be shipped from the two factories to the three stores so that the overall shipping costs are minimized?

[1]The term trans-shipment refers to the shipment of goods or containers to an intermediate destination, then to yet another destination.

2. *Formulate the optimization model corresponding to the previous question. Use the general parameters (i.e.,* s_A*,* d_2*,* c_{B3} *etc.) instead of concrete values.*

3. *Imagine the company established a warehouse as shown in the following figure. The company can either directly serve its stores, resulting in costs* c_{ij} *as before, or ship laptops to the warehouse (at costs* e_i *per laptop) and subsequently to the stores (at costs* f_j *per laptop). Moreover, at most* $\overline{x}_{ij}$ *laptops can be shipped from factory* $i \in \{A, B\}$ *to store* $j \in \{1, 2, 3\}$*, while the shipments through the warehouse are unrestricted. Formulate the cost minimization problem, using general parameters as in the previous question.*



Solution

1. We see that factory A can serve stores 1 and 3 at low costs, whereas factory B is better for serving store 2. Given the particular cost matrix, A solely serves stores 1 and 3 in every optimal solution, and B satisfies the remaining demands. All such solutions result in total costs of 7 pounds per year.

2. We introduce variables x_{ij}, $i \in \{A, B\}$ and $j \in \{1, 2, 3\}$ with the meaning that x_{ij} laptops should be shipped from factory i to store j. Then the model becomes:

$$\min \sum_{i \in \{A,B\}} \sum_{j \in \{1,2,3\}} c_{ij} x_{ij}$$

subject to

$$\begin{aligned} x_{Aj} + x_{Bj} &= d_j && \text{for } j \in \{1, 2, 3\} \\ x_{i1} + x_{i2} + x_{i3} &\leq s_i && \text{for } i \in \{A, B\} \\ x_{ij} &\geq 0 && \text{for } i \in \{A, B\}, \, j \in \{1, 2, 3\}. \end{aligned}$$

The objective function minimizes the overall costs. The first equation makes sure that each store receives exactly the desired amount of laptops. The second constraint ensures that a factory does not ship more laptops than it actually produces. The last constraint forbids negative amounts of laptops to be shipped.

3. The model follows exactly the same logic as before. Again we have variables x_{ij} with $i \in \{A, B\}$ and $j \in \{1, 2, 3\}$ for direct trans-shipments from factories to stores. Additionally we introduce variables x_{iw} with $i \in \{A, B\}$ for shipments to the warehouse and x_{wj} with $j \in \{1, 2, 3\}$ for shipments from the warehouse. The model then becomes:

$$\min \sum_{i \in \{A,B\}} \sum_{j \in \{1,2,3\}} c_{ij} x_{ij} + \sum_{i \in \{A,B\}} e_i x_{iw} + \sum_{j \in \{1,2,3\}} f_j x_{wj}$$

subject to

$$
\begin{aligned}
x_{Aj} + x_{Bj} + x_{wj} &= d_j && \text{for } j \in \{1,2,3\} \\
x_{i1} + x_{i2} + x_{i3} + x_{iw} &\le s_i && \text{for } i \in \{A,B\} \\
x_{Aw} + x_{Bw} &= x_{w1} + x_{w2} + x_{w3} && \\
x_{ij} &\le \overline{x}_{ij} && \text{for } i \in \{A,B\}\,,\ j \in \{1,2,3\} \\
x_{ij} &\ge 0 && \text{for } i \in \{A,B\}\,,\ j \in \{1,2,3\} \\
x_{iw} &\ge 0 && \text{for } i \in \{A,B\} \\
x_{wj} &\ge 0 && \text{for } j \in \{1,2,3\}\,.
\end{aligned}
$$

Note the third constraint, which is often called "flow conservation": every laptop that is shipped to the warehouse must also leave the warehouse to one of the stores.

Exercise 3. *Find a solution to the following system of linear equations (if one exists):*

$$
\begin{array}{rcrcrcrcr}
x_1 & - & x_2 & + & x_3 & + & x_4 & = & 6 \\
3x_1 & + & x_2 & + & 2x_3 & + & 2x_4 & = & 12 \\
x_1 & + & 4x_2 & - & x_3 & - & 4x_4 & = & 3 \\
2x_1 & & & - & 2x_3 & + & 3x_4 & = & -8
\end{array}
$$

1. *What is the rank of A?*
2. *If this system is used to define the feasible set for a LP, would this LP be feasible?*
3. *How would the answer change if the LP is assumed to be in standard form?*

Solution 1. The rank of A is 4, i.e., there exists a unique solution. This can be directly verified by Gaussian elimination (check!). The solution to $A\,x = b$ is found to be:

$$
\begin{aligned}
x_1 &= 3 \\
x_2 &= -1 \\
x_3 &= 4 \\
x_4 &= -2
\end{aligned}
$$

2. No specific assumption is taken on the bounds of the decision variables. Therefore, the solution to $A\,x = b$ found in the previous question, even though it has negative values, would be in the feasible set. The resulting LP would be feasible, even though a degenerate one, since its feasible set has a unique point.

3. If the LP is in standard form, one would first need to change the sign of the last equation by multiplying both sides by -1. After this operation, the solution of the system of linear equations is the same. However, the assumption of a standard form implies that the decision variables must be non-negative, therefore $(3, -1, 4, -2)$ would not be anymore in the feasible set. Since this system has a unique solution, we conclude that the feasible set would be empty.

Exercise 4 (Shift Scheduling Problem *(Adapted from Exam 2016, Q3b)*)**.** *A Police Department uses work shifts in which officers work 5 out of the 7 days of the week, with 2 successive days off. For example, a shift might work Sunday through Thursday and then have Friday and Saturday off. The following constraints are in place:*

- *At least 6 officers must be on duty Monday, Tuesday, Wednesday, and Thursdays;*
- *At least 10 officers are required on Friday and Saturday;*
- *Exactly 8 officers are needed on Sunday*

The Police Department wants to meet these staffing needs incurring the minimum cost. For each day on duty an officer receives £100, except on Saturdays and Sundays where the pay is £80 in each day.

Assuming that the variables can be approximately treated as continuous quantities, formulate a shift scheduling program to minimize the cost for the Police Department.

Solution The decision variables are

$x_1 :=$ number of officers working 5 days starting on Monday
$x_2 :=$ number of officers working 5 days starting on Tuesday
$\vdots$
$x_7 :=$ number of officers working 5 days starting on Sunday

Linear programming formulation:

$$\min z = 500x_1 + 480(x_2 + x_7) + 460(x_3 + x_4 + x_5 + x_6)$$

subject to

$$\begin{aligned}
x_1 + x_4 + x_5 + x_6 + x_7 &\geq 6 \quad (Monday) \\
x_1 + x_2 + x_5 + x_6 + x_7 &\geq 6 \quad (Tuesday) \\
x_1 + x_2 + x_3 + x_6 + x_7 &\geq 6 \quad (Wednesday) \\
x_1 + x_2 + x_3 + x_4 + x_7 &\geq 6 \quad (Thursday) \\
x_1 + x_2 + x_3 + x_4 + x_5 &\geq 10 \quad (Friday) \\
x_2 + x_3 + x_4 + x_5 + x_6 &\geq 10 \quad (Saturday) \\
x_3 + x_4 + x_5 + x_6 + x_7 &= 8 \quad (Sunday) \\
x_j \geq 0, \; j &= 1, \ldots, 7
\end{aligned}$$

Exercise 5. *Consider a rectangular matrix with n columns and m rows, where $n > m$. Assume rows to be linearly independent, does this mean that also columns are linearly independent?*

Solution No, for example the matrix

$$\begin{bmatrix} 1 & 2 & 4 \\ 7 & 2 & 4 \end{bmatrix}$$

has just two linearly dependent columns, even though the rows are linearly independent. In general, if the rank is $m < n$ there will be at least m independent columns, but not necessarily n.

Exercise 6. *Indicate, among the following problems, which ones you think can be solved using linear programming (i.e., linear objective, linear constraints, continuous variables). Justify your answers (you are not asked to write the actual optimization program).*

- Schedule tasks such that they do not overlap and precedences are met.

  Solution *Yes, this is a LP. Times are continuous variables. Precedences can be expressed as linear constraints between start times, e.g., $t_{j+1} \geq t_j + d$ where d is a given constant indicating the minimum time that must elapse between the start time of activity j and the start of activity $j+1$.*

- Choose the next move in a chess game.

Solution *No, there is a discrete set of possibilities, so we need discrete decision variables. This problem may be tackled using integer linear programming, but not linear programming.*

- Optimise a linear function $f(x)$ where x takes values in a triangular domain.

Solution *Yes, a triangle is the intersection of three linear inequalities.*

- Optimise a linear function $f(x)$ where x takes values in two disjoint triangular domains.

Solution *No. If the two triangles are disjoint, we cannot write a system of linear inequalities that represents the two regions at the same time. This would require a way to express that either we choose the point from one triangular domain or from the other. This requires boolean decision variables.*

- Find the optimal route on the London tube to a given destination.

Solution *No, we are asked again to pick from a discrete set of possible routes (albeit a very large one!). This cannot be done using linear programming, but can be done using integer linear programming, assuming that the admissible routes are finite in number (i.e. no infinite cycles).*

- Find an $a \geq 0$ such that the curve $f(x) = a^3 x$ has $f(2) \geq 7$.

Solution *Yes. Taking the logarithm of $f(x)$ we can write the condition*

$$3 \log a + \log 2 \geq \log 7$$

Calling $x_1 = \log a$ this becomes the linear constraint $3x_1 \geq \log 7 - \log 2$, where x_1 is a free variable since $a \geq 0$. Of course, this is a very simple problem that can be solved by hand, but similar considerations apply to fitting a curve that passes through a set of points.

- Fit the probability density function of a Gaussian (i.e., normal) distribution to data.

Solution *No. The normal p.d.f. requires two parameters, mean μ and standard deviation σ. The probability density function of a Gaussian distribution is*

$$\Pr[X = x] = \frac{1}{\sqrt{2\sigma^2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

An optimization program that searches for σ and μ using this formula will be non-linear, since there is no simple transformation to get rid of all the non-linearities in the expression (squaring, square root, exponential, ratios).

- Find a set of probabilities $\Pr[X = k]$, $1 \leq k \leq n$, for a discrete random variable X such that the mean satisfies $E[X] = m$, where m is a given constant.

Solution *Yes,* $E[X] = \sum_{k=1}^{n} k \Pr[X = k]$*, therefore we can impose the mean using a linear constraint. The linear program would look like*

$$
\begin{aligned}
&\min z = 1 \\
s.t. \quad & \\
&\sum_{k=1}^{n} k x_k = m, \\
&\sum_{k=1}^{n} x_k = 1, \\
&x_k \geq 0, \qquad \forall k
\end{aligned}
$$

where $x_k = \Pr[X = k]$*. Here we use a constant objective since we are not attempting to minimise or maximise anything in particular. We just seek for a feasible allocation of the variables* x_k*. A problem where we just ask for a feasible assignment of the variables is called a* feasibility problem.