60016 OPERATIONS RESEARCH

Cutting Plane Algorithms

23 November 2020

Problem

How to solve ILPs?

- Can we reuse or extend LP algorithms?
  - Yes: cutting plane algorithm
- Can we define ILP-specific algorithms?
  - Yes: branch-and-bound algorithm

Other algorithms exist

- branch-and-cut
- genetic algorithms
- simulated annealing
- ...

Key Idea: Continuous Relaxation

LP relaxation: LP program obtained by replacing all integer variables $x_j \in \mathbb{N}_0$ in a ILP with continuous variables $x_j \in \mathbb{R}$.

- LP relaxation has better or same optimal value as ILP!

Outline of solution procedure:

- Solve a LP relaxation.
  - Contains all originally feasible solutions, plus others.
- If optimal solution is integer, we are done.
- Otherwise, *tighten* the LP relaxation and repeat.

Tightening: restrict feasible set of the LP relaxation without excluding the optimum solution of the ILP.

Cutting Plane Algorithm

Step 0. Write the ILP in standard form.

Step 1. Solve the LP relaxation.

Step 2. If the resulting optimal solution x^* is integer, stop $\Rightarrow$ optimal solution found.

Step 3. Generate a cut, a constraint satisfied by all feasible integer solutions, but not by previous solution x^* with non-integer components.

Step 4. Add cut to the LP relaxation and go back to Step 1.

The algorithm terminates after finite number of iterations. The resulting x^* is integer and optimal.

Example

Consider the following problem:

$$\max \ y = 5x_1 + 8x_2$$

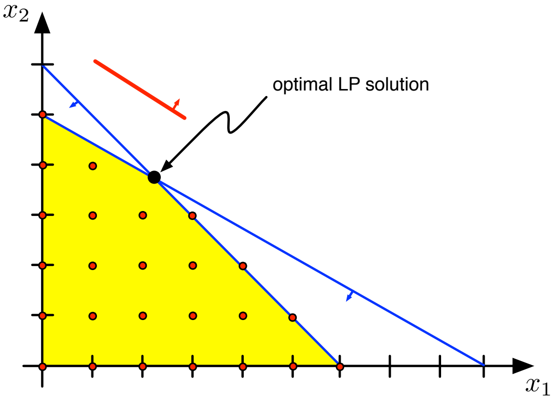
subject to

$$\begin{aligned} x_1 + x_2 &\leq 6 \\ 5x_1 + 9x_2 &\leq 45 \\ x_1, x_2 &\geq 0 \\ x_1, x_2 &\in \mathbb{N}_0. \end{aligned}$$

Step 0. Rewrite in standard form.

Example

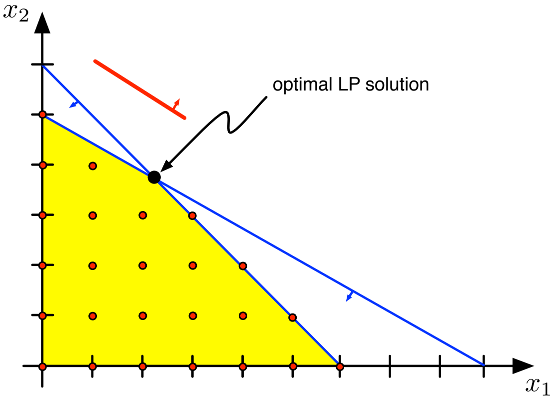
Step 1. Solve the LP relaxation.



Sanity Check. For maximisation, how is the optimal value of the LP relaxation y^*_{LP} related to the optimal value of the ILP y^*_{ILP}?

Example

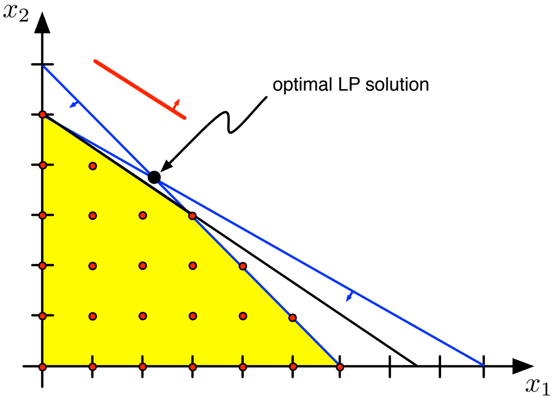
Step 2. If the resulting optimal solution x^* is integer, stop.



Resulting solution is $x^* = (2.25, 3.75)$ and hence *not* integer.

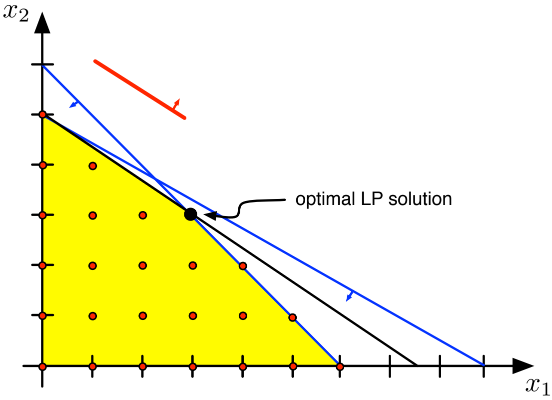
Example

Step 3. Generate a cut, in this example $2x_1 + 3x_2 \leq 15$.



Example

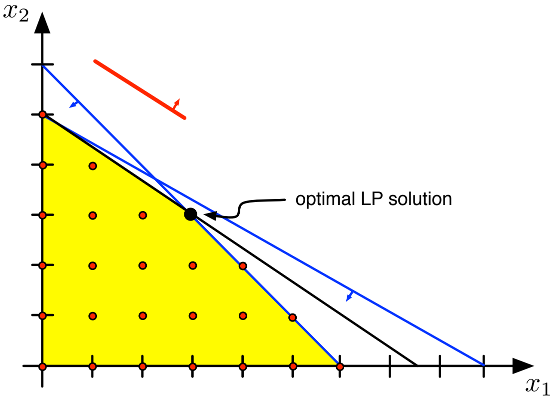
Step 4. Add cut to the LP relaxation and go back to Step 1.



New optimal solution is $x^* = (3, 3)$.

Example

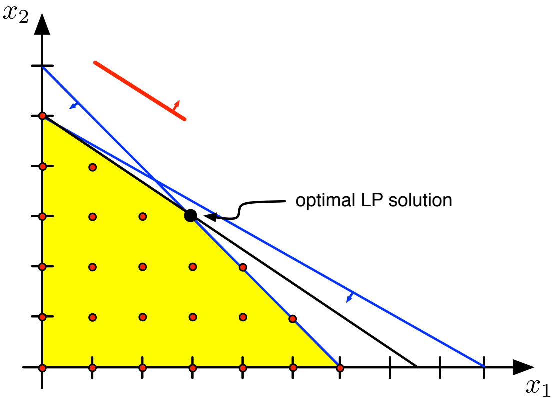
Step 2. If the resulting optimal solution x^* is integer, stop.



$x^* = (3, 3)$ is integer $\Rightarrow$ optimal solution found.

Example

Remark. The cut only removed non-integer solutions. Cuts never cut off feasible solutions of the original ILP!



Importance of cutting planes

Bixby & Rothberg (*Ann Oper Res*, 2007)

Disabled cut	Year	Degradation
Gomory mixed-integer	**1960**	**2.52X**
Mixed-integer rounding	2001	1.83X
Knapsack cover	**1983**	**1.40X**
Flow cover	1985	1.22X
Implied bound	1991	1.19X
Flow path	1985	1.04X
Clique	1983	1.02X
GUB cover	1998	1.02X

Mean performance degradation from turning off various cutting planes in CPLEX 8.0

C343 studies **Gomory mixed-integer** and **knapsack cover** cuts.

Gomory Cut

- Previous example illustrated a Gomory cut.
- Assume $x_1, \dots, x_n \geq 0$ and integer.
- Let $\lfloor c \rfloor = \max\{a \in \mathbb{Z} : a \leq c\}$ be the floor function
  - $\lfloor -2.7 \rfloor = -3$
  - $\lfloor 3.2 \rfloor = \lfloor 3 \rfloor = 3$
- Thus, any real number c can be written as $c = \lfloor c \rfloor + (c - \lfloor c \rfloor)$

Gomory Cut

Setup: we computed x^* non-integer, and we know it to live on the boundary of the polytope.

- We show how to construct a Gomory Cut for

$$a_1 x_1 + \ldots + a_n x_n = b,$$

  where $a_j, b \in \mathbb{R}$ (not necessarily integer).

- The constraint can be written as

$$(\lfloor a_1 \rfloor + \underbrace{(a_1 - \lfloor a_1 \rfloor)}_{f_1})x_1 + \ldots + (\lfloor a_n \rfloor + \underbrace{(a_n - \lfloor a_n \rfloor)}_{f_n})x_n$$
$$= \lfloor b \rfloor + \underbrace{(b - \lfloor b \rfloor)}_{f},$$

- Rearranging terms we get

$$f_1 x_1 + \ldots + f_n x_n - f = \lfloor b \rfloor - \lfloor a_1 \rfloor x_1 - \ldots - \lfloor a_n \rfloor x_n.$$

Gomory Cut

Theorem. For all $x \in \mathbb{N}_0^n$ satisfying $a_1 x_1 + \cdots + a_n x_n = b$, it is

$$f_1 x_1 + \ldots + f_n x_n \geq f.$$

Proof. Consider

$$f_1 x_1 + \ldots + f_n x_n - f = \lfloor b \rfloor - \lfloor a_1 \rfloor x_1 - \ldots - \lfloor a_n \rfloor x_n.$$

- As $x \in \mathbb{N}_0^n$, right-hand side is integer.
- Thus left-hand side (LHS) must be an integer too.
- Since $x \geq 0$, $0 \leq f_i < 1$, $\forall i$

$$f_1 x_1 + \cdots + f_n x_n - f \geq 0 + \cdots + 0 - f > -1$$

- Since LHS can only take integer values, it can only be ≥ 0
- Therefore $f_1 x_1 + \ldots + f_n x_n - f \geq 0$

Gomory Cut

Suppose Step 1 of our cutting plane algorithm gives a non-integer x^*. Then there is a row in the last Simplex tableau that has

$$x_i^* + \sum_{j \notin I} y_{ij} x_j^* = y_{i0} \qquad \text{(Row)}$$

with $y_{i0} \notin \mathbb{N}_0$. Note: the summation is on the non-basic variables.
Gomory Cut. Setting $f_j := y_{ij} - \lfloor y_{ij} \rfloor$, $f := y_{i0} - \lfloor y_{i0} \rfloor$:

$$\sum_{j \notin I} f_j x_j \geq f. \qquad \text{(GC)}$$

(GC) is violated by a non-integer x^* since $x_j^* = 0$ if $j \notin I$, thus

$$\sum_{j \notin I} f_j x_j^* = 0 < f$$

Sanity Check. What if no row in the last tableau satisfies (Row)?

Gomory Cut Example [1/12]

Consider the following problem:

$$\max \ y = 3x_1 + 4x_2$$

subject to

$$\begin{aligned} \frac{2}{5}x_1 + x_2 &\leq 3 \\ \frac{2}{5}x_1 - \frac{2}{5}x_2 &\leq 1 \\ x_1, x_2 &\geq 0 \\ x_1, x_2 &\in \mathbb{N}_0. \end{aligned}$$

Gomory Cut Example [2/12]

Step 1. Convert maximisation objective into minimisation.

$$\min \; z = -3x_1 - 4x_2$$

subject to

$$\begin{aligned} \frac{2}{5}x_1 + x_2 &\leq 3 \\ \frac{2}{5}x_1 - \frac{2}{5}x_2 &\leq 1 \\ x_1, x_2 &\geq 0 \\ x_1, x_2 &\in \mathbb{N}_0. \end{aligned}$$

Gomory Cut Example [3/12]

Step 1. Scale the equations of the problem.

$$\min\ z = -3x_1 - 4x_2$$

subject to

$$\begin{aligned} \frac{2}{5}x_1 + x_2 &\le 3 \qquad (*5) \\ \frac{2}{5}x_1 - \frac{2}{5}x_2 &\le 1 \qquad (*5) \\ x_1, x_2 &\ge 0 \\ x_1, x_2 &\in \mathbb{N}_0. \end{aligned}$$

Gomory Cut Example [4/12]

Step 1. Scale the equations of the problem.

$$\min \; z = -3x_1 - 4x_2$$

subject to

$$\begin{aligned} 2x_1 + 5x_2 &\leq 15 \\ 2x_1 - 2x_2 &\leq 5 \\ x_1, x_2 &\geq 0 \\ x_1, x_2 &\in \mathbb{N}_0. \end{aligned}$$

Gomory Cut Example [5/12]

Step 1. Insert integer slack variables.

$$\min\ z = -3x_1 - 4x_2$$

subject to

$$\begin{aligned}
2x_1 + 5x_2 + x_3 \qquad &= 15 \\
2x_1 - 2x_2 \qquad + x_4 &= 5 \\
x_1, x_2, x_3, x_4 &\geq 0 \\
x_1, x_2, x_3, x_4 &\in \mathbb{N}_0.
\end{aligned}$$

Gomory Cut Example [6/12]

Step 1. Solve LP relaxation of problem.

BV	x_1	x_2	x_3	x_4	RHS
z	3	4			0
x_3	2	5	1		15
x_4	2	-2		1	5

Gomory Cut Example [7/12]

Step 1. Solve LP relaxation of problem.

BV	x_1	x_2	x_3	x_4	RHS
z	3	4			0
x_3	2	5	1		15
x_4	2	-2		1	5

The optimal solution has the tableau:

BV	x_1	x_2	x_3	x_4	RHS
z			-1	$-\frac{1}{2}$	$-\frac{35}{2}$
x_2		1	$\frac{1}{7}$	$-\frac{1}{7}$	$\frac{10}{7}$
x_1	1		$\frac{1}{7}$	$\frac{5}{14}$	$\frac{55}{14}$

Step 2. Solution is not integer, go to Step 3.

Gomory Cut Example [8/12]

Step 3. Generate cut based, e.g., on x_1 row.

$$x_1 + \frac{1}{7}x_3 + \frac{5}{14}x_4 = \frac{55}{14}$$

- $f_1 = 1 - \lfloor 1 \rfloor = 0$ (basic, does not appear in GC)
- $f_3 = \frac{1}{7}$ (non-basic)
- $f_4 = \frac{5}{14}$ (non-basic)
- $f = \frac{55}{14} - \lfloor \frac{55}{14} \rfloor = \frac{13}{14}$

Gomory Cut (GC1):

$$\frac{1}{7}x_3 + \frac{5}{14}x_4 \geq \frac{13}{14} \implies 2x_3 + 5x_4 \geq 13.$$

Q: how to write in original variables?

Gomory Cut Example [9/12]

Step 4. Add cut to the LP relaxation and go back to Step 1.

Standardise (GC1) introducing excess $x_5 \geq 0$:

$$2x_3 + 5x_4 - x_5 = 13.$$

LP relaxation solution is $x_3^* = x_4^* = 0 \Rightarrow$ (GC1) is infeasible!

We need to solve a problem similar to Simplex Phase 1 to find an initial BFS for Step 1, thus we add the artificial variable ξ_1:

$$2x_3 + 5x_4 - x_5 + \xi_1 = 13.$$

Sanity Check. The LP relaxation solution is now infeasible. Is this typical?

Gomory Cut Example [10/12]

Step 1.

$$\zeta = \xi_1 = 13 - 2x_3 - 5x_4 + x_5$$

BV	x_1	x_2	x_3	x_4	x_5	ξ_1	RHS
ζ			2	5	-1		13
x_2		1	$\frac{1}{7}$	$-\frac{1}{7}$			$\frac{10}{7}$
x_1	1		$\frac{1}{7}$	$\frac{5}{14}$			$\frac{55}{14}$
ξ_1			2	$\boxed{5}$	-1	1	13

Pivot on (x_4, ξ_1) based on reduced costs of ζ.

Gomory Cut Example [11/12]

Step 1. After removing both ζ and ξ_1, add z back to the basic representation, and solve the new LP relaxation (Simplex Phase 2).

BV	x_1	x_2	x_3	x_4	x_5	RHS
z			-1		$-\frac{1}{10}$	$-\frac{81}{5}$
x_2		1	$\frac{1}{5}$		-$\frac{1}{70}$	$\frac{9}{5}$
x_1	1				$\frac{1}{14}$	3
x_4			$\frac{2}{5}$	1	$-\frac{1}{5}$	$\frac{13}{5}$

Solution optimal; Simplex stops.

Step 2. Solution is not integer, go to Step 3.

Gomory Cut Example [12/12]

Step 3. Generate cut based, e.g., on x_2 row.

$$x_2 + \frac{1}{5}x_3 - \frac{1}{70}x_5 = \frac{9}{5}$$

- $f_2 = 0$ (basic, does not appear in GC)
- $f_3 = \frac{1}{5}$ (non-basic)
- $f_5 = -\frac{1}{70} - \lfloor -\frac{1}{70} \rfloor = -\frac{1}{70} + 1 = \frac{69}{70}$ (non-basic)
- $f = \frac{9}{5} - \lfloor \frac{9}{5} \rfloor = \frac{9}{5} - 1 = \frac{4}{5}$

Gomory Cut (GC2):

$$\frac{1}{5}x_3 + \frac{69}{70}x_5 \geq \frac{4}{5}.$$

Outlook on Gomory Cuts

- Developed in 1950's and considered impractical for 40 years due to: Poor convergence properties, saturation, bad numerical behavior, etc.
- A very important paper published in the late 1990s changed the perception of Gomory cuts:
  - Balas, Ceria, Cornuéjols, Natraj. Gomory cuts revisited. *Operations Research Letters*, 1996.
- The strategies recommended in this paper contributed to a big jump in the capability of MILP solvers.

Example: Knapsack Cover Cuts

These cuts are derived from logic about packing problems

$$3x_1 + \mathbf{5x_2} + \mathbf{4x_3} + 2x_4 + 7x_5 \leq \mathbf{8}$$
$$x_1, x_2, x_3, x_4, x_5 \in \{0, 1\}$$

$5 + 4 > 8 \implies x_2$ **and** x_3 **cannot simultaneously be equal to 1**

Cover Cut. $x_2 + x_3 \leq 1$

Sanity Check. What are other knapsack cover cuts?

Recall from Last Lecture: The Knapsack Problem

- Consider n items of weight w_j, $j \in \{1, \dots, n\}$ and a knapsack of weight capacity W.
- Item j has value v_j, but not all items may fit the knapsack.
- How to maximise the total value of the knapsack?

$$
\begin{aligned}
\max_{x} \quad & z = \sum_{j=1}^{n} v_j x_j \\
\text{s.t.} \quad & \sum_{j=1}^{n} w_j x_j \leq W \\
& x_j \in \{0, 1\} \qquad\qquad\qquad \forall j \in \{1, \dots, n\}
\end{aligned}
$$

Knapsack Cover Cuts

A set S of items in a knapsack problem is called a **cover** if:

$$\sum_{j \in S} w_j > W$$

If S is a cover, then the corresponding **knapsack cover cut** is:

$$\sum_{j \in S} x_j \leq |S| - 1$$

Usually, we want a **minimal cover constraint**, that is, a cover constraint such that for all proper subsets T of S:

$$\sum_{j \in T} w_j \leq W$$

Sanity Check. What are the minimal cover cuts from the previous example?

Outlook on Cutting Planes (valid inequalities)

- **Typical approach** Find & exploit useful cutting planes
- **Pure cutting plane approach** Typically very difficult
  - Too many constraints
  - It's difficult to find some constraints
- **Usually preferred** Branch & bound
  - State-of-the-art approaches hybridise cutting planes and branch & bound