IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2015-2016

BEng Honours Degree in Computing Part III
BEng Honours Degree in Electronic and Information Engineering Part III
MEng Honours Degree in Electronic and Information Engineering Part III
MEng Honours Degree in Mathematics and Computer Science Part IV
BEng Honours Degree in Mathematics and Computer Science Part III
MEng Honours Degree in Mathematics and Computer Science Part III
MEng Honours Degrees in Computing Part III
MSc in Computing Science (Specialist)
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute

PAPER C343

OPERATIONS RESEARCH

Wednesday 16 December 2015, 10:00
Duration: 120 minutes

Answer THREE questions

Paper contains 4 questions
Calculators required

1 a Consider the integer programming problem (ILP)

$$\max y = -x_1 + 2x_2$$

subject to

$$\begin{aligned} -4x_1 + 5x_2 &\leq 9 \\ x_1 + x_2 &\leq 3 \\ x_1, x_2 &\geq 0 \\ x_1 \text{ and } x_2 &\text{ integers} \end{aligned}$$

i) Solve this problem using the branch-and-bound method of integer programming.

[Hints: if the above problem is solved as a linear program without the integer restrictions on x_1 and x_2, we have the solution $x_1 = \frac{2}{3}$, $x_2 = \frac{7}{3}$, and $y = 4$. Branch first on x_1.]

ii) Could the ILP above be an instance of a bin packing problem? Explain.

iii) How does the branch-and-cut method differ from the branch-and-bound method?

b A company is considering seven large capital investments. The investments differ in the estimated long-run profit (net present value) they will generate as well as in the amount of capital required, as shown by the following table (in units of millions of Pounds):

	Investment Opportunity						
	A	B	C	D	E	F	G
Estimated long-run profit:	17	10	15	19	7	13	9
Capital required:	43	28	34	48	17	32	23

The total amount of capital available for these investments is £100 million. Investment opportunities A and B are mutually exclusive, and so are C and D. Furthermore, neither C nor D can be undertaken unless either A or B is undertaken. There are no such restrictions on investment opportunities E, F, G. The objective is to select the combination of capital investments that will maximise the total estimated long run profit. Write the 0-1 integer programming formulation for this problem. (Do *not* solve the integer programming problem.)

The two parts carry, respectively, 60% and 40% of the marks.

2a You are given the following linear programming (LP) problem:

$$\max 4x_1 + x_2$$

subject to

$$\begin{aligned} x_1 + 2x_2 &= 6 \\ x_1 - x_2 &\geq 3 \\ 2x_1 + x_2 &\leq 10 \\ x_1, x_2 &\geq 0 \end{aligned}$$

i) Solve it using the two-phase simplex method.

ii) Suppose you obtained at the end of Phase 1 of the simplex method a basic variable corresponding to an artificial variable. Explain how you would proceed to Phase 2.

iii) Formulate the dual of the LP. (Do *not* solve the dual problem.)

iv) State and prove the weak-duality theorem of linear programming.

b Totally unimodular matrices frequently arise in integer linear programs for supply chain management problems. Give an example of a totally unimodular matrix and explain why this property is important for the efficient solution of these problems.

The two parts carry, respectively, 80% and 20% of the marks.

3a i) Formulate a linear programming problem for finding (x_1, x_2) such that

$$|x_1 + x_2 - 1| + |x_1 + x_2 - 2|$$

is as small as possible. (Do *not* solve the linear programming problem.)

ii) Given an example of min-min problem. Then prove that min-min problems can be solved exactly using linear programming.

b A Police Department uses work shifts in which officers work 5 out of the 7 days of the week, with 2 successive days off. For example, a shift might work Sunday through Thursday and then have Friday and Saturday off. A total of 6 officers must be on duty Monday, Tuesday, Wednesday, and Thursdays; 10 are required on Friday and Saturday; and 8 are needed on Sunday. The Police Department wants to meet these staffing needs with the minimum total number of officers. For each day on duty an officer receives £100, except on Saturdays and Sundays where the pay is £80 in each day.

i) Formulate a shift scheduling program to minimize the cost for the Police Department.

ii) Suppose now that the Police Department wants to treat the constraint on the Wednesday shift as optional, how would you change the program? Justify your answer.

iii) Write the Phase I linear program for this problem. Indicate the initial basis for this linear program. (Do *not* solve the linear program).

The two parts carry, respectively, 35% and 65% of the marks.

4 a Colonel Rogers has three divisions to defend two mountain passes. He will defend successfully against equal or smaller strength, but lose against superior forces. The enemy has two divisions. The battle is lost if either pass is captured. Neither side has advance information on the disposition of the opponent's divisions.

i) Formulate this situation as a two-player zero-sum game, giving the payoff matrix for Colonel Rogers. Does the problem have dominated strategies?

ii) Express the problem of finding Colonel Rogers's optimal mixed strategy as a max-min program. Then propose an equivalent linear programming formulation. (Do *not* solve the linear program.)

iii) State the Minimax theorem and explain its implications for the theory of zero-sum games.

b Consider the following basic representation

$$
\begin{aligned}
z &= -\frac{35}{5} + \frac{3}{5}x_3 + \frac{4}{5}x_4 \\
x_1 &= \frac{8}{5} - \frac{1}{5}x_3 + \frac{2}{5}x_4 \\
x_2 &= \frac{3}{5} + \frac{3}{10}x_3 - \frac{1}{10}x_4
\end{aligned}
$$

Assume this representation to be optimal for the initial LP relaxation of an integer minimisation problem with variable x_1 required to be a nonnegative integer.

i) Derive a Gomory cut for x_1.

ii) Give the initial tableau for the next problem that needs to be solved by the cutting plane algorithm. (Do *not* solve the problem!). Discuss the termination condition for the cutting plane algorithm.

The two parts carry equal marks.