GLPK Case Study 3 - 60016 Operations Research

In this case study, we study a famous result in operations research. In 1972, Klee and Minty[1] showed that, in the worst-case, the simplex algorithm needs to visit all the vertices of the feasible set before finding the optimal one. A feasible set of this kind is called a *Klee-Minty cube*.

Consider the following Klee-Minty problem:

$$\min z = \; -10^{n-1}x_1 - 10^{n-2}x_2 - \ldots - 10x_{n-1} - x_n$$

subject to

$$\begin{array}{rcrcccccrcl}
x_1 & & & & & & & & & \leq & 1 \\
20x_1 & + & x_2 & & & & & & & \leq & 100 \\
200x_1 & + & 20x_2 & + & x_3 & & & & & \leq & 100^2 \\
\vdots & & & & & & & & & & \\
2 \cdot 10^{n-1}x_1 & + & 2 \cdot 10^{n-2}x_2 & + & \ldots & + & 2 \cdot 10x_{n-1} & + & x_n & \leq & 100^{n-1}
\end{array}$$

and

$$x_1 \geq 0,\; x_2 \geq 0,\; \ldots,\; x_n \geq 0$$

The problem has n constraints and n variables.

1. Write a GMPL model to solve the problem for arbitrary n. Call the resulting file `klee.mod`.
2. To show the behavior of a "textbook" simplex algorithm implementation on this problem, run `glpsol -m klee.mod -o klee.out --norelax --nosteep --noscale --nopresol` for $n = 2, 4, 8$. Determine the general structure of the optimal solution. Then comment on the number of iterations performed by the solver. Is the simplex algorithm visiting all the vertices?
3. Run `glpsol` with the same parameters for $n = 20, 50, 500$. Discuss qualitatively the behavior of the solver.
4. Repeat the experiment for $n = 20$ after appending the `--exact` parameter. What happened?
5. The *steepest edge* technique is a criteria to select the NBV that enters the basis in the pivoting operation of the simplex algorithm. It is an alternative to the techniques we see in the course and it is enabled by default in `glpsol`. It work as follows. Given the current vertex V (current BFS), it seeks for the neighbouring vertex V' such that the edge VV' forms the smallest angle with the direction of improvement of the objective function, i.e., the gradient of the objective. Remove the `--nosteep` parameter, thus re-enabling the steepest edge technique in `glpsol`, and evaluate $n = 2, 4, 8$.
6. The *interior point method* is an algorithm to solve LPs searching through the interior of the feasible set. Its main advantage is that, even in the worst case, for a problem with n variables and n constraints, it requires a computational effort that is proportional to a polynomial in

[1] Klee, Victor; Minty, George J. (1972). "How good is the simplex algorithm?". In Shisha, Oved. Inequalities III (Proceedings of the Third Symposium on Inequalities held at the University of California, Los Angeles, Calif., September 1–9, 1969, dedicated to the memory of Theodore S. Motzkin). New York-London: Academic Press. pp. 159–175.

n (e.g., $n^3 - 3n^2$). For the simplex algorithm, instead, this effort can grow exponentially with n and thus become much larger than in the interior point method. To run the interior point method, use `glpsol -m klee.mod -o klee.out --interior --noscale --nopresol`. Compare the number of iterations of the simplex algorithm and of the interior point method for $n = 2, 4, 8$.