60016 OPERATIONS RESEARCH

Game Theory Background
Zero Sum Games & Pure Strategies

13 November 2020

Game Theory

Optimization (e.g. LP) $\to$ problems with a single decision-maker:

- LP: diet optimisation, newsvendor problem, ...
- ILP: supply chain design, airline crew scheduling, ...

Game theory $\to$ problems with multiple decision-makers

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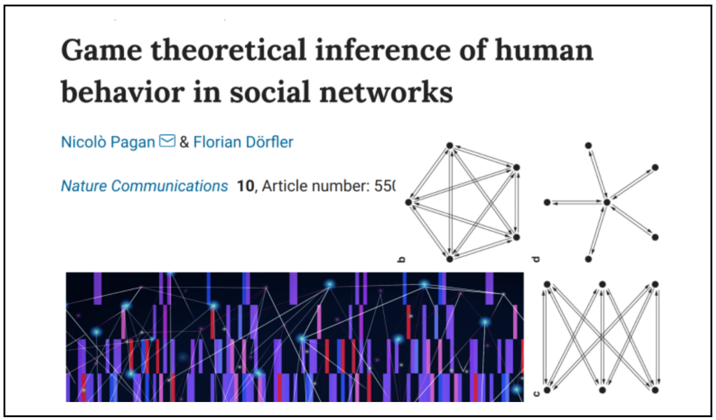

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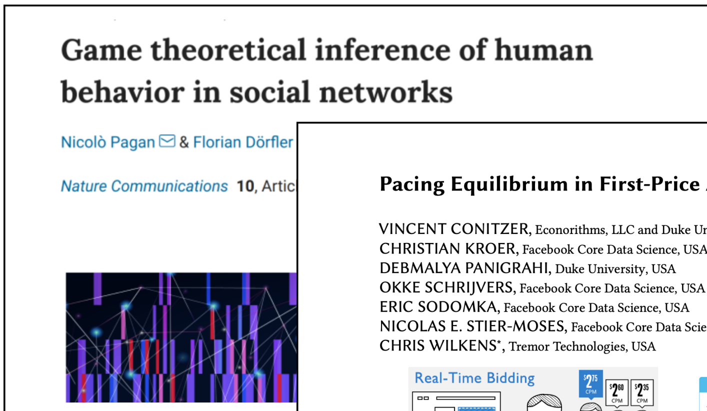
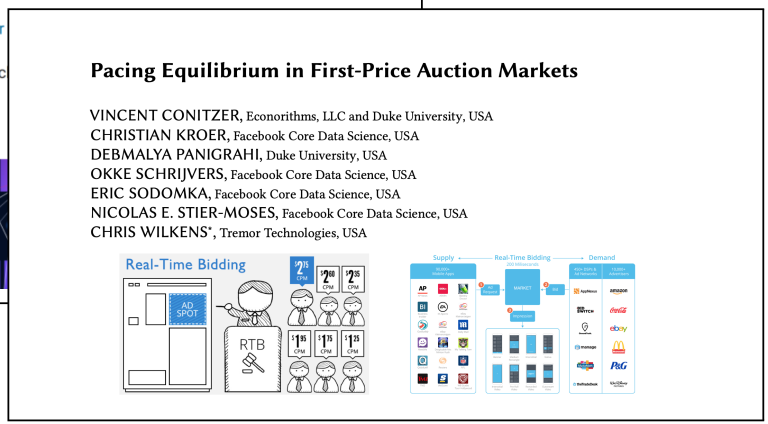

Game theoretical inference of human behavior in social networks

Nicolò Pagan & Florian Dörfler

Nature Communications **10**, Artic


Pacing Equilibrium in First-Price Auction Markets

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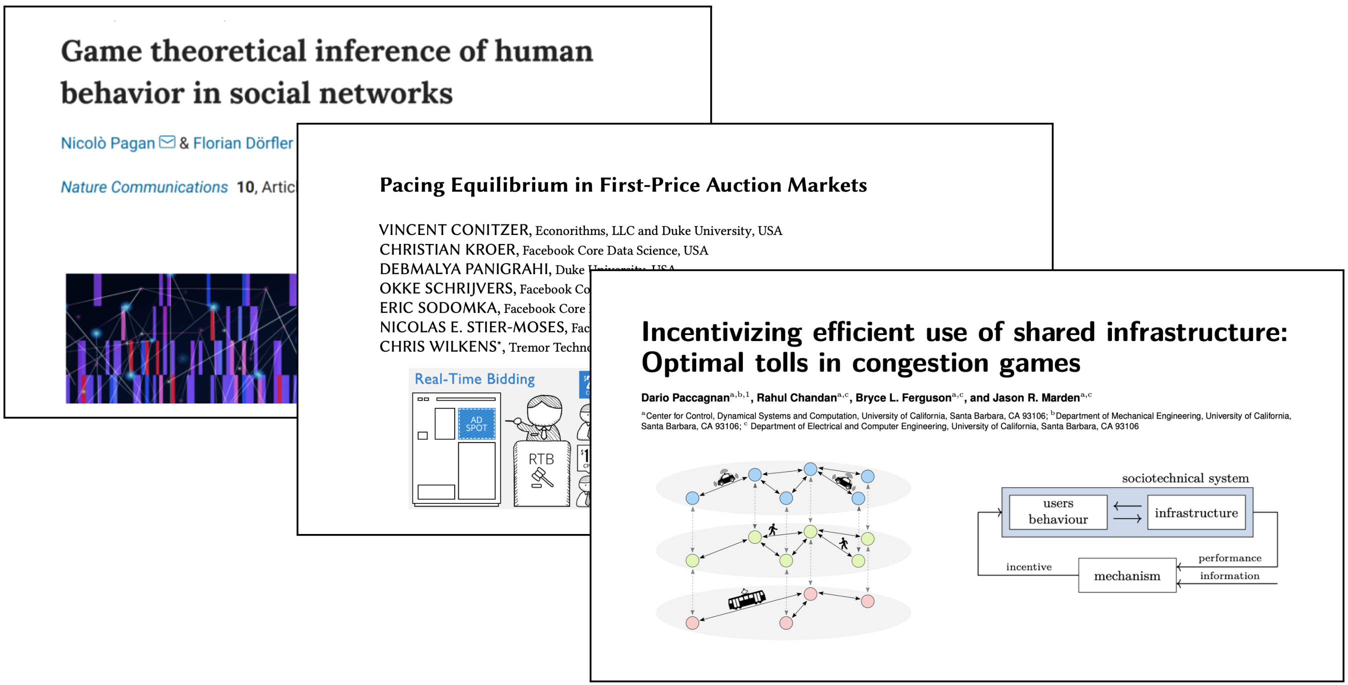

Games

Key ingredients:

- Every player takes own decision
- Payoff each player receives depends on choice of all players

In "math language":

- Each player $i = 1, \dots, n$ has a set of actions $x_i \in \mathcal{X}_i$
- Player i receives a payoff $J_i(x_1, \dots, x_i, \dots, x_n)$

Very special class: two-Person Zero-Sum Games w finite actions

- Two players: row player (RP) and column player (CP)
- RP chooses one out of m strategies (row strategies)
- CP chooses one out of n strategies (column strategies)
- Zero-Sum: RP wins whatever CP loses and viceversa

Payoff Matrix

- Payoff matrix: descriptor of a two-player zero-sum game
- If RP plays strategy i and CP plays strategy j, then CP pays a_{ij} to RP

$$
\begin{array}{cc|cccc|}
 & & & CP & & \\
 & & \text{Strategy } 1 & \text{Strategy } 2 & \dots & \text{Strategy } n \\
\hline
 & \text{Strategy } 1 & a_{11} & a_{12} & \dots & a_{1n} \\
RP & \text{Strategy } 2 & a_{21} & a_{22} & \dots & a_{2n} \\
 & \vdots & \vdots & \vdots & \ddots & \vdots \\
 & \text{Strategy } m & a_{m1} & a_{m2} & \dots & a_{mn} \\
\hline
\end{array}
$$

Example: Odds-and-Evens

- Both players simultaneously show "1" or "2" fingers
- If the sum of both numbers is even: CP gives £1 to RP
- If the sum of both numbers is odd: RP gives £1 to CP

		CP	
		1 Finger	2 Fingers
RP	1 Finger	1	-1
	2 Fingers	-1	1

Example: Rock-Paper-Scissors

- Both players simultaneously play "rock", "paper" or "scissors"
- Rock defeats scissor, scissor defeats paper, paper defeats rock. All other combinations are draws.
- If a player is defeated, s/he gives £1 to the other player

		CP		
		Rock	Paper	Scissors
RP	Rock	0	-1	1
	Paper	1	0	-1
	Scissors	-1	1	0

Two-Person Zero-Sum Games

Assumptions of Two-Person Zero-Sum Games:

1. Each player knows the game setting (available strategies to RP and CP, values of payoff matrix)
2. Both players simultaneously choose their strategy, that is, without knowing what their opponent chooses
3. Each player chooses a strategy that enables him/her to do best, reasoning as if the opponent could anticipate his/her strategy
4. Both players are rational:
   - They try to maximise their utility
   - They show no compassion for their opponent

Running Example

Elections game:

1. Two players: RP (row) and CP (column)
2. Both players have three strategies:
   - L: campaign the last two days in London
   - B: campaign the last two days in Birmingham
   - S: split the last two days, campaign one day in London and one day in Birmingham
3. Payoffs: how many voters does RP acquire from CP ?

Running Example

Consider the following setting of the Elections Game:

$$
\begin{array}{cc|ccc|}
 & & & CP & \\
 & & \text{L} & \text{B} & \text{S} \\
\hline
 & \text{L} & 1 & 2 & 4 \\
RP & \text{B} & 1 & 0 & 5 \\
 & \text{S} & 0 & 1 & -1 \\
\hline
\end{array}
$$

We want to find the strategies that will be played by RP and CP.

Running Example

$$
\begin{array}{ccccc}
 & & & \text{MIN} & \\
 & & & CP & \\
 & & L & B & S \\
\text{MAX} & & & & \\
 & L & 1 & 2 & 4 \\
RP & B & 1 & 0 & 5 \\
 & S & \cancel{0} & \cancel{1} & \cancel{-1}
\end{array}
$$

Observation:
Strategy L ("London only") is always better for RP than Strategy S ("London and Birmingham").

Conclusion:
RP will never play strategy S. Both players will realise this and we can ignore it.

Running Example

MIN

MAX

		CP		
		L	B	S
RP	L	1	2	~~4~~
	B	1	0	~~5~~

Observation:
Both strategy L ("London only") and strategy B ("Birmingham only") are always better (less to pay) for CP than strategy S ("Split between London and Birmingham").

Conclusion:
CP will never play strategy S. Both players will realise this and we can ignore it.

Running Example

			CP	
			L	B
RP	L		1	2
	B		1	0

Observation:
Strategy L ("London only") is no worse for RP than strategy B ("Birmingham only") and can be better (if CP plays B).

Conclusion:
RP will never play strategy B. Both players will realise this and we can ignore it.

Running Example

		CP	
		L	B
RP	L	1	2

Observation:
Strategy L ("London only") is always better for CP than strategy B ("Birmingham only").

Conclusion:
CP will never play strategy B. Both players will realise this and we can ignore it.

Running Example

$$
\begin{array}{ccc}
 & & CP \\
 & & \text{L} \\
RP & \text{L} & \boxed{1}
\end{array}
$$

Dominant Strategy Equilibrium:
Both RP and CP will campaign in London.

Dominance

Dominated row strategy: Row strategy i is dominated by row strategy i' if $a_{i'j} \geq a_{ij}$ for all column strategies $j = 1, \ldots, n$ and $a_{i'j} > a_{ij}$ for some j.

Dominated column strategy: Column strategy j is dominated by column strategy j' if $a_{ij'} \leq a_{ij}$ for all row strategies $i = 1, \ldots, m$ and $a_{ij'} < a_{ij}$ for some i.

- A rational player will never play a dominated strategy
- A rational opponent knows this

Dominant Strategy Equilibria

Dominant Strategy Equilibrium: If a repeated removal of dominated strategies leads to a game where each player has just one strategy left, then this strategy pair is a dominant strategy equilibrium.

Properties:

- If a dominant strategy equilibrium exists, then it is unique.
- If a dominant strategy equilibrium exists, then any rational players will play the associated equilibrium strategies.

Running Example

Consider a different payoff matrix for the Elections Game:

$$
\begin{array}{cc|ccc|}
 & & & CP & \\
 & & \text{L} & \text{B} & \text{S} \\
\hline
 & \text{L} & -3 & -2 & 6 \\
RP & \text{B} & 2 & 0 & 2 \\
 & \text{S} & 5 & -2 & -4 \\
\hline
\end{array}
$$

- This game has no dominated strategies
- Hence, there is no dominant strategy equilibrium

Security strategy over rows

Assumption: "Each player chooses a strategy that enables him/her to do best in face of worst-case opponent"

- α_i: payoff of row strategy i, when facing worst-case opponent

ROW IS MAX

$$\alpha_i = \min_{j=1,\ldots,n} a_{ij}$$

- Thus, the RP will pick the strategy i that maximizes the worst-case payoff

$$\max_{i=1,\ldots,n} \min_{j=1,\ldots,n} a_{ij}$$

Example: security strategy (rows)

MAX

		CP L	CP B	CP S	α_i
RP	L	-3	-2	6	-3
RP	B	2	0	2	0
RP	S	5	-2	-4	-4

- Strategy B is the best for the RP

Security strategy over columns

- We repeat the reasoning for the CP.
- β_j: cost of column strategy j, when facing worst-case opponent

Col is MIN

$$\beta_j = \max_{i=1,\ldots,n} a_{ij}$$

- Thus, the CP will pick the strategy j that minimizes the worst-case cost

$$\min_{j=1,\ldots,n} \max_{i=1,\ldots,n} a_{ij}$$

Example: Security strategy (columns)

		CP		
		L	B	S
RP	L	-3	-2	6
	B	2	0	2
	S	5	-2	-4
	β_j	5	0	6

- Strategy B is also best for the CP

Informal: Nash Equilibrium

			CP		
		L	B	S	α_i
RP	L	-3	-2	6	-3
	B	2	0	2	0
	S	5	-2	-4	-4
	β_j	5	0	6	

- Rational outcome for both players is to play (B,B)
- Strategy pair (B,B) is a pure strategy Nash equilibrium.
- The (B,B) payoff (0) is called value of the game
- Players have no incentive to change their strategies

Nash Equilibrium in Pure Strategies

Definition: a Nash Equilibrium is a strategy pair $(i^\star, j^\star)$ such that no player has an incentive to unilaterally deviate from his/her chosen strategy if told the strategy of the other player.

Note: a Nash equilibrium may not always exist in pure strategies.

Properties:

- If $(i^\star, j^\star)$ is a Nash equilibrium, then $\alpha_{i^\star} = \beta_{j^\star}$
- The payoff of the Nash equilibrium's strategy pair $\alpha_{i^\star} = \beta_{j^\star}$ is called the value of the game.

Example

		CP			
		L	B	S	α_i
RP	L	0	-1	2	-1
	B	5	4	-3	-3
	S	2	3	-4	-4
	β_j	5	4	2	

Example

		CP			
		L	B	S	α_i
RP	L	0	-1	2	-1
	B	5	4	-3	-3
	S	2	3	-4	-4
	β_j	5	4	2	

- (L,S) is not a Nash equilibrium in pure strategies
- If told RP's strategy, CP would change its strategy to B. This violates the definition of Nash equilibrium.

Tutorial 5 – 60016 Operations Research

Duality & Shadow Prices

Exercise 1. Consider the following optimisation problem.

$$
\begin{array}{ll}
\underset{\mathbf{x},\mathbf{y},\mathbf{z}}{\text{minimise}} & \mathbf{c_1}^\top \mathbf{x} + \mathbf{c_2}^\top \mathbf{y} + \mathbf{c_3}^\top \mathbf{z} \\
\text{subject to} & \mathrm{A}_1\mathbf{x} + \mathrm{B}_1\mathbf{y} + \mathrm{C}_1\mathbf{z} \leq \mathbf{b_1} \\
& \mathrm{A}_2\mathbf{x} + \mathrm{B}_2\mathbf{y} + \mathrm{C}_2\mathbf{z} = \mathbf{b_2} \\
& \mathrm{A}_3\mathbf{x} + \mathrm{B}_3\mathbf{y} + \mathrm{C}_3\mathbf{z} \geq \mathbf{b_3} \\
& \mathbf{x}_1, \ldots, \mathbf{x}_k \geq 0 \\
& \mathbf{y}_1, \ldots, \mathbf{y}_k \in \mathbb{R} \\
& \mathbf{z}_1, \ldots, \mathbf{z}_k \leq 0
\end{array}
$$

Where capital letters denote matrices and bold lowercase letters denote vectors and

$$\mathbf{x} \in \mathbb{R}^k, \mathbf{y} \in \mathbb{R}^k, \mathbf{z} \in \mathbb{R}^k.$$

You can think about this whole problem as having $3k$ variables. Also

$$\mathbf{b_1} \in \mathbb{R}^l, \mathbf{b_2} \in \mathbb{R}^l, \mathbf{b_3} \in \mathbb{R}^l.$$

You may assume all other vectors and matrices are of the appropriate dimensions. Dualise this problem.

Exercise 2. Write the dual formulation of the following (primal) linear program:

$$
\begin{array}{ll}
\text{minimise} & -x_1 + x_2 \\
\text{subject to} & 2x_1 + x_2 \leq 10 \\
& 3x_1 + 7x_2 \geq 20 \\
& x_1, x_2 \geq 0
\end{array}
$$

Now write the dual formulation of the following (primal) linear program. What is the objective value for the new primal and dual problem? How are the primal solutions related? How are the dual solutions related?

$$
\begin{array}{ll}
\text{minimise} & -x_1 + x_2 \\
\text{subject to} & 4x_1 + 2x_2 \leq 20 \\
& 3x_1 + 7x_2 \geq 20 \\
& x_1, x_2 \geq 0
\end{array}
$$

Generalising, what is the dual formulation of the following (primal) linear program if $\mu > 0$? What is the new objective value and how are the primal and dual solutions related?

$$
\begin{array}{ll}
\text{minimise} & -x_1 + x_2 \\
\text{subject to} & 2 \cdot \mu x_1 + \mu x_2 \leq \mu \cdot 10 \\
& 3x_1 + 7x_2 \geq 20 \\
& x_1, x_2 \geq 0
\end{array}
$$

Exercise 3. Consider the following (primal) linear program (P):

$$\begin{array}{ll} \max z = & x_4 \\ \text{subject to} & x_4 \leq x_2 - x_3 \\ & x_4 \leq -x_1 + x_3 \\ & x_4 \leq x_1 - x_2 \\ & x_1 + x_2 + x_3 = 1 \\ & x_1, x_2, x_3 \geq 0 \\ & x_4 \text{ free} \end{array}$$

Derive the dual linear program (D).

Exercise 4. Let:

$$v(b) = \min \left\{ c^T x \mid Ax = b,\, x \geq 0 \right\},$$

with $c,\, x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$. If the vector b is perturbed by a vector $\epsilon \in \mathbb{R}^m$, we consider the perturbed problem:

$$v(b+\epsilon) \min \left\{ c^T x \mid Ax = b + \epsilon,\, x \geq 0 \right\}.$$

In general, the solution of the latter problem does not have the same basic representation, and hence the optimal basis matrix, as the former. Using the optimal basis matrix of the former, define its shadow prices and use the shadow prices to establish a relationship between $v(b)$ and $v(b+\epsilon)$.

Exercise 5. Construct an example of a primal problem that has no feasible solutions and whose corresponding dual also has no feasible solutions.

Exercise 6. Let $A \subseteq B \subseteq \mathbb{R}^n$, and $c, x \in \mathbb{R}^n$. Prove that

$$\min_{x \in B} c^T x \leq \min_{x \in A} c^T x.$$

Hint: using the definition of minimum and the properties of subsets is sufficient.