Tutorial 5 - 60016 Operations Research

Duality & Shadow prices

Exercise 1. Consider the following optimisation problem.

$$\begin{array}{ll}
\underset{\mathbf{x},\mathbf{y},\mathbf{z}}{\text{minimise}} & \mathbf{c_1}^\top \mathbf{x} + \mathbf{c_2}^\top \mathbf{y} + \mathbf{c_3}^\top \mathbf{z} \\
\text{subject to} & \mathrm{A}_1\mathbf{x} + \mathrm{B}_1\mathbf{y} + \mathrm{C}_1\mathbf{z} \leq \mathbf{b_1} \\
& \mathrm{A}_2\mathbf{x} + \mathrm{B}_2\mathbf{y} + \mathrm{C}_2\mathbf{z} = \mathbf{b_2} \\
& \mathrm{A}_3\mathbf{x} + \mathrm{B}_3\mathbf{y} + \mathrm{C}_3\mathbf{z} \geq \mathbf{b_3} \\
& \mathbf{x}_1, \ldots, \mathbf{x}_k \geq 0 \\
& \mathbf{y}_1, \ldots, \mathbf{y}_k \in \mathbb{R} \\
& \mathbf{z}_1, \ldots, \mathbf{z}_k \leq 0
\end{array}$$

Where capital letters denote matrices and bold lowercase letters denote vectors and

$$\mathbf{x} \in \mathbb{R}^k, \mathbf{y} \in \mathbb{R}^k, \mathbf{z} \in \mathbb{R}^k.$$

You can think about this whole problem as having $3k$ variables. Also

$$\mathbf{b_1} \in \mathbb{R}^l, \mathbf{b_2} \in \mathbb{R}^l, \mathbf{b_3} \in \mathbb{R}^l.$$

You may assume all other vectors and matrices are of the appropriate dimensions. Dualise this problem using the direct method.

Solution. Throughout we use ? as a placeholder for elements currently not known to us.

For every primal constraint, create one dual variable. There are $3l$ primal constraints, so we have to create $3l$ dual variables. We create 3 <u>vector</u> dual variables $\mathbf{u} \in \mathbb{R}^l, \mathbf{v} \in \mathbb{R}^l, \mathbf{w} \in \mathbb{R}^l$.

For every primal variable, create one dual constraint. There are $3k$ primal variables, so the dual problem will have $3k$ constraints. So far, the dual problem is:

$$\begin{array}{ll}
\underset{\mathbf{u},\mathbf{v},\mathbf{w}}{?} & ?^\top \mathbf{u} + ?^\top \mathbf{v} + ?^\top \mathbf{w} \\
\text{subject to} & ? \qquad (3k \text{ constraints}) \\
& \mathbf{u}_1, \ldots, \mathbf{u}_l \quad ? \quad ? \\
& \mathbf{v}_1, \ldots, \mathbf{v}_l \quad ? \quad ? \\
& \mathbf{w}_1, \ldots, \mathbf{w}_l \quad ? \quad ?
\end{array}$$

Transpose the primal coefficient matrix. Former right-hand sides become new objective coefficients. Former objective coefficients become new right-hand sides.

$$\begin{array}{ll}
\underset{\mathbf{u},\mathbf{v},\mathbf{w}}{?} & \mathbf{b_1}^\top \mathbf{u} + \mathbf{b_2}^\top \mathbf{v} + \mathbf{b_3}^\top \mathbf{w} \\
\text{subject to} & \mathrm{A}_1^\top \mathbf{u} + \mathrm{A}_2^\top \mathbf{v} + \mathrm{A}_3^\top \mathbf{w} \quad ? \quad \mathbf{c_1} \\
& \mathrm{B}_1^\top \mathbf{u} + \mathrm{B}_2^\top \mathbf{v} + \mathrm{B}_3^\top \mathbf{w} \quad ? \quad \mathbf{c_2} \\
& \mathrm{C}_1^\top \mathbf{u} + \mathrm{C}_2^\top \mathbf{v} + \mathrm{C}_3^\top \mathbf{w} \quad ? \quad \mathbf{c_3} \\
& \mathbf{u}_1, \ldots, \mathbf{u}_l \quad ? \quad ? \\
& \mathbf{v}_1, \ldots, \mathbf{v}_l \quad ? \quad ? \\
& \mathbf{w}_1, \ldots, \mathbf{w}_l \quad ? \quad ?
\end{array}$$

The primal is a minimisation problem, therefore the dual problem is a maximisation. Dual variables $\mathbf{u}$ for primal $\leq$ constraints become non-positive. Dual variables $\mathbf{v}$ for primal $=$ constraints become free. Dual variables $\mathbf{w}$ for primal $\geq$ constraints become non-negative.

Dual constraints for primal non-negative variables become $\leq$ constraints. Dual constraints for primal free variables become $=$ constraints. Dual constraints for primal non-positive variables become $\geq$ constraints.

$$
\begin{array}{ll}
\underset{\mathbf{u},\mathbf{v},\mathbf{w}}{\text{maximise}} & \mathbf{b_1}^\top \mathbf{u} + \mathbf{b_2}^\top \mathbf{v} + \mathbf{b_3}^\top \mathbf{w} \\
\text{subject to} & \mathrm{A}_1^\top \mathbf{u} + \mathrm{A}_2^\top \mathbf{v} + \mathrm{A}_3^\top \mathbf{w} \leq \mathbf{c_1} \\
 & \mathrm{B}_1^\top \mathbf{u} + \mathrm{B}_2^\top \mathbf{v} + \mathrm{B}_3^\top \mathbf{w} = \mathbf{c_2} \\
 & \mathrm{C}_1^\top \mathbf{u} + \mathrm{C}_2^\top \mathbf{v} + \mathrm{C}_3^\top \mathbf{w} \geq \mathbf{c_3} \\
 & \mathbf{u}_1, \ldots, \mathbf{u}_l \leq 0 \\
 & \mathbf{v}_1, \ldots, \mathbf{v}_l \in \mathbb{R} \\
 & \mathbf{w}_1, \ldots, \mathbf{w}_l \geq 0
\end{array}
$$

Exercise 2. Write the dual formulation of the following (primal) linear program:

$$
\begin{array}{ll}
\text{minimise} & -x_1 + x_2 \\
\text{subject to} & 2x_1 + x_2 \leq 10 \\
 & 3x_1 + 7x_2 \geq 20 \\
 & x_1, x_2 \geq 0
\end{array}
$$

Now write the dual formulation of the following (primal) linear program. What is the objective value for the new primal and dual problem? How are the primal solutions related? How are the dual solutions related?

$$
\begin{array}{ll}
\text{minimise} & -x_1 + x_2 \\
\text{subject to} & 4x_1 + 2x_2 \leq 20 \\
 & 3x_1 + 7x_2 \geq 20 \\
 & x_1, x_2 \geq 0
\end{array}
$$

Generalising, what is the dual formulation of the following (primal) linear program if $\mu > 0$? What is the new objective value and how are the primal and dual solutions related?

$$
\begin{array}{ll}
\text{minimise} & -x_1 + x_2 \\
\text{subject to} & 2 \cdot \mu x_1 + \mu x_2 \leq \mu \cdot 10 \\
 & 3x_1 + 7x_2 \geq 20 \\
 & x_1, x_2 \geq 0
\end{array}
$$

Solution. Solving the first optimisation problem: $z^P \approx -3.63, x_1 \approx 4.54, x_2 \approx 0.91$. The dual is:

$$
\begin{array}{lll}
\text{maximise} & 10y_1 + 20y_2 & \\
\text{subject to} & 2y_1 + 3y_2 & \leq -1 \\
 & y_1 + 7y_2 & \leq 1 \\
 & y_1 \leq 0, y_2 \geq 0 &
\end{array}
$$

and the solution to the dual problem is: $z^P = -40/11 \approx -3.63, y_1 = -10/11 \approx -0.91, y_2 = 3/11 \approx 0.27$.

Solving the second optimisation problem: $z^P \approx -3.63, x_1 \approx 4.54, x_2 \approx 0.91$. The dual is:

$$
\begin{array}{lll}
\text{maximise} & 20y_1 + 20y_2 & \\
\text{subject to} & 4y_1 + 3y_2 & \leq -1 \\
 & 2y_1 + 7y_2 & \leq 1 \\
 & y_1 \leq 0, y_2 \geq 0 &
\end{array}
$$

and the solution to the dual problem is: $z^P = -40/11 =\approx -3.63, y_1 = -5/11 \approx -0.45, y_2 = 3/11 \approx 0.27$. So the objective values have remained the same and the primal variables have remained the same, but the value of the dual variable associated with the constraint that was multplied by 2 is now divided by 2.

Solving the third optimisation problem: $z^P \approx -3.63, x_1 \approx 4.54, x_2 \approx 0.91$. The dual is:

$$
\begin{array}{lll}
\text{maximise} & 10 \cdot \mu y_1 + 20y_2 & \\
\text{subject to} & 2\mu y_1 + 3y_2 & \leq -1 \\
 & \mu y_1 + 7y_2 & \leq 1 \\
 & y_1 \leq 0, y_2 \geq 0 &
\end{array}
$$

and the solution to the dual problem is: $z^P = -40/11 =\approx -3.63, y_1 = -10/(11\mu), y_2 = 3/11 \approx 0.27$.

Similarly to the first exercise, a good answer will comment explaining the duality rules used to derive the dual formulation.

Exercise 3. Consider the following (primal) linear program ($\underline{\text{P}}$):

$$
\begin{array}{ll}
\max z = & x_4 \\
\text{subject to} & x_4 \leq x_2 - x_3 \\
 & x_4 \leq -x_1 + x_3 \\
 & x_4 \leq x_1 - x_2 \\
 & x_1 + x_2 + x_3 = 1 \\
 & x_1, x_2, x_3 \geq 0 \\
 & x_4 \text{ free}
\end{array}
$$

Derive the dual linear program ($\underline{\text{D}}$).

Solution. Using the standard transformations we get

$$
\begin{array}{ll}
\min z = & y_4 \\
\text{subject to} & y_4 \geq -y_2 + y_3 \\
 & y_4 \geq y_1 - y_3 \\
 & y_4 \geq -y_1 + y_2 \\
 & y_1 + y_2 + y_3 = 1 \\
 & y_1, y_2, y_3 \geq 0 \\
 & y_4 \text{ free}
\end{array}
$$

Similarly to the first exercise, a good answer will comment explaining the duality rules used to derive the dual formulation.

Exercise 4. Let:

$$v(b) = \min \left\{ c^T x \mid Ax = b, \, x \geq 0 \right\},$$

with $c,\, x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m\times n}$. If the vector b is perturbed by a vector $\epsilon \in \mathbb{R}^m$, we consider the perturbed problem:

$$v(b+\epsilon) \min \left\{c^T x \mid Ax = b+\epsilon,\, x \geq 0\right\}.$$

In general, the solution of the latter problem does not have the same basic representation, and hence the optimal basis matrix, as the former. Using the optimal basis matrix of the former, define its shadow prices and use the shadow prices to establish a relationship between $v(b)$ and $v(b+\epsilon)$.

Solution.

$$\begin{aligned}
v(b+\epsilon) &= \min \left\{c^T x \mid Ax = b+\epsilon,\, x \geq 0\right\} \\
&= \min \left\{c^T x - \Pi^T \left(Ax - b - \epsilon\right) \mid Ax = b+\epsilon,\, x \geq 0\right\} \\
&\geq \min \left\{c^T x - \Pi^T \left(Ax - b - \epsilon\right) \mid x \geq 0\right\} \\
&= \min \left\{\left(c^T - \Pi^T A\right) x + \Pi^T \left(b+\epsilon\right) \mid x \geq 0\right\} \\
&\geq \Pi^T \left(b+\epsilon\right) \\
&= \Pi^T b + \Pi^T \epsilon \\
&= c_B^T B^{-1} b + \Pi^T \epsilon \\
&= v(b) + \Pi^T \epsilon
\end{aligned}$$

Exercise 5. Construct an example of a primal problem that has no feasible solutions and whose corresponding dual also has no feasible solutions.

Solution.

$$\begin{aligned}
\max_{x_1,x_2} z = \quad & 2x_1 + x_2 \\
\text{subject to} \quad & -x_1 + x_2 \leq -4 \\
& x_1 - x_2 \leq 2 \\
& x_1, x_2 \geq 0
\end{aligned}$$

$$\begin{aligned}
\min_{y_1,y_2} z = \quad & -4y_1 + 2y_2 \\
\text{subject to} \quad & -y_1 + y_2 \geq 2 \\
& y_1 - y_2 \geq 1 \\
& y_1, y_2 \geq 0
\end{aligned}$$

Exercise 6. Let $A \subseteq B \subseteq \mathbb{R}^n$, and $c, x \in \mathbb{R}^n$. Assume that $\min_{x\in B} c^T x$ and $\min_{x\in A} c^T x$ exist. Prove that

$$\min_{x\in B} c^T x \leq \min_{x\in A} c^T x.$$

Solution. We prove the claim by contradiction. Towards this goal, we begin assuming that

$$\min_{x\in B} c^T x > \min_{x\in A} c^T x.$$

This means that there exists $x^* \in A$ such that

$$c^T x^* = \min_{x\in A} c^T x < \min_{x\in B} c^T x \leq c^T x \quad \forall x \in B, \tag{1}$$

where the last inequality holds by definition of minimum. Nevertheless, we observe that $x^* \in A$ also belongs to B since $B \supseteq A$. Therefore, selecting $x = x^\star$ in the RHS of (1) we conclude that $c^T x^* < c^T x^*$, a contradiction. This completes the proof.