

Tue: (Today)
+
Friday

COMPUTATIONAL FINANCE: 422

Mean-Variance Portfolio Theory Next Tues.
Tutorial
(2h).

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This Lecture

- Asset returns
- Portfolio returns
- Variance as a risk measure
- Mean-variance diagrams
 - Feasible set
 - Minimum-variance set
 - Efficient frontier
- Markowitz problem
- Parameter estimation

Variance
min Risk \sim
s.t. Expected Returns $\sim 10\%$
Average / Expectation
Return
single - per 100

Further reading:

- D.G. Luenberger: *Investment Science*, Chapters 6 & 8

Asset Return I

Bonds, stock, mortgages, ...

- Asset: investment instrument that can be bought/sold.
- If you buy an asset today at a price X_0 and sell it in 1 year at a price X_1 , then the **total return** R on your investment is defined to be

$$R = \frac{X_1}{X_0}.$$

X_0 in \$

- Similarly, the **rate of return** r is defined as

$$r = \frac{X_1 - X_0}{X_0}.$$

$$r = R - 1$$
$$R = 1 + r$$

- The expression 'return' is used both for the **total return** and the **rate of return**. The context should make clear which interpretation is meant.

Asset Return II

- By definition, we have

$$R = 1 + r$$

$$\Rightarrow X_1 = (1 + r)X_0.$$

Thus, the **rate of return** acts much like an **interest rate**.

- If X_1 is uncertain, then **r must be uncertain**, as well. In contrast, current interest rates are always certain.

Short Sales

- Sometimes it is possible to **sell an asset that you do not own**. This process is called **short selling** or **shorting**.
- How does it work?
 1. You **borrow** the asset from someone who owns it.
 2. You **sell** the asset to someone else at its current price X_0 .
 3. At a later date, you **buy** the asset for X_1 .
 4. You **return** the asset to your lender.
- Your **overall profit** is $X_0 - X_1 \Rightarrow$ short selling is profitable if the asset price declines.
- The **potential loss** of a short sale is **unbounded!**
 \Rightarrow Short selling is often **restricted or avoided**.

Portfolios

- Suppose n different assets are available.
- We form a master asset or portfolio by apportioning an amount X_0 among the assets.
- We select amounts X_{0i} , $i = 1, 2, \dots, n$, such that

$\theta (\$, \pounds, \text{€})$

$$\sum_{i=1}^n X_{0i} = X_0,$$

Budget
Budget Constraint

where X_{0i} represents the amount invested in asset i .

- If short selling is allowed, some X_{0i} 's can be negative; otherwise we require $X_{0i} \geq 0$.

- The X_{0i} can be expressed as $X_{0i} = w_i X_0$, $i = 1, 2, \dots, n$, where w_i is the weight of asset i in the portfolio.

$$X_0 = \sum_{i=1}^n X_{0i} = \sum_{i=1}^n w_i X_0 \Rightarrow \sum_{i=1}^n w_i = 1$$

Portfolio Return

Budget
Constraint

- The **asset weights** sum to 1, that is, $\sum_{i=1}^n w_i = 1$.
- R_i = **total return of asset i** . \Rightarrow The amount of money generated at the end of the period by the i th asset is

X_{0i}
 X_{1i}

$$R_i X_{0i} = R_i w_i X_0.$$

- Thus, the **total value of the portfolio** after the period is

Value
after 1 period \rightarrow

$$\sum_{i=1}^n R_i w_i X_0.$$

X_0 ✓ Value
today

\Rightarrow The portfolio's **total return** and **rate of return** are

Portfolio
Expected
Return
Constraint

$$R = \frac{\sum_{i=1}^n w_i R_i X_0}{X_0} = \sum_{i=1}^n w_i R_i$$

$$\text{and } r = \sum_{i=1}^n w_i r_i.$$

Describing a Portfolio

Assume that there are n assets in which you can invest.

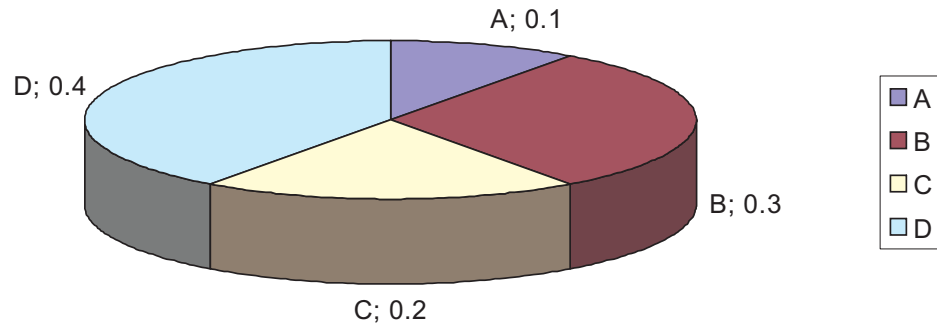
Expected

Asset	£ invested	% invested	Return
1	X_{01}	$w_1 = X_{01}/X_0$	R_1
2	X_{02}	$w_2 = X_{02}/X_0$	R_2
\vdots	\vdots	\vdots	\vdots
n	X_{0n}	$w_n = X_{0n}/X_0$	R_n
Total:	$X_0 = \sum_{i=1}^n X_{0i}$	$1 = \sum_{i=1}^n w_i$	$R = \sum_{i=1}^n w_i R_i$

A portfolio can be described by £ invested or by portfolio weights. Using weights facilitates the calculation of the portfolio return.

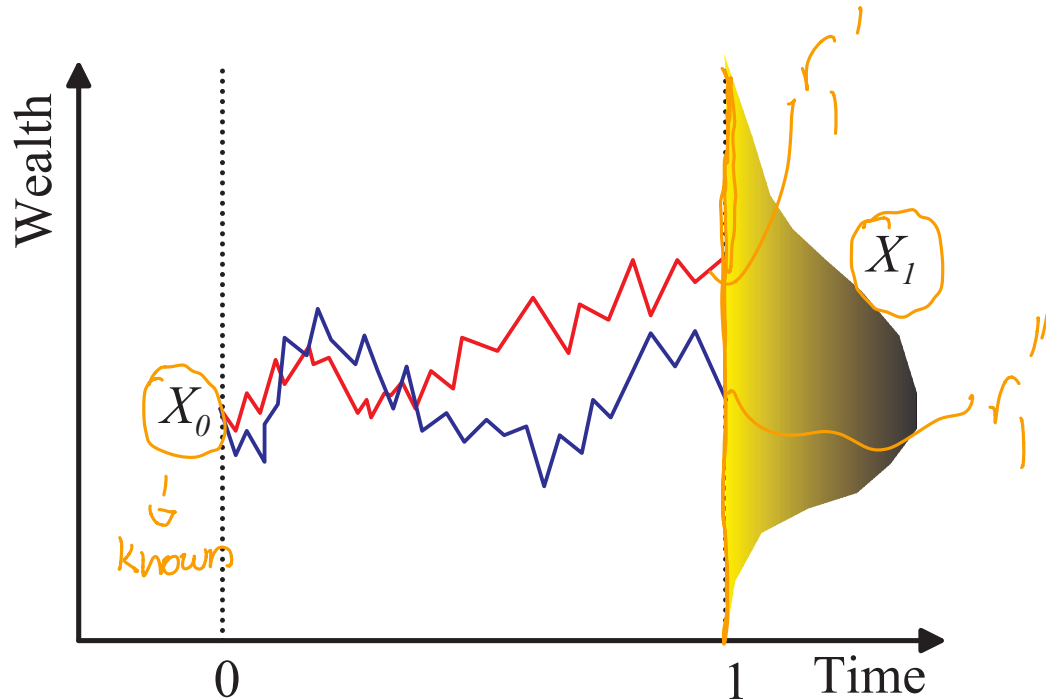
Example

Security	£ Amount	% Weight	Return	
	X_{oi}	w_i	r_i	$w_i r_i$
A	100	0.1	1.1	0.11
B	300	0.3	1.2	0.36
C	200	0.2	1.05	0.21
D	400	0.4	1.25	0.5
Total	1000	1		1.18



Randomness I

- For any asset, today's value X_0 is **deterministic**, while the future value X_1 is **random**.
- ⇒ The **total return R** and the **rate of return r** are **random**.

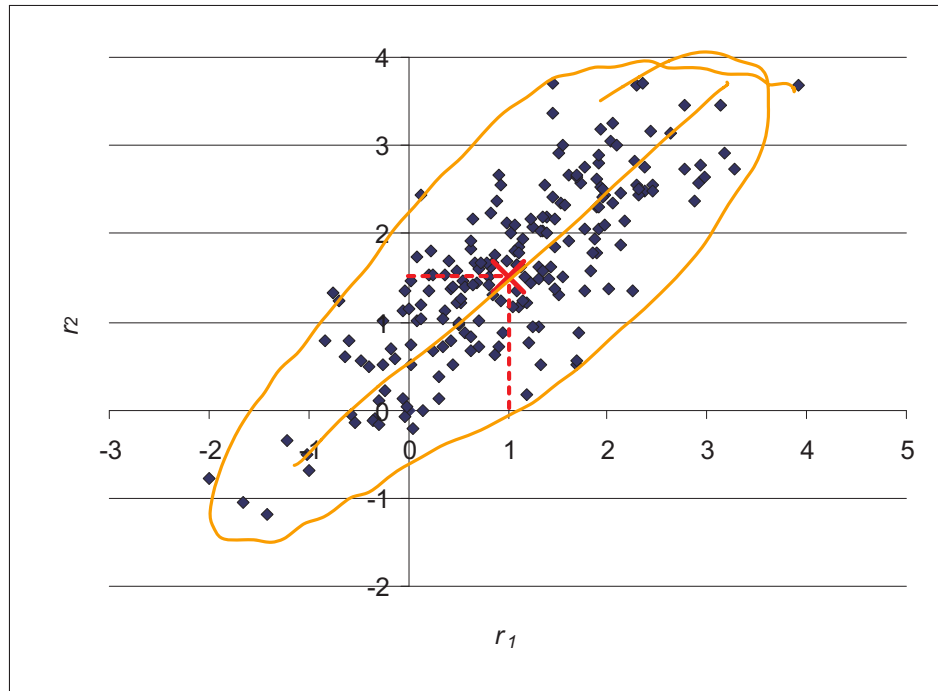


Randomness II

- Suppose there are n assets with random rates of return r_1, r_2, \dots, r_n .
- These have expected values $E(r_1) = \bar{r}_1, E(r_2) = \bar{r}_2, \dots, E(r_n) = \bar{r}_n$.
- We denote the variance of r_i by σ_i^2 and the covariance of r_i with r_j by σ_{ij} ($\Rightarrow \sigma_{ii} = \sigma_i^2$).
- Otherwise, we make no assumptions about the (joint) distribution of r_1, r_2, \dots, r_n .

Randomness III

- Scatter plot of two jointly normally distributed returns r_1 and r_2 with $\bar{r}_1 = 1$, $\bar{r}_2 = 1.5$, $\sigma_1^2 = \sigma_2^2 = 1$, and $\sigma_{12} = 0.8$.



Mean and Variance of Portfolio Return

- The return of a portfolio is given by $r = \sum_{i=1}^n w_i r_i$.
 return of portfolio to be computed
 parameter
- The expected (or mean) return of a portfolio is given by

$$\bar{r} = E(r) = E\left(\sum_{i=1}^n w_i r_i\right) = \sum_{i=1}^n w_i E(r_i) = \sum_{i=1}^n w_i \bar{r}_i$$

deterministic
 expected return / performance constraint

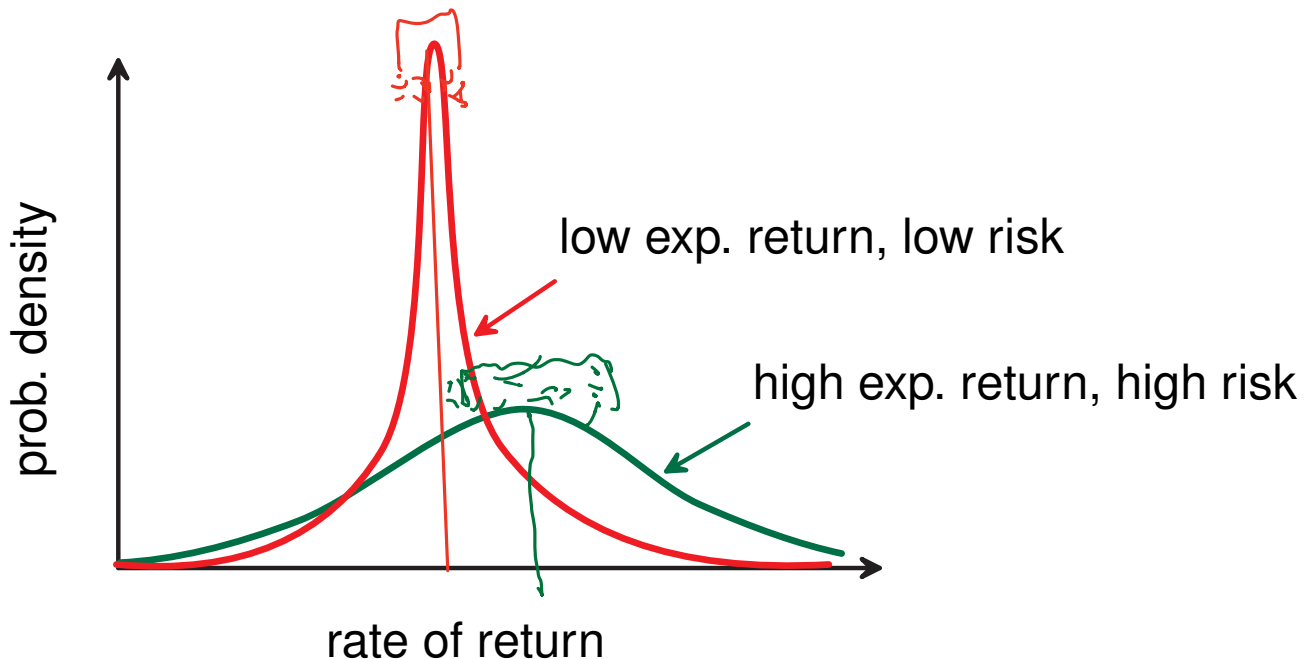
- The variance of the return of a portfolio is given by

$$\begin{aligned} \sigma^2 &= \text{var}(r) = E[(r - \bar{r})^2] = E\left[\left(\sum_{i=1}^n w_i r_i - \sum_{i=1}^n w_i \bar{r}_i\right)^2\right] \\ &= E\left[\left(\sum_{i=1}^n w_i (r_i - \bar{r}_i)\right)\left(\sum_{j=1}^n w_j (r_j - \bar{r}_j)\right)\right] \\ &= E\left[\sum_{i,j=1}^n w_i w_j (r_i - \bar{r}_i)(r_j - \bar{r}_j)\right] = \sum_{i,j=1}^n w_i \sigma_{ij} w_j \end{aligned}$$

Handwritten notes:
 $E\left(\sum_{i=1}^n w_i r_i\right)^2 \neq E\left(\sum_{i=1}^n w_i r_i\right) E\left(\sum_{j=1}^n w_j r_j\right)$
 $E(r_i - \bar{r}_i)(r_j - \bar{r}_j)$

Variance as a Risk Measure σ_{ij}

- The **variance of the return** can be interpreted as a measure of the **risk** associated with an asset/portfolio.



Diversification

Q: Why should we form portfolios?

A: Portfolios can **reduce risk (variance)** w/o sacrificing **mean return**.

Example: Consider n assets with **independent and identically distributed (iid)** returns, that is,

$$\bar{r}_i = \bar{r} \quad \text{and} \quad \sigma_i^2 = \sigma^2 \quad \forall i = 1, 2, \dots, n.$$

$$\sigma_{ij} = 0$$

What are the **mean and variance** \bar{r}_P and σ_P^2 of the portfolio with $w_1 = w_2 = \dots = w_n = 1/n$?

$$\sum_{i=1}^n w_i = 1 \quad w_i = \frac{1}{n}$$

$$\sigma_P^2 = \sum_{i,j} w_i w_j \sigma_{ij}$$
$$\begin{cases} \bar{r}_P = \sum_{i=1}^n w_i \bar{r}_i = \sum_{i=1}^n \frac{1}{n} \bar{r} = \bar{r} \\ \sigma_P^2 = \sum_{i=1}^n w_i^2 \sigma_i^2 = \sum_{i=1}^n \frac{1}{n^2} \sigma^2 = \frac{\sigma^2}{n} \end{cases}$$

⇒ **Portfolios reduce risk!**

Portfolio Diagrams

If $\rho = -1$ $\sigma_p = \sqrt{(\alpha\sigma_1 - (1-\alpha)\sigma_2)^2} = |\alpha\sigma_1 - (1-\alpha)\sigma_2|$

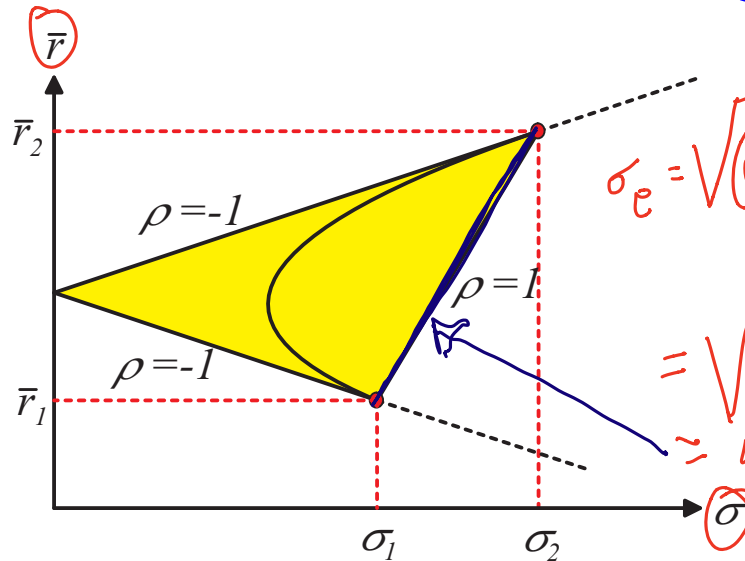
Two assets in a mean-standard deviation diagram:

$$\sigma_p^2 = \sum_{i,j}^N w_i w_j \sigma_{ij}$$

(N=2)

$$= w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1 w_2 \sigma_{12}$$

$w_1 = \alpha, w_2 = 1 - \alpha$



$$\sigma_e = \sqrt{(\alpha^2 \sigma_1^2 + 2\alpha(1-\alpha)\sigma_1\sigma_2 + (1-\alpha)^2 \sigma_2^2)}$$

$$= \sqrt{(\alpha\sigma_1 + (1-\alpha)\sigma_2)^2}$$

$$= \alpha\sigma_1 + (1-\alpha)\sigma_2$$

$w_1 + w_2 = 1$

The portfolio with $w_1 = \alpha$ and $w_2 = 1 - \alpha$ has:

mean return: $\bar{r}_P = \alpha\bar{r}_1 + (1 - \alpha)\bar{r}_2$

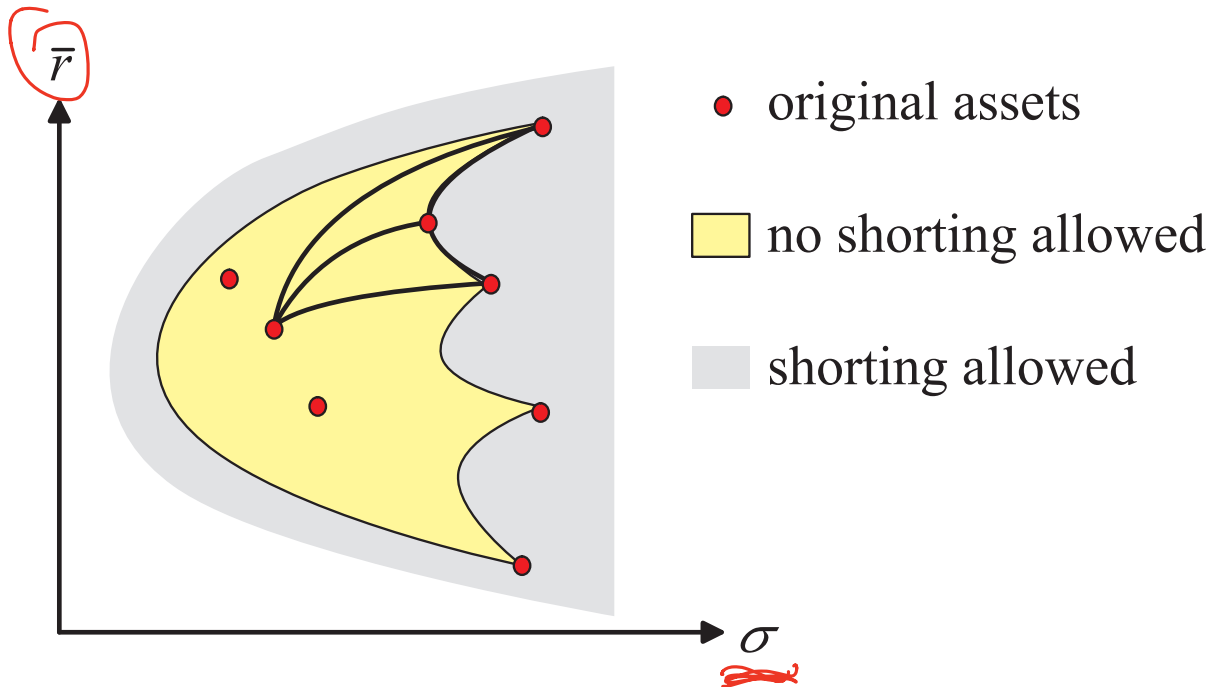
variance: $\sigma_P^2 = \alpha^2 \sigma_1^2 + 2\alpha(1 - \alpha)\sigma_{12} + (1 - \alpha)^2 \sigma_2^2$

standard deviation: $\sigma_P = \sqrt{\alpha^2 \sigma_1^2 + 2\alpha(1 - \alpha)\rho\sigma_1\sigma_2 + (1 - \alpha)^2 \sigma_2^2}$

correlation $\rho = \frac{\sigma_{12}}{\sigma_1 \sigma_2}$ ← covariance of (1,2) $-1 \leq \rho \leq 1$

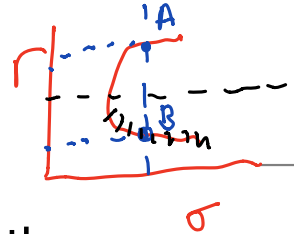
The Feasible Set

Given n assets, what does the set of all possible portfolios look like in the (σ, \bar{r}) plane?



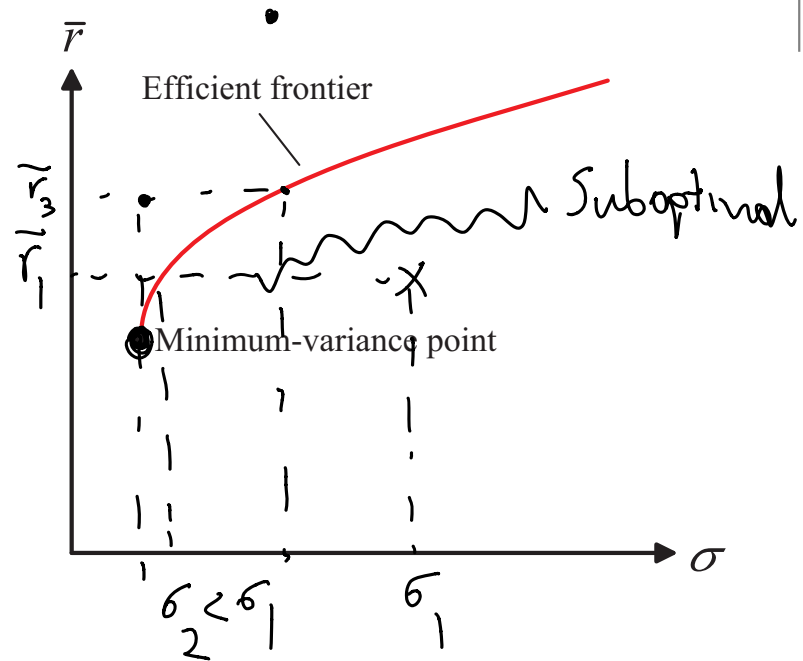
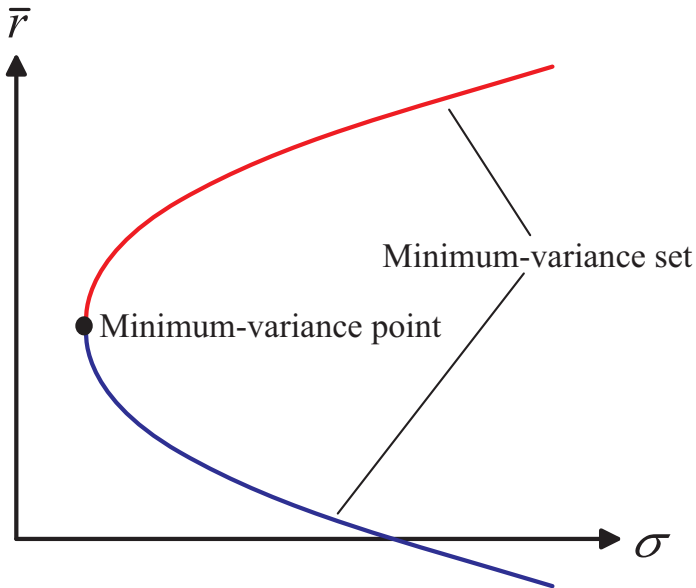
The feasible set defined with short selling allowed contains the one defined without short selling.

Minimum-Variance Set



- The **left boundary** of the feasible set is called the **minimum variance set**.
 - The point with lowest possible variance is called the **minimum variance point**.
 - Decision criteria:
 - Given 2 portfolios with the same mean return, a **risk-averse investor** will prefer the one with the smaller risk (variance).
 - Given 2 portfolios with the same risk (variance), a **greedy investor** will prefer the one with the higher mean return.
- ⇒ Only the **upper half of the minimum variance set** is of interest to investors. This set is termed **efficient frontier**.

Efficient Frontier



- The **minimum-variance set** is obtained by **minimizing the risk/variance for any given mean return**.
- The **efficient frontier** is the top portion of the minimum-variance set.

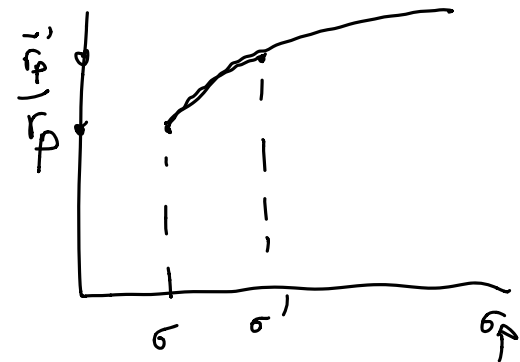
Harry Markowitz

- **Harry Max Markowitz** (born August 24, 1927) won the Nobel Prize in Economics in 1990 for his pioneering work on portfolio theory.



The Markowitz Model I

- Markowitz formulated the problem to **determine the efficient frontier** as a mathematical optimization problem.
- Assume there are n **risky assets** with
 - mean returns $\bar{r}_1, \bar{r}_2, \dots, \bar{r}_n$
 - covariances σ_{ij} for $i, j = 1, 2, \dots, n$.
- The **portfolio** with weights w_1, w_2, \dots, w_n has
 - mean return $\bar{r}_P = \sum_{i=1}^n w_i \bar{r}_i$
 - variance $\sigma_P^2 = \sum_{i,j=1}^n w_i \sigma_{ij} w_j$.



The Markowitz Model II

QP = Quadratic Program, Quad obj, Linear constraints

minimize $\frac{1}{2} \sum_{i,j=1}^n w_i \sigma_{ij} w_j = \frac{1}{2} \sigma_P^2$

subject to $\sum_{i=1}^n w_i \bar{r}_i = \bar{r}_P$

$\sum_{i=1}^n w_i = 1$

$\min_x \alpha f(x)$, $\alpha > 0$
 $= \alpha \min f(x)$

$\nabla f(x) = 0$
 (performance / target) $\alpha \nabla f(x) = 0$
 = weights sum to 1
 (Budget constraint)

- In this formulation, **short selling is allowed**.
- The solution of the problem depends on the **return target parameter** \bar{r}_P .
- The **minimum-variance set** is obtained by plotting the minimal σ_P^2 for different parameter values \bar{r}_P .

Solution of the Markowitz Model I

minimize $\frac{1}{2} \sum_{i,j=1}^n w_i \sigma_{ij} w_j$ Quadratic Objective

subject to $\begin{cases} \sum_{i=1}^n w_i \bar{r}_i - \bar{r}_P = 0 & \text{constraint \#1} \\ \sum_{i=1}^n w_i - 1 = 0 & \text{constraint \#2} \end{cases}$

Linear constraints short-selling allowed.

Lagrange multipliers: λ μ { scalars. }

The associated Lagrangian function L is given by

$$L(\mathbb{R}^n, \mathbb{R}, \mathbb{R}) \rightarrow \mathbb{R}$$

$$L = \underbrace{\frac{1}{2} \sum_{i,j=1}^n w_i \sigma_{ij} w_j}_{\text{objective}} - \lambda \left(\sum_{i=1}^n w_i \bar{r}_i - \bar{r}_P \right) - \mu \left(\sum_{i=1}^n w_i - 1 \right).$$

- $\lambda \times \text{const \#1}$ - $\mu \times (\text{constraint \#2})$

Solution of the Markowitz Model II

Differentiate the **Lagrangian** w.r.t. $w_1, w_2, \dots, w_n, \lambda$, and μ , and set all derivatives = 0:

$$w_i : \quad \sum_{j=1}^n \sigma_{ij} w_j - \lambda \bar{r}_i - \mu = 0 \quad \text{for } i = 1, 2, \dots, n$$

$$\lambda : \quad \sum_{i=1}^n w_i \bar{r}_i = \bar{r}_P \quad L = \frac{1}{2} \sum_{i,j=1}^n w_i \sigma_{ij} w_j - \lambda \left(\sum_{i=1}^n w_i \bar{r}_i - \bar{r}_P \right) - \mu \left(\sum_{i=1}^n w_i - 1 \right)$$

$$\mu : \quad \sum_{i=1}^n w_i = 1 \quad L : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$$

$\Rightarrow n + 2$ equations for $n + 2$ unknowns $w_1, w_2, \dots, w_n, \lambda, \mu$.

These equations characterize the efficient portfolios!

Vector Notation

Define

- $\mathbf{w} = (w_1, w_2, \dots, w_n) \in \mathbb{R}^n$ vector of portfolio weights;
- $\bar{\mathbf{r}} = (\bar{r}_1, \bar{r}_2, \dots, \bar{r}_n) \in \mathbb{R}^n$ vector of exp. asset returns;
- $\mathbf{e} = (1, 1, \dots, 1) \in \mathbb{R}^n$ vector of 1's;
- $\mathbf{0} = (0, 0, \dots, 0) \in \mathbb{R}^n$ vector of 0's;
- covariance matrix of asset returns

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{pmatrix} \in \mathbb{R}^{n \times n}.$$

Markowitz Revisited

In vectorial notation, the **Markowitz problem** reads:

$$\begin{aligned}
 &\text{minimize} && \frac{1}{2} \mathbf{w}^\top \Sigma \mathbf{w} = \frac{1}{2} \sum_{i,j} w_i \underbrace{\sum_j \Sigma_{ij} w_j}_{\text{entry of } \Sigma} \\
 &\text{subject to} && \mathbf{w}^\top \bar{\mathbf{r}} - \bar{r}_P = 0 \quad \sum_N w_i \bar{r}_i = \bar{r}_P \\
 &&& \mathbf{w}^\top \mathbf{e} - 1 = 0 \quad \sum_{i=1} w_i = 1
 \end{aligned}$$

The associated **Lagrangian function** can be rewritten as

$$L(\mathbf{w}, \lambda, \mu) = \underbrace{\frac{1}{2} \mathbf{w}^\top \Sigma \mathbf{w}}_{\text{objective}} - \underbrace{\lambda (\mathbf{w}^\top \bar{\mathbf{r}} - \bar{r}_P)}_{-\lambda \times \text{constraint \#1}} - \underbrace{\mu (\mathbf{w}^\top \mathbf{e} - 1)}_{-\mu \times \text{constraint \#2}}$$

while the **optimality conditions** become

$$\Sigma \mathbf{w} - \lambda \bar{\mathbf{r}} - \mu \mathbf{e} = \mathbf{0}, \quad \bar{\mathbf{r}}^\top \mathbf{w} = \bar{r}_P \quad \text{and} \quad \mathbf{e}^\top \mathbf{w} = 1.$$

$$\frac{1}{2} (\Sigma + \Sigma^\top) \mathbf{w} - \lambda \bar{\mathbf{r}} - \mu \mathbf{e} = \mathbf{0} \quad \Sigma + \Sigma^\top = 2\Sigma \quad (\Sigma \text{ is symmetric})$$

Solution of Optimality Conditions

The optimality conditions

$$\Sigma w - \lambda \bar{r} - \mu e = 0, \quad \bar{r}^\top w = \bar{r}_P \quad \text{and} \quad e^\top w = 1$$

can be written as one vectorial equation

$$\underbrace{\begin{pmatrix} \Sigma & -\bar{r} & -e \\ -\bar{r}^\top & 0 & 0 \\ -e^\top & 0 & 0 \end{pmatrix}}_{\substack{(n+2) \times (n+2) \\ A}} \underbrace{\begin{pmatrix} w \\ \lambda \\ \mu \end{pmatrix}}_{(n+2)} = \begin{pmatrix} 0 \\ -\bar{r}_P \\ -1 \end{pmatrix}.$$

This is solvable if Σ has full rank and \bar{r} is not a multiple of e .

$$\Rightarrow \begin{pmatrix} w \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} \Sigma & -\bar{r} & -e \\ -\bar{r}^\top & 0 & 0 \\ -e^\top & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ -\bar{r}_P \\ -1 \end{pmatrix}$$

A has linearly independent rows,

Markowitz Model w/o Short Selling

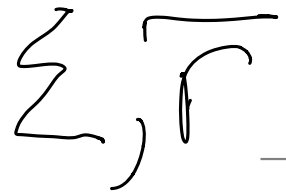
$$\begin{array}{ll}
 \text{minimize} & \frac{1}{2} \sum_{i,j=1}^n w_i \sigma_{ij} w_j \\
 \text{subject to} & \sum_{i=1}^n w_i \bar{r}_i = \bar{r}_P \\
 & \sum_{i=1}^n w_i = 1 \\
 & w_i \geq 0 \quad \text{for } i = 1, 2, \dots, n
 \end{array}$$

λ
 μ
 $K; i=1..n$

long \equiv buy
 short \equiv sell.

- This problem cannot be reduced to the solution of a set of linear equations. It is termed a **quadratic program**.
- Such problems can be solved via **special computer programs** (use e.g. the function 'quadprog' in Matlab).

Parameter Estimation



- Means, variances, and covariances of the asset returns must be **estimated from historical data**.
- Select a **basic period length** p (e.g. $p = 1/12$ for monthly periods).
- For a given asset, assume that we have n **samples** of returns r_1, r_2, \dots, r_n corresponding to **non-overlapping** periods of length p .
- Assume that these returns are
 - **independent** and
 - **identically distributed** with common mean value \bar{r} and variance σ^2 .

Estimation of \bar{r}

- The best estimate $\hat{\bar{r}}$ of the (unknown) mean rate of return \bar{r} is obtained by **averaging the samples**:

$$\hat{\bar{r}} = \frac{1}{n} \sum_{i=1}^n r_i$$

$\hat{\bar{r}} \equiv \text{estimate of } \bar{r}$

- Note: the value $\hat{\bar{r}}$ is itself **random**! If we used a different set of n samples, we would obtain a different $\hat{\bar{r}}$.
- What are the mean and variance of $\hat{\bar{r}}$?

- $E(\hat{\bar{r}}) = E\left(\frac{1}{n} \sum_{i=1}^n r_i\right) = \bar{r}$ because $E(r_i) = \bar{r}$

- $\text{var}(\hat{\bar{r}}) = E[(\hat{\bar{r}} - \bar{r})^2] = E\left[\frac{1}{n} \sum_{i=1}^n (r_i - \bar{r})\right]^2 = \frac{1}{n} \sigma^2$ (use i.i.d. assumption)

$\Rightarrow \hat{\bar{r}}$ is an **unbiased estimator** for \bar{r} .

$\frac{1}{n^2} E\left[\sum_{i=1}^n (r_i - \bar{r})\right]^2 = \frac{1}{n^2} \left[\sum_{i=1}^n E[(r_i - \bar{r})^2] + 2 \sum_{i < j} E[(r_i - \bar{r})(r_j - \bar{r})] \right]$

$= \frac{1}{n^2} \left[\sum_{i=1}^n \sigma^2 + 2 \sum_{i < j} 0 \right] = \frac{1}{n} \sigma^2$

Estimation of σ^2 I

Diagonal matrix
since $\sigma_{ij} = 0$ (i.i.d)

- An estimate $\hat{\sigma}^2$ of the (unknown) variance of the rate of return σ^2 is given by the **sample variance**:

$$\hat{\sigma}^2 = \frac{1}{n-1} \sum_{i=1}^n (r_i - \hat{r})^2.$$

- Note that this formula uses the **sample mean** \hat{r} as an input.
- The value of $\hat{\sigma}^2$ is also **random**!

Estimation of σ^2 II

$\hat{\sigma}^2$ is an unbiased estimator of σ^2 : \hat{r}

$$\begin{aligned}
 E(\hat{\sigma}^2) &= E\left(\frac{1}{n-1} \sum_{i=1}^n [r_i - \frac{1}{n} \sum_{j=1}^n r_j]^2\right) \\
 &= E\left(\frac{1}{n-1} \sum_{i=1}^n [(r_i - \bar{r}) - \frac{1}{n} \sum_{j=1}^n (r_j - \bar{r})]^2\right) \quad (+/- \bar{r}) \\
 &= E\left(\frac{1}{n-1} \sum_{i=1}^n [(r_i - \bar{r})^2 - \frac{2}{n} \sum_{j=1}^n (r_i - \bar{r})(r_j - \bar{r}) + \frac{1}{n^2} \sum_{j,k=1}^n (r_j - \bar{r})(r_k - \bar{r})]\right) \quad (\text{expand square}) \\
 &= \frac{1}{n-1} \left(\sum_{i=1}^n E[(r_i - \bar{r})^2] - \frac{1}{n} \sum_{i,j=1}^n E[(r_i - \bar{r})(r_j - \bar{r})] \right) \\
 &= \frac{1}{n-1} (n\sigma^2 - \frac{1}{n} n\sigma^2) \quad \hookrightarrow \text{Bring in } \sum_i, \text{ use } \sigma_{ij}=0 \\
 &= \sigma^2
 \end{aligned}$$

If the returns are normally distributed, it can be shown that

$$\boxed{\text{var}(\hat{\sigma}^2) = \frac{2\sigma^4}{n-1}} \rightarrow 0 \text{ as } n \rightarrow \infty$$

Estimation of Covariances

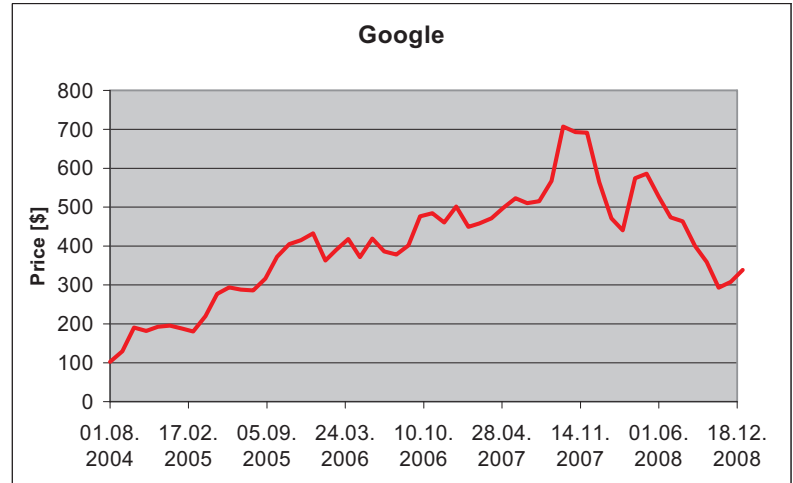
- Assume that $r_{A,1}, r_{A,2}, \dots, r_{A,n}$ and $r_{B,1}, r_{B,2}, \dots, r_{B,n}$ are the **returns of assets A and B** over non-overlapping periods of length p .
- An estimate $\hat{\sigma}_{AB}$ of the (unknown) covariance σ_{AB} is given by the **sample covariance**:

$$\hat{\sigma}_{AB} = \frac{1}{n-1} \sum_{i=1}^n (r_{A,i} - \hat{r}_A)(r_{B,i} - \hat{r}_B).$$

- Note that this formula uses the **sample means \hat{r}_A and \hat{r}_B** as inputs.
- The value of $\hat{\sigma}_{AB}$ is **random**!
- It can be shown that **$E(\hat{\sigma}_{AB}) = \sigma_{AB}$** .

Monthly Returns of Google Stock Price

Date	Close	Return	Squared Err.
Feb 09	340.45		
Jan 09	338.53	5.67E-03	5.97E-04
Dec-08	307.65	1.00E-01	4.94E-03
Nov 08	292.96	5.01E-02	4.02E-04
Oct-08	359.36	-1.85E-01	4.62E-02
Sep 08	400.52	-1.03E-01	1.77E-02
Aug 08	463.29	-1.35E-01	2.74E-02
Jul 08	473.75	-2.21E-02	2.72E-03
Jun 08	526.42	-1.00E-01	1.69E-02
May-08	585.8	-1.01E-01	1.73E-02
Apr 08	574.29	2.00E-02	1.01E-04
Mar-08	440.47	3.04E-01	7.49E-02
Feb 08	471.18	-6.52E-02	9.08E-03
Jan 08	564.3	-1.65E-01	3.81E-02
Dec-07	691.48	-1.84E-01	4.58E-02
Nov 07	693	-2.19E-03	1.04E-03
Oct-07	707	-1.98E-02	2.49E-03
Sep 07	567.27	2.46E-01	4.67E-02
Aug 07	515.25	1.01E-01	5.02E-03
Jul 07	510	1.03E-02	3.92E-04
Jun 07	522.7	-2.43E-02	2.96E-03



Sample average	3.01E-02
Sample variance	1.68E-02
Sample std. deviation	1.30E-01

\uparrow
 Date

\uparrow
 Price

\downarrow
 Return.

#5 (Leave things as they are + CAPM in C.W)

#19. (Release C.W on 2/09, same deadline) same Qs.