IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2019-2020

MSc in Computing Science (Specialist)
MEng Honours Degrees in Computing Part III
MEng Honours Degree in Mathematics and Computer Science Part III
BEng Honours Degree in Mathematics and Computer Science Part III
MEng Honours Degree in Mathematics and Computer Science Part IV
MEng Honours Degree in Electronic and Information Engineering Part IV
BEng Honours Degree in Electronic and Information Engineering Part III
MEng Honours Degree in Electronic and Information Engineering Part III
BEng Honours Degree in Computing Part III
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute

PAPER C343

OPERATIONS RESEARCH

Friday 13th December 2019, 14:00
Duration: 120 minutes

Answer THREE questions

Paper contains 4 questions
Calculators required

1 a i) Formulate a linear programming problem for finding a vector satisfying

$$3x_1 + x_2 \leq 5 \quad \text{and} \quad x_1 \geq 0, x_2 \geq 0$$

and having the maximum of

$$2x_1 - x_2 \quad \text{and} \quad -3x_1 + 2x_2$$

as small as possible.

ii) Find a solution of the linear program using the simplex algorithm, justifying at each step the chosen pivot.

iii) Indicate which pivots in your previous answer are degenerate, if any. Would the sequence of pivots change if you were to use Bland's rule? Explain.

b Assume that, $\forall i = 1, \ldots, n$, there exists a basic solution $\hat{x}$ with $\hat{x}_i \neq 0$. Prove that if a basic solution x is degenerate then it is associated with more than one index set.

c Suppose that at the end of Phase 1 of the two-phase simplex method the following tableau is obtained, in which x_1, x_2, x_3, x_4 are standard variables, ζ is the Phase-1 objective function, and ξ_1, ξ_2 are artificial variables. Write the initial tableau for Phase 2.

BV	x_1	x_2	x_3	x_4	ξ_1	ξ_2	RHS
ζ	0	0	1	0	-2	0	0
x_2	0	1	-2	-1	2	0	3
x_1	1	0	1	1	-1	0	0
ξ_2	0	0	1	0	-1	1	0

d Suppose that we are given two sets of points x_i, $i = 1, \ldots, N$, and y_j, $j = 1, \ldots, M$ in a d-dimensional space. Explain what the linear separability problem is and why it can be efficiently solved using linear programming.

The four parts carry, respectively, 45%, 20%, 20%, and 15% of the marks.

2 a A post office requires different numbers of full-time employees on different days of the week. The number of full-time employees required on each day is given in the following table:

Monday	17
Tuesday	13
Wednesday	15
Thursday	19
Friday	14
Saturday	16
Sunday	11

Union rules state that each full-time employee must work five consecutive days and then receive two days off. For example, an employee who works Monday to Friday must be off on Saturday and Sunday. The post office wants to meet its daily requirements using only full-time employees. The daily pay for a full-time employee is £100 Monday to Friday and £200 on Saturday and Sunday.

Formulate an LP that the post office can use to minimize the expenditure for the full-time employees who must be hired.

b A two person zero-sum game with an $n \times n$ payoff matrix A is called a symmetric game if A is related to its transpose A^T through the expression $A = -A^T$.

i) Formulate the (mixed) strategies of the row and column players using linear programming.

ii) It is possible to show that a symmetric game must have a value of zero. Using this observation, show that in a symmetric game if $(x_1^*, x_2^*, ..., x_n^*)$ is an optimal strategy for the row player, then $(x_1^*, x_2^*, ..., x_n^*)$ is also the optimal strategy for the column player.

iii) Consider the following symmetric game

$$\begin{bmatrix} 0 & -1 & +1 \\ +1 & 0 & -1 \\ -1 & +1 & 0 \end{bmatrix}$$

Does this game have a Nash equilibrium in pure strategies? If so, determine it. Otherwise, explain why it does not exist.

The two parts carry, respectively, 40% and 60% of the marks.

3 **Duality**

a In following LP, $c \in \mathbb{R}$ is constant and $x_1, x_2 \in \mathbb{R}$ are variables:

$$\begin{aligned} \max_{x_1,x_2} z_{\text{LP}} = & c \cdot x_1 - 4 \cdot x_2 \\ & x_1 - \quad x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned} \qquad \text{(LP)}$$

i) Construct the dual LP.

The dual LP does not have to be expressed in standard form.

ii) Suppose that $c = 2$. Determine the optimal solution of the dual LP. Use the optimal solution of the dual LP to get the optimal primal value z^*_{LP} and point (x^*_1, x^*_2).

It's fine to answer this question by inspection. There's no need to get the problem into standard form or use the simplex method.

iii) For which values of c does the primal LP have a feasible solution? For which values of c does the dual LP have a feasible solution?

b Let:

$$v(\ b) = \min \left\{ c^T x \mid Ax = \ b,\, x \geq 0 \right\}, \qquad \text{(P1)}$$

with $c,\, x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$. If each element in the vector b is multiplied by 3, we consider the perturbed problem:

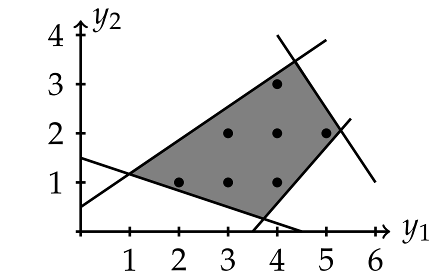
$$v(3b) = \min \left\{ c^T x \mid Ax = 3b,\, x \geq 0 \right\}. \qquad \text{(P2)}$$

In general, the solution of Problem (P2) does not have the same basic representation, and hence the optimal basis matrix, as Problem (P1). Using the optimal basis matrix of Problem (P1), define its shadow prices $\Pi \in \mathbb{R}^m$ and use the shadow prices to establish a relationship between $v(b)$ and $v(3b)$. Please justify each step.

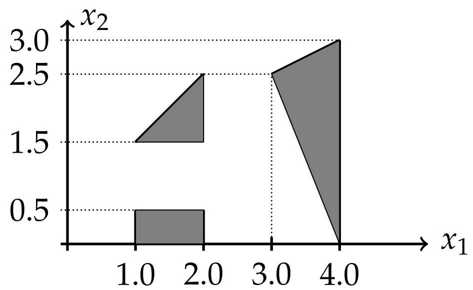
The two parts carry, respectively, 60% and 40% of the marks.

4 **Integer Programming**

a An ILP, defined on $y_1 \in \{1,2,3,4,5,6\}$, $y_2 \in \{1,2,3,4\}$, is constrained by the illustrated inequalities, i.e. the black dots are the feasible points. Justify which of the following are valid ILP inequalities? Which of the valid inequalities would be useful cutting planes, i.e. eliminate part of the relaxed feasible space? (i) $y_1 \le 4$, (ii) $y_1 + y_2 \ge 3$, (iii) $y_1 + y_2 \le 10$.



b Develop constraints restricting the feasible set to the shaded region shown in the figure. In words, variable x_1 is either between $[1.0, 2.0]$ or $[3.0, 4.0]$. On $x_1 \in [1.0, 2.0]$, x_2 is restricted by either $x_2 \in [0.0, 0.5]$ or $x_2 \ge 1.5$ and $x_2 \le 2x_1 - 1.5$. On $x_1 \in [3.0, 4.0]$, x_2 is restricted by $x_2 \ge -2.5x_1 + 10$ and $x_2 \le 0.5x_1 + 1$. Formulate the mixed-integer linear optimization constraints but do not solve it. Justify the values of any big-M parameters.



c Use the branch-and-bound algorithm to find an optimal integer solution to the following ILP. Please justify each branch-and-bound step. There is no need to show the simplex algorithm.

Hint: The relaxed LP solution is $z^*_{\text{LP}} = 165$, $x^*_{1,\text{LP}} = 2.25$, $x^*_{2,\text{LP}} = 8$.

$$
\begin{aligned}
\max_{x_1,x_2} z = \quad & 20x_1 + 15x_2 \\
\text{s.t.} \quad & 2\ x_1 + \quad x_2 \le 12.5 \\
& 2\ x_1 - \quad x_2 \le 0 \\
& \qquad\qquad x_2 \le 8 \\
& x_1, x_2 \ge 0, x_1, x_2 \in \mathbb{N}_0
\end{aligned}
$$

The three parts carry, respectively, 20%, 50%, and 30% of the marks.