IMPERIAL COLLEGE LONDON

TIMED REMOTE ASSESSMENTS 2020-2021

BEng Honours Degree in Computing Part III
BEng Honours Degree in Electronic and Information Engineering Part III
MEng Honours Degree in Electronic and Information Engineering Part III
MEng Honours Degree in Electronic and Information Engineering Part IV
MEng Honours Degree in Mathematics and Computer Science Part IV
BEng Honours Degree in Mathematics and Computer Science Part III
MEng Honours Degree in Mathematics and Computer Science Part III
MEng Honours Degrees in Computing Part III
MSc in Computing (Specialism)
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant assessments for the Associateship of the City and Guilds of London Institute

PAPER COMP60016=COMP96025=COMP96026

OPERATIONS RESEARCH

Monday 7 December 2020, 14:00
Duration: 140 minutes
Includes 20 minutes for access and submission

Answer ALL TWO questions
Open book assessment

By completing and submitting work for this assessment, candidates confirm that the submitted work is entirely their own and they have not (i) used the services of any agency or person(s) providing specimen, model or ghostwritten work in the preparation of the work they have submitted for this assessment, (ii) given assistance in accessing this paper or in providing specimen, model or ghostwritten answers to other candidates submitting work for this assessment.

Paper contains 2 questions

1 a Solve the following linear program using the two-phase simplex algorithm, justifying at each step the chosen pivot.

$$\max y = x_1 + x_2$$

subject to

$$\begin{aligned} -x_1 + x_2 &\leq 2 \\ -2x_1 + 3x_2 &\geq 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

b Consider the linear programming problem

$$\min\{c^T x \mid Ax = b, x \geq 0\}$$

and consider N optimal basic feasible solutions $x^{(1)}$, $x^{(2)}$, ..., $x^{(N)}$. A weighted average of these N solutions is defined to be a vector

$$x_a = \sum_{k=1}^{N} \alpha_k x^{(k)}$$

where the weights α_k are such that $0 \leq \alpha_k \leq 1$; $k = 1, \ldots, N$; and $\sum_{k=1}^{N} \alpha_k = 1$. Show that every weighted average of the optimal basic feasible solutions must also be <u>both</u> feasible and optimal. (*Hint*: If there are more than one optima for a linear programming problem, all optima have the same objective function value.)

c For each of the next statements, state if it is *True* or *False*, giving a brief <u>justification</u>.

i) The worst-case complexity of the simplex algorithm is exponential in the number of variables n.

ii) A zero reduced cost associated to a non-basic variable x_q implies that the objective cannot ever increase by changing the value of x_q.

iii) Basic solutions are points where at least one slack variable becomes zero.

iv) The objective function of the phase-1 simplex algorithm is zero only after the artificial variables become non-basic.

d Write a linear program in standard form to solve the following problem (Note: You are <u>not</u> asked to solve the linear program):

$$\min \frac{\max\{x_1 + 3x_2, 1 - 2x_2\}}{x_1 + x_2}$$

subject to

$$\begin{aligned} x_1 + 3x_2 &= 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

e A company is considering seven large capital investments. The investments differ in the estimated long-run profit they will generate as well as in the amount of capital required, as shown by the following table (in units of millions of Pounds):

	Investment Opportunity						
	A	B	C	D	E	F	G
Estimated long-run profit:	2	1	5	9	7	3	2
Capital required:	4	2	3	4	1	3	2

The total amount of capital available for these investments is £10 million. Investment opportunities A and C are mutually exclusive. Similarly, opportunities B, D and E are mutually exclusive. Furthermore, neither B nor D can be undertaken unless either A or C is undertaken. Further, if G is undertaken then also A must be undertaken. The objective is to select the combination of capital investments that will maximise the total estimated long run profit while satisfying the budget constraint on the total capital. Write the 0-1 integer programming formulation for this problem. (Do *not* solve the integer programming problem.)

The five parts carry equal marks.

2 a Consider the following parametric linear program

$$V_{LP}(p) = \max 5x_1 + 6x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &\leq p \\ 5x_1 + 9x_2 &\leq 45 \\ x_1, x_2 &\geq 0. \end{aligned} \tag{1}$$

i) Verify that an optimal basis matrix is given by $B = \begin{bmatrix} 1 & 1 \\ 5 & 9 \end{bmatrix}$ when $p = 6$.

ii) Using the shadow prices, determine the expression of the program's value $V_{LP}(p)$ for all real values of the parameter $5 \leq p \leq 9$.

b Consider the integer version of (1) where the variables x_1, x_2 are additionally constrained to $x_1, x_2 \in \mathbb{N}_0$, and denote with $V_{IP}(p)$ its value. State if each of the following sentences is *True* or *False*, briefly justifying each answer.

i) $V_{LP}(p) \geq V_{IP}(p)$ for all real values of $5 \leq p \leq 9$.

ii) $V_{LP}(p_1) \leq V_{LP}(p_2)$ for all real values of $5 \leq p_1 \leq p_2 \leq 9$.

iii) There exist $5 \leq p_1 \leq p_2 \leq 9$ such that $V_{LP}(p_1) < V_{IP}(p_2)$.

c Consider the integer version of (1) where x_1, x_2 are additionally constrained to $x_1, x_2 \in \mathbb{N}_0$. Fix $p = 6$ and determine the Gomory cut corresponding to x_2. How does this constraint read in the original variables x_1, x_2?

d For any fixed $\alpha \in \mathbb{R}$, $\alpha \geq 0$ consider the following linear program

$$\max x_2$$

subject to

$$\begin{aligned} x_1 &\leq 1/2 \\ -2(\alpha+1)x_1 + x_2 &\leq 0 \\ x_1, x_2 &\geq 0. \end{aligned} \tag{2}$$

i) Verify that $x^{LP} = (1/2, \alpha + 1)$ is the unique solution of the program.

ii) Denote with x^{IP} the solution of the corresponding integer program where x_1, x_2 are additionally constrained to $x_1, x_2 \in \mathbb{N}_0$. Determine x^{IP}.

iii) What is the difference $x_2^{LP} - x_2^{IP}$ as a function of α?
[*Note:* x_2^{LP} denote the second component of x^{LP}, similarly for x_2^{IP}]

iv) Assume $\alpha = 10000$, is x^{LP} a good approximation of x^{IP}? Discuss.

The four parts carry, respectively, 40%, 20%, 20%, and 20% of the marks.