IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2015

MEng Honours Degree in Mathematics and Computer Science Part IV
MEng Honours Degrees in Computing Part IV
MSc in Advanced Computing
MSc in Computing Science
MSc in Computing Science (Specialist)
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute

PAPER C422

COMPUTATIONAL FINANCE

Thursday 26 March 2015, 14:00
Duration: 120 minutes

Answer THREE questions

Paper contains 4 questions
Calculators required

1 **Fixed-Income Securities.**

a Assume that there is a constant ideal bank which offers a yearly interest rate of 6% compounded monthly. Consider an annuity whose equal cash flows of size £100 are paid at the end of each month over a period of ten years. Compute the size of a single cash flow arising after five years that is equivalent to the annuity.

b Let s_k be the spot rate associated with maturity k, that is, the present value of one pound to be received after k years under yearly compounding. Show explicitly that if the spot rate curve is flat (with $s_k = r$ for all $k \in \mathbb{N}$), then all forward rates $f_{k,l}$, $k < l \in \mathbb{N}$, are also equal to r.

c A corporation issues a 10%, 20-year bond at a time when yields are 10%. The bond has a call provision that allows the corporation to force a bond holder to redeem his or her bond at face value plus 5%. After 5 years the corporation finds that exercise of this call provision is advantageous. What can you deduce about the yield at that time? (Assume one coupon payment per year).

d Let $s^0 = (s_1^0, s_2^0, s_3^0, \ldots, s_n^0)$ be the initial spot rate curve sequence (based on m periods per year). Let $s(\lambda) = (s_1, s_2, s_3, \ldots, s_n)$ be spot rates parameterised by λ, where

$$1 + \frac{s_k}{m} = e^{\lambda/m}(1 + s_k^0/m)$$

for $k = 1, 2, \ldots, n$. Suppose a bond price $P(\lambda)$ is determined be these spot rates. Show that

$$-\frac{1}{P}\frac{dP}{d\lambda} = D$$

is a pure duration. In other words, find D and describe its meaning.

The four parts carry, respectively, 15%, 20%, 25%, and 40% of the marks.

2 **Mean-Variance Portfolio Theory.**

a Suppose that to short a stock you are required to deposit an amount equal to the initial price X_0 of the stock. At the end of 1 year the stock price is X_1 and you liquidate your position. You receive your profit from shorting equal to $X_0 - X_1$ and you recover your original deposit. If R is the total return of the stock, what is the total return of your short?

b Two stocks are available. The corresponding rates of return are $\bar{r}_1$ and $\bar{r}_2$; the corresponding variances and covariances are σ_1^2, σ_2^2 and σ_{12}. What percentages of total investment should be invested in each of the two stocks to minimise the total variance of the rate of return of the resulting portfolio? What is the mean rate of return of this portfolio?

c Describe the assumptions of the one-fund theorem. Describe the implications of this theorem in terms of the funds investors should consider in a market that satisfies these assumptions.

d Suppose that a market consists of n uncorrelated assets. You may invest in any one, or in a combination of them. The mean rate of return is $\bar{r}$ and is the same same for each asset, but the variances are different. The return of asset i has a variance of σ_i^2, for $i = 1, \ldots, n$. Find the minimum variances point. Express your result in terms of

$$\bar{\sigma}^2 = \left(\sum_{i=1}^{n} \frac{1}{\sigma_i^2} \right)^{-1} .$$

The four parts carry, respectively, 20%, 35%, 15%, and 30% of the marks.

3 **General Principles of Risk.**

a Suppose an investor has exponential utility function $U(x) = -e^{ax}$ and an initial wealth level of W. The investor is faced with an opportunity to invest an amount $w \leq W$ and obtain a random payoff x. Show that the evaluation of the investor of this investment opportunity is independent of W.

b Suppose an investor has utility function U. There are n risky assets with rates of return r_i, $i = 1, 2, \ldots, n$ and one risk-free asset with rate of return r_f. The investor has initial wealth W. State the optimisation problem the investor will solve.

Suppose that the optimal portfolio for this investor has a (random) payoff of x^*. Show that

$$\mathrm{E}\left[\frac{dU(x^*)}{dx}(r_i - r_f)\right] = 0$$

for $i = 1, \ldots, n$.

c Suppose an investor uses the quadratic utility function $U(x) = x - \frac{1}{2}cx^2$. Suppose that there are n risky assets and one risk free asset with total return R. Let R_M be the total return on the optimal portfolio of risky assets. Show that the expected total return of any asset i is given by the formula,

$$\overline{R}_i = R + \beta_i(\overline{R}_M - R)$$

where $\beta_i = \mathrm{cov}(R_M, R_i)/\sigma_M^2$.
Hint: Use the result in (b).

The three parts carry, respectively, 20%, 30%, and 50% of the marks.

4 **Pricing a Butterfly Spread.** The current price of stock A is £120. Every month, the stock price either increases by a factor 1.25 with probability $4/7$ or it decreases by a factor 0.8 with probability $3/7$. To keep the calculations simple, we assume that interest rates are zero. Consider a butterfly spread written on stock A. The butterfly spread is formed by buying two call options, one with strike price K_1 and another with strike price K_3, and by selling two units of a call option with strike price K_2, where $K_1 < K_2 < K_3$. All options are written on stock A, have the same expiration date, and are of European style.

a Determine an explicit mathematical expression for the payoff of the butterfly spread at expiration as a function of the price of stock A.

b Assume that the three call options from the definition of the butterfly spread expire in three months from now and have strike prices $K_1 = £60$, $K_2 = £100$, and $K_3 = £140$. Compute the arbitrage-free price of the butterfly spread.

c Assume now that the butterfly spread considered in part (b) may be exercised early, that is, all call options from the definition of the butterfly spread may be exercised early (but at the same time!). Compute the arbitrage-free price of this modified butterfly spread with an early exercise right.

The three parts carry, respectively, 20%, 50%, and 30% of the marks.