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IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2017-2018

BEng Honours Degree in Computing Part III
BEng Honours Degree in Electronic and Information Engineering Part III
MEng Honours Degree in Electronic and Information Engineering Part III
MEng Honours Degree in Mathematics and Computer Science Part IV
BEng Honours Degree in Mathematics and Computer Science Part III
MEng Honours Degree in Mathematics and Computer Science Part III
MEng Honours Degrees in Computing Part III
MSc in Computing Science (Specialist)
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute

PAPER C343

OPERATIONS RESEARCH

Friday 15 December 2017, 14:00
Duration: 120 minutes

Answer THREE questions

Paper contains 4 questions
Calculators not required

1 a You are given the following linear program:

$$\max y = x_1 + x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &= 12 \\ 1.5x_1 + x_2 &\leq 15 \\ 0.5x_1 + x_2 &\leq 10 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solve the linear program using the simplex algorithm.

b i) Give an example of optimal simplex tableau for a linear program with *multiple* optimal basic feasible solutions. Justify your answer.

ii) In linear programming show that a variable that becomes nonbasic in one iteration of the simplex method cannot become basic in the next iteration. *Hint*: Consider the sign of the coefficient in the objective row for the variable that becomes nonbasic.

The two parts carry equal marks.

2a A furniture production company sells two products: a console and a table. A console is sold at 400£ and requires 40 hours to produce. A table is sold at 300£ and requires 60 hours to produce. The company wants to schedule production for two periods. Both periods have $20,000$ hours of available manual work each. The period demand is as follows:

Demand	Period 1	Period 2
Console	200	150
Table	100	300

When production exceeds demand in a period, the unsold products are stored in an inventory. The inventory can store at most 200 consoles and 100 tables. It is possible to both produce and sell the products in the same period.

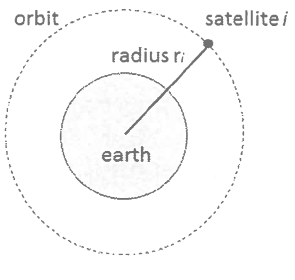
The company wants to use an integer linear program (ILP) to maximizes profit at the end of period 2. Assume that the inventory is empty at the start of period 1. The ILP must require that the inventory is empty again at the end of period 2. *Note*: you are *not* requested to solve the ILP.

i) Define the decision variables for the ILP, explaining their meaning.

ii) Specify the constraints for the ILP.

iii) Specify the objective function of the ILP.

b A space agency wants to deliver n artificial satellites in orbit. The agency needs to assign the radius r_i of the orbit of each satellite i, $i = 1, \ldots, n$. As shown, for a satellite i its orbit radius is defined as the distance from the centre of the Earth.



The lowest acceptable radius for satellite i is given and equal to d_i miles. The cost to deliver satellite i increases linearly with the radius r_i at rate m_i pounds per mile. A solution must ensure that the minimum distance between any two satellites is at least 10 miles.

Formulate this decision problem as a linear program in standard form. (Do *not* solve the linear program.)

The two parts carry equal marks.

3 Game Theory & Duality

a Consider the payoff matrix of a two-player zero-sum game. The two players are Hawkeye (row player) and Black Widow (column player). The elements correspond to the reward of the row player (Hawkeye).

		Black Widow				
		1	2	3	4	5
Hawkeye	1	1.35	1.2	1.3	1.4	1.5
	2	1.5	1.35	1.3	1.4	1.5
	3	1.4	1.4	1.35	1.4	1.5
	4	1.3	1.3	1.3	1.35	1.5
	5	1.2	1.2	1.2	1.2	1.35

i) Use the minimax criterion to find the best strategy for each side.

ii) Is there a Nash equilibrium to this game?

b Consider the primal linear programming problem (P):

$$\begin{aligned} \min_{\mathbf{x}} \; z = \;& \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}, \end{aligned} \qquad \text{(P)}$$

with variables $\mathbf{x} \in \mathbb{R}^n$ and parameters $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$, and $\mathbf{c} \in \mathbb{R}^n$. Assume that both the primal (P) and its dual (D) are feasible. Let $\mathbf{y}_\mathbf{D}^\star$ be a known solution to the dual.

i) Write down the dual linear programming problem (D).

ii) What is the dimension of $\mathbf{y}_\mathbf{D}^\star$, i.e. how many variables are in the vector?

iii) Suppose that the kth equation of the primal (P) is multiplied by $\mu \neq 0$. Find a solution $\hat{\mathbf{y}}_\mathbf{D}^\star$ corresponding to the new dual problem.

iv) Suppose that, in the original primal (P), we add μ times the kth equation to the rth equation. Find a solution $\tilde{\mathbf{y}}_\mathbf{D}^\star$ corresponding to the new dual problem.

The two parts carry, respectively, 30% and 70% of the marks.

4 Integer Programming

a Consider the linear program (Q). Reformulate (Q) so that x_1 and x_2 can only take finite values: $x_1 \in \{0, 30, 60\}, x_2 \in \{0, 15, 30\}$.

$$\begin{array}{lll}
\max\limits_{x_1,x_2,x_3} & 18x_1 + 3x_2 + 9x_3 & \\
\text{s.t.} & 2x_1 + \ \ x_2 + 7x_3 \leq 150 & \\
& 0 \leq x_1 \leq 60 & \qquad\qquad\qquad \text{(Q)} \\
& 0 \leq x_2 \leq 30 & \\
& 0 \leq x_3 \leq 20. &
\end{array}$$

Do not solve the integer program.

b Solve the following integer program. You may use any combination of: branch & bound, Gomory cuts, and/or knapsack cover cuts.

$$\begin{array}{ll}
\max\limits_{x_1,x_2,x_3} & 15x_1 + 10x_2 + 20x_3 \\
\text{s.t.} & 6x_1 + \ 3x_2 + 10x_3 \ \leq 12 \\
& x_1, x_2, x_3 \in \{0, 1\}.
\end{array}$$

c Consider the optimisation problem (IP). Let $z_{IP}^\star$ be the optimal objective value, if it has an optimal solution. Let $z_{LP}^\star$ be the optimal value of the linear programming (LP) relaxation of (IP), if it has an optimal solution.

$$\begin{array}{llll}
\min\limits_{\mathbf{x}} \ z = & \mathbf{c}^T\mathbf{x} & & \\
\text{s.t.} & \mathbf{A}\mathbf{x} = \mathbf{b} & & \\
& x_j \geq 0 & j \in N = \{1, \ldots, n\} & \qquad \text{(IP)} \\
& x_j \in \mathbb{N}_0 = \{0, 1, 2, \ldots\} & j \in Z \subseteq N. &
\end{array}$$

For each of the following, could the statement be true? Please justify.

i) Integer program (IP) is feasible but its LP relaxation is infeasible.

ii) (IP) is infeasible but its LP relaxation has a feasible solution.

iii) $z_{IP}^\star = z_{LP}^\star$

iv) $z_{IP}^\star > z_{LP}^\star$

v) $z_{IP}^\star < z_{LP}^\star$

d Discuss the role of totally unimodular matrices in optimisation.

The four parts carry, respectively, 20%, 30%, 35%, and 15% of the marks.