The fundamental theorem of Linear Programming

LP in standard form:

$$\min_x c^\top x \quad \text{subject to}$$
$$Ax = b$$
$$x \geq 0, \ (b \geq 0).$$

Theorem

Let A be an m-by-n matrix of rank m.

(i) *If there is a feasible solution then there is a basic feasible solution.*

(ii) *If there is an optimal feasible solution then there is an optimal basic feasible solution.*

Proof. Let x be a feasible solution. Then $x \geq 0$, and we may write

$$Ax = \sum_{i \in I} a_i x_i = b, \qquad I \subset \{1, \ldots, n\}, \quad x_i > 0,$$

i.e. indices i such that $x_i = 0$ are deleted. There are two cases:

(a) $\{a_i\}_{i \in I}$ are linearly independent. Then x is already a feasible basic solution. Both (i) and (ii) proven!

(b) $\{a_i\}_{i \in I}$ are linearly dependent.

Then $\sum_{i \in I} a_i y_i = 0$ for some y_i's not all vanishing. Can assume $y_p > 0$ for at least one $p \in I$. Define the vector y with components y_i as above for $i \in I$ and $y_i = 0$ for $i \notin I$. Define $x^\epsilon = x - \epsilon y$. Then,

$$Ax^\epsilon = Ax - \epsilon Ay = b - \epsilon \sum_{i \in I} a_i y_i = b.$$

That is, $Ax^\epsilon = b$ and x^ϵ is feasible for small positive or negative ϵ since

$$x_i^\epsilon = \begin{cases} \overbrace{x_i}^{>0} - \epsilon y_i & i \in I \\ 0 & i \notin I \end{cases}$$

At least for one $p \in I$, $x_p^\epsilon = x_p - \epsilon y_p = 0$ for $\epsilon = x_p/y_p$. Choose the smallest such ϵ and associated p. The resulting x^ϵ is then feasible and

$$Ax^\epsilon = \sum_{i \in I^\epsilon} a_i x_i^\epsilon, \quad I^\epsilon = I \setminus \{ p \},$$

that is, we have found another feasible solution spanned by one less of the columns of A. Continue the same procedure until only linearly independent columns remain. Then, back to case (a) and part (i) is shown.

To show (ii), we also need to show that x^ϵ is optimal if x is optimal. Since x^ϵ is feasible for small positive or negative ϵ, $c^T x^\epsilon = c^T x - \epsilon c^T y < c^T x$ for small ϵ of the same sign as $c^T y$ if $c^T y \neq 0$. Since x is optimal, must have $c^T y = 0$. Thus $c^T x^\epsilon = c^T x$ and x^ϵ is optimal. □