COMPUTATIONAL FINANCE: 422

General Principles

Fri 13^{th} Lecture
Tue 2h Tutorial

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This Lecture

Evaluation of random cash flows:

- direct evaluation using risk measures
  - Utility functions
  - Risk aversion
- indirect evaluation by reducing the flow to a combination of flows which have already been evaluated
  - Linear pricing
  - Portfolio choice
  - Risk-neutral pricing

Further reading:

- D.G. Luenberger: *Investment Science*, Chapter 9

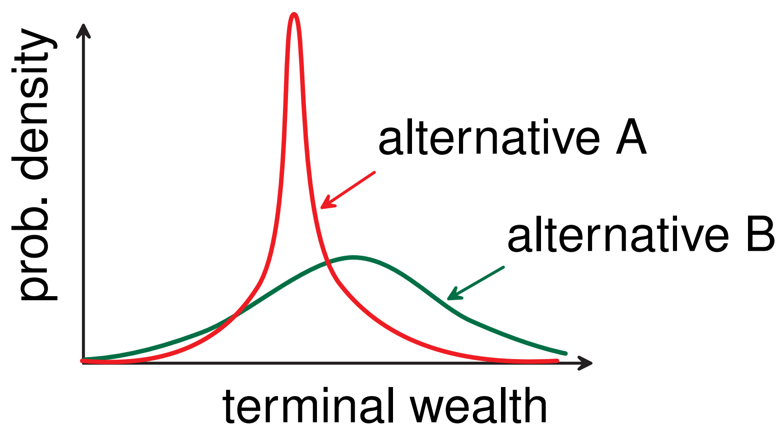
Utility Functions I

- 100 ← for sure
- 100.01 ← on average.

Assume that today there are different investment opportunities which lead to different wealth levels after one year.

General aim: maximize wealth at the end of the year.

- Certain outcomes: select the alternative that produces the highest wealth.
- Random outcomes: not obvious how to rank choices.



Utility Functions II

We need a procedure for ranking random wealth levels.

Utility function U:

- defined on the real line (possible wealth levels);
- gives a real value (utility index).

w_1 $U(w_1)$
w_2 $U(w_2)$
w_3 $U(w_3)$

For a given utility function, alternative random wealth levels are ranked by evaluating their expected utility values.

$\Rightarrow$ we compare random wealth variables x and y by comparing $\mathrm{E}[U(x)]$ and $\mathrm{E}[U(y)]$; the larger value is preferred.

Utility functions vary among decision makers, depending on

- their risk tolerance;
- their individual financial environment.

Utility Functions III

The simplest utility function is $U(x) = x$

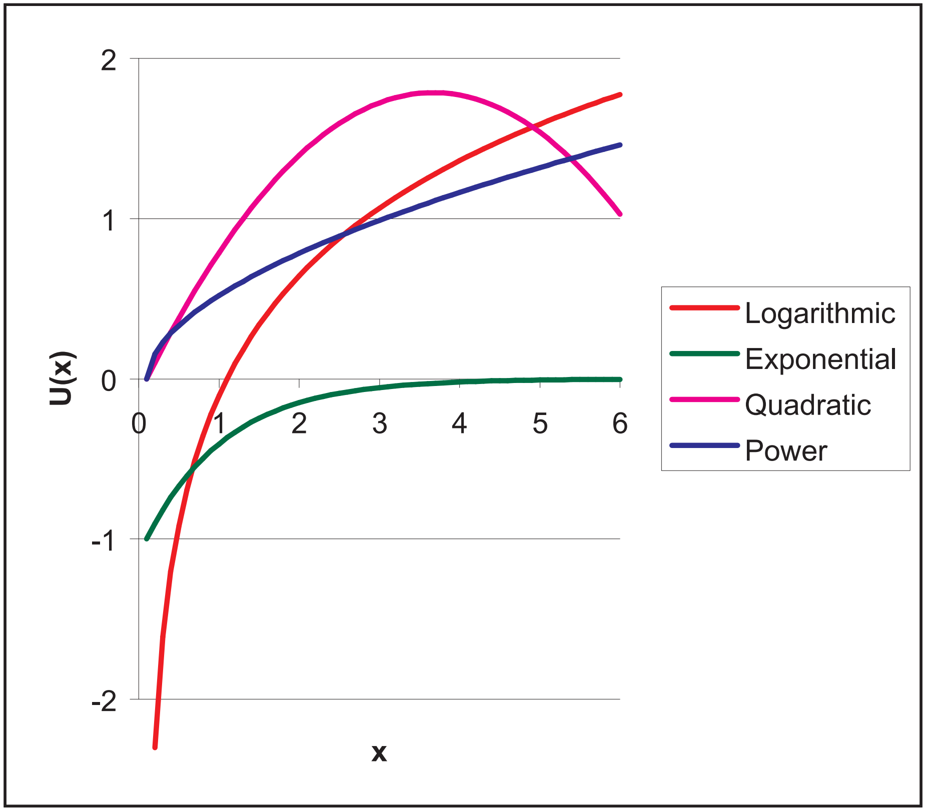
$\Rightarrow$ ranking by expected values!

Individuals using this utility function are called risk neutral.

Some of the most commonly used utility functions:

- Exponential: $U(x) = -e^{-ax}$ for some $a > 0$;
- Logarithmic: $U(x) = \ln(x)$; defined only for $x > 0$;
- Power: $U(x) = bx^b$ for some $b \leq 1$, $b \neq 0$;
- Quadratic: $U(x) = x - bx^2$ for some $b > 0$; this function is increasing only for $x < 1/(2b)$.

Utility Functions IV



Venture Capitalist

Sybil, a venture capitalist, considers two investment alternatives for next year:

1. buy treasury bills, which give $\$6$M for sure;
2. invest in a start-up company; this will produce wealth levels $\$10$M, $\$5$M, and $\$1$M with probabilities 0.2, 0.4, and 0.4, respectively.

Sybil uses $U(x) = x^{1/2}$ (where x is in millions of dollars):

1. the treasury bills have an expected utility of $\sqrt{6} = 2.45$;
2. the start-up company has expected utility of

$$0.2 \times \sqrt{10} + 0.4 \times \sqrt{5} + 0.4 \times \sqrt{1} = 1.93\,.$$

$\Rightarrow$ The first alternative is preferred to the second!

Equivalent Utility Functions

Since a utility function is merely used to rank different choices, its actual numerical value has no real meaning.

Utility functions can be modified without changing the ranking by:

1. adding a constant $b \in \mathbb{R}$: $U(x) \rightarrow V(x) = U(x) + b$;
2. multiplying by a constant $a > 0$: $U(x) \rightarrow V(x) = aU(x)$.

It can be shown that the combined transformation

$$U(x) \rightarrow V(x) = aU(x) + b \quad \text{for} \quad a > 0$$

is the only transformation that preserves the rankings of all random outcomes.

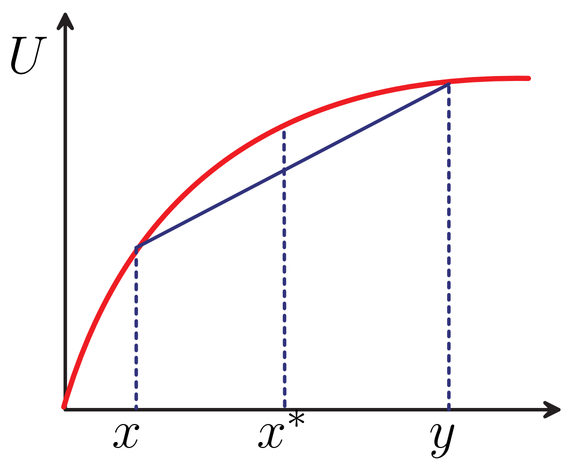
Risk Aversion I

Definition: A function $U : [a, b] \to \mathbb{R}$ is said to be concave if for any α in $[0, 1]$ and for any x and y in $[a, b]$ there holds

$$U[\alpha x + (1 - \alpha)y] \geq \alpha U(x) + (1 - \alpha)U(y)\,.$$

A utility fct. U is called risk averse if it is concave on $[a, b]$.

$\Rightarrow$ The straight line drawn between two points on the function must lie below or on the function itself.



$$x^* = \alpha x + (1 - \alpha)y$$

Risk Aversion II

Assume that we have two alternatives for future wealth:

1. we obtain x with probability α or y with probability $1-\alpha$;
2. we obtain $x^* = \alpha x + (1-\alpha)y$ with certainty.

Both alternatives have the same expected wealth x^*. However, the expected utility of the first alternative is

$$\alpha U(x) + (1-\alpha)U(y)\,,$$

while the expected utility of the second alternative is

$$U[\alpha x + (1-\alpha)y]\,.$$

$\Rightarrow$ The risk-free (second) alternative is preferred if U is concave.

Risk Aversion III

The properties of a utility function relate to its derivatives:

- $U(x)$ is strictly increasing in $x \iff U'(x) > 0$;
- $U(x)$ is strictly concave in $x \iff U''(x) < 0$.

Most people are greedy. From a set of deterministic wealth levels they prefer the highest one $\Rightarrow$ typical utility functions are increasing. Most people are also risk-averse $\Rightarrow$ typical utility functions are concave. Exceptions:

- people accept unfavorable bets with a high potential reward if the initial investment is small (lotteries);
- imagine that a mafia thug threatens to shoot you if you fail to pay \$10M; if you only own \$1M, you may go to a casino and put all your money on one number.

Risk Aversion IV

The degree of risk aversion implied by a utility function is related to the magnitude of the curvature of the function.

Arrow-Pratt absolute risk aversion coefficient:

$$a(x) = -\frac{U''(x)}{U'(x)}$$

- $a(x)$ shows how risk-aversion changes with wealth;
- usually, risk-aversion decreases as wealth grows;
- $a(x)$ is the same for all equivalent utility functions.

Example: $U(x) = ae^{-ax}$ (exponential utility) $\Rightarrow a(x) = a$;
$U(x) = \ln x$ (logarithmic utility) $\Rightarrow a(x) = 1/x$.

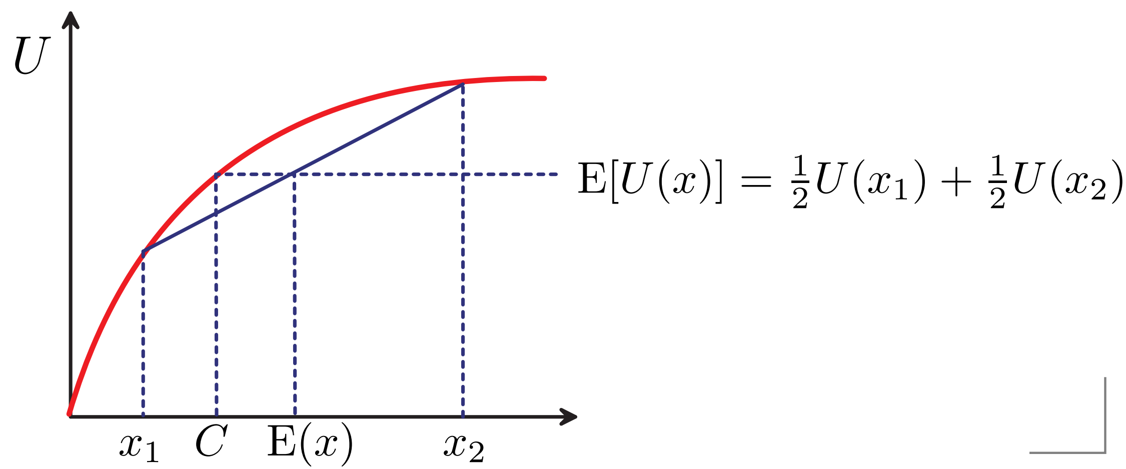
Certainty Equivalent

Definition: The certainty equivalent C of a random wealth variable x is the amount of certain (deterministic) wealth that has a utility level equal to the expected utility of x.

$$\Rightarrow \quad U(C) = \mathrm{E}[U(x)]$$

Note that C is the same for all equivalent utility functions.

Example: assume x takes values x_1 and x_2 with probability $\frac{1}{2}$



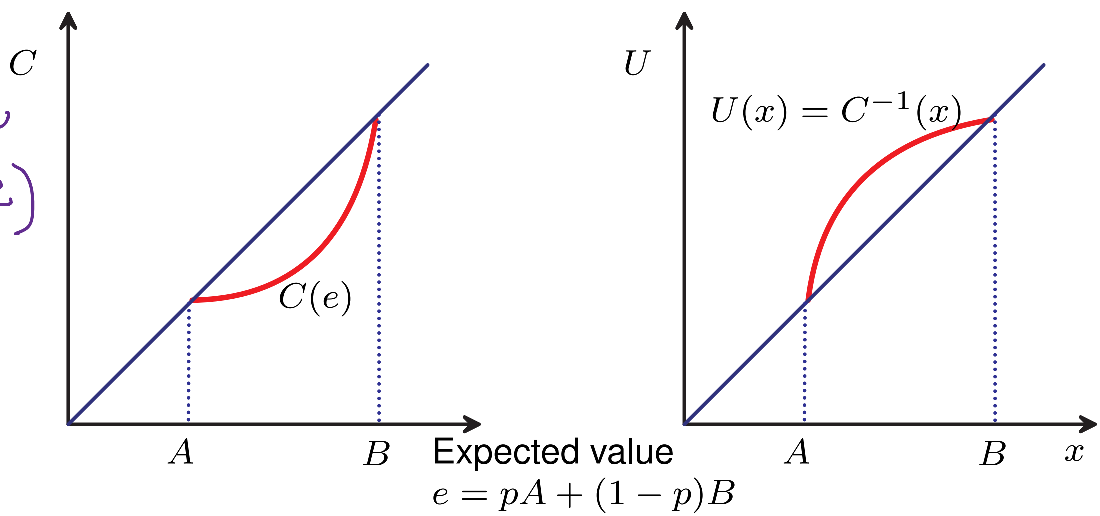
Measuring Utility Functions I

A way to measure an investor's utility function is as follows:

1. select fixed wealth levels A and B (reference points);
2. propose a lottery that has outcome A with probability p and outcome B with probability $1-p$;
3. for $p \in [0,1]$ the investor is asked how much certain wealth C he or she would accept in place of the lottery.

$U(C) = e$

$C = U^{-1}(e)$



Measuring Utility Functions II

Another method to assign utility functions is to select a parameterized family of functions and determine suitable parameter values:

- one often assumes $U(x) = -e^{-ax}$ (exponential utility);
- only the risk aversion parameter a must be determined;
- this can be done by evaluating a single lottery in certainty equivalent terms.

Example: Ask an investor how much he or she would accept in place of a lottery that offers a 50-50 chance of winning $\$1$M or $\$100,000$. If the investor feels that the certainty equivalent wealth is $\$400,000$, then we set

$$-e^{-400,000a} = -0.5e^{-1,000,000a} - 0.5e^{-100,000a}\,.$$

Numerical solution: $a = 1/\$623,426$.

Measuring Utility Functions III

The risk aversion characteristics of a person depend on the person's

- feelings about risk;
- current financial situation;
- the prospects for financial gains or requirements (such as college expenses);
- age.

An investor's attitude toward risk and toward type of investment might be inferred from responses to a questionnaire; see e.g. *Investment Science* p. 238.

Connection to Mean-Variance Criterion

Return(?) Variance(?)

- Quadratic utility function $U(x) = ax - \frac{1}{2}bx^2$ for $a, b > 0$.
- Meaningful range of U: $\boxed{x \le a/b}$ (where U is increasing).
- All random variables are assumed to lie in this range.
- Since $\boxed{b > 0,}$ U is concave $\Rightarrow$ risk aversion.

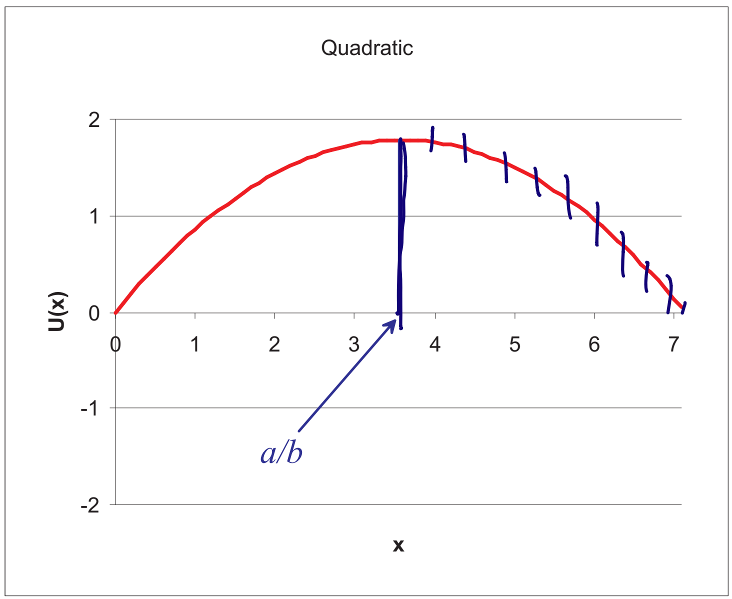
$u(x) = ax - \frac{1}{2}bx^2$

$u'(x) = a - bx \geq 0$

$x \leq a/b$

$u''(x) \leq 0$

$b > 0$



Connection to Mean-Variance Criterion

max E(u) same as
min Var(y)

max $-f$ = $-$ min f

- Suppose a portfolio has random wealth level y.
  Evaluate the expected utility of this portfolio:

quadratic u.f.

a, b deterministic (parameters of utility function)

max y

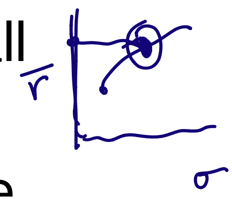
$$\begin{aligned} \mathrm{E}[U(y)] &= \mathrm{E}\Big(\underbrace{ay - \frac{1}{2}by^2}\Big) = a\,\mathrm{E}(y) - \frac{1}{2}b\,\underbrace{\mathrm{E}(y^2)}_{=\mathrm{var}(y) + [\mathrm{E}(y)]^2} \\ &= \boxed{a\,\mathrm{E}(y) - \frac{1}{2}b\,\mathrm{E}(y)^2} - \frac{1}{2}b\,\mathrm{var}(y)\,. \end{aligned}$$

fixed in Markowitz model

$\left[\mathrm{var}(y) = \mathrm{E}(y - \mathrm{E}y)^2 = \mathrm{E}(y^2) - (\mathrm{E}y)^2\right]$

$\mathrm{E}y = \bar{y}$

- The optimal portfolio maximizes this value w.r.t. all feasible choices of y.



- If initial wealth $= 1$, then $y =$ portfolio return. If the optimal solution has $\mathrm{E}(y) = 1 + \bar{r}_{\mathrm{P}}$, then y has minimum variance w.r.t. all feasible y's with $\mathrm{E}(y) = 1 + \bar{r}_{\mathrm{P}}$.

$\Rightarrow$ The solution is a mean-variance efficient point!

Securities

Definition: A security is a random payoff variable d. The payoff is revealed and obtained at the end of the period (d can be interpreted as a dividend). Associated with a security is a price P.

Examples:

- imagine a security that pays $d = \$10$ if it rains tomorrow or $d = \$ - 10$ if it is sunny, with zero initial price (this is a $\$10$ bet that it will rain tomorrow);
- a share of IBM stock whose value at the end of the year is unknown.

Note: the payoff d is a random variable, while the price P is a real number.

Type A Arbitrage

≡ make money without risk (for sure).

Definition: A type A arbitrage is an investment that produces an immediate positive reward with no future payoff.

$\Rightarrow$ A type A arbitrage is a security with $P < 0$ and $d = 0$.

Reasonable assumption: there is no market-traded security which is a type A arbitrage since

- the market price of a security settles in such a way as to equalize the quantity demanded by buyers and the quantity supplied by sellers;
- nobody would want to sell a type A arbitrage, while everybody would want to buy it $\Rightarrow$ no equilibrium of demand and supply is possible for a type A arbitrage.

Portfolios

- Suppose that there are n securities with payoffs $d_1, d_2, \ldots, d_n$ and prices $P_1, P_2, \ldots, P_n$;
- a portfolio is represented by an n-dimensional vector $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$;
- the ith component θ_i represents the number of securities of type i in the portfolio;
- the payoff of the portfolio is

amount of i × return of i

$$\boxed{d = \sum_{i=1}^{n} \theta_i d_i}$$

- the total price of the portfolio is

$$\boxed{P = \sum_{i=1}^{n} \theta_i P_i}$$

$\Rightarrow$ $\boxed{\text{Linearity of pricing}}$

Linearity of Pricing I

Linearity of pricing means that

1. the price of the sum of two securities is the sum of their prices;
2. the price of a multiple of an asset is the same multiple of the price.

In an ideal market, the absence of type A arbitrage opportunities implies linear pricing.

A market is ideal if

- securities can be arbitrarily divided; (% wealth)
- there are no transaction costs; (can be dealt with)
- short sales are allowed.

Linearity of Pricing II

Theorem 1. *In an ideal market, the absence of type A arbitrage opportunities implies linear pricing.*

Proof. Let d be a security with price P. Consider the security $d' = 2d$ with price P'. $P = \{A\}$, $P' = \{2A\}$ $P' = 2P$

- If $P' < 2P$, we would buy d' and sell short two units of d. We would obtain an immediate profit $2P - P'$ and have no further obligation. This is a type A arbitrage! $\Rightarrow P' \geq 2P$.
- The reverse argument shows that $P' \leq 2P \Rightarrow \boxed{P' = 2P.}$

Similarly, we can show that for any $\alpha \in \mathbb{R}$ the price of αd is αP. $\square$

If $P' < 2P$ (try and buy "cheap" sell the "expensive" one)

	$t=0$	$t=1$
Buy (d')	$-P'$	$d' = 2d$
Sell (d)	$2P$	$-2d$
	$2P - P' > 0$	0

$\leftarrow$ Type A arbitrage $\Rightarrow P' \geq 2P$

Linearity of Pricing II

Theorem 2. *In an ideal market, the absence of type A arbitrage opportunities implies linear pricing.*

Proof. Let d_1 and d_2 be securities with prices P_1 and P_2. Consider the security $d' = d_1 + d_2$ with price P'. $P' = P_1 + P_2$ ← Linearity of pricing.

- If $P' < P_1 + P_2$, we would buy d' and sell short one unit of d_1 and d_2 each. We would obtain an immediate profit $P_1 + P_2 - P'$ and have no further obligation. This is a type A arbitrage!
  $\Rightarrow P' \geq P_1 + P_2$.
- The reverse argument shows that $P' \leq P_1 + P_2 \Rightarrow P' = P_1 + P_2$.

Therefore, in general, the price of $\alpha d_1 + \beta d_2$ must be $\alpha P_1 + \beta P_2$. $\square$

	$t=0$	$t=1$
Buy P'	$-P'$	$d' = d_1 + d_2$
Short P_1+P_2	P_1+P_2	$-(d_1+d_2)$
	>0	0

→ Type A arbitrage if $P' \ngeq P_1 + P_2$

Type B Arbitrage

Break until 2:05 pm.

Definition: A type B arbitrage is an investment that has

- nonpositive cost,
- positive probability of yielding a positive payoff,
- and no probability of yielding a negative payoff.

$\Rightarrow$ A type B arbitrage is a security with

- $P \leq 0$,
- $d \geq 0$,
- and $\mathrm{Prob}(d > 0) > 0$.

Example: a free lottery ticket.

Below we assume that neither type A nor type B arbitrage is possible.

Portfolio Problem I

An investor with utility function U and initial wealth W solves the problem

Utility Maximization problem.

$$\begin{array}{ll} \underset{\theta \in \mathbb{R}^n}{\text{maximize}} & \boxed{\mathrm{E}[U(x)]} \quad \text{objective constraint} \\ \text{subject to} & \sum_{i=1}^n \theta_i d_i = x \quad \text{Performance constraint} \\ & \sum_{i=1}^n \theta_i P_i \leq W \quad \text{Budget constraint.} \end{array} \qquad (\mathcal{P})$$

- Investor maximizes expected utility of final wealth.
- Final wealth is described by the random variable x.
- The portfolio may not cost more than W.

$\begin{pmatrix} \text{IBM} \\ \text{Gift} \end{pmatrix}$

Portfolio Problem II

θ^0 valid if risk-free bank.

Theorem 3. *Assume that $U(x)$ is continuous, $U(x) \to +\infty$ as $x \to +\infty$, and there is a portfolio θ^0 such that $\sum_{i=1}^n \theta_i^0 d_i > 0$. Then:*

$$\boxed{\mathcal{P} \text{ has a solution} \iff \text{there is no arbitrage possibility.}}$$

Proof. $\Rightarrow$:

- If $\exists$ type A arbitrage $\Rightarrow$ using the arbitrage we can generate money to buy an arbitrary amount of portfolio θ^0. Thus, $\mathrm{E}[U(x)]$ is unbounded, and there exists no optimal portfolio.
- If $\exists$ type B arbitrage with payoff $\bar{x} \Rightarrow$ we can buy (at zero or negative cost) an arbitrary amount of this arbitrage to increase $\mathrm{E}[U(x)]$ arbitrarily (recall that $\mathrm{Prob}(\bar{x} > 0) > 0$).

$\Rightarrow$ If $\exists$ a solution for $\mathcal{P}$, then there can be no type A or B arbitrage. $\square$

Portfolio Problem III

To solve $\mathcal{P}$ we note that $\boxed{\sum_{i=1}^{n} \theta_i P_i = W}$ at the optimum.

Thus, we study the simplified problem

substituted x.

$$\underset{\theta \in \mathbb{R}^n}{\text{maximize}} \quad \mathrm{E}\left[U\left(\sum_{i=1}^{n} \theta_i d_i\right)\right]$$

$$\text{subject to} \quad \boxed{\sum_{i=1}^{n} \theta_i P_i = W}\,,$$

$\lambda \in \mathbb{R}$

whose Lagrangian function reads

$$L(\theta, \lambda) = \underbrace{\mathrm{E}\left[U\left(\sum_{i=1}^{n} \theta_i d_i\right)\right]}_{\text{objective}} - \underbrace{\lambda}_{\text{Lagrange multiplier}} \underbrace{\left(\sum_{i=1}^{n} \theta_i P_i - W\right)}_{\text{constraint}}.$$

$U : \mathbb{R} \to \mathbb{R}$

$\frac{\partial}{\partial\theta} E(U(x)) = E \frac{\partial}{\partial\theta}(U(X))$

Portfolio Problem IV

$$\frac{\partial L}{\partial \theta_i} = E\Big[U'\big(\sum_i \theta_i d_i\big) d_i\Big] - \lambda P_i = 0$$

Differentiating L w.r.t. θ_i gives the optimality conditions:

$$\boxed{\mathrm{E}[U'(x^*)d_i] = \lambda P_i} \quad \text{for} \quad i = 1, 2, \ldots, n\,, \tag{1}$$

where $\boxed{x^* = \sum_{i=1}^{n} \theta_i^* d_i}$ and θ^* is an optimal portfolio for $\mathcal{P}$.

The optimality conditions (1) and the budget constraint $\boxed{\sum_{i=1}^{n} \theta_i P_i = W}$ represent $n+1$ equations for the $n+1$ unknowns $\theta_1, \theta_2, \ldots, \theta_n$, and λ. It can be shown that $\lambda > 0$.

The equations (1) serve two roles:

- they can be used to solve $\mathcal{P}$;
- they provide a characterization of the securities prices under the assumption of no arbitrage.

Portfolio Problem V

Theorem 4. *If* $\boxed{x^* = \sum_{i=1}^n \theta_i^* d_i}$ *solves* $\mathcal{P}$*, then*

$$\mathrm{E}[U'(x^*)d_i] = \underbrace{\lambda}\underbrace{P_i}_{=1} \quad \textit{for} \quad i = 1, 2, \ldots, n\,, \tag{1}$$

λ

where $\lambda > 0$. *If there is a risk-free asset with total return* R*, then*

$$\boxed{\frac{\mathrm{E}[U'(x^*)d_i]}{R\,\underbrace{\mathrm{E}[U'(x^*)]}_{=\lambda}} = P_i \quad \textit{for} \quad i = 1, 2, \ldots, n\,.}$$

(1+r)

Proof. The risk-free asset has price $\underbrace{P_i = 1}$ and payoff $\underbrace{d_i = R}$. The optimality condition for this asset implies

$$\boxed{\lambda = \mathrm{E}[U'(x^*)]R\,.}$$ (d_i = R, deterministic),

Substituting this expression for λ into (1) proves the theorem. □

A Film Venture I

There are two 'securities' with a two year horizon:

- a risk free asset yielding 20%;
- a film venture with three possible return outcomes.

$X = 3\theta_1 + 1.2\theta_2$

X

	Return	Probability
High success	3.0	0.3
Moderate success	1.0	0.4
Failure	0.0	0.3
Risk free	1.2	1.0

$E(U(X)) = p_1 U(X_1) + p_2 U(X_2) + p_3 U(X_3)$

An investor with utility $U(x) = \ln x$ and capital W selects the amounts θ_1 and θ_2 of the two securities (both have price 1).

$E(U(X)) \approx$

$$\text{maximize } [.3 \ln(\underbrace{3\theta_1 + 1.2\theta_2}_{x_1}) + .4 \ln(\underbrace{\theta_1 + 1.2\theta_2}_{x_2}) + .3 \ln(\underbrace{1.2\theta_2}_{x_3})]$$

$$\text{subject to } \theta_1 + \theta_2 = W\,.$$

A Film Venture II

The optimality conditions (1) translate to

$$
\boxed{
\begin{aligned}
\frac{.9}{3\theta_1 + 1.2\theta_2} + \frac{.4}{\theta_1 + 1.2\theta_2} &= \lambda \\
\frac{.36}{3\theta_1 + 1.2\theta_2} + \frac{.48}{\theta_1 + 1.2\theta_2} + \frac{.36}{1.2\theta_2} &= \lambda\,.
\end{aligned}
}
$$

Solving these two equations together with the constraint $\boxed{\theta_1 + \theta_2 = W}$ yields the optimal portfolio choice:

$$
\boxed{\theta_1 = .089W\,, \quad \theta_2 = .911W\,, \quad \lambda = 1/W\,.}
$$

$\Rightarrow$ The investor should commit $\boxed{8.9\%}$ of his/her wealth to the film venture and the rest to the risk free asset.