60016 OPERATIONS RESEARCH

Simplex Algorithm

Last Lecture

- BS's and BFS's
  - Set $n - m$ variables to zero and solve for the others
  - # of BS's $\leq \binom{n}{m}$
- Algebra vs. geometry
  - BFS's $\simeq$ vertices of the feasible set
- Fundamental theorem of linear programming
  - $\exists$ FS $\Rightarrow$ $\exists$ BFS
  - $\exists$ optimal FS $\Rightarrow$ $\exists$ optimal BFS
- Basic representations

This Lecture

- Idea of the simplex algorithm
  - Move from BFS to BFS to improve objective value
- Simplex tableau
- Pivoting
- Simplex algorithm

Idea of the Simplex Algorithm

$$\begin{array}{ll} \min & z = c^T x \\ \text{s.t.} & Ax = b,\ x \geq 0 \end{array} \qquad (\mathcal{LP})$$

1. Among the FS to $\mathcal{LP}$, an important finite subset are the BFS.
2. A BFS is just the set of coordinates of a vertex. Thus at least one BFS will be optimal.
3. Each BFS is associated with a basic representation, i.e., a set of equations equivalent to $z = c^T x$, $A\,x = b$, that expresses the BV's in terms of the NBV's.
4. The basic representation also tells us the reduced costs, which indicate the best NBV to increase to improve the objective.
5. We can iteratively apply this idea until finding the optimal solution.

Simplex Tableau

- Consider a basic representation of the form

$$\left.\begin{array}{rcrcl} z & - & r^T x_N & = & c_B^T B^{-1} b \\ x_B & + & B^{-1} N x_N & = & B^{-1} b \end{array}\right\} \qquad (*)$$

  where $r = c_N - N^T B^{-T} c_B$ is the reduced cost vector.

- To simplify notation we represent $(*)$ as a tableau

BV	z	x_B^T	x_N^T	RHS
z	1	0^T	$-r^T$	$c_B^T B^{-1} b$
x_B	0	I	$B^{-1}N$	$B^{-1}b$

  where I is the $m \times m$ identity matrix.

  The tableau is useful to read the properties of the current BS.

Example: Simplex Tableau

Basic representation (4) from Example 1 ($I = \{1, 2, 5\}$).

Explicit formulation:

$$
\begin{array}{ccccccccccccc}
z & & & & - & \frac{2}{5}x_3 & - & \frac{1}{5}x_4 & & & = & -8 \\
 & & x_2 & & - & \frac{1}{5}x_3 & + & \frac{2}{5}x_4 & & & = & 5 \\
 & & & & - & \frac{3}{5}x_3 & + & \frac{1}{5}x_4 & + & x_5 & = & 1 \\
 & x_1 & & & + & \frac{3}{5}x_3 & - & \frac{1}{5}x_4 & & & = & 3
\end{array}
$$

Example: Simplex Tableau

Basic representation (4) from Example 1 ($I = \{1, 2, 5\}$).

Removing variables and mathematical operators:

$$
\begin{array}{cccccc|c}
1 & & & -\frac{2}{5} & -\frac{1}{5} & & -8 \\
 & & 1 & -\frac{1}{5} & \frac{2}{5} & & 5 \\
 & & & -\frac{3}{5} & \frac{1}{5} & 1 & 1 \\
 & 1 & & \frac{3}{5} & -\frac{1}{5} & & 3
\end{array}
$$

Example: Simplex Tableau

Basic representation (4) from Example 1 ($I = \{1, 2, 5\}$).

Appending zeroes:

$$\begin{array}{cccccc|c}
1 & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & 0 & -8 \\
0 & 0 & 1 & -\frac{1}{5} & \frac{2}{5} & 0 & 5 \\
0 & 0 & 0 & -\frac{3}{5} & \frac{1}{5} & 1 & 1 \\
0 & 1 & 0 & \frac{3}{5} & -\frac{1}{5} & 0 & 3
\end{array}$$

Example: Simplex Tableau

Basic representation (4) from Example 1 ($I = \{1, 2, 5\}$).

Introducing labels:

BV	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	0	0	$-\frac{2}{5}$	$-\frac{1}{5}$	0	-8
x_2	0	0	1	$-\frac{1}{5}$	$\frac{2}{5}$	0	5
x_5	0	0	0	$-\frac{3}{5}$	$\frac{1}{5}$	1	1
x_1	0	1	0	$\frac{3}{5}$	$-\frac{1}{5}$	0	3

Rows are indexed by the respective basic variables.
Column labels are fixed.

Properties of the Simplex Tableau (cont)

The tableau is a practical way to analyse the BS associated to the basic representation:

- the RHS of the objective row is the objective value of the current BS
- the RHS's of the other rows are the values of the basic variables at the current BS;
- the coefficients of the nonbasic variables in the objective row are the negative reduced costs.
- The current BS is feasible if and only if the RHS's are nonnegative in all rows (except objective row).

General Tableau Notation

Tableau for a feasible index set I (with $p \in I$, $q \notin I$):

BV	z	x_1	$\cdots$	x_p	$\cdots$	x_q	$\cdots$	x_n	RHS
z	1	β_1	$\cdots$	β_p	$\cdots$	β_q	$\cdots$	β_n	β_0
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$	$\vdots$
x_p	0	y_{p1}	$\cdots$	y_{pp}	$\cdots$	y_{pq}	$\cdots$	y_{pn}	y_{p0}
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$	$\vdots$

- $y_{ii} = 1 \ \forall i \in I$ and $y_{ji} = 0 \ \forall i \in I, \ j \in I \backslash \{i\}$
- $\beta_i = -r_i \ \forall i \notin I, \ i \neq 0$ (negative reduced cost)
- $\beta_i = 0 \ \forall i \in I$

Pivoting

Simplex Algorithm Idea: if vertex x for index set I is not optimal, then in its neighbouring vertices there is one with a better objective value.

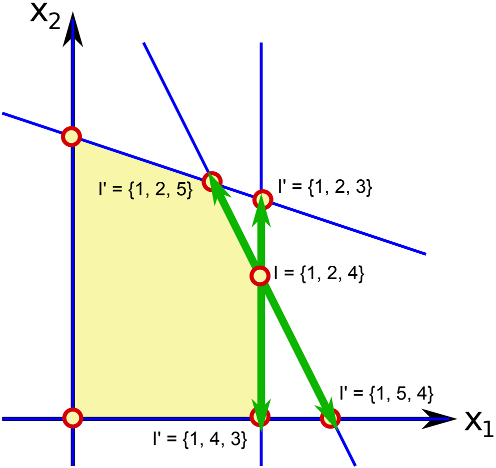
Neighbouring vertices are obtained by swapping a basic variable x_p with a non-basic variable x_q, obtaining a new index set I'. We say that x_p leaves the basis, while x_q enters the basis.

We use a technique called pivoting to efficiently compute the new basic representation for I' by updating I

- Similar to applying elementary row operations in Gaussian elimination
- The pair (p, q) is often referred to as the pivot

Pivoting

Consider the BS for $I = \{1, 2, 4\}$ in Example 1.



We can pivot on:

- $(x_p, x_q) = (x_4, x_5)$
  $\Rightarrow I' = \{1, 2, 5\}$
- $(x_p, x_q) = (x_4, x_3)$
  $\Rightarrow I' = \{1, 2, 3\}$
- $(x_p, x_q) = (x_2, x_3)$
  $\Rightarrow I' = \{1, 4, 3\}$
- $(x_p, x_q) = (x_2, x_5)$
  $\Rightarrow I' = \{1, 5, 4\}$

Pivoting (cont)

To swap x_p and x_q we:

1. divide row p by the pivot element y_{pq}, and relabel it as row q:

$$y'_{qj} = \frac{y_{pj}}{y_{pq}} \quad \forall j = 0, \ldots, n \tag{1}$$

2. subtract row p multiplied by $\frac{y_{iq}}{y_{pq}}$ from row $i \in I \backslash \{p\}$:

$$y'_{ij} = y_{ij} - \frac{y_{iq}}{y_{pq}} y_{pj} \quad \forall j = 0, \ldots, n \tag{2}$$

3. subtract row p multiplied by $\frac{\beta_q}{y_{pq}}$ from the objective row:

$$\beta'_j = \beta_j - \frac{\beta_q}{y_{pq}} y_{pj} \quad \forall j = 0, \ldots, n \tag{3}$$

Pivoting is possible if and only if $y_{pq} \neq 0$.

Pivoting (Example)

Basic representation for $I = \{1, 2, 3\}$:

BV	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	0	0	1	2	1	1
x_1	0	1	0	0	1	1	-1	5
x_2	0	0	1	0	2	-3	1	3
x_3	0	0	0	1	-1	2	-1	-1

Find the neighbouring basic representation for $I' = \{4, 2, 3\}$.

Pivoting (Example)

Exchange: x_1 becomes nonbasic, x_4 becomes basic.

BV	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	0	0	1	2	1	1
x_1	0	1	0	0	1	1	-1	5
x_2	0	0	1	0	2	-3	1	3
x_3	0	0	0	1	-1	2	-1	-1

Pivot Element: $y_{14} = 1$

Pivoting (Example)

Exchange: x_1 becomes nonbasic, x_4 becomes basic.

BV	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	0	0	1	2	1	1
x_1	0	1	0	0	1	1	-1	5
x_2	0	0	1	0	2	-3	1	3
x_3	0	0	0	1	-1	2	-1	-1
z								
x_4	0	1	0	0	1	1	-1	5
x_2								
x_3								

Equation (1) $\Rightarrow$ row for x_1 remains unchanged,
but now relabelled as x_4.

Pivoting (Example)

Exchange: x_1 becomes nonbasic, x_4 becomes basic.

BV	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	0	0	1	2	1	1
x_1	0	1	0	0	1	1	-1	5
x_2	0	0	1	0	2	-3	1	3
x_3	0	0	0	1	-1	2	-1	-1
z								
x_4	0	1	0	0	1	1	-1	5
x_2	0	-2	1	0	0	-5	3	-7
x_3								

Equation (2) $\Rightarrow$ subtract row for x_1 twice from row for x_2,
no relabelling necessary.

Pivoting (Example)

BV	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	0	0	1	2	1	1
x_1	0	1	0	0	1	1	-1	5
x_2	0	0	1	0	2	-3	1	3
x_3	0	0	0	1	-1	2	-1	-1
z	1	-1	0	0	0	1	2	-4
x_4	0	1	0	0	1	1	-1	5
x_2	0	-2	1	0	0	-5	3	-7
x_3	0	1	0	1	0	3	-2	4

Equation (3) $\Rightarrow$ subtract row for x_1 from objective row, no relabelling necessary.

Pivoting (Example)

Exchange: x_1 becomes nonbasic, x_4 becomes basic.

BV	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	0	0	1	2	1	1
x_1	0	1	0	0	1	1	-1	5
x_2	0	0	1	0	2	-3	1	3
x_3	0	0	0	1	-1	2	-1	-1
z	1	-1	0	0	0	1	2	-4
x_4	0	1	0	0	1	1	-1	5
x_2	0	-2	1	0	0	-5	3	-7
x_3	0	1	0	1	0	3	-2	4

Pivot equations ensure that the other basic columns and the objective column remain unchanged!

Pivot Selection

The crucial question is, how do we decide the pivot?

We impose the following rules:

1. Feasibility: The new BS must be feasible, i.e., it must be a vertex! Thus we need to check in I' that $y'_{i0} \geq 0$, $\forall i \in I'$.
   - This criterion is used to choose x_p (leaving variable).

2. Non-Inferiority: The new vertex must have a better objective value than the current vertex, $\beta'_0 \leq \beta_0$.
   - This criterion is used to choose x_q (entering variable).

Choosing the variable x_q which enters the basis

The objective row of the simplex tableau is equivalent to:

$$z + \sum_{i=1}^{n} \beta_i x_i = \beta_0 \quad \Longleftrightarrow \quad z = \beta_0 - \sum_{i \notin I} \beta_i x_i$$

By definition, we have

- $\beta_i = 0$ for all $i \in I$ (basic variables);
- $\beta_i = -r_i$ for all $i \notin I$ (nonbasic variables).

In principle, any nonbasic x_i with $\beta_i > 0$ can enter the basis and be x_q, since each of these will decrease z.

- If there exist only a single x_i with $\beta_i > 0$, pick this as x_q.
- If several x_i have $\beta_i > 0$, pick the x_i with maximum β_i.
- If several x_i's achieve the largest β_i, break the tie by picking the smallest index i.

Choosing the variable x_p which leaves the basis

- Once x_q enters the basis and it is set to a value $x_q > 0$, the constraints will force some basic variables to change value.
- To ensure that all variables remain feasible, it must be

$$x_i = y_{i0} - y_{iq} x_q \geq 0 \iff \begin{cases} x_q \leq \overline{x}_{iq} \triangleq \dfrac{y_{i0}}{y_{iq}} & \text{if } y_{iq} > 0, \\ x_q \leq \overline{x}_{iq} \triangleq \infty & \text{if } y_{iq} \leq 0. \end{cases}$$

$\Rightarrow$ The feasibility requirement is equivalent to asking that x_q is set so that these bounds are all simultaneously satisfied, i.e.,

$$x_q \leq \min_{i \in I} \overline{x}_{iq}. \qquad (**)$$

Choosing the variable x_p which leaves the basis

Suppose first that the bounds are trivial, i.e., $\min_{i \in I} \overline{x}_{iq} = \infty$, then

- The entering variables x_q can grow indefinitely.
- As $\beta_q > 0$, the objective value $z = \beta_0 - \beta_q x_q$ can drop indefinitely.
- Then the LP is unbounded below and the optimization stops.
- In this case, there's no need to choose the x_p variable.

Choosing the variable x_p which leaves the basis

Suppose now that the bounds are non-trivial, i.e., $\min_{i \in I} \overline{x}_{iq} < \infty$

- The best value of the objective is obtained by making x_q as large as possible, i.e., we set $x_q = \min_{i \in I} \overline{x}_{iq}$.
- Call p the row such that $\overline{x}_{pq} = \min_{i \in I} \overline{x}_{iq}$, i.e., the row that constraints the most the increase in value of x_q.
- After p is chosen with this criterion, we are sure that feasibility is preserved if we set

$$x_q = \overline{x}_{pq} = \frac{y_{p0}}{y_{pq}}$$

- But then $x_p = y_{p0} - y_{pq} x_q = 0$ becomes nonbasic. So this assignment defines a BFS, i.e., we moved to another vertex!
- If there are several $p \in \arg\min_{i \in I} \overline{x}_{iq}$, we choose the one with the smallest index.

Simplex Algorithm (Minimisation)

- Step 0: Find initial BFS and its basic representation.
- Step 1: If $\beta_i \leq 0$ for all $i \notin I$:
  STOP — the current BFS is optimal.
- Step 2: If $\exists j \notin I$ with $\beta_j > 0$ and $y_{ij} \leq 0$ for all $i \in I$:
  STOP — no finite minimum exists.
- Step 3: Choose x_q with largest $\beta_q > 0$
  Entry criterion — x_q enters the basis.
- Step 4: Choose $p \in \arg\min_{i \in I} \overline{x}_{iq}$
  Exit criterion — x_p leaves the basis.
- Step 5: Pivot on y_{pq} and go back to Step 1.

Example 1 (revisited)

After standardising Example 1, this can be rewritten as:

$$
\begin{array}{ccccccccccccc}
z & + & x_1 & + & x_2 & & & & & & & = & 0 \\
 & & 2x_1 & + & x_2 & + & x_3 & & & & & = & 11 \\
 & & x_1 & + & 3x_2 & & & + & x_4 & & & = & 18 \\
 & & x_1 & & & & & & & + & x_5 & = & 4
\end{array} \qquad (1)
$$

Note that this is a basic representation where the slack variables are the basic variables ($I = \{3, 4, 5\}$).

BV: $\{x_3, x_4, x_5\}$ NBV: $\{x_1, x_2\}$

Example 1 (revisited)

In equations (1) set NBV's to zero:

$$x_1 = x_2 = 0$$

"Solve" for the remaining BV's

$$z = 0,\ x_3 = 11,\ x_4 = 18,\ x_5 = 4$$

The corresponding BS is:

$$(z, \underline{x_1}, \underline{x_2}, x_3, x_4, x_5) = (0, 0, 0, 11, 18, 4)$$

It is also a BFS!

Example 1 (revisited)

For this BFS the objective function is $z = 0$.

In order to find a better BFS, search for a nonbasic variable x_j such that increasing x_j improves z.

Looking at the objective function

$$z = -x_1 - x_2$$

we see that we can decrease z either by increasing x_1 or x_2 (increasing both simultaneously gets too complicated).

Example 1 (revisited)

E.g. consider increasing x_1 to λ and leaving $x_2 = 0$.

$\Rightarrow$ The decrease for z will be: $z = -x_1 - x_2 = -\lambda$

As we increase λ, we need to make sure that the largest the basic variables must remain feasible ($x_i \geq 0 \; \forall i \in I$):

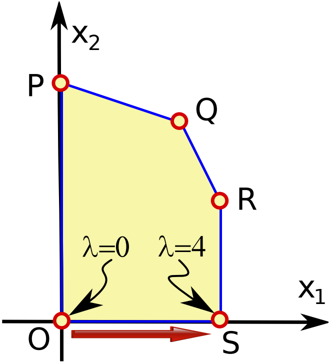
$$\begin{array}{rclccc} x_3 & = & 11 - 2\lambda & \geq 0 & \Rightarrow & \lambda \leq \frac{11}{2} \\ x_4 & = & 18 - \lambda & \geq 0 & \Rightarrow & \lambda \leq 18 \\ x_5 & = & 4 - \lambda & \geq 0 & \Rightarrow & \lambda \leq 4 \end{array} \qquad (1')$$

We cannot increase λ indefinitely, at some point at least one basic variable will get negative! Still we want the largest λ, since the objective function decreases with λ.

What is the largest λ that satisfies (1') ? $\lambda = 4$.

Example 1 (revisited)

From (1') we see that λ takes values between 0 and 4.
Any solution defined by (1'), i.e., $x_1 = \lambda, x_2 = 0$,
corresponds to a point along OS.



Example 1 (revisited)

The original BFS had been:

$$(z, \underline{x_1}, \underline{x_2}, x_3, x_4, x_5) = (0, 0, 0, 11, 18, 4)$$

corresponding to point O.

Setting $x_1 = \lambda = 4$ implies also $z = -4$.
The values of x_3, x_4, and x_5 change too.

The new BS is given by:

$$(z, x_1, \underline{x_2}, x_3, x_4, \underline{x_5}) = (-4, 4, 0, 3, 14, 0)$$

corresponding to point S.

Example 1 (revisited)

The new BS:

$$(z, x_1, \underline{x_2}, x_3, x_4, \underline{x_5}) = (-4, 4, 0, 3, 14, 0)$$

is also a BFS to (1).

The new basic and nonbasic variables are:

BV: $\{x_1, x_3, x_4\}$ NBV: $\{x_2, x_5\}$

Task: Obtain a new basic representation: i.e., transform (1) to express z, x_1, x_3, and x_4 in terms of x_2, x_5.

Systematic approach: Pivoting (discussed later)

Example 1 (revisited)

Rearranging the equations (1) by using elementary row operations (ERO), we obtain a new basic representation, corresponding to $I = \{1, 3, 4\}$.

$$
\begin{array}{ccccccccccccc}
z & & + & x_2 & & & & & - & x_5 & = & -4 \\
 & & & x_2 & + & x_3 & & & - & 2x_5 & = & 3 \\
 & & & 3x_2 & & & + & x_4 & - & x_5 & = & 14 \\
 & x_1 & & & & & & & + & x_5 & = & 4
\end{array} \qquad (2)
$$

Any solution to (2) is also a solution to (1) and vice versa.

Example 1 (revisited)

Looking at the new objective function

$$z = -4 - x_2 + x_5$$

we see that we can decrease z by increasing x_2.

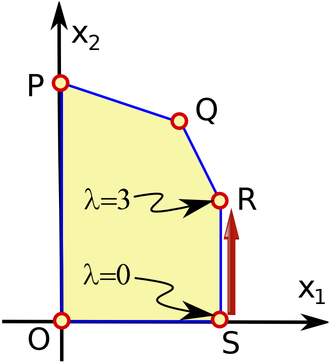
$\Rightarrow$ Increase x_2 to λ and keep $x_5 = 0$, making sure that the basic variables x_3, x_4 and x_1 remain feasible.

$$\begin{array}{rcclcl}
z & = & -4-\lambda & & & \\
x_3 & = & 3-\lambda & \geq 0 & \Rightarrow & \lambda \leq 3 \\
x_4 & = & 14-3\lambda & \geq 0 & \Rightarrow & \lambda \leq \frac{14}{3} \\
x_1 & = & 4 & \geq 0 & \Rightarrow & \lambda \leq \infty
\end{array} \qquad (2')$$

We want the best (largest) λ satisfying (2') $\Rightarrow \lambda = 3$

Example 1 (revisited)

From (2') we see that λ takes values between 0 and 3.
Any solution defined by (2'), i.e., $x_1 = 4, x_2 = \lambda$,
corresponds to a point along SR.



Example 1 (revisited)

The previous BFS had been

$$(z, x_1, \underline{x_2}, x_3, x_4, \underline{x_5}) = (-4, 4, 0, 3, 14, 0)$$

corresponding to point S.

Setting $x_2 = \lambda = 3$ implies $z = -7$.
The values of x_3, and x_4 change too.

The new BS is given by:

$$(z, x_1, x_2, \underline{x_3}, x_4, \underline{x_5}) = (-7, 4, 3, 0, 5, 0)$$

corresponding to point R.

Example 1 (revisited)

The new BS

$$(z, x_1, x_2, \underline{x_3}, x_4, \underline{x_5}) = (-7, 4, 3, 0, 5, 0)$$

is also a BFS to (1).

The new basic and nonbasic variables are:

BV: $\{x_1, x_2, x_4\}$ NBV: $\{x_3, x_5\}$

Task: Obtain a new basic representation, i.e., transform (2) to express z, x_1, x_2, and x_4 in terms of x_3, x_5.

Example 1 (revisited)

Rearranging the equations (2) by using EROs, we obtain a new basic representation corresponding to $I = \{1, 2, 4\}$.

$$
\begin{array}{rrrrrrrrrrrl}
z & & & - & x_3 & & & + & x_5 & = & -7 & \\
& & x_2 & + & x_3 & & & - & 2x_5 & = & 3 & \\
& & & - & 3x_3 & + & x_4 & + & 5x_5 & = & 5 & \qquad (3) \\
& x_1 & & & & & & + & x_5 & = & 4 &
\end{array}
$$

Any solution to (3) also solves (1) and (2) and vice versa.

Example 1 (revisited)

Looking at the new objective function

$$z = -7 + x_3 - x_5$$

we see that we can decrease z by increasing x_5.

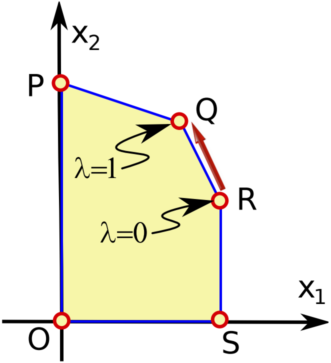
$\Rightarrow$ Increase x_5 to λ and keep $x_3 = 0$, making sure that the basic variables x_2, x_4 and x_1 remain feasible.

$$\begin{array}{rcrclcl} z & = & -7-\lambda & & & & \\ x_2 & = & 3+2\lambda & \geq 0 & \Rightarrow & \lambda \leq \infty \\ x_4 & = & 5-5\lambda & \geq 0 & \Rightarrow & \lambda \leq 1 \\ x_1 & = & 4-\lambda & \geq 0 & \Rightarrow & \lambda \leq 4 \end{array} \qquad (3')$$

We want the best (largest) λ satisfying (3') $\Rightarrow \lambda = 1$

Example 1 (revisited)

From (3') we see that λ takes values between 0 and 1.
Any solution defined by (3'), i.e., $x_1 = 4 - \lambda, x_2 = 3 + 2\lambda$, corresponds to a point along RQ.



Example 1 (revisited)

The previous BFS had been

$$(z, x_1, x_2, \underline{x_3}, x_4, \underline{x_5}) = (-7, 4, 3, 0, 5, 0)$$

corresponding to point R.

Setting $x_5 = \lambda = 1$ implies $z = -8$.
The values of x_1, x_2, and x_3 change too.

The new BS is given by

$$(z, x_1, x_2, \underline{x_3}, \underline{x_4}, x_5) = (-8, 3, 5, 0, 0, 1)$$

corresponding to point Q.

Example 1 (revisited)

$$(z, x_1, x_2, \underline{x_3}, \underline{x_4}, x_5) = (-8, 3, 5, 0, 0, 1)$$

is also a BFS to (1).

The new basic and nonbasic variables are:

BV: $\{x_1, x_2, x_5\}$ NBV: $\{x_3, x_4\}$

Task: Obtain a new basic representation, i.e., transform (3) to express z, x_1, x_2, and x_5 in terms of x_3, x_4.

Example 1 (revisited)

Rearranging the equations (3) by using EROs, we obtain a new basic representation, corresponding to $I = \{1, 2, 5\}$.

$$
\begin{array}{cccccccccccc}
z & & & - & \frac{2}{5}x_3 & - & \frac{1}{5}x_4 & & & = & -8 \\
& & x_2 & - & \frac{1}{5}x_3 & + & \frac{2}{5}x_4 & & & = & 5 \\
& & & - & \frac{3}{5}x_3 & + & \frac{1}{5}x_4 & + & x_5 & = & 1 \\
& x_1 & & + & \frac{3}{5}x_3 & - & \frac{1}{5}x_4 & & & = & 3
\end{array}
\qquad (4)
$$

Any solution to (4) also solves (1), (2), (3) and vice versa.

Example 1 (revisited)

From the first equation in (4) we deduce that any solution to (1), (2), (3), or (4) has to satisfy

$$z = -8 + \frac{2}{5}x_3 + \frac{1}{5}x_4.$$

Any FS further satisfies $x_3,\ x_4 \geq 0$.
$\Rightarrow$ Thus, (4) implies that $z \geq -8$ for any FS.

The BFS corresponding to (5), $(z, x_1, x_2, x_3, x_4, x_5) =$
$= (-8, 3, 5, 0, 0, 1)$, has objective value $z = -8$.

This must be a minimal solution!

Note: The optimal value of the original max problem is $+8$!