COMPUTATIONAL FINANCE: 422

Basic Options Theory

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Today (20/11)

2-Lecture

Tues. 24/11

• finish slides

(Fri) 27/11: Tutorial

Cover remaining Tutorials

(Tue) 1/12: Revision/CW feedback (1-3pm)

This Lecture

Option Price = { Arbitrage, Binomial Model, } (Stochastic Differential Equation) → $\boxed{\text{Black-Scholes Model.}}$

Introduction to options theory:

- Basic concepts
  - Definitions, terminology
  - Payoff diagrams
- Single period binomial options theory
- Multiperiod binomial options theory

Further reading:

- D.G. Luenberger: *Investment Science*, Chapter 12
- D.J. Higham: *Financial Option Valuation*, Chapters 1–2

European Call Options

An asset is any financial object whose value is known at present but is liable to change in the future.

Asset or underlying asset

Typical examples are:

- shares of a company,
- commodities such as gold, oil, or electricity,
- currencies such as the value of USD 100 in GBP.

Definition: A European call option gives its $\boxed{\text{holder}}$ the right (but not the obligation) to purchase form the writer a prescribed asset for a prescribed exercise price (or strike price) at a prescribed time in the future. → Maturity Date

Example I/II

Wilma (the writer) writes a European call option that gives Henry (the holder) the right to buy 100 shares of Google for £250 in three months. — Maturity/Expiry Date.

= underlying asset, asset,

→ strike price

After three months, Henry takes one of two actions:

Henry

$S_T > 250$

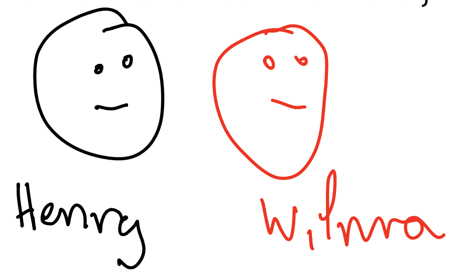
1. if the value of 100 Google shares is more than £250, Henry will exercise the option. e.g $S_T = 260$ $(260 - 250) \times 100$

Wilma –

2. if the value of 100 Google shares is less than £250, Henry will not exercise the option.

0

e.g $S_T = 240$



Example II/II

- Since Henry is not obliged to buy the shares, he will never lose money at expiry. In case 1 he gains money and in case 2 he neither gains nor loses.
- Wilma, on the other hand, will never gain money at expiry. In case 1 she loses money and in case 2 she neither gains nor loses.
- To compensate for this imbalance, when the option is agreed (today) Henry has to pay Wilma an amount of money, which is known as the value of the option/option premium.

Other Standard Options

- A European put option gives its holder the right (but not the obligation) to sell to the writer a prescribed asset for a fixed price at a prescribed future time.

- An American call (put) option gives its holder the right to buy from (sell to) the writer a prescribed asset for a fixed price any time before a prescribed future time.

Question:

How much should the option holder pay the writer for the privilege to hold the option?

What are Options Used for?

Options / Derivatives

- Risk management
  - Assume that you own a British company.
  - This company is committed to buy an American factory in 3 months for a prescribed price in USD.

  $\Rightarrow$ Buy a European put option that makes a profit when the GBP drops in value against the USD! This will protect you against exchange rate risk!
- Speculation
  - You believe that the Google shares will increase.

  $\Rightarrow$ You may speculate by buying a suitable call option.

  $\Rightarrow$ If Google goes up, you make a greater profit relative to your initial payout than by purchasing the shares.

How to Read an Option Table?

Stk	Exp	P/C	Vol	Bid	Ask	Opint
360networks (TSIX)						20.15
20	Feb	C	3	1.00	1.25	26
22	Mar	C	10	1.60	1.85	138
24	Mar	C	2	1.05	1.25	366
18	June	P	2	2.50	2.75	11
20	June	C	12	4.05	4.30	83
24	June	C	1	2.65	2.90	77

The strike price (Stk) is the price per share for which the underlying stock (360networks) may be purchased (call option) or sold (put option) upon exercise.

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The column with the label ‘Exp’ shows the month in which the option expires. The exact expiration date during that month is the Saturday following the third Friday.

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20	June	C	12	4.05	4.30	83
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The P/C column determines whether the option is a put (P) or a call (C) option.

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20	Feb	C	3	1.00	1.25	26
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20	June	C	12	4.05	4.30	83
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The Vol column shows the traded volume, that is, the total number of options traded on the reported day.

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20	Feb	C	3	1.00	1.25	26
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18	June	P	2	2.50	2.75	11
20	June	C	12	4.05	4.30	83
24	June	C	1	2.65	2.90	77

The Bid column shows the bid price, that is, the price at which the market maker will buy the option from you.

How to Read an Option Table?

Stk	Exp	P/C	Vol	Bid	Ask	Opint
360networks (TSIX)						20.15
20	Feb	C	3	1.00	1.25	26
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18	June	P	2	2.50	2.75	11
20	June	C	12	4.05	4.30	83
24	June	C	1	2.65	2.90	77

The Ask column shows the ask price, that is, the price at which the market maker will sell the option to you.

How to Read an Option Table?

Stk	Exp	P/C	Vol	Bid	Ask	Opint
360networks (TSIX)						20.15
20	Feb	C	3	1.00	1.25	26
22	Mar	C	10	1.60	1.85	138
24	Mar	C	2	1.05	1.25	366
18	June	P	2	2.50	2.75	11
20	June	C	12	4.05	4.30	83
24	June	C	1	2.65	2.90	77

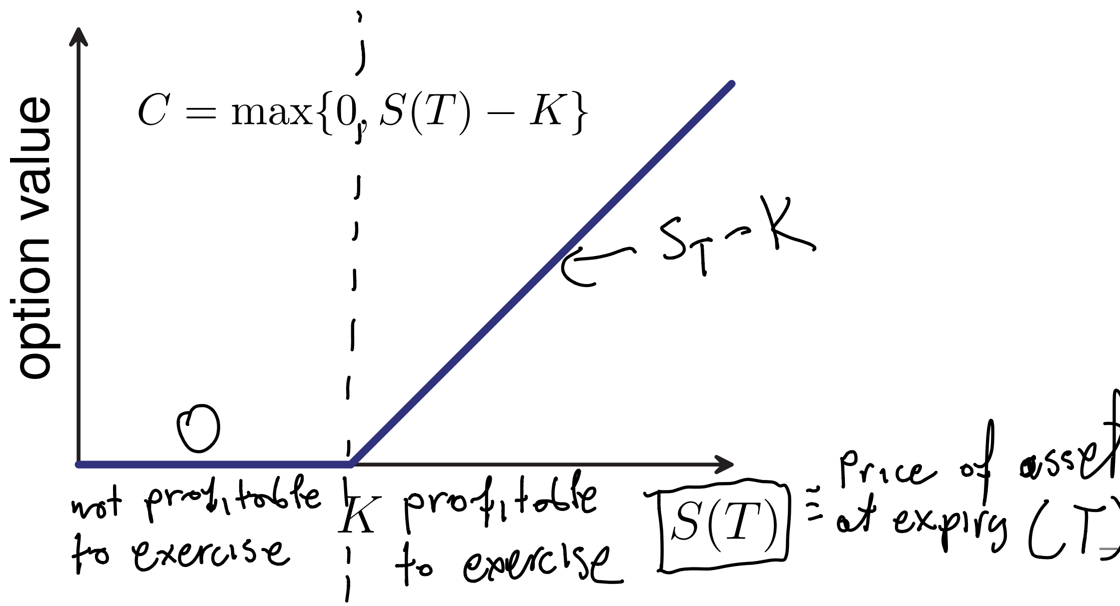
The Opint column shows the open interest, that is, the number of option contracts which have not yet expired or have not been exercised.

Payoff Diagrams I

You own a European call option with strike price K and expiration date T. Suppose that at T the price of the underlying stock is $S(T)$. What is the option value C at T?

- If $S(T) < K$: option is not exercised $\Rightarrow C = 0$;
- If $S(T) > K$: exercise the option $\Rightarrow C = S(T) - K$.

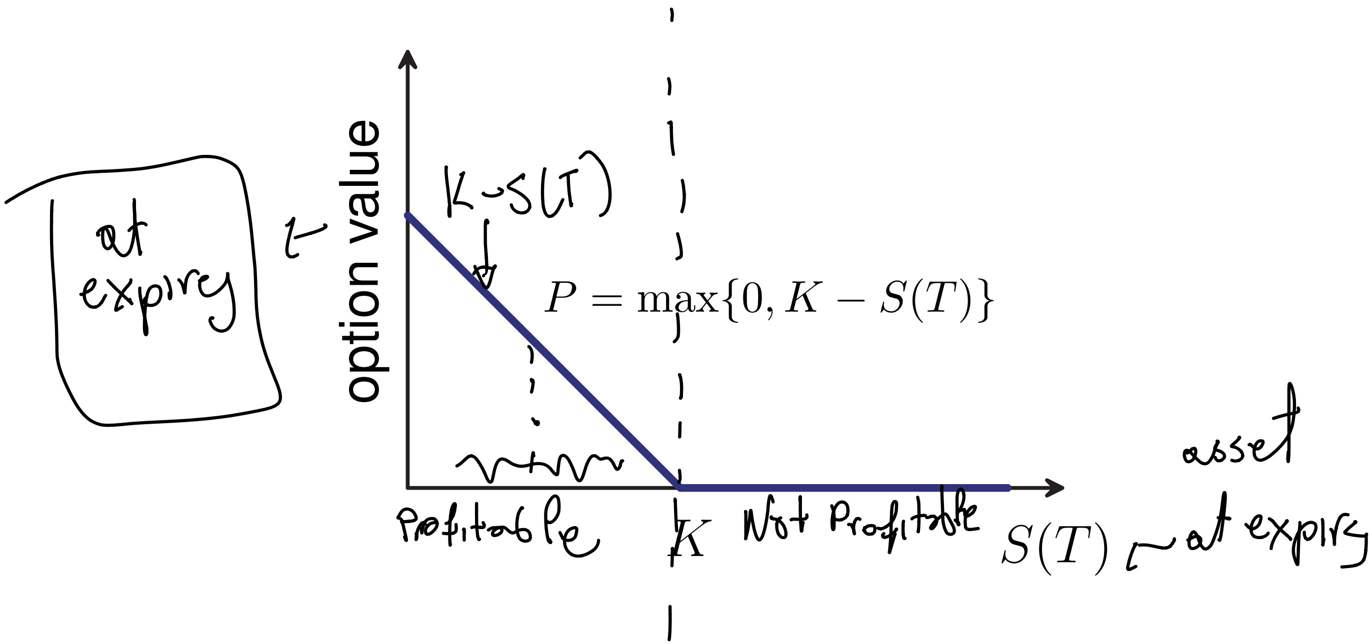
$C(T)$
$= \max\{S_T - K, 0\}$
$= [S_T - K]^+$



Payoff Diagrams II

You own a European put option with strike price K and expiration date T. Suppose that at T the price of the underlying stock is $S(T)$. What is the option value P at T?

- If $S(T) < K$: exercise the option $\Rightarrow P = K - S(T)$;
- If $S(T) > K$: option is not exercised $\Rightarrow P = 0$.



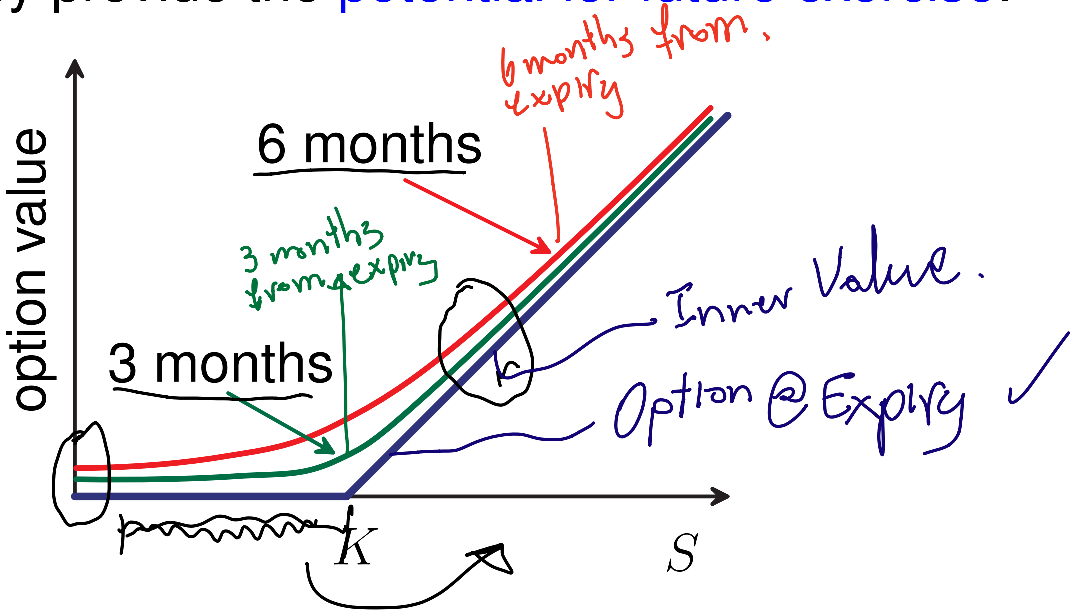
Payoff Diagrams III

Some terminology:

- The payoff curves of call and put options are sometimes referred to as hockey sticks;
- We say that a call option is in the money, at the money, or out of the money depending on whether $S(t) > K$, $S(t) = K$, or $S(t) < K$, respectively ($t \leq T$).
- Puts have the reverse terminology since the payoffs at exercise are positive if $S(t) < K$.
- The inner value of an (American or European) option is the value that it would have if it was exercised immediately (at any $t \leq T$).

Time Value of Options I

The value of an option at expiration is derived from its basic specification. However, options have also a value at earlier times since they provide the potential for future exercise.



Even if an option is out of the money 3 months before T, say, it is still possible to end up in the money at T. Thus, the option value at time $t < T$ is larger than its inner value.

Time Value of Options II

- At $t < T$, the value of a call option is a smooth curve rather than a decidedly kinked curve.
- This smooth curve can be determined by estimation, using data of actual option prices.
- The curve gets higher with increasing length to expiration since additional time increases the chance for the stock to rise in value.

Option Combinations

- It is common to invest in combinations of options and the stock in order to implement special hedging or speculative strategies.
- By forming such combinations, it is possible to approximate virtually any payoff function by a sequence of straight line segments.
- The cost of such a payoff is then just the sum of the costs of the individual components.

Example: Butterfly Spread

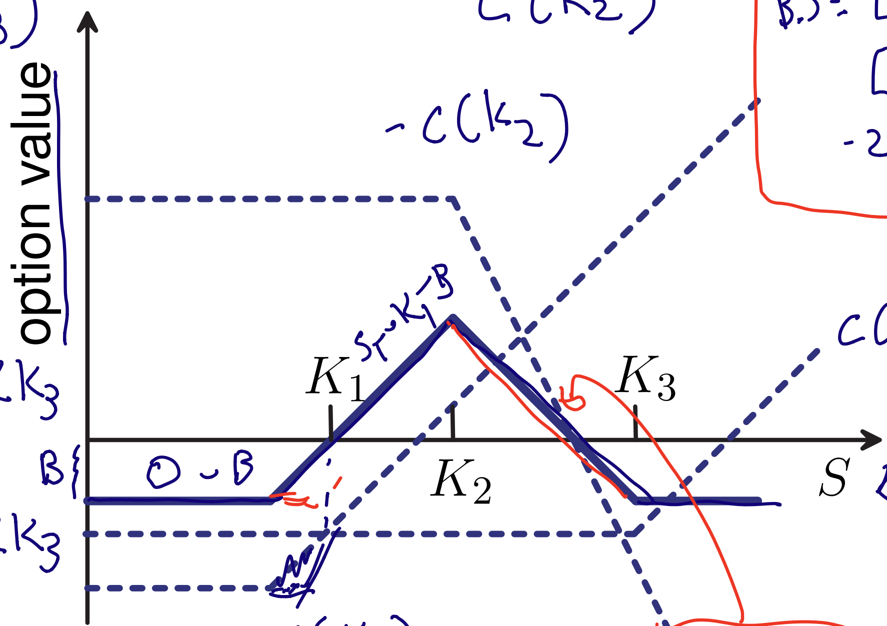
$C(K_1)$ $C(K_3)$

Buy two calls with different strike prices K_1 and K_3 and sell two calls with equal strike price K_2, where $K_1 < K_2 < K_3$.

$C(K_1) + C(K_3) - 2C(K_2)$

$[\cdot]^+ = \max\{\cdot, 0\}$

$\boxed{\begin{aligned} B.S = {} & [S_T - K_1]^+ \\ & [S_T - K_3]^+ \\ & -2[S_T - K_2]^+ \end{aligned}}$



Cases:

1. $S_T < K_1 < K_2 < K_3$
   $B.S = 0$
2. $K_1 < S_T < K_2 < K_3$
   $B.S = S_T - K_1$
3. $K_1 < K_2 < S_T < K_3$ $= B.S = S_T - K_1 - 2S_T + K_2 = \boxed{K_2 - K_1 - S_T}$

All curves show profits (including payoffs and initial costs).
The butterfly spread yields a positive profit for $S(T) \approx K_2$.

4. $K_1 < K_2 < K_3 < S_T$

Put-Call Parity I

For European options the following holds: a suitable combination of a put, a call, and a risk-free loan has a payoff identical to that of the underlying stock.

Construction:

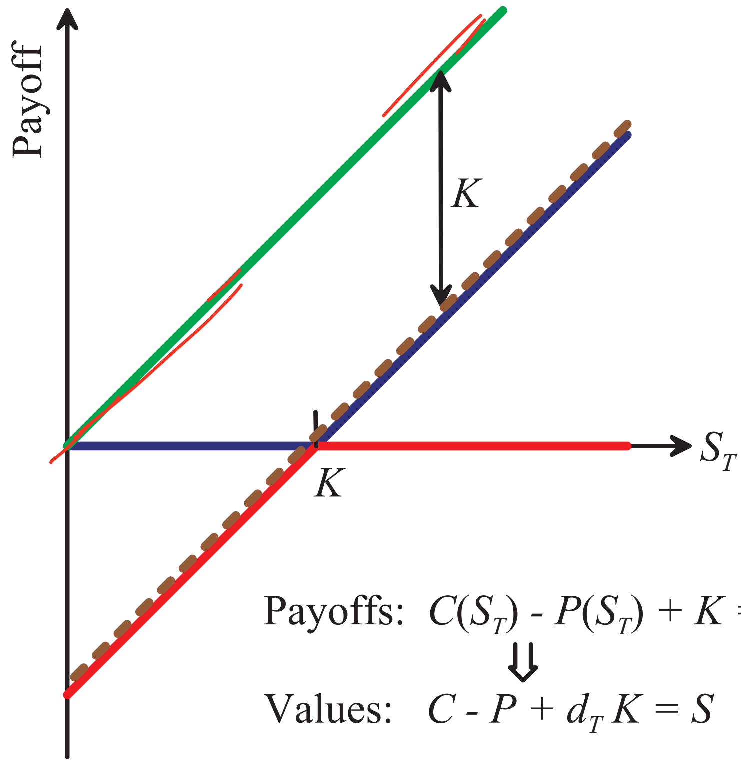
- buy one call, sell one put (with the same strike price K), and lend an amount $d_T K$;
- the combination of the call and the put has a payoff that is a line at 45°, passing through K on the horizontal axis;
- by lending $d_T K$, we lift the payoff line up so that it passes through the origin. This final payoff is exactly that of the stock!

$$\Rightarrow \quad C - P + d_T K = S$$

Put-Call Parity II

Long = Buy
Short = Sell.

C(t)



← extra asset

C − P

Payoffs: $C(S_T) - P(S_T) + K = S_T$

$$\Downarrow$$

Values: $C - P + d_T K = S$

Options Pricing Theory

(Arbitrage Pricing Theory)

Options pricing theory calculates the theoretical value of an option. There are several approaches based on different assumptions about

- the market,
- the dynamics of stock prices,
- individual preferences.

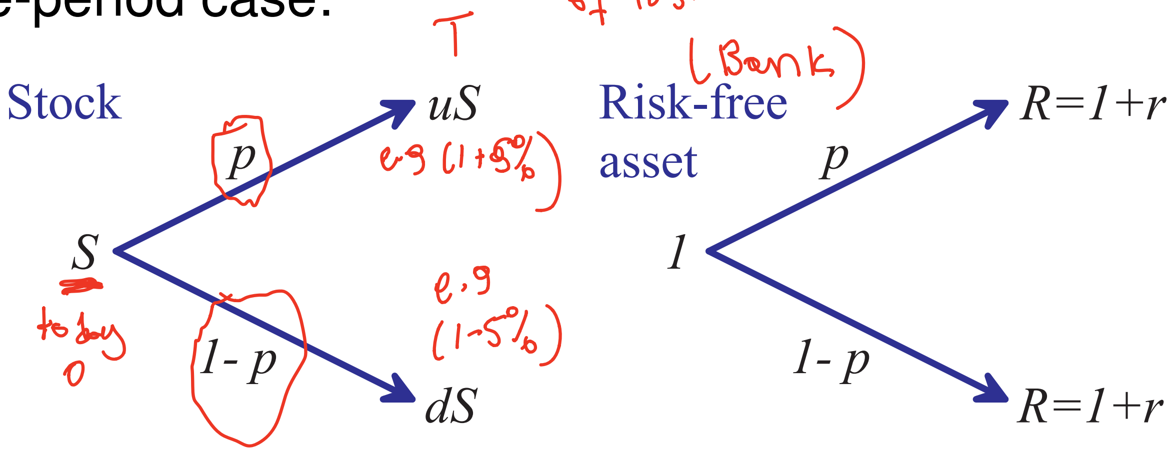
The most important theories are based on the no arbitrage principle, which can be applied when the stock price dynamics take certain forms.

The simplest of these theories is based on the binomial model of stock price fluctuations.

Binomial Options Theory I

e.g use u,d from Last slide of last lecture.

Single-period case:



To avoid arbitrage, we must require $\boxed{u > R > d}$.

Proof: Assume that $R \geq u > d$ and $0 < p < 1$. By shorting £1.00 of the stock and loaning the proceeds, one obtains a profit of either $R - u$ or $R - d$. The initial cost is zero, but the profit is positive in both events. This is a type B arbitrage!

A similar argument rules out $u > d \geq R$. $\square$

Binomial Options Theory II

(Derivatives)

Consider a call option with strike price K and expiration at the end of the (single) period. The binomial lattices for the stock price, the value of a risk-free asset, and the value of the option have common arcs:

$C(T) = \max\{S(T) - K, 0\}$

T = expiry date, K - strike

$S(T)$ = asset at time T.

$$\begin{tikzcd} & uS \\ \boxed{S} \arrow[ur] \arrow[dr] & \\ & dS \end{tikzcd}$$

$$\begin{tikzcd} & R \\ \boxed{1} \arrow[ur] \arrow[dr] & \\ & R \end{tikzcd}$$

$T=1$

$$\begin{tikzcd} & \max\{uS - K, 0\} \\ \boxed{C} = ? \arrow[ur] \arrow[dr] & \\ & \max\{dS - K, 0\} \end{tikzcd}$$

Payoffs at expiry $T = 1$

$$S(T) = \begin{cases} uS & \text{w.p } p \\ dS & \text{w.p } 1-p \end{cases}$$

Binomial Options Theory III

By combining suitable proportions of the stock and the risk free asset, we can construct any pattern of outcomes.

C? $\to$ C_u, C_d

The outcome pattern of the option is:

$$\boxed{C_u = \max\{uS - K, 0\}} \quad \text{and} \quad \boxed{C_d = \max\{dS - K, 0\}\,.}$$

The portfolio that invests x dollars in the stock and b dollars in the risk-free asset pays off either $ux + Rb$ or $dx + Rb$, depending on the realized scenario.

Π $\to$ $\Pi_u = C_u$, $\Pi_d = C_d$

then by no arbitrage $\Pi = C$

To match the option, we require

$$\Pi_u = \boxed{ux + Rb = C_u} \quad \text{and} \quad \Pi_d = \boxed{dx + Rb = C_d\,.}$$

$\Pi_u - \Pi_d$: find x

$$\Rightarrow \quad \boxed{x = \frac{C_u - C_d}{u - d}}\,, \quad \boxed{b = \frac{C_u - ux}{R}} = \boxed{\frac{uC_d - dC_u}{R(u-d)}}$$

$(u-d)x = C_u - C_d$

Binomial Options Theory IV

The value of the replicating portfolio thus amounts to

$$x+b=\underbrace{\frac{C_u-C_d}{u-d}+\frac{uC_d-dC_u}{R(u-d)}}=\boxed{\frac{1}{R}\left(\frac{R-d}{u-d}C_u+\frac{u-R}{u-d}C_d\right)}.$$

- The replicating portfolio has the same payoff as the call option in every scenario.

$\Rightarrow$ The portfolio value must equal the price of the call.

- If $C > x+b$ ($C < x+b$), we could buy (sell) the portfolio and sell (buy) the call for an immediate gain and no future costs $\Rightarrow$ type A arbitrage!

>0 >0

$d<R<u$
from no arbitrage

$$\Rightarrow \quad \boxed{C=\frac{1}{R}\left(\boxed{\frac{R-d}{u-d}}C_u+\boxed{\frac{u-R}{u-d}}C_d\right)}.$$

~q (1-q), q>0

Binomial Options Theory V

$\Pi = x + b$

- The replicating portfolio made up of the stock and the risk-free asset duplicates the outcome of the option.
- This replicating idea can be used to find the value of any security defined on the same lattice; that is, any derivative of the stock.
- We can also write

$$\boxed{C = \frac{1}{R}\left[qC_u + (1-q)C_d\right]} \quad \text{where} \quad \boxed{q = \frac{R-d}{u-d}}. \tag{1}$$

$\Rightarrow$ C is found by taking the expected value of the option payoff using the artificial probability q, and then discounting this value at the risk-free rate.

Note that $0 < q < 1$ since $d < R < u$.

Risk-Neutral Valuation

We can rewrite (1) as

$$\boxed{C(T-1) = \frac{1}{R}\hat{\mathrm{E}}[C(T)]},$$

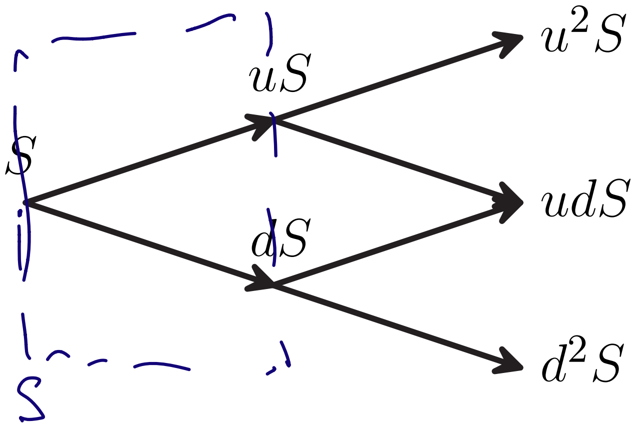
where $C(T)$ and $C(T-1)$ are the call values at T and $T-1$, respectively, and $\hat{\mathrm{E}}$ denotes expectation with respect to the risk neutral probabilities q and $1-q$.

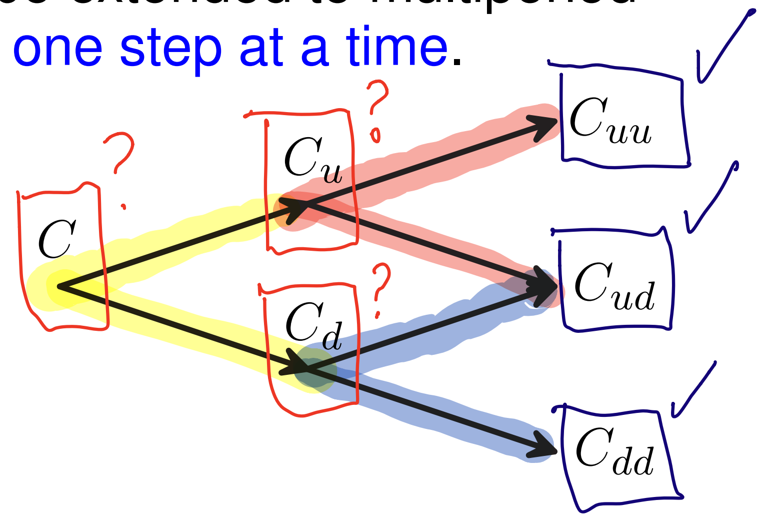
The pricing formula is is independent of the probability p!

The value C is found by matching the outcomes of the option with a combination of the stock and the risk-free asset. Probability never enters this calculation.

Multiperiod Options I

The one-period solution can be extended to multiperiod options by working backward one step at a time.





By the basic structure of a call with strike price K, we find

$$
\begin{aligned}
C_{uu} &= \max\{u^2S - K, 0\} \\
C_{ud} &= \max\{udS - K, 0\} \\
C_{dd} &= \max\{d^2S - K, 0\}
\end{aligned}
$$

Multiperiod Options II

Again, we define the risk-neutral probability

$$q = \frac{R-d}{u-d}\,.$$

Repeating the single-period calculations, we find

$$\boxed{C_u = \frac{1}{R}\left[qC_{uu} + (1-q)C_{ud}\right]}$$
$$\boxed{C_d = \frac{1}{R}\left[qC_{ud} + (1-q)C_{dd}\right]\,.}$$

Applying the risk-neutral valuation formula once again gives

$$\boxed{C = \frac{1}{R}\left[qC_u + (1-q)C_d\right]\,.}$$

5 Month Call I

Assume stock is a log-normal random variable. (Black & Scholes)

Consider a non-dividend paying stock with a volatility of its logarithm of $\sigma = 20\%$ and with initial price £62. The current interest rate is 10% and compounded monthly. We want to price a European call with $K = £60$ and expiration date 5 months from now.

K

T

We choose a period length of 1 month, that is, $\boxed{\Delta t = 1/12}$.

see previous lecture.

$$\Rightarrow \quad u = e^{\sigma\sqrt{\Delta t}} = 1.05943$$

$$d = e^{-\sigma\sqrt{\Delta t}} = 0.94390$$

$$R = 1 + 0.1/12 = 1.00833$$

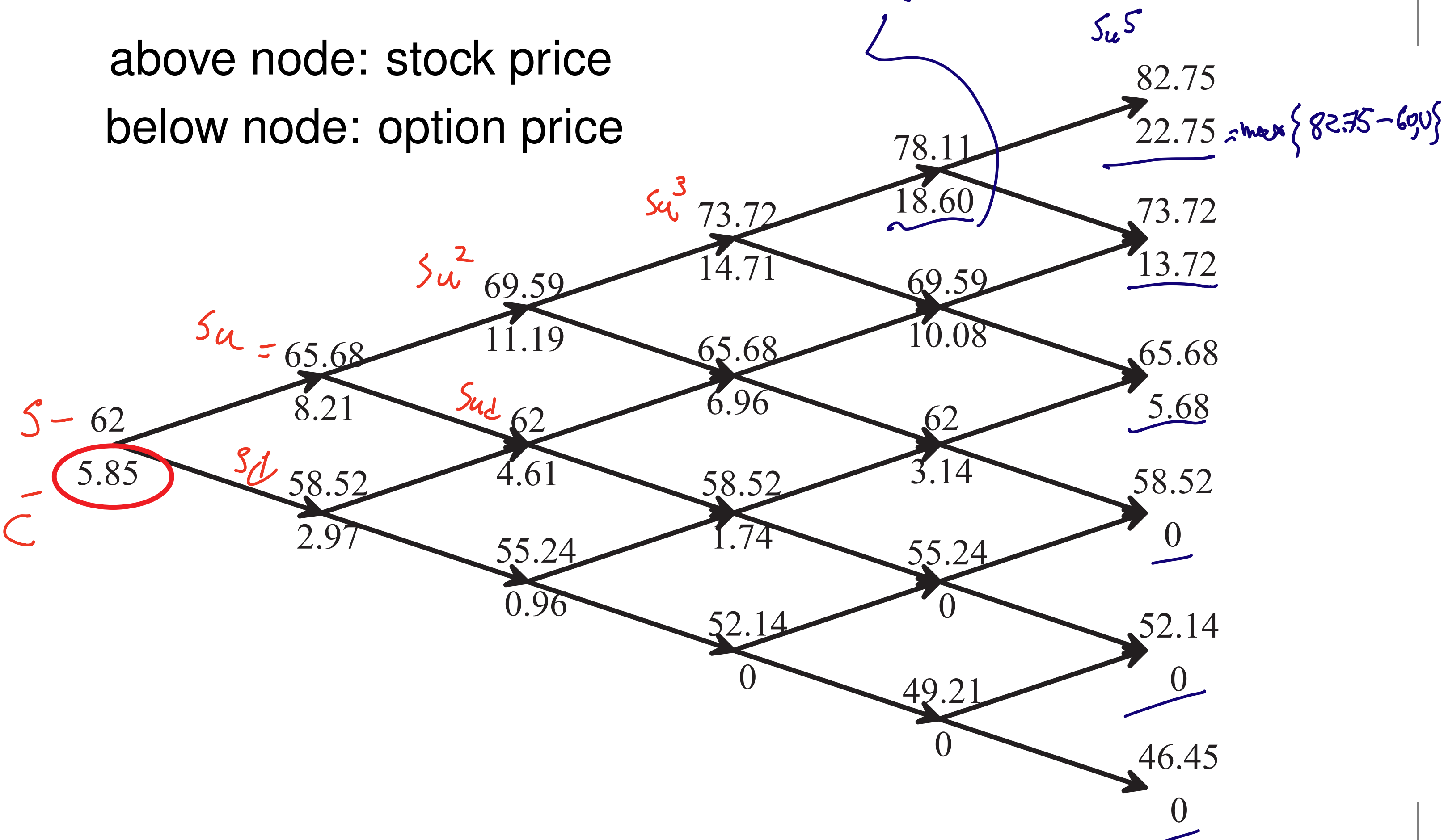
$$\Rightarrow \quad q = (R-d)/(u-d) = 0.55770\,.$$

5 Month Call II

$18.6 = \frac{1}{R}[q \times 22.75 + (1-q)13.72]$

above node: stock price

below node: option price



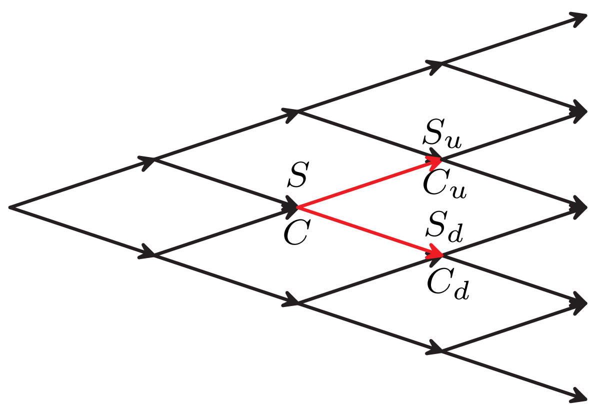
American Options I

Recursion for European call options:

$$\boxed{C = \frac{1}{R}\left[qC_u + (1-q)C_d\right] .}$$

Recursion for American call options:

$$\underline{C} = \max\left\{\underbrace{S - K}, \underbrace{\frac{1}{R}\left[qC_u + (1-q)C_d\right]}\right\}$$



American Options II

Theorem 1. *American call options are never exercised early.* American Call = European Call.

Proof. Fix any non-terminal node S in the stock price lattice and consider its successor nodes $\boxed{S_u = uS}$ and $\boxed{S_d = dS}$. As the option value is always larger than the value of immediate exercise, we have

$$\boxed{C_u \geq S_u - K} \quad \text{and} \quad \boxed{C_d \geq S_d - K\,.}$$

Hence, we find

$$\boxed{C \;\geq\; \frac{1}{R}\left[q(\underbrace{S_u}_{Su} - K) + (1-q)(S_d - K)\right]}$$

$$= \frac{1}{R}\left[(qu + \underbrace{[1-q]d}_{\hookrightarrow\, q = (u-R)/u-d})S - K\right] = S - \frac{K}{R} \geq \underbrace{S - K}_{\text{since } R>1}.$$

Sd

$\Rightarrow$ the option is not exercised in node S. Since S is an arbitrary non-terminal node, the option is never exercised before the expiration date. $\square$

American Options III

Early exercise is optimal for American put options! Consider a 5 month put with $K = £60$ on the same binomial lattice as before. Bold face entries indicate points of early exercise.

Stock Price					
62.00	65.68	69.59	73.72	78.11	82.75
	58.52	62.00	65.68	69.59	73.72
		55.24	58.52	$\boxed{62.00}$	65.68
			52.14	55.24	58.52
				49.21	52.14
					46.45

Stock Price

American Put Option					
1.56	0.61	0.12	0.00	0.00	0.00
	2.79	1.23	0.28	0.00	0.00
		4.80	2.45	$\boxed{0.65}$	0.00
			7.86	**4.76**	**1.48**
				10.79	**7.86**
					13.55

American Put

q=	0.55775
R=	1.008333

$\max\{\underset{60}{K} - \underset{62}{S}, \underbrace{\tfrac{1}{R}(q C_u + (1-q) C_d)}\}$

$[K - S_T, 0]^+$

Simplico Gold Mine I

(See materials for a spreadsheet)

- Gold can be extracted from a mine at a rate of $10,000$ ounces per year at a cost of $£200$ per ounce.
- Today, the market price of gold is $£400$ per ounce, and each year it increases either by a factor of $\boxed{1.2}$ (with probability 0.75) or decreases by a factor of $\boxed{0.9}$. (σ)
  don't need this.
- The risk-free interest rate is 10% (compounded yearly).

We assume that gold mined in a given year is sold at the price at the beginning of the year, but the corresponding cash flow occurs at the end of the year.

What is the fair value of a 10 year lease of this mine?

Simplico Gold Mine II

The gold mine is a derivative whose underlying security is gold. In fact, the value of the mine can only be a function of the gold price.

We represent future gold prices by a binomial lattice, and the value of the lease is calculated for each of its nodes:

- at the end of the 10 years, the values are zero since the mine must be returned to the owners;
- for an earlier node, the value of the lease is the sum of the profit made in that year and the risk-neutral expected value of the lease in the next period.

Risk neutral probabilities: $\boxed{q = \frac{1.1-0.9}{1.2-0.9} = \frac{2}{3},\ 1 - q = \frac{1}{3}.}$

At nodes where the price of gold is $\leq$ £200 we do not mine.

Simplico Gold Mine III

Gold Price										
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
S= 400.0	Su 480.0	576.0	691.2	829.4	995.3	1194.4	1433.3	1719.9	2063.9	2476.7
	Sd 360.0	432.0	518.4	622.1	746.5	895.8	1075.0	1289.9	1547.9	1857.5
		324.0	388.8	466.6	559.9	671.8	806.2	967.5	1161.0	1393.1
			291.6	349.9	419.9	503.9	604.7	725.6	870.7	1044.9
				262.4	314.9	377.9	453.5	544.2	653.0	783.6
					236.2	283.4	340.1	408.1	489.8	587.7
						212.6	255.1	306.1	367.3	440.8
							191.3	229.6	275.5	330.6
								172.2	206.6	247.9
									155.0	186.0
										139.5

Price of the gold mine lease (millions of $)										
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
24.1	27.8	31.2	34.2	36.5	37.7	37.1	34.1	27.8	16.9	0.0
	17.9	20.7	23.3	25.2	26.4	26.2	24.3	20.0	12.3	0.0
		12.9	15.0	16.7	17.9	18.1	17.0	14.1	8.7	0.0
			8.8	10.4	11.5	12.0	11.5	9.7	6.1	0.0
				5.6	6.7	7.4	7.4	6.4	4.1	0.0
q=	0.66667				3.2	4.0	4.3	3.9	2.6	0.0
						1.4	2.0	2.1	1.5	0.0
							0.4	0.7	0.7	0.0
								0.0	0.1	0.0
									0.0	0.0
										0.0

$$10.000 \, (2063.9 - 200) \frac{1}{R}$$

Real Options

Sometimes, options are associated with investment opportunities that are not financial instruments:

- if you own a factory, you have the option of hiring additional employees or buying new equipment;
- if you own a piece of land, you have the option to drill for oil, and, if oil is found, you have the option to extract it.

Operational options of this type are termed real options to emphasize that they involve real activities or real commodities, as opposed to purely financial ones.

Options theory can also be used to analyze real options!

General Risk-Neutral Pricing I

Let the price S of an asset be described by a binomial lattice, and suppose that f is a security whose cash flow f_k at any time k is only a function of the node at time k.

$\Rightarrow$ The arbitrage-free price f_{val} of the asset is

$$\boxed{f_{\mathrm{val}} = \hat{\mathrm{E}}\left(\sum_{k=0}^{N} d_k \, f_k\right)}.$$

- d_k is the risk-free discount factor as seen at time 0; 1/R
- f_k is the period cash-flow which is node-dependent (i.e. random); (any function of S :).
- expectation $\hat{\mathrm{E}}(\cdot)$ is taken w.r.t. risk-neutral probabilities.

(q, 1-q)

General Risk-Neutral Pricing II

Example: In the case of a European call option with strike price K, there is only a single cash-flow $\max(S_N - K, 0)$ occurring at the final time.

In general, the cash-flow stream can be influenced by our actions. We may have the opportunity to

- exercise an option early,
- decide how much gold to mine,
- or add enhancements.

In such cases, the general pricing formula becomes

$$f_{\text{val}} = \underline{\max_{\text{our actions}}} \hat{\mathrm{E}}\left(\sum_{k=0}^{N} d_k\, f_k\right) .$$

Stochastic Dynamic Programming