Tutorial 7 – 60016 Operations Research

Cutting Plane Methods, Branch & Bound

Exercise 1

Solve the following problem with the Gomory cutting plane approach

$$\begin{aligned}
\underset{x_1,x_2}{\text{maximise}} \quad & y = x_1 + 4x_2 \\
\text{subject to} \quad & 2x_1 + 4x_2 \leq 7 \\
& 10x_1 + 3x_2 \leq 14 \\
& x_1, x_2 \geq 0,\ x_1, x_2 \in \mathbb{N}_0
\end{aligned}$$

Solution 1

The original problem is

$$\begin{aligned}
\underset{x_1,x_2}{\text{maximise}} \quad & x_1 + 4x_2 \\
\text{subject to} \quad & 2x_1 + 4x_2 \leq 7 \\
& 10x_1 + 3x_2 \leq 14 \\
& x_1, x_2 \geq 0,\ x_1, x_2 \in \mathbb{N}_0
\end{aligned}$$

Change the problem from maximisation to minimisation

$$\begin{aligned}
\underset{x_1,x_2}{\text{minimise}} \quad & -x_1 - 4x_2 \\
\text{subject to} \quad & 2x_1 + 4x_2 \leq 7 \\
& 10x_1 + 3x_2 \leq 14 \\
& x_1, x_2 \geq 0,\ x_1, x_2 \in \mathbb{N}_0
\end{aligned}$$

Convert the 1st constraint to an equality constraint by introducing an integer slack variable x_3

$$\begin{aligned}
\underset{x_1,x_2,x_3}{\text{minimise}} \quad & -x_1 - 4x_2 \\
\text{subject to} \quad & 2x_1 + 4x_2 + x_3 = 7 \\
& 10x_1 + 3x_2 \leq 14 \\
& x_1, x_2, x_3 \geq 0,\ x_1, x_2, x_3 \in \mathbb{N}_0
\end{aligned}$$

Convert the 2nd constraint to an equality constraint by introducing an integer slack variable x_4

$$\begin{aligned}
\underset{x_1,x_2,x_3,x_4}{\text{minimise}} \quad & -x_1 - 4x_2 \\
\text{subject to} \quad & 2x_1 + 4x_2 + x_3 = 7 \\
& 10x_1 + 3x_2 + x_4 = 14 \\
& x_1, x_2, x_3, x_4 \geq 0,\ x_1, x_2, x_3, x_4 \in \mathbb{N}_0
\end{aligned}$$

Initial basic representation

BV	x_1	x_2	x_3	x_4	RHS
z	1	4			0
x_3	2	4	1		7
x_4	10	3		1	14

Pivoting on row 2, col 2

BV	x_1	x_2	x_3	x_4	RHS
z	1	4			0
x_3	2	4	1		7
x_4	10	3		1	14

Perform the pivoting operations

BV	x_1	x_2	x_3	x_4	RHS
z	-1		-1		-7
x_2	$\frac{1}{2}$	1	$\frac{1}{4}$		$\frac{7}{4}$
x_4	$\frac{17}{2}$		$-\frac{3}{4}$	1	$\frac{35}{4}$

Simplex Stops, optimal solution found.
Not integral, so need to generate Gomory cut.
Can generate cuts for variables x_2, x_4. Picking x_2

$$\tfrac{1}{2}x_1 + x_2 + \tfrac{1}{4}x_3 = \tfrac{7}{4} \iff \left[0 + \tfrac{1}{2}\right] x_1 + [1 + 0]\, x_2 + \left[0 + \tfrac{1}{4}\right] x_3 = \left[1 + \tfrac{3}{4}\right]$$

$$\tfrac{1}{2}x_1 + \tfrac{1}{4}x_3 \geq \tfrac{3}{4} \rightsquigarrow \tfrac{1}{2}x_1 + \tfrac{1}{4}x_3 - x_5 + \xi_1 = \tfrac{3}{4}$$

Add temporary objective $\zeta = \xi_1$ expressed as function of the NBVs:

BV	x_1	x_2	x_3	x_4	x_5	ξ_1	RHS
z	-1		-1				-7
x_2	$\frac{1}{2}$	1	$\frac{1}{4}$				$\frac{7}{4}$
x_4	$\frac{17}{2}$		$-\frac{3}{4}$	1			$\frac{35}{4}$
ζ	$\frac{1}{2}$		$\frac{1}{4}$		-1		$\frac{3}{4}$
ξ_1	$\frac{1}{2}$		$\frac{1}{4}$		-1	1	$\frac{3}{4}$

Pivoting on row 3, col 1

BV	x_1	x_2	x_3	x_4	x_5	ξ_1	RHS
z	-1		-1				-7
x_2	$\frac{1}{2}$	1	$\frac{1}{4}$				$\frac{7}{4}$
x_4	$\frac{17}{2}$		$-\frac{3}{4}$	1			$\frac{35}{4}$
ζ	$\frac{1}{2}$		$\frac{1}{4}$		-1		$\frac{3}{4}$
ξ_1	$\frac{1}{2}$		$\frac{1}{4}$		-1	1	$\frac{3}{4}$

Perform the pivoting operations

BV	x_1	x_2	x_3	x_4	x_5	ξ_1	RHS
z			$-\frac{37}{34}$	$\frac{2}{17}$			$-\frac{203}{34}$
x_2		1	$\frac{5}{17}$	$-\frac{1}{17}$			$\frac{21}{17}$
x_1	1		$-\frac{3}{34}$	$\frac{2}{17}$			$\frac{35}{34}$
ζ			$\frac{5}{17}$	$-\frac{1}{17}$	-1		$\frac{4}{17}$
ξ_1			$\frac{5}{17}$	$-\frac{1}{17}$	-1	1	$\frac{4}{17}$

Pivoting on row 5, col 3

BV	x_1	x_2	x_3	x_4	x_5	ξ_1	RHS
z			$-\frac{37}{34}$	$\frac{2}{17}$			$-\frac{203}{34}$
x_2		1	$\frac{5}{17}$	$-\frac{1}{17}$			$\frac{21}{17}$
x_1	1		$-\frac{3}{34}$	$\frac{2}{17}$			$\frac{35}{34}$
ζ			$\frac{5}{17}$	$-\frac{1}{17}$	-1		$\frac{4}{17}$
ξ_1			$\frac{5}{17}$	$-\frac{1}{17}$	-1	1	$\frac{4}{17}$

Perform the pivoting operations

BV	x_1	x_2	x_3	x_4	x_5	ξ_1	RHS
z				$-\frac{1}{10}$	$-\frac{37}{10}$	$\frac{37}{10}$	$-\frac{51}{10}$
x_2		1			1	-1	1
x_1	1			$\frac{1}{10}$	$-\frac{3}{10}$	$\frac{3}{10}$	$\frac{11}{10}$
ζ						-1	0
x_3			1	$-\frac{1}{5}$	$-\frac{17}{5}$	$\frac{17}{5}$	$\frac{4}{5}$

Simplex Stops, optimal solution found.
Removing the ζ row and ξ_1 column

BV	x_1	x_2	x_3	x_4	x_5	RHS
z				$-\frac{1}{10}$	$-\frac{37}{10}$	$-\frac{51}{10}$
x_2		1			1	1
x_1	1			$\frac{1}{10}$	$-\frac{3}{10}$	$\frac{11}{10}$
x_3			1	$-\frac{1}{5}$	$-\frac{17}{5}$	$\frac{4}{5}$

Simplex Stops, optimal solution found.
Not integral, so need to generate Gomory cut.
Can generate cuts for variables x_1, x_3. Picking x_1

$$x_1 + \tfrac{1}{10}x_4 - \tfrac{3}{10}x_5 = \tfrac{11}{10} \iff [1+0]\,x_1 + \left[0 + \tfrac{1}{10}\right] x_4 + \left[-1 + \tfrac{7}{10}\right] x_5 = \left[1 + \tfrac{1}{10}\right]$$

$$\tfrac{1}{10}x_4 + \tfrac{7}{10}x_5 \geq \tfrac{1}{10} \rightsquigarrow \tfrac{1}{10}x_4 + \tfrac{7}{10}x_5 - x_6 + \xi_1 = \tfrac{1}{10}$$

Add temporary objective $\zeta = \xi_1$ expressed as function of the NBVs:

BV	x_1	x_2	x_3	x_4	x_5	x_6	ξ_1	RHS
z				$-\frac{1}{10}$	$-\frac{37}{10}$			$-\frac{51}{10}$
x_2		1			1			1
x_1	1			$\frac{1}{10}$	$-\frac{3}{10}$			$\frac{11}{10}$
x_3			1	$-\frac{1}{5}$	$-\frac{17}{5}$			$\frac{4}{5}$
ζ				$\frac{1}{10}$	$\frac{7}{10}$	-1		$\frac{1}{10}$
ξ_1				$\frac{1}{10}$	$\frac{7}{10}$	-1	1	$\frac{1}{10}$

Pivoting on row 6, col 5

BV	x_1	x_2	x_3	x_4	x_5	x_6	ξ_1	RHS
z				$-\frac{1}{10}$	$-\frac{37}{10}$			$-\frac{51}{10}$
x_2		1			1			1
x_1	1			$\frac{1}{10}$	$-\frac{3}{10}$			$\frac{11}{10}$
x_3			1	$-\frac{1}{5}$	$-\frac{17}{5}$			$\frac{4}{5}$
ζ				$\frac{1}{10}$	$\frac{7}{10}$	-1		$\frac{1}{10}$
ξ_1				$\frac{1}{10}$	$\frac{7}{10}$	-1	1	$\frac{1}{10}$

Perform the pivoting operations

BV	x_1	x_2	x_3	x_4	x_5	x_6	ξ_1	RHS
z				$\frac{3}{7}$		$-\frac{37}{7}$	$\frac{37}{7}$	$-\frac{32}{7}$
x_2		1		$-\frac{1}{7}$		$\frac{10}{7}$	$-\frac{10}{7}$	$\frac{6}{7}$
x_1	1			$\frac{1}{7}$		$-\frac{3}{7}$	$\frac{3}{7}$	$\frac{8}{7}$
x_3			1	$\frac{2}{7}$		$-\frac{34}{7}$	$\frac{34}{7}$	$\frac{9}{7}$
ζ							-1	0
x_5				$\frac{1}{7}$	1	$-\frac{10}{7}$	$\frac{10}{7}$	$\frac{1}{7}$

Simplex Stops, optimal solution found.
Removing the ζ row and ξ_1 column

BV	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z				$\frac{3}{7}$		$-\frac{37}{7}$	$-\frac{32}{7}$
x_2		1		$-\frac{1}{7}$		$\frac{10}{7}$	$\frac{6}{7}$
x_1	1			$\frac{1}{7}$		$-\frac{3}{7}$	$\frac{8}{7}$
x_3			1	$\frac{2}{7}$		$-\frac{34}{7}$	$\frac{9}{7}$
x_5				$\frac{1}{7}$	1	$-\frac{10}{7}$	$\frac{1}{7}$

Pivoting on row 5, col 4

BV	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z				$\frac{3}{7}$		$-\frac{37}{7}$	$-\frac{32}{7}$
x_2		1		$-\frac{1}{7}$		$\frac{10}{7}$	$\frac{6}{7}$
x_1	1			$\frac{1}{7}$		$-\frac{3}{7}$	$\frac{8}{7}$
x_3			1	$\frac{2}{7}$		$-\frac{34}{7}$	$\frac{9}{7}$
x_5				$\frac{1}{7}$	1	$-\frac{10}{7}$	$\frac{1}{7}$

Perform the pivoting operations

BV	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z					-3	-1	-5
x_2		1			1		1
x_1	1				-1	1	1
x_3			1		-2	-2	1
x_4				1	7	-10	1

Simplex Stops, optimal solution found.
Termination - all variables are integral.
Optimal Solutions $x_1 = 1, x_2 = 1, x_3 = 1, x_4 = 1, x_5 = 0, x_6 = 0$, giving an objective value of -5.

Exercise 2

Consider the knapsack constraint set:

$$\begin{aligned} 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 &\leq 19 \\ \mathbf{x} &\in \{0,1\}^7. \end{aligned}$$

For each of the following inequalities, identify whether or not they are valid knapsack cover cuts, and explain why. For all valid knapsack covers, identify whether or not they are minimal.

- $x_4 + x_5 + x_6 \leq 2$
- $x_1 + x_2 + x_6 \leq 2$
- $x_2 + x_3 + x_6 + x_7 \leq 3$
- $x_2 + x_4 + x_5 + x_6 \leq 3$
- $x_1 + x_3 + x_4 + x_5 \leq 3$
- $x_2 + x_3 + x_4 + x_5 + x_6 \leq 4$

Solution 2

- $x_4 + x_5 + x_6 \leq 2$
  The coefficients associated with x_4, x_5, x_6 are $5, 5, 4$. Since $5+5+4 \not> 19$, this is NOT a valid knapsack cover.

- $x_1 + x_2 + x_6 \leq 2$
  The coefficients associated with x_1, x_2, x_6 are $11, 6, 4$. Since $11 + 6 + 4 = 21 > 19$, this is a valid knapsack cover. This knapsack cover is a minimal cover because every proper subset of $\{11, 6, 4\}$ has summation less than 19.

- $x_2 + x_3 + x_6 + x_7 \leq 3$
  The coefficients associated with x_2, x_3, x_6, x_7 are $6, 6, 4, 1$. Since $6+6+4+1 \not> 19$, this is NOT a valid knapsack cover.

- $x_2 + x_4 + x_5 + x_6 \leq 3$
  The coefficients associated with x_2, x_4, x_5, x_6 are $6, 5, 5, 4$. Since $6+5+5+4 = 20 > 19$, this is a valid knapsack cover. This knapsack cover is a minimal cover because every proper subset of $\{6, 5, 5, 4\}$ has summation less than 19.

- $x_1 + x_3 + x_4 + x_5 \leq 3$
  The coefficients associated with x_1, x_3, x_4, x_5 are $11, 6, 5, 5$. Since $11 + 6 + 5 + 5 = 27$, this is a valid knapsack cover. It is not a minimal cover because there are subsets, e.g. $\{11, 6, 5\}$ that are also knapsack covers.
- $x_2 + x_3 + x_4 + x_5 + x_6 \leq 4$
  This must be a valid knapsack cover because $x_2 + x_4 + x_5 + x_6 \leq 3$ is a valid cover. It is not a minimal cover because there are proper subsets with summation strictly greater than 19.

Exercise 3

Solve the following problem using the branch and bound method

$$\begin{aligned} \underset{x_1,x_2}{\text{maximise}} \quad & y = x_1 + 2x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \leq 7 \\ & -x_1 + x_2 \leq 3 \\ & x_1, x_2 \geq 0,\ x_1, x_2 \in \mathbb{N}_0 \end{aligned}$$

Solution 3

The original problem is

$$\begin{aligned} \underset{x_1,x_2}{\text{maximise}} \quad & x_1 + 2x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \leq 7 \\ & -x_1 + x_2 \leq 3 \\ & x_1, x_2 \geq 0,\ x_1, x_2 \in \mathbb{N}_0 \end{aligned}$$

Change the problem from maximisation to minimisation

$$\begin{aligned} \underset{x_1,x_2}{\text{minimise}} \quad & -x_1 - 2x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \leq 7 \\ & -x_1 + x_2 \leq 3 \\ & x_1, x_2 \geq 0,\ x_1, x_2 \in \mathbb{N}_0 \end{aligned}$$

Convert the 1st constraint to an equality constraint by introducing slack variable x_3

$$\begin{aligned} \underset{x_1,x_2,x_3}{\text{minimise}} \quad & -x_1 - 2x_2 \\ \text{subject to} \quad & 2x_1 + x_2 + x_3 = 7 \\ & -x_1 + x_2 \leq 3 \\ & x_1, x_2, x_3 \geq 0,\ x_1, x_2 \in \mathbb{N}_0 \end{aligned}$$

Convert the 2nd constraint to an equality constraint by introducing slack variable x_4

$$\begin{aligned} \underset{x_1,x_2,x_3,x_4}{\text{minimise}} \quad & -x_1 - 2x_2 \\ \text{subject to} \quad & 2x_1 + x_2 + x_3 = 7 \\ & -x_1 + x_2 + x_4 = 3 \\ & x_1, x_2, x_3, x_4 \geq 0,\ x_1, x_2 \in \mathbb{N}_0 \end{aligned}$$

Step 1 - Initialisation

$J = \{x_1, x_2\}$. Problem List is $\{\ P_0 = []\ \}$. P_0 is the continuous relaxation of the original MILP problem (that is, $J = \emptyset$). Go to step 2.

Step 2 - Node Selection

OPT	Problem List	Feasible?	$c^\top x^*(P)$	$c^\top x^*(P) < \text{OPT}$
∞	$P_0 = []$	yes	-10.0	yes

Select problem P_0 because it is feasible and satisfies $c^\top x^*(P) < \text{OPT}$. Go to step 3.

Step 3 - Branching Rule

x_j, $j \in J$	$x_j^*(P_0)$	$x_j^*(P_0) \in \mathbb{N}_0$
x_1	1.33333	no
x_2	4.33333	no

Select variable x_1 because $x_1 \notin \mathbb{N}_0$ and go to step 4.

Step 4 - Branching

Create new problem P_1 with $x_1 \leq \lfloor x_1^*(P) \rfloor = \lfloor 1.33333 \rfloor = 1$ and add it to the problem list.
Create new problem P_2 with $x_1 \geq \lceil x_1^*(P) \rceil = \lceil 1.33333 \rceil = 2$ and add it to the problem list.
Go back to step 2.

Step 2 - Node Selection

OPT	Problem List	Feasible?	$c^\top x^*(P)$	$c^\top x^*(P) < \text{OPT}$
∞	$P_1 = [x_1 \leq 1]$	yes	-9.0	yes
	$P_2 = [x_1 \geq 2]$	yes	-8.0	yes

Select problem P_1 because it is feasible and satisfies $c^\top x^*(P) < \text{OPT}$. Go to step 3.

Step 3 - Branching Rule

x_j, $j \in J$	$x_j^*(P_1)$	$x_j^*(P_1) \in \mathbb{N}_0$
x_1	1.0	yes
x_2	4.0	yes

All $x_j \in \mathbb{N}_0$ so we set $\text{OPT} = \min\{\text{OPT}, c^\top x^*(P_1)\} = -9.0$ and go to step 2.

Step 2 - Node Selection

OPT	Problem List	Feasible?	$c^\top x^*(P)$	$c^\top x^*(P) < \text{OPT}$
-9.0	$P_1 = [x_1 \leq 1]$	yes	-9.0	no
	$P_2 = [x_1 \geq 2]$	yes	-8.0	no

All problems are infeasible or do not satisfy $c^\top x^*(P) < \text{OPT}$. Go to step 5.

Step 5 - Termination

$\text{OPT} = -9.0 < \infty$, therefore the MILP is feasible and -9.0 is the optimal value. Optimal solution is given by problem P_1 and is $[x_1, x_2, x_3, x_4] = [1.0, 4.0, 1.0, -0.0]$.

Exercise 4

Solve the following problem with the Gomory cutting plane approach

$$\begin{aligned} \underset{x_1,x_2}{\text{maximise}} \quad & y = 3x_1 + 4x_2 \\ \text{subject to} \quad & \tfrac{2}{5}x_1 + x_2 \leq 3 \\ & \tfrac{2}{5}x_1 - \tfrac{2}{5}x_2 \leq 1 \\ & x_1, x_2 \geq 0,\ x_1, x_2 \in \mathbb{N}_0 \end{aligned}$$

Solution 4

The original problem is

$$\begin{aligned} \underset{x_1,x_2}{\text{maximise}} \quad & 3x_1 + 4x_2 \\ \text{subject to} \quad & \tfrac{2}{5}x_1 + x_2 \leq 3 \\ & \tfrac{2}{5}x_1 - \tfrac{2}{5}x_2 \leq 1 \\ & x_1, x_2 \geq 0,\ x_1, x_2 \in \mathbb{N}_0 \end{aligned}$$

Change the problem from maximisation to minimisation

$$\begin{aligned} \underset{x_1,x_2}{\text{minimise}} \quad & -3x_1 - 4x_2 \\ \text{subject to} \quad & \tfrac{2}{5}x_1 + x_2 \leq 3 \\ & \tfrac{2}{5}x_1 - \tfrac{2}{5}x_2 \leq 1 \\ & x_1, x_2 \geq 0,\ x_1, x_2 \in \mathbb{N}_0 \end{aligned}$$

Multiply the 1st constraint by 5 to make all coefficients integral

$$\begin{aligned} \underset{x_1,x_2}{\text{minimise}} \quad & -3x_1 - 4x_2 \\ \text{subject to} \quad & 2x_1 + 5x_2 \leq 15 \\ & \tfrac{2}{5}x_1 - \tfrac{2}{5}x_2 \leq 1 \\ & x_1, x_2 \geq 0,\ x_1, x_2 \in \mathbb{N}_0 \end{aligned}$$

Convert the 1st constraint to an equality constraint by introducing an integer slack variable x_3

$$\begin{aligned} \underset{x_1,x_2,x_3}{\text{minimise}} \quad & -3x_1 - 4x_2 \\ \text{subject to} \quad & 2x_1 + 5x_2 + x_3 = 15 \\ & \tfrac{2}{5}x_1 - \tfrac{2}{5}x_2 \leq 1 \\ & x_1, x_2, x_3 \geq 0,\ x_1, x_2, x_3 \in \mathbb{N}_0 \end{aligned}$$

Multiply the 2nd constraint by 5 to make all coefficients integral

$$\begin{aligned} \underset{x_1,x_2,x_3}{\text{minimise}} \quad & -3x_1 - 4x_2 \\ \text{subject to} \quad & 2x_1 + 5x_2 + x_3 = 15 \\ & 2x_1 - 2x_2 \leq 5 \\ & x_1, x_2, x_3 \geq 0,\ x_1, x_2, x_3 \in \mathbb{N}_0 \end{aligned}$$

Convert the 2nd constraint to an equality constraint by introducing an integer slack variable x_4

$$\begin{aligned} \underset{x_1,x_2,x_3,x_4}{\text{minimise}} \quad & -3x_1 - 4x_2 \\ \text{subject to} \quad & 2x_1 + 5x_2 + x_3 = 15 \\ & 2x_1 - 2x_2 + x_4 = 5 \\ & x_1, x_2, x_3, x_4 \geq 0,\ x_1, x_2, x_3, x_4 \in \mathbb{N}_0 \end{aligned}$$

Initial basic representation

BV	x_1	x_2	x_3	x_4	RHS
x_0	3	4			0
x_3	2	5	1		15
x_4	2	-2		1	5

Pivoting on row 2, col 2

BV	x_1	x_2	x_3	x_4	RHS
x_0	3	4			0
x_3	2	5	1		15
x_4	2	-2		1	5

Perform the pivoting operations

BV	x_1	x_2	x_3	x_4	RHS
x_0	$\frac{7}{5}$		$-\frac{4}{5}$		-12
x_2	$\frac{2}{5}$	1	$\frac{1}{5}$		3
x_4	$\frac{14}{5}$		$\frac{2}{5}$	1	11

Pivoting on row 3, col 1

BV	x_1	x_2	x_3	x_4	RHS
x_0	$\frac{7}{5}$		$-\frac{4}{5}$		-12
x_2	$\frac{2}{5}$	1	$\frac{1}{5}$		3
x_4	$\frac{14}{5}$		$\frac{2}{5}$	1	11

Perform the pivoting operations

BV	x_1	x_2	x_3	x_4	RHS
x_0			-1	$-\frac{1}{2}$	$-\frac{35}{2}$
x_2		1	$\frac{1}{7}$	$-\frac{1}{7}$	$\frac{10}{7}$
x_1	1		$\frac{1}{7}$	$\frac{5}{14}$	$\frac{55}{14}$

Simplex Stops, optimal solution found.
Not integral, so need to generate Gomory cut.
Can generate cuts for variables x_1, x_2. Picking x_1

$$x_1 + \frac{1}{7}x_3 + \frac{5}{14}x_4 = \frac{55}{14} \iff [1+0]\,x_1 + \left[0+\frac{1}{7}\right]x_3 + \left[0+\frac{5}{14}\right]x_4 = \left[3+\frac{13}{14}\right]$$

$$\frac{1}{7}x_3 + \frac{5}{14}x_4 \geq \frac{13}{14} \rightsquigarrow \frac{1}{7}x_3 + \frac{5}{14}x_4 - x_5 + \xi_1 = \frac{13}{14}$$

Minimize new (temporary) objective $\zeta = \xi_1$
New tableau is

BV	x_1	x_2	x_3	x_4	x_5	ξ_1	RHS
x_0			-1	$-\frac{1}{2}$			$-\frac{35}{2}$
x_2		1	$\frac{1}{7}$	$-\frac{1}{7}$			$\frac{10}{7}$
x_1	1		$\frac{1}{7}$	$\frac{5}{14}$			$\frac{55}{14}$
ζ						-1	0
ξ_1			$\frac{1}{7}$	$\frac{5}{14}$	-1	1	$\frac{13}{14}$

Perform one operation row ζ = row ζ + row ξ_1

BV	x_1	x_2	x_3	x_4	x_5	ξ_1	RHS
x_0			-1	$-\frac{1}{2}$			$-\frac{35}{2}$
x_2		1	$\frac{1}{7}$	$-\frac{1}{7}$			$\frac{10}{7}$
x_1	1		$\frac{1}{7}$	$\frac{5}{14}$			$\frac{55}{14}$
ζ						-1	0
ξ_1			$\frac{1}{7}$	$\frac{5}{14}$	-1	1	$\frac{13}{14}$

Result

BV	x_1	x_2	x_3	x_4	x_5	ξ_1	RHS
x_0			-1	$-\frac{1}{2}$			$-\frac{35}{2}$
x_2		1	$\frac{1}{7}$	$-\frac{1}{7}$			$\frac{10}{7}$
x_1	1		$\frac{1}{7}$	$\frac{5}{14}$			$\frac{55}{14}$
ζ			$\frac{1}{7}$	$\frac{5}{14}$	-1		$\frac{13}{14}$
ξ_1			$\frac{1}{7}$	$\frac{5}{14}$	-1	1	$\frac{13}{14}$

Pivoting on row 5, col 4

BV	x_1	x_2	x_3	x_4	x_5	ξ_1	RHS
x_0			-1	$-\frac{1}{2}$			$-\frac{35}{2}$
x_2		1	$\frac{1}{7}$	$-\frac{1}{7}$			$\frac{10}{7}$
x_1	1		$\frac{1}{7}$	$\frac{5}{14}$			$\frac{55}{14}$
ζ			$\frac{1}{7}$	$\frac{5}{14}$	-1		$\frac{13}{14}$
ξ_1			$\frac{1}{7}$	$\frac{5}{14}$	-1	1	$\frac{13}{14}$

Perform the pivoting operations

BV	x_1	x_2	x_3	x_4	x_5	ξ_1	RHS
x_0			$-\frac{4}{5}$		$-\frac{7}{5}$	$\frac{7}{5}$	$-\frac{81}{5}$
x_2		1	$\frac{1}{5}$		$-\frac{2}{5}$	$\frac{2}{5}$	$\frac{9}{5}$
x_1	1				1	-1	3
ζ						-1	0
x_4			$\frac{2}{5}$	1	$-\frac{14}{5}$	$\frac{14}{5}$	$\frac{13}{5}$

Simplex Stops, optimal solution found.
Removing the ζ row and ξ_1 column

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_0			$-\frac{4}{5}$		$-\frac{7}{5}$	$-\frac{81}{5}$
x_2		1	$\frac{1}{5}$		$-\frac{2}{5}$	$\frac{9}{5}$
x_1	1				1	3
x_4			$\frac{2}{5}$	1	$-\frac{14}{5}$	$\frac{13}{5}$

Simplex Stops, optimal solution found.
Not integral, so need to generate Gomory cut.
Can generate cuts for variables x_2, x_4. Picking x_2

$$x_2 + \tfrac{1}{5}x_3 - \tfrac{2}{5}x_5 = \tfrac{9}{5} \iff [1+0]\,x_2 + \left[0 + \tfrac{1}{5}\right]x_3 + \left[-1 + \tfrac{3}{5}\right]x_5 = \left[1 + \tfrac{4}{5}\right]$$

$$\tfrac{1}{5}x_3 + \tfrac{3}{5}x_5 \geq \tfrac{4}{5} \rightsquigarrow \tfrac{1}{5}x_3 + \tfrac{3}{5}x_5 - x_6 + \xi_1 = \tfrac{4}{5}$$

Minimize new (temporary) objective $\zeta = \xi_1$
New tableau is

BV	x_1	x_2	x_3	x_4	x_5	x_6	ξ_1	RHS
x_0			$-\frac{4}{5}$		$-\frac{7}{5}$			$-\frac{81}{5}$
x_2		1	$\frac{1}{5}$		$-\frac{2}{5}$			$\frac{9}{5}$
x_1	1				1			3
x_4			$\frac{2}{5}$	1	$-\frac{14}{5}$			$\frac{13}{5}$
ζ							-1	0
ξ_1			$\frac{1}{5}$		$\frac{3}{5}$	-1	1	$\frac{4}{5}$

Perform one operation row ζ = row ζ + row ξ_1

BV	x_1	x_2	x_3	x_4	x_5	x_6	ξ_1	RHS
x_0			$-\frac{4}{5}$		$-\frac{7}{5}$			$-\frac{81}{5}$
x_2		1	$\frac{1}{5}$		$-\frac{2}{5}$			$\frac{9}{5}$
x_1	1				1			3
x_4			$\frac{2}{5}$	1	$-\frac{14}{5}$			$\frac{13}{5}$
ζ							-1	0
ξ_1			$\frac{1}{5}$		$\frac{3}{5}$	-1	1	$\frac{4}{5}$

Result

BV	x_1	x_2	x_3	x_4	x_5	x_6	ξ_1	RHS
x_0			$-\frac{4}{5}$		$-\frac{7}{5}$			$-\frac{81}{5}$
x_2		1	$\frac{1}{5}$		$-\frac{2}{5}$			$\frac{9}{5}$
x_1	1				1			3
x_4			$\frac{2}{5}$	1	$-\frac{14}{5}$			$\frac{13}{5}$
ζ			$\frac{1}{5}$		$\frac{3}{5}$	-1		$\frac{4}{5}$
ξ_1			$\frac{1}{5}$		$\frac{3}{5}$	-1	1	$\frac{4}{5}$

Pivoting on row 6, col 5

BV	x_1	x_2	x_3	x_4	x_5	x_6	ξ_1	RHS
x_0			$-\frac{4}{5}$		$-\frac{7}{5}$			$-\frac{81}{5}$
x_2		1	$\frac{1}{5}$		$-\frac{2}{5}$			$\frac{9}{5}$
x_1	1				1			3
x_4			$\frac{2}{5}$	1	$-\frac{14}{5}$			$\frac{13}{5}$
ζ			$\frac{1}{5}$		$\frac{3}{5}$	-1		$\frac{4}{5}$
ξ_1			$\frac{1}{5}$		$\frac{3}{5}$	-1	1	$\frac{4}{5}$

Perform the pivoting operations

BV	x_1	x_2	x_3	x_4	x_5	x_6	ξ_1	RHS
x_0			$-\frac{1}{3}$			$-\frac{7}{3}$	$\frac{7}{3}$	$-\frac{43}{3}$
x_2		1	$\frac{1}{3}$			$-\frac{2}{3}$	$\frac{2}{3}$	$\frac{7}{3}$
x_1	1		$-\frac{1}{3}$			$\frac{5}{3}$	$-\frac{5}{3}$	$\frac{5}{3}$
x_4			$\frac{4}{3}$	1		$-\frac{14}{3}$	$\frac{14}{3}$	$\frac{19}{3}$
ζ							-1	0
x_5			$\frac{1}{3}$		1	$-\frac{5}{3}$	$\frac{5}{3}$	$\frac{4}{3}$

Simplex Stops, optimal solution found.
Removing the ζ row and ξ_1 column

BV	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_0			$-\frac{1}{3}$			$-\frac{7}{3}$	$-\frac{43}{3}$
x_2		1	$\frac{1}{3}$			$-\frac{2}{3}$	$\frac{7}{3}$
x_1	1		$-\frac{1}{3}$			$\frac{5}{3}$	$\frac{5}{3}$
x_4			$\frac{4}{3}$	1		$-\frac{14}{3}$	$\frac{19}{3}$
x_5			$\frac{1}{3}$		1	$-\frac{5}{3}$	$\frac{4}{3}$

Simplex Stops, optimal solution found.
Not integral, so need to generate Gomory cut.
Can generate cuts for variables x_1, x_2, x_4, x_5. Picking x_1

$$x_1 - \frac{1}{3}x_3 + \frac{5}{3}x_6 = \frac{5}{3} \iff [1+0]\,x_1 + \left[-1+\frac{2}{3}\right]x_3 + \left[1+\frac{2}{3}\right]x_6 = \left[1+\frac{2}{3}\right]$$

$$\frac{2}{3}x_3 + \frac{2}{3}x_6 \geq \frac{2}{3} \rightsquigarrow \frac{2}{3}x_3 + \frac{2}{3}x_6 - x_7 + \xi_1 = \frac{2}{3}$$

Minimize new (temporary) objective $\zeta = \xi_1$
New tableau is

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	ξ_1	RHS
x_0			$-\frac{1}{3}$			$-\frac{7}{3}$			$-\frac{43}{3}$
x_2		1	$\frac{1}{3}$			$-\frac{2}{3}$			$\frac{7}{3}$
x_1	1		$-\frac{1}{3}$			$\frac{5}{3}$			$\frac{5}{3}$
x_4			$\frac{4}{3}$	1		$-\frac{14}{3}$			$\frac{19}{3}$
x_5			$\frac{1}{3}$		1	$-\frac{5}{3}$			$\frac{4}{3}$
ζ								-1	0
ξ_1			$\frac{2}{3}$			$\frac{2}{3}$	-1	1	$\frac{2}{3}$

Perform one operation row ζ = row ζ + row ξ_1

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	ξ_1	RHS
x_0			$-\frac{1}{3}$			$-\frac{7}{3}$			$-\frac{43}{3}$
x_2		1	$\frac{1}{3}$			$-\frac{2}{3}$			$\frac{7}{3}$
x_1	1		$-\frac{1}{3}$			$\frac{5}{3}$			$\frac{5}{3}$
x_4			$\frac{4}{3}$	1		$-\frac{14}{3}$			$\frac{19}{3}$
x_5			$\frac{1}{3}$		1	$-\frac{5}{3}$			$\frac{4}{3}$
ζ								-1	0
ξ_1			$\frac{2}{3}$			$\frac{2}{3}$	-1	1	$\frac{2}{3}$

Result

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	ξ_1	RHS
x_0			$-\frac{1}{3}$			$-\frac{7}{3}$			$-\frac{43}{3}$
x_2		1	$\frac{1}{3}$			$-\frac{2}{3}$			$\frac{7}{3}$
x_1	1		$-\frac{1}{3}$			$\frac{5}{3}$			$\frac{5}{3}$
x_4			$\frac{4}{3}$	1		$-\frac{14}{3}$			$\frac{19}{3}$
x_5			$\frac{1}{3}$		1	$-\frac{5}{3}$			$\frac{4}{3}$
ζ			$\frac{2}{3}$			$\frac{2}{3}$	-1		$\frac{2}{3}$
ξ_1			$\frac{2}{3}$			$\frac{2}{3}$	-1	1	$\frac{2}{3}$

Pivoting on row 7, col 3

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	ξ_1	RHS
x_0			$-\frac{1}{3}$			$-\frac{7}{3}$			$-\frac{43}{3}$
x_2		1	$\frac{1}{3}$			$-\frac{2}{3}$			$\frac{7}{3}$
x_1	1		$-\frac{1}{3}$			$\frac{5}{3}$			$\frac{5}{3}$
x_4			$\frac{4}{3}$	1		$-\frac{14}{3}$			$\frac{19}{3}$
x_5			$\frac{1}{3}$		1	$-\frac{5}{3}$			$\frac{4}{3}$
ζ			$\frac{2}{3}$			$\frac{2}{3}$	-1		$\frac{2}{3}$
ξ_1			$\frac{2}{3}$			$\frac{2}{3}$	-1	1	$\frac{2}{3}$

Perform the pivoting operations

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	ξ_1	RHS
x_0						-2	$-\frac{1}{2}$	$\frac{1}{2}$	-14
x_2		1				-1	$\frac{1}{2}$	$-\frac{1}{2}$	2
x_1	1					2	$-\frac{1}{2}$	$\frac{1}{2}$	2
x_4				1		-6	2	-2	5
x_5					1	-2	$\frac{1}{2}$	$-\frac{1}{2}$	1
ζ								-1	0
x_3			1			1	$-\frac{3}{2}$	$\frac{3}{2}$	1

Simplex Stops, optimal solution found.
Removing the ζ row and ξ_1 column

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_0						-2	$-\frac{1}{2}$	-14
x_2		1				-1	$\frac{1}{2}$	2
x_1	1					2	$-\frac{1}{2}$	2
x_4				1		-6	2	5
x_5					1	-2	$\frac{1}{2}$	1
x_3			1			1	$-\frac{3}{2}$	1

Simplex Stops, optimal solution found.
Termination - all variables are integral.
Optimal Solutions $x_1 = 2, x_2 = 2, x_3 = 1, x_4 = 5, x_5 = 1, x_6 = 0, x_7 = 0$, giving an objective value of -14.

Exercise 5

Solve the following problem with the Gomory cutting plane approach. Note that both the coefficients and right-hand sides are not integral.

$$\begin{aligned}\underset{x_1,x_2}{\text{maximise}} \quad & y = 5x_1 + 6x_2 \\ \text{subject to} \quad & 0.2x_1 + 0.3x_2 \leq 1.8 \\ & 0.2x_1 + 0.1x_2 \leq 1.2 \\ & 0.3x_1 + 0.3x_2 \leq 2.4 \\ & x_1, x_2 \geq 0,\ x_1, x_2 \in \mathbb{N}_0\end{aligned}$$

Solution 5

The original problem is

$$\begin{aligned}\underset{x_1,x_2}{\text{maximise}} \quad & 5x_1 + 6x_2 \\ \text{subject to} \quad & \tfrac{1}{5}x_1 + \tfrac{3}{10}x_2 \leq \tfrac{9}{5} \\ & \tfrac{1}{5}x_1 + \tfrac{1}{10}x_2 \leq \tfrac{6}{5} \\ & \tfrac{3}{10}x_1 + \tfrac{3}{10}x_2 \leq \tfrac{12}{5} \\ & x_1, x_2 \geq 0,\ x_1, x_2 \in \mathbb{N}_0\end{aligned}$$

Change the problem from maximisation to minimisation

$$\begin{aligned}
\underset{x_1,x_2}{\text{minimise}} \quad & -5x_1 - 6x_2 \\
\text{subject to} \quad & \tfrac{1}{5}x_1 + \tfrac{3}{10}x_2 \leq \tfrac{9}{5} \\
& \tfrac{1}{5}x_1 + \tfrac{1}{10}x_2 \leq \tfrac{6}{5} \\
& \tfrac{3}{10}x_1 + \tfrac{3}{10}x_2 \leq \tfrac{12}{5} \\
& x_1, x_2 \geq 0,\ x_1, x_2 \in \mathbb{N}_0
\end{aligned}$$

Multiply the 1st constraint by 10 to make all coefficients integral

$$\begin{aligned}
\underset{x_1,x_2}{\text{minimise}} \quad & -5x_1 - 6x_2 \\
\text{subject to} \quad & 2x_1 + 3x_2 \leq 18 \\
& \tfrac{1}{5}x_1 + \tfrac{1}{10}x_2 \leq \tfrac{6}{5} \\
& \tfrac{3}{10}x_1 + \tfrac{3}{10}x_2 \leq \tfrac{12}{5} \\
& x_1, x_2 \geq 0,\ x_1, x_2 \in \mathbb{N}_0
\end{aligned}$$

Convert the 1st constraint to an equality constraint by introducing an integer slack variable x_3

$$\begin{aligned}
\underset{x_1,x_2,x_3}{\text{minimise}} \quad & -5x_1 - 6x_2 \\
\text{subject to} \quad & 2x_1 + 3x_2 + x_3 = 18 \\
& \tfrac{1}{5}x_1 + \tfrac{1}{10}x_2 \leq \tfrac{6}{5} \\
& \tfrac{3}{10}x_1 + \tfrac{3}{10}x_2 \leq \tfrac{12}{5} \\
& x_1, x_2, x_3 \geq 0,\ x_1, x_2, x_3 \in \mathbb{N}_0
\end{aligned}$$

Multiply the 2nd constraint by 10 to make all coefficients integral

$$\begin{aligned}
\underset{x_1,x_2,x_3}{\text{minimise}} \quad & -5x_1 - 6x_2 \\
\text{subject to} \quad & 2x_1 + 3x_2 + x_3 = 18 \\
& 2x_1 + x_2 \leq 12 \\
& \tfrac{3}{10}x_1 + \tfrac{3}{10}x_2 \leq \tfrac{12}{5} \\
& x_1, x_2, x_3 \geq 0,\ x_1, x_2, x_3 \in \mathbb{N}_0
\end{aligned}$$

Convert the 2nd constraint to an equality constraint by introducing an integer slack variable x_4

$$\begin{aligned}
\underset{x_1,x_2,x_3,x_4}{\text{minimise}} \quad & -5x_1 - 6x_2 \\
\text{subject to} \quad & 2x_1 + 3x_2 + x_3 = 18 \\
& 2x_1 + x_2 + x_4 = 12 \\
& \tfrac{3}{10}x_1 + \tfrac{3}{10}x_2 \leq \tfrac{12}{5} \\
& x_1, x_2, x_3, x_4 \geq 0,\ x_1, x_2, x_3, x_4 \in \mathbb{N}_0
\end{aligned}$$

Multiply the 3rd constraint by 10 to make all coefficients integral

$$\begin{aligned}
\underset{x_1,x_2,x_3,x_4}{\text{minimise}} \quad & -5x_1 - 6x_2 \\
\text{subject to} \quad & 2x_1 + 3x_2 + x_3 = 18 \\
& 2x_1 + x_2 + x_4 = 12 \\
& 3x_1 + 3x_2 \leq 24 \\
& x_1, x_2, x_3, x_4 \geq 0,\ x_1, x_2, x_3, x_4 \in \mathbb{N}_0
\end{aligned}$$

Convert the 3rd constraint to an equality constraint by introducing an integer slack variable x_5

$$\begin{aligned}
\underset{x_1,x_2,x_3,x_4,x_5}{\text{minimise}} \quad & -5x_1 - 6x_2 \\
\text{subject to} \quad & 2x_1 + 3x_2 + x_3 = 18 \\
& 2x_1 + x_2 + x_4 = 12 \\
& 3x_1 + 3x_2 + x_5 = 24 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0,\ x_1, x_2, x_3, x_4, x_5 \in \mathbb{N}_0
\end{aligned}$$

Initial basic representation

BV	x_1	x_2	x_3	x_4	x_5	RHS
z	5	6				0
x_3	2	3	1			18
x_4	2	1		1		12
x_5	3	3			1	24

Pivoting on row 2, col 2

BV	x_1	x_2	x_3	x_4	x_5	RHS
z	5	6				0
x_3	2	3	1			18
x_4	2	1		1		12
x_5	3	3			1	24

Perform the pivoting operations

BV	x_1	x_2	x_3	x_4	x_5	RHS
z	1		-2			-36
x_2	$\frac{2}{3}$	1	$\frac{1}{3}$			6
x_4	$\frac{4}{3}$		$-\frac{1}{3}$	1		6
x_5	1		-1		1	6

Pivoting on row 3, col 1

BV	x_1	x_2	x_3	x_4	x_5	RHS
z	1		-2			-36
x_2	$\frac{2}{3}$	1	$\frac{1}{3}$			6
x_4	$\frac{4}{3}$		$-\frac{1}{3}$	1		6
x_5	1		-1		1	6

Perform the pivoting operations

BV	x_1	x_2	x_3	x_4	x_5	RHS
z			$-\frac{7}{4}$	$-\frac{3}{4}$		$-\frac{81}{2}$
x_2		1	$\frac{1}{2}$	$-\frac{1}{2}$		3
x_1	1		$-\frac{1}{4}$	$\frac{3}{4}$		$\frac{9}{2}$
x_5			$-\frac{3}{4}$	$-\frac{3}{4}$	1	$\frac{3}{2}$

Simplex Stops, optimal solution found.
Not integral, so need to generate Gomory cut.
Can generate cuts for variables x_1, x_5. Picking x_1

$$x_1 - \frac{1}{4}x_3 + \frac{3}{4}x_4 = \frac{9}{2} \iff [1+0]\, x_1 + \left[-1 + \frac{3}{4}\right] x_3 + \left[0 + \frac{3}{4}\right] x_4 = \left[4 + \frac{1}{2}\right]$$

$$\frac{3}{4}x_3 + \frac{3}{4}x_4 \geq \frac{1}{2} \rightsquigarrow \frac{3}{4}x_3 + \frac{3}{4}x_4 - x_6 + \xi_1 = \frac{1}{2}$$

Add temporary objective $\zeta = \xi_1$ expressed as function of the NBVs:

BV	x_1	x_2	x_3	x_4	x_5	x_6	ξ_1	RHS
z			$-\frac{7}{4}$	$-\frac{3}{4}$				$-\frac{81}{2}$
x_2		1	$\frac{1}{2}$	$-\frac{1}{2}$				3
x_1	1		$-\frac{1}{4}$	$\frac{3}{4}$				$\frac{9}{2}$
x_5			$-\frac{3}{4}$	$-\frac{3}{4}$	1			$\frac{3}{2}$
ζ			$\frac{3}{4}$	$\frac{3}{4}$		-1		$\frac{1}{2}$
ξ_1			$\frac{3}{4}$	$\frac{3}{4}$		-1	1	$\frac{1}{2}$

Pivoting on row 6, col 3

BV	x_1	x_2	x_3	x_4	x_5	x_6	ξ_1	RHS
z			$-\frac{7}{4}$	$-\frac{3}{4}$				$-\frac{81}{2}$
x_2		1	$\frac{1}{2}$	$-\frac{1}{2}$				3
x_1	1		$-\frac{1}{4}$	$\frac{3}{4}$				$\frac{9}{2}$
x_5			$-\frac{3}{4}$	$-\frac{3}{4}$	1			$\frac{3}{2}$
ζ			$\frac{3}{4}$	$\frac{3}{4}$		-1		$\frac{1}{2}$
ξ_1			$\frac{3}{4}$	$\frac{3}{4}$		-1	1	$\frac{1}{2}$

Perform the pivoting operations

BV	x_1	x_2	x_3	x_4	x_5	x_6	ξ_1	RHS
z				1		$-\frac{7}{3}$	$\frac{7}{3}$	$-\frac{118}{3}$
x_2		1		-1		$\frac{2}{3}$	$-\frac{2}{3}$	$\frac{8}{3}$
x_1	1			1		$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{14}{3}$
x_5					1	-1	1	2
ζ							-1	0
x_3			1	1		$-\frac{4}{3}$	$\frac{4}{3}$	$\frac{2}{3}$

Simplex Stops, optimal solution found.
Removing the ζ row and ξ_1 column

BV	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z				1		$-\frac{7}{3}$	$-\frac{118}{3}$
x_2		1		-1		$\frac{2}{3}$	$\frac{8}{3}$
x_1	1			1		$-\frac{1}{3}$	$\frac{14}{3}$
x_5					1	-1	2
x_3			1	1		$-\frac{4}{3}$	$\frac{2}{3}$

Pivoting on row 5, col 4

BV	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z				1		$-\frac{7}{3}$	$-\frac{118}{3}$
x_2		1		-1		$\frac{2}{3}$	$\frac{8}{3}$
x_1	1			1		$-\frac{1}{3}$	$\frac{14}{3}$
x_5					1	-1	2
x_3			1	1		$-\frac{4}{3}$	$\frac{2}{3}$

Perform the pivoting operations

BV	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z			-1			-1	-40
x_2		1	1			$-\frac{2}{3}$	$\frac{10}{3}$
x_1	1		-1			1	4
x_5					1	-1	2
x_4			1	1		$-\frac{4}{3}$	$\frac{2}{3}$

Simplex Stops, optimal solution found.
Not integral, so need to generate Gomory cut.
Can generate cuts for variables x_2, x_4. Picking x_2

$$x_2 + x_3 - \frac{2}{3}x_6 = \frac{10}{3} \iff [1+0]\,x_2 + [1+0]\,x_3 + \left[-1+\frac{1}{3}\right]x_6 = \left[3+\frac{1}{3}\right]$$

$$\frac{1}{3}x_6 \geq \frac{1}{3} \rightsquigarrow \frac{1}{3}x_6 - x_7 + \xi_1 = \frac{1}{3}$$

Add temporary objective $\zeta = \xi_1$ expressed as function of the NBVs:

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	ξ_1	RHS
z			-1			-1			-40
x_2		1	1			$-\frac{2}{3}$			$\frac{10}{3}$
x_1	1		-1			1			4
x_5					1	-1			2
x_4			1	1		$-\frac{4}{3}$			$\frac{2}{3}$
ζ						$\frac{1}{3}$	-1		$\frac{1}{3}$
ξ_1						$\frac{1}{3}$	-1	1	$\frac{1}{3}$

Pivoting on row 7, col 6

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	ξ_1	RHS
z			-1			-1			-40
x_2		1	1			$-\frac{2}{3}$			$\frac{10}{3}$
x_1	1		-1			1			4
x_5					1	-1			2
x_4			1	1		$-\frac{4}{3}$			$\frac{2}{3}$
ζ						$\frac{1}{3}$	-1		$\frac{1}{3}$
ξ_1						$\frac{1}{3}$	-1	1	$\frac{1}{3}$

Perform the pivoting operations

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	ξ_1	RHS
z			-1				-3	3	-39
x_2		1	1				-2	2	4
x_1	1		-1				3	-3	3
x_5					1		-3	3	3
x_4			1	1			-4	4	2
ζ								-1	0
x_6						1	-3	3	1

Simplex Stops, optimal solution found.
Removing the ζ row and ξ_1 column

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z			-1				-3	-39
x_2		1	1				-2	4
x_1	1		-1				3	3
x_5					1		-3	3
x_4			1	1			-4	2
x_6						1	-3	1

Simplex Stops, optimal solution found.
Termination - all variables are integral.
Optimal Solutions $x_1 = 3, x_2 = 4, x_3 = 0, x_4 = 2, x_5 = 3, x_6 = 1, x_7 = 0$, giving an objective value of -39.

Exercise 6

Consider the following problem

$$\begin{aligned} \underset{x_1,x_2}{\text{maximise}} \quad & y = 5x_1 + x_2 \\ \text{subject to} \quad & -x_1 + 2x_2 \le 4 \\ & x_1 - x_2 \le 1 \\ & 4x_1 + x_2 \le 12 \\ & x_1, x_2 \ge 0,\ x_1, x_2 \in \mathbb{N}_0 \end{aligned}$$

- Solve this problem graphically.
- Solve LP relaxation. Round this solution to the nearest integer solution and check whether it is feasible. Then enumerate all the rounded solutions, check them for feasibility and calculate y for those that are feasible. Are any of these feasible rounded solutions optimal for the IP problem?

Solution 6

- The feasible region of the IP problem is shown in the next figure.

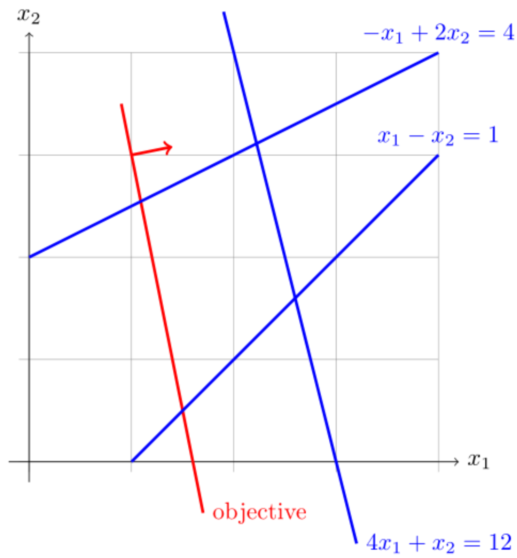

Figure 1: Graph of the feasible region.

It can be seen that the following pairs of integers are in the feasible region:

$$\begin{matrix} (0,0) & (0,1) & (0,2) \\ (1,0) & (1,1) & (1,2) \\ (2,1) & (2,2) & (2,3), \end{matrix}$$

and the optimal solution is $x^* = (13, 2, 3)$.

- Solving the LP relaxation we obtain $x^* = (14.6, 2.6, 1.6)$.

  Rounding the optimal solution of the LP relaxation 4 pairs of integers are obtained:

$$\begin{matrix} (3,2) & (3,1) \\ (2,2) & (2,1). \end{matrix}$$

For each of the four pairs we whether they are feasible, and if yes, the objective function value:

rounded solutions	Constraints violated	x_0
(3,2)	3rd	–
(3,1)	2nd, 3rd	–
(2,2)	none	12
(2,1)	none	11

It can be seen that none of the rounded solutions are optimal for the IP problem.