$\min \xi = x_6 = 1 - x_2 + x_5$

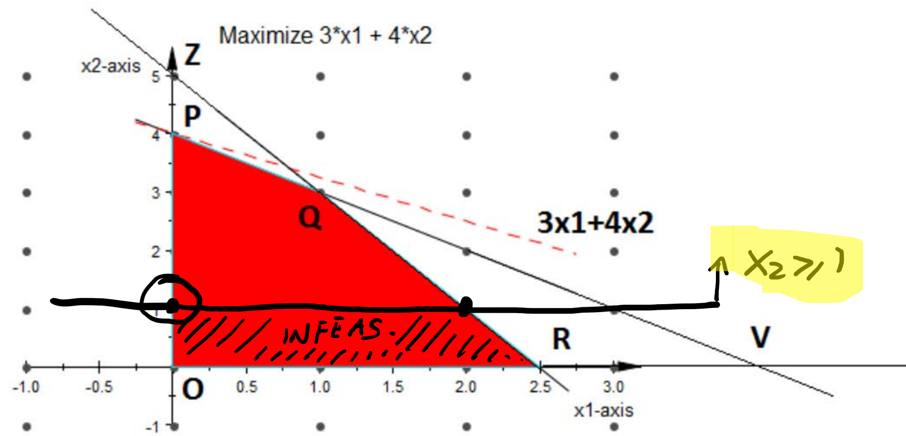
$-\min Z = -3x_1 - 4x_2$

s.t. $x_1 + x_2 + x_3 = 4$

$2x_1 + x_2 + x_4 = 5$

$x_2 - x_5 + x_6 = 1$ ($x_6 = 0$)

$x_6, x_1, x_2, x_3, x_4, x_5 \geq 0$



$I = \{3, 4, 6\}$

PHASE I

BV	ξ	x_1	x_2	x_3	x_4	x_5	x_6	RHS	RATIO
ξ	1	0	1	0	0	-1	0	1	
x_3	0	1	1	1	0	0	0	4	4/1 = 4
x_4	0	2	1	0	1	0	0	5	5/1 = 5
x_6	0	0	1	0	0	-1	1	1	1/1 = 1 ←
~~ξ~~	~~1~~	~~0~~	~~0~~	~~0~~	~~0~~	~~0~~	-1	~~0~~	
x_3	0	1	0	1	0	1	-1	3	
x_4	0	2	0	0	1	1	-1	4	
x_2	0	0	1	0	0	-1	1	1	
Z	1	3	0	0	0	4		-4	

PHASE II

$(x_1, x_2, x_3, x_4, x_5) = (0, 1, 3, 4, 0)$

$Z = -3x_1 - 4x_2 = -3x_1 - 4 - 4x_5$

60016 OPERATIONS RESEARCH

Extensions of linear programming

Last Lecture

- Initial BFS
  - "All slack basis"
  - Artificial variables
- Two phase simplex algorithm
  - Systematically finding initial BFS's
  - Detecting infeasibility

This Lecture

Linearization methods

- Min-max problems
- Min-min problems
- Fractional linear programming

Linearization methods

- Linear programming is useful, but often fairly restrictive.
- Nonlinear optimisation problems arise very commonly in applications.
  - Euclidean distance, chemical reactions, queueing problems, ...
- Nonlinearities can affect objective, constraints, or both.
  - $z = x_1^2$, $z = x \bmod 2$, ...
  - $|x_1 - x_2| \leq 3$, $x_1 x_2 + x_3 \leq 5$, ...
- Nonlinear programs can sometimes be reformulated
  - as a single LP problem
  - or as a sequence of LP problems
- We see the following cases:
  - min-max models (important for game theory!)
  - min-min models
  - fractional linear programming

Min-Max Problems

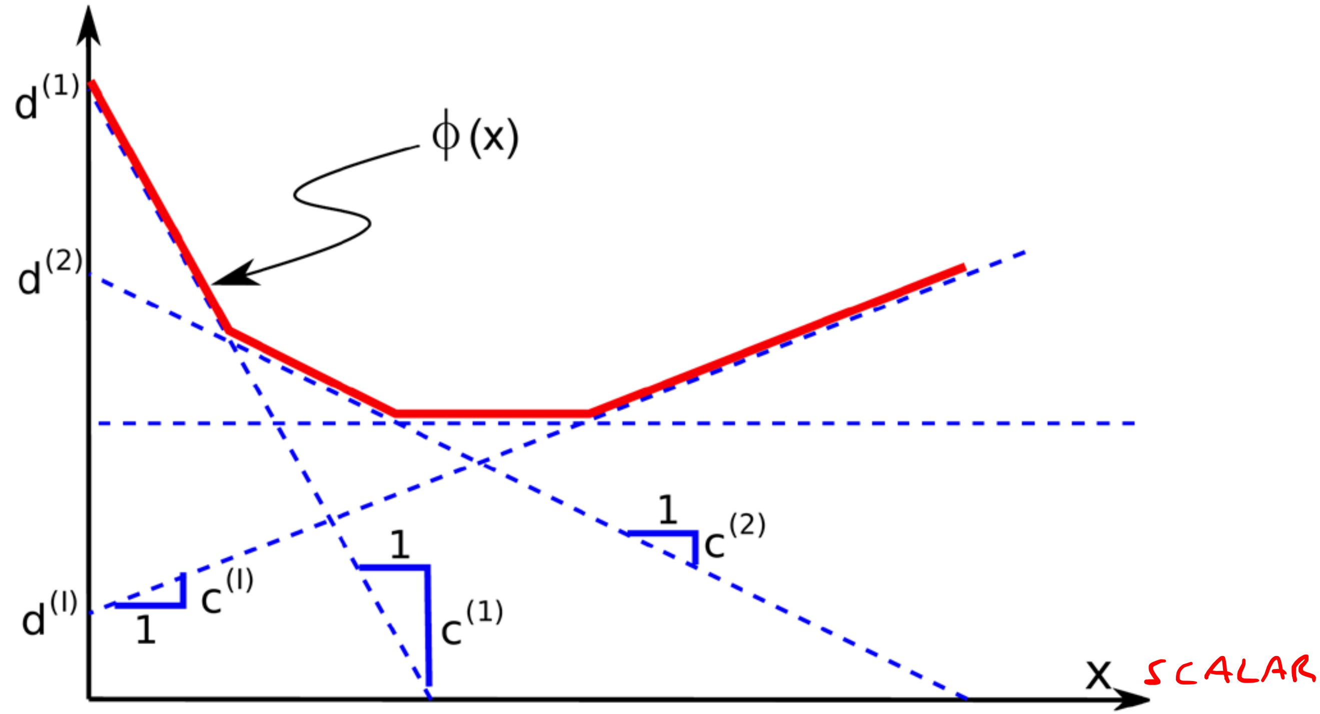
Consider a family of linear functions: $y_i(x) = c(i)^T x + d(i)$.

Set $\phi(x) = \max_{i=1,\ldots,I} \{c(i)^T x + d(i)\}$ for $c(i) \in \mathbb{R}^n$, $d(i) \in \mathbb{R}$.

Then,

$$\begin{aligned} \min \quad & \phi(x) \\ \text{s.t.} \quad & Ax = b, \; x \geq 0 \end{aligned} \tag{MM}$$

is called a min-max problem.



Min-Max Problems (cont)

Perhaps surprisingly, Min-Max problems can be solved by LPs.

Theorem. Consider the following linear program:

$$\begin{aligned} \min \quad & z \\ \text{s.t.} \quad & z \geq c(i)^T x + d(i) \ \forall i = 1, \ldots, I \\ & Ax = b \\ & x \geq 0, \ z \text{ free} \end{aligned} \tag{LP}$$

If (x^*_{LP}, z^*_{LP}) is an optimal solution of this LP, then x^*_{LP} is also an optimal solution of MM, and MM has optimal value $\phi(x^*_{LP}) = z^*_{LP}$.

Min-Max Problems (cont)

Proof by contradiction:

1. Let (x^*_{LP}, z^*_{LP}) be optimal in LP. x^*_{LP} is *feasible* in MM since in LP it is $Ax^*_{LP} = b$ and $x^*_{LP} \geq 0$.
2. Assume that x^*_{LP} is not *optimal* in MM, then there exist a x^*_{MM} such that $\phi(x^*_{MM}) < \phi(x^*_{LP})$.
3. But then $(x^*_{MM}, \phi(x^*_{MM}))$ must be *feasible* in LP, since in MM we require $Ax^*_{MM} = b$, $x^*_{MM} \geq 0$, and for all $i = 1, \ldots, I$

$$\phi(x^*_{MM}) = \max_{j=1,..,I} \left\{ c(j)^T x^*_{MM} + d(j) \right\} \geq c(i)^T x^*_{MM} + d(i)$$

4. By the constraints in LP

$$z^*_{LP} \geq \max_{i=1,\ldots,I} \{c(i)^T x^*_{LP} + d(i)\} \;\Rightarrow\; z^*_{LP} \geq \phi(x^*_{LP})$$

and since $z^*_{LP} \geq \phi(x^*_{LP}) > \phi(x^*_{MM})$, x^*_{MM} would achieve in LP a better objective than z^*_{LP}. Thus x^*_{MM} cannot exist and x^*_{LP} must be optimal also for MM.

MN

$$\min \ \phi(x)$$
$$\text{s.t.} \ Ax = b$$
$$x \geq 0$$

LP

$$\min \ z$$
$$z \geq c(i)^T x + d(i) \quad \forall i$$
$$Ax = b$$
$$x \geq 0, \quad z \text{ free}$$

x^*_{LP} is feasible

$\exists \ x^*_{MN} : \ \phi(x^*_{MN}) < \phi(x^*_{LP})$

$\phi(x^*_{MN}) \equiv \phi^*_{MN}$

$(x^*_{LP}, \underline{z^*_{LP}})$

$\longrightarrow \quad (x^*_{MN}, \phi^*_{MN})$

$$\phi^*_{MN} = \max_{j=1...I} \{ c(j)^T x^*_{MN} + d(j) \} \geq$$
$$c(i)^T x^*_{MN} + d(i) \quad \forall i$$

$$z^*_{LP} \geq \max_{i=1...I} \{ c(i)^T x^*_{LP} + d(i) \}$$
$$= \phi(x^*_{LP})$$

$$z^*_{LP} \geq \phi(x^*_{LP}) > \phi(x^*_{MN})$$

$$\min \; \phi(x)$$

$$\text{s.T.} \quad Ax = b$$

$$x \geq 0$$

$$\phi(x) = \max_{i=1,\dots,I} \left\{ c^T(i)\, x \right\} \qquad c(i) \in \mathbb{R}^n \qquad A \in \mathbb{R}^{m \times n} \qquad b \in \mathbb{R}^m$$

LP $\quad I = 2$

$$A = \begin{bmatrix} 10 & 5 \\ 5 & 9 \end{bmatrix} \qquad b = \begin{bmatrix} -2 \\ 5 \end{bmatrix} \qquad c(i) = [5-i,\ 3-i]$$

$$\min \; z$$

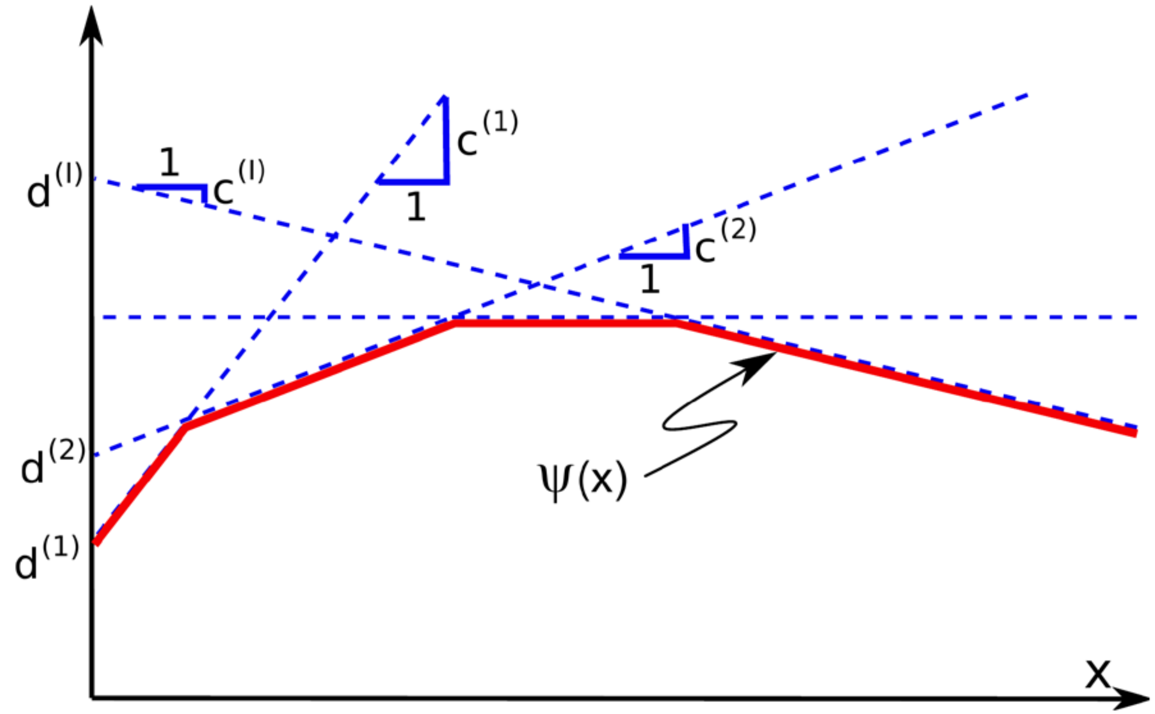
$$\begin{aligned} \text{s.t.} \quad & z \geq 4x_1 + 2x_2 \\ & z \geq 3x_1 + x_2 \\ & 10x_1 + 5x_2 = -2 \\ & 5x_1 + 9x_2 = 5 \\ & x_1, x_2 \geq 0 \qquad z \text{ free} \end{aligned}$$

Min-Min Problems

Set $\psi(x) = \min_{i=1,\ldots,I} \{c(i)^T x + d(i)\}$ for $c(i) \in \mathbb{R}^n$, $d(i) \in \mathbb{R}$. Then,

$$\begin{array}{ll} \min & \psi(x) \\ \text{s.t.} & Ax = b,\ x \geq 0 \end{array} \qquad \text{(MM')}$$

is called a min-min problem.



Min-Min Problems (cont)

Theorem: Consider the following linear programs.

$$\begin{array}{ll} \min & z_i = c^T(i)x(i) + d(i) \\ \text{s.t.} & Ax(i) = b \\ & x(i) \geq 0 \end{array} \qquad (\mathrm{LP}(i))$$

Let z_i^* be the optimal solution to LP(i). Let LP(j) be the LP that has the minimal objective, i.e.,

$$z_j^* = \min_{i=1,\ldots,I} z_i$$

and let $x^*(j)$ be its optimal solution. Then $x^*(j)$ is optimal in MM' and $\psi^* = \psi(x^*(j)) = z_j^*$.

Interchangeability of Min-Operations

Lemma: Let X and Y be arbitrary sets, and let $f : X \times Y \to \mathbb{R}$ be an arbitrary function defined on $X \times Y$. Then,

$$\min_{x \in X} \min_{y \in Y} f(x,y) = \min_{y \in Y} \min_{x \in X} f(x,y) \tag{1}$$

Proof of Min-Min Theorem: The Lemma implies that

$$\left.\min_{i=1,\ldots,I} \begin{array}{ll} \min_x & c(i)^T x + d(i) \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array} \right\} = \min_{i=1,\ldots,I} \text{LP}(i)$$

is equal to

$$\left.\begin{array}{ll} \min_x & \min_{i=1,\ldots,I} c(i)^T x + d(i) \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array} \right\} = \text{MM'}.$$

Thus, the claim follows. $\square$

Fractional Linear Programming

Consider the fractional linear program

$$\min\left\{\frac{\alpha_0+\alpha_1x_1+\ldots+\alpha_nx_n}{\beta_0+\beta_1x_1+\ldots+\beta_nx_n} \mid Ax=b;\ x\geq 0\right\}. \tag{FLP}$$

We assume that

- the feasible set of FLP is bounded, i.e.,

$$\exists L>0 \quad \text{with} \quad \|x\|\leq L \quad \forall x \ :\ Ax=b,\ x\geq 0;$$

- the denominator of the objective function is strictly positive, i.e., $\beta_0+\beta_1x_1+\ldots+\beta_nx_n>0$ for all feasible x.

Homogenisation

- Introduce new variables $y_i \geq 0$, $i = 1, \ldots, n$ and $y_0 > 0$.
- Setting $x_i = \frac{y_i}{y_0}$, $i = 1, \ldots, n$, we can homogenise the fractional linear program as follows.

$$\left.\begin{array}{ll} \min & \dfrac{\alpha_0 y_0 + \alpha_1 y_1 + \alpha_2 y_2 + \ldots + \alpha_n y_n}{\beta_0 y_0 + \beta_1 y_1 + \beta_2 y_2 + \ldots + \beta_n y_n} \\ \text{s.t.} & b_i y_0 - \displaystyle\sum_{j=1}^{n} a_{ij} y_j = 0 \quad \forall\, i = 1, \ldots, m \\ & y_0 > 0, y_1 \geq 0, \ldots, y_n \geq 0 \end{array}\right\} \qquad \text{(HFLP)}$$

Homogenisation

- For any $(y_0, y_1, \ldots, y_n)$ feasible in HFLP, $\lambda(y_0, y_1, \ldots, y_n)$ with $\lambda > 0$ is also feasible and has the same objective value.

- For each $(y_0, y_1, ..., y_n)$ we can then find a λ such that

$$\beta_0 y_0 + \ldots + \beta_n y_n = 1.$$

and the scaled point will have identical objective. Thus we can restrict our attention to these points and still find an optimal solution. In other words:

$\Rightarrow$ The denominator in the objective of HFLP can always be normalised to unity.

Normalised Problem

Solve the normalised problem

$$
\begin{aligned}
\min \quad & \alpha_0 y_0 + \alpha_1 y_1 + \alpha_2 y_2 + \ldots + \alpha_n y_n \\
\text{s.t.} \quad & \beta_0 y_0 + \sum_{j=1}^{n} \beta_j y_j = 1 \\
& b_i y_0 - \sum_{j=1}^{n} a_{ij} y_j = 0 \quad \forall\ i = 1, \ldots, m \\
& y_0 > 0, y_1 \geq 0, \ldots, y_n \geq 0
\end{aligned}
$$

where the first constraint forces the denominator of the objective of HFLP to be equal to 1.

$\Rightarrow$ This is an LP!

Constructing a Solution for FLP

- Denote by $(y_0^*, y_1^*, \ldots, y_n^*)$ the optimal solution of the normalised problem.
- Then, $(\frac{y_1^*}{y_0^*}, \ldots, \frac{y_n^*}{y_0^*})$ is an optimal solution for FLP.
- Note: this construction only works if $y_0^* \neq 0$.
- However, we can show that under the assumptions y_0 cannot be zero at optimality.

Relaxing the Lower Bound on y_0

- Let us now assume to use in HFLP $y_0 \geq 0$ instead of $y_0 > 0$.
- Suppose $y_0^* = 0$ and set $y^* = (y_1^*, \ldots, y_n^*)^T$. Since $b_i y_0^* = 0$

$$Ay^* = 0, \; y^* \geq 0$$

- Here $y^* \neq 0$ since otherwise the denominator of the objective in both HFLP and FLP is not strictly positive as assumed.
- We now note that, if x is feasible in FLP, then $x + \lambda y^*$ is also feasible in FLP $\forall \lambda > 0$ since given that $Ay^* = 0$ it is

$$A(x + \lambda y^*) = Ax + \lambda A y^* = b,$$

- This would contradict that FLP has a bounded feasible set, so y_0^* cannot be zero and we can use $y_0 \geq 0$ in HFLP.

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$$A(x + \lambda y^*) = Ax + \lambda Ay^* = b,$$

- This would contradict that FLP has a bounded feasible set, so y_0^* cannot be zero and we can use $y_0 \geq 0$ in HFLP.

$$\min \frac{x_1 + 2x_2}{4x_1 + 3x_2 + 3}$$

s.t.

$$x_1 + x_2 \leq 2$$
$$-x_1 + x_2 \leq 1$$
$$x_1, x_2 \geq 0$$

BOUNDED

$$x_1 = \frac{y_1}{y_0} \qquad x_2 = \frac{y_2}{y_0}$$

$$\min \quad y_1 + 2y_2$$
$$4y_1 + 3y_2 + 3y_0 = 1$$

$$y_1 + y_2 + s_1 = 2y_0$$
$$-y_1 + y_2 + s_2 = y_0$$
$$y_1, y_2 \geq 0$$
$$y_0 \geq 0$$
$$s_1, s_2 \geq 0$$

$$x^* = \left(\frac{y_1^*}{y_0^*}, \frac{y_2^*}{y_0^*} \right)$$