Tutorial 1 - 60016 Operations Research

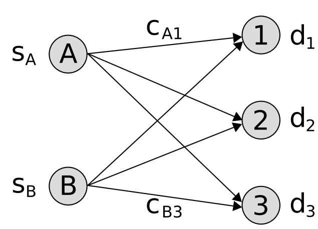
Linear Programming

Exercise 1. *Consider the following linear programming (LP) problem:*

$$
\begin{array}{ll}
\textit{maximise} & y = 2x_1 + x_2 \\
\textit{subject to} & x_1 - 4x_2 \leq 1 \\
& -x_1 - 5x_2 \leq -3 \\
& x_1, x_2 \geq 0
\end{array}
$$

Write the LP in standard form.

Exercise 2 (Transportation and Trans-shipment Problems)**.** *In this problem*[1]*, an IT company produces laptops at two factories A and B. In factory A, s_A laptops are produced per year, whereas the output of factory B is s_B laptops/year. The company owns the three stores 1, 2 and 3. At store 1, d_1 laptops are sold every year. The corresponding numbers for stores 2 and 3 are d_2 and d_3, respectively. The costs of shipping one laptop from factory $i \in \{A, B\}$ to store $j \in \{1, 2, 3\}$ is c_{ij} pounds. Assume that $s_A + s_B \geq d_1 + d_2 + d_3$, i.e., the demand of all stores can be satisfied.*



1. *Assume that $(s_A, s_B) = (3, 3)$ and $(d_1, d_2, d_3) = (2, 2, 2)$. Furthermore, supposed that matrix (c_{ij}) is as follows:*

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix},$$

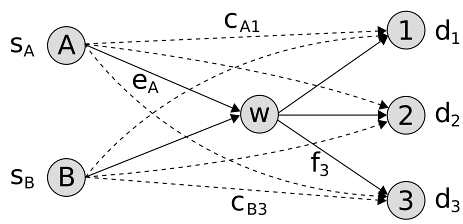
   where the first row corresponds to factory A, the second row to factory B, and the columns correspond to the different stores. The costs c_{B2}, for example, are one pound per laptop.

   How should the laptops be shipped from the two factories to the three stores so that the overall shipping costs are minimized?

2. *Formulate the optimization model corresponding to the previous question. Use the general parameters (i.e., s_A, d_2, c_{B3} etc.) instead of concrete values.*

3. *Imagine the company established a warehouse as shown in the following figure. The company can either directly serve its stores, resulting in costs c_{ij} as before, or ship laptops to the warehouse (at costs e_i per laptop) and subsequently to the stores (at costs f_j per laptop). Moreover, at most $\overline{x}_{ij}$ laptops can be shipped from factory $i \in \{A, B\}$ to store $j \in \{1, 2, 3\}$, while the shipments through the warehouse are unrestricted. Formulate the cost minimization problem, using general parameters as in the previous question.*

[1]The term trans-shipment refers to the shipment of goods or containers to an intermediate destination, then to yet another destination.



Exercise 3. *Find a solution to the following system of linear equations (if one exists):*

$$\begin{array}{rcrcrcrcr} x_1 & - & x_2 & + & x_3 & + & x_4 & = & 6 \\ 3x_1 & + & x_2 & + & 2x_3 & + & 2x_4 & = & 12 \\ x_1 & + & 4x_2 & - & x_3 & - & 4x_4 & = & 3 \\ 2x_1 & & & - & 2x_3 & + & 3x_4 & = & -8 \end{array}$$

1. *What is the rank of A?*

2. *If this system is used to define the feasible set for a LP, would this LP be feasible?*

3. *How would the answer change if the LP is assumed to be in standard form?*

Exercise 4 (Shift Scheduling Problem *(Adapted from Exam 2016, Q3b)*)**.** *A Police Department uses work shifts in which officers work 5 out of the 7 days of the week, with 2 successive days off. For example, a shift might work Sunday through Thursday and then have Friday and Saturday off. The following constraints are in place:*

- *At least 6 officers must be on duty Monday, Tuesday, Wednesday, and Thursdays;*
- *At least 10 officers are required on Friday and Saturday;*
- *Exactly 8 officers are needed on Sunday*

The Police Department wants to meet these staffing needs incurring the minimum cost. For each day on duty an officer receives £100, except on Saturdays and Sundays where the pay is £80 in each day.

Assuming that the variables can be approximately treated as continuous quantities, formulate a shift scheduling program to minimize the cost for the Police Department.

Exercise 5. *Consider a rectangular matrix with n columns and m rows, where $n > m$. Assume rows to be linearly independent, does this mean that also columns are linearly independent?*

Exercise 6. *Indicate, among the following problems, which ones you think can be solved using linear programming (i.e., linear objective, linear constraints, continuous variables). Justify your answers (you are not asked to write the actual optimization program).*

- Schedule tasks such that they do not overlap and precedences are met.
- Choose the next move in a chess game.
- Optimise a linear function $f(x)$ where x takes values in a triangular domain.
- Optimise a linear function $f(x)$ where x takes values in two disjoint triangular domains.
- Find the optimal route on the London tube to a given destination.
- Find an $a \geq 0$ such that the curve $f(x) = a^3 x$ has $f(2) \geq 7$.
- Fit the probability density function of a Gaussian (i.e., normal) distribution to data.
- Find a set of probabilities $\Pr[X = k]$, $1 \leq k \leq n$, for a discrete random variable X such that the mean satisfies $E[X] = m$, where m is a given constant.