60016 OPERATIONS RESEARCH

Sensitivity Analysis

09 November 2020

Last Lecture

- Duality

This Lecture

- Value function
- Shadow prices

In a nutshell: how does solution of LP depend on parameters?

Example 1 (perturbed)

Assume that p_1, the availability of machine X, is not precisely known.

$$
\begin{array}{lll}
\max & y = x_1 + x_2 & : \text{ objective function} \\
\text{s.t.} & 2x_1 + x_2 \leq p_1 & : \text{ constraint on availability of machine X} \\
& x_1 + 3x_2 \leq 18 & : \text{ constraint on availability of machine Y} \\
& x_1 \leq 4 & : \text{ constraint on demand of } x_1 \\
& x_1, x_2 \geq 0 & : \text{ non-negativity constraints}
\end{array}
$$

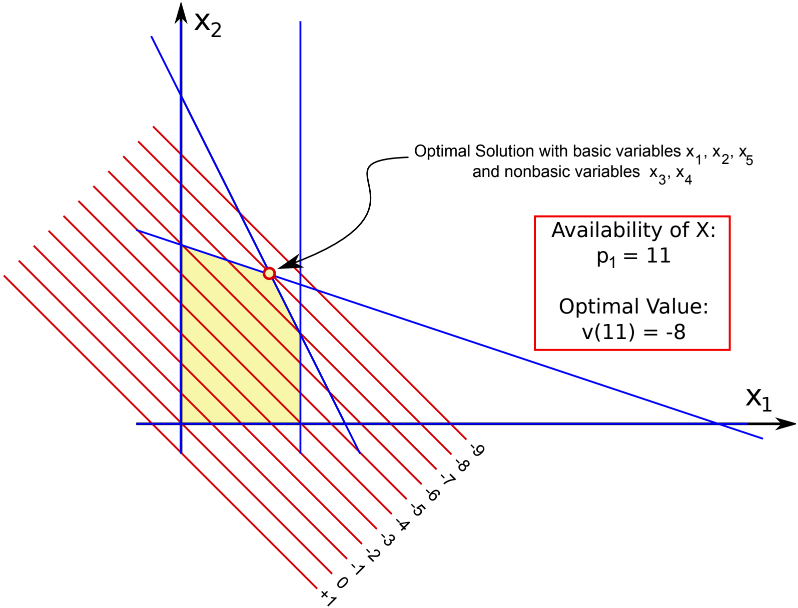
$$
\begin{array}{ll}
-\min & -x_1 - x_2 \\
\text{s.t.} & 2x_1 + x_2 + x_3 = p_1 \\
& x_1 + 3x_2 + x_4 = 18 \\
& x_1 + x_5 = 4 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0
\end{array}
$$

Example 1 (perturbed)

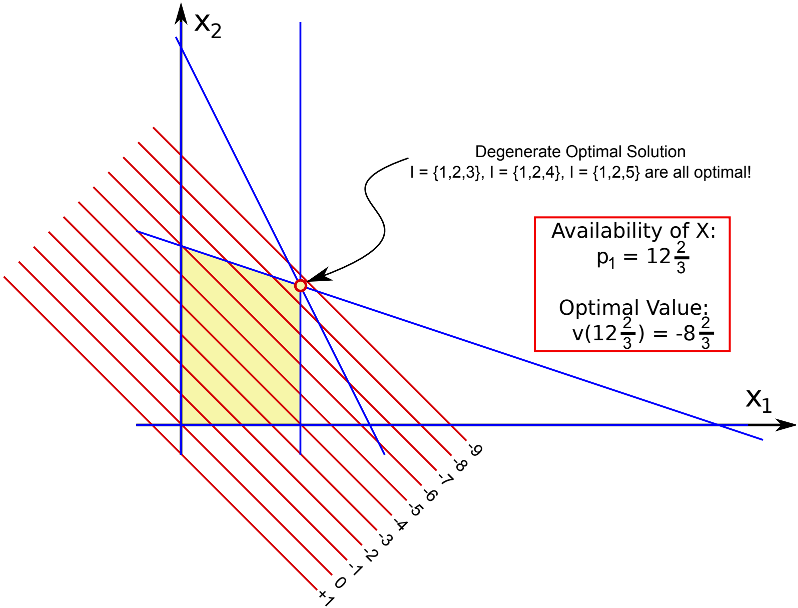
The value function $v(p_1)$ expresses the optimal value of the LP as a function of the unknown availability parameter p_1.

$$
\begin{array}{rcll}
v(p_1) & = & \min & -x_1 - x_2 \\
 & & \text{s.t.} & \quad 2x_1 + x_2 + x_3 = p_1 \\
 & & & \quad x_1 + 3x_2 + x_4 = 18 \\
 & & & \quad x_1 + x_5 = 4 \\
 & & & \quad x_1, x_2, x_3, x_4, x_5 \geq 0
\end{array}
$$

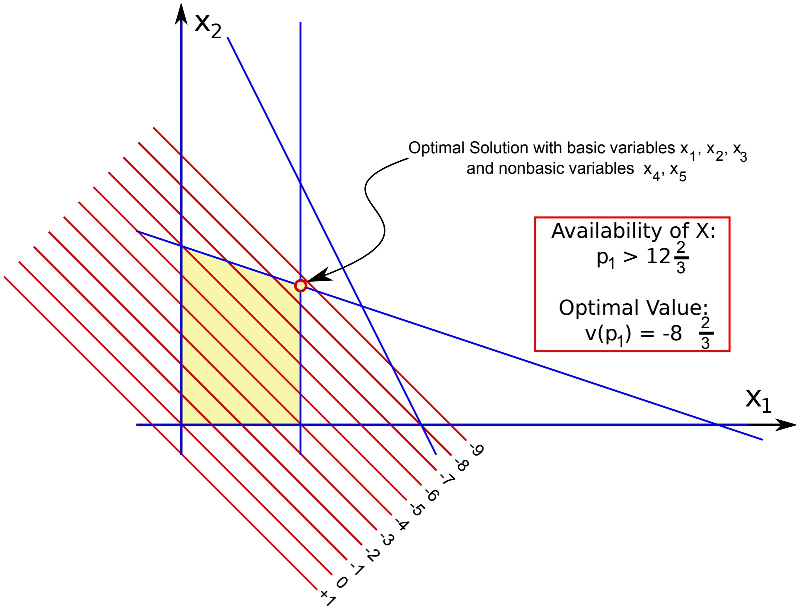
Example 1 (perturbed)



Example 1 (perturbed)



Example 1 (perturbed)



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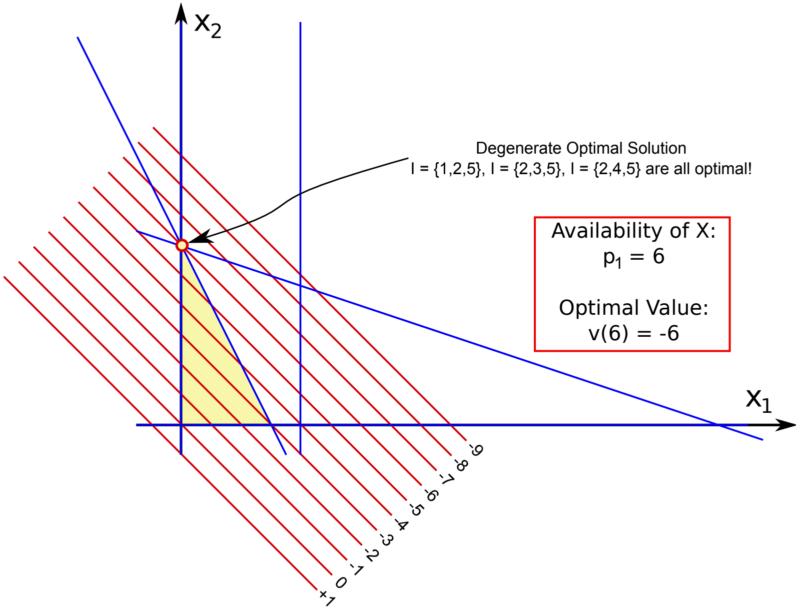
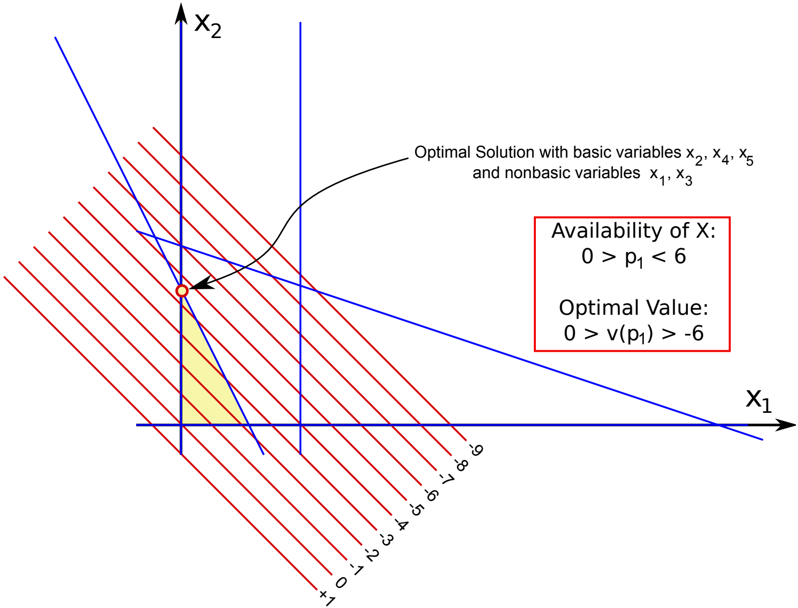
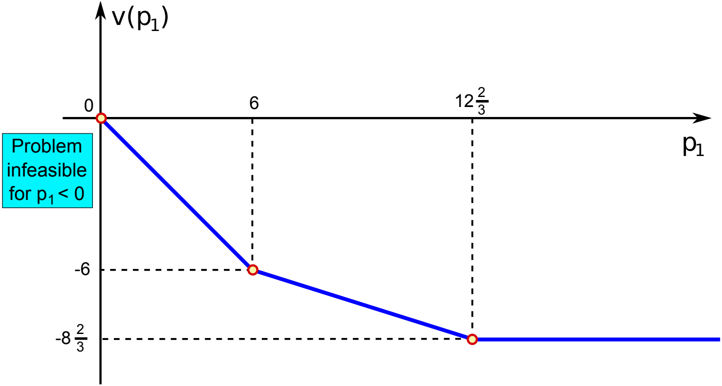


Example 1 (perturbed)



Example 1 (perturbed)

Note: $v(p_1)$ is non-increasing, convex and piecewise linear.



Perturbation

Let $p \in \mathbb{R}^m$ denote a general RHS and define the value function $v(p) : \mathbb{R}^m \to \mathbb{R}$ by:

$$v(p) = \min \left\{ z = c^T x \mid A\,x = p; x \geq 0 \right\}$$

Solving the original LP (the reference problem)

$$\min \left\{ z = c^T x \mid A\,x = b, x \geq 0 \right\}$$

thus computes $v(b)$.

Q: what do we learn on $v(p)$ from $v(b)$?

Shadow Prices

Suppose we have solved the reference problem

$$\min \left\{ z = c^T x \mid Ax = b, x \geq 0 \right\}$$

and found an optimal basis matrix B satisfying

$$x_B = B^{-1} b \geq 0 \quad \text{(Feasibility)}$$

and

$$r = c_N - N^T (B^{-1})^T c_B \geq 0 \quad \text{(Optimality)}.$$

Shadow Prices (cont)

Definition: The vector of shadow prices $\Pi \in \mathbb{R}^m$ is defined as

$$\Pi = (B^{-1})^T c_B,$$

where $B = B(I)$ is an optimal basis.

Note that there can be more than one optimal basis
$\Rightarrow$ The shadow prices need not be unique.

The shadow prices give information about the sensitivity of the value function $v(p)$ at $p = b$.

Behaviour of Value Function

Theorem: $v(p) = v(b) + \Pi^T(p-b)$ for all $p \in \mathbb{R}^m$ with $B^{-1}p \geq 0$.

Proof:

- If $B^{-1}p \geq 0$, then B remains the optimal basis for

$$\min\{z = c^T x : Ax = p, x \geq 0\}$$

  since r is not affected by changing b to p.

- Thus, we find

$$\begin{aligned} v(p) &= c_B^T B^{-1} p \\ &= c_B^T B^{-1} b + c_B^T B^{-1}(p-b) \\ &= v(b) + \Pi^T(p-b) \qquad \square \end{aligned}$$

In general: $v(p) \geq v(b) + \Pi^T(p-b)$ for all $p \in \mathbb{R}^m$.

Global Behaviour of Value Function

Theorem: $v(p) \geq v(b) + \Pi^T(p-b)$ for all $p \in \mathbb{R}^m$.

Proof:

$$
\begin{array}{rcl}
v(p) & = & \min_{x \geq 0; Ax=p} \left\{ c^T x \right\} \\
& = & \min_{x \geq 0; Ax=p} \left\{ c^T x - \Pi^T(Ax-p) \right\} \\
& \geq & \min_{x \geq 0} \left\{ c^T x - \Pi^T(Ax-p) \right\} \\
& = & \min_{x \geq 0} \left\{ (c^T - \Pi^T A)x + \Pi^T p \right\} \\
& = & \Pi^T p + \underbrace{\min_{x \geq 0} \left\{ (c^T - \Pi^T A)x \right\}}_{\geq 0,\ \text{proof left as an exercise (see next slide)}}
\end{array}
$$

Global Behaviour of Value Function

$$
\begin{aligned}
\left[c^T - \Pi^T A\right]\ x &= \left(\left[c_B^T \mid c_N^T\right] - c_B^T B^{-1}\ [B \mid N]\right) \begin{bmatrix} x_B \\ x_N \end{bmatrix} \\
&= \left[c_B^T \mid c_N^T\right] \begin{bmatrix} x_B \\ x_N \end{bmatrix} - c_B^T \left[I \mid B^{-1} N\right] \begin{bmatrix} x_B \\ x_N \end{bmatrix} \\
&= c_B^T\ x_B - c_B^T\ x_B + \left(c_N^T - c_B^T B^{-1} N\right) x_N \\
&= r^T x_N \\
&\geq 0 \qquad\qquad (\text{as } r \geq 0, \text{ and } x_N \geq 0)
\end{aligned}
$$

$$
\Rightarrow \qquad \min_{x \geq 0} \left\{(c^T - \Pi^T A)x\right\} \geq 0
$$

Global Behaviour of Value Function

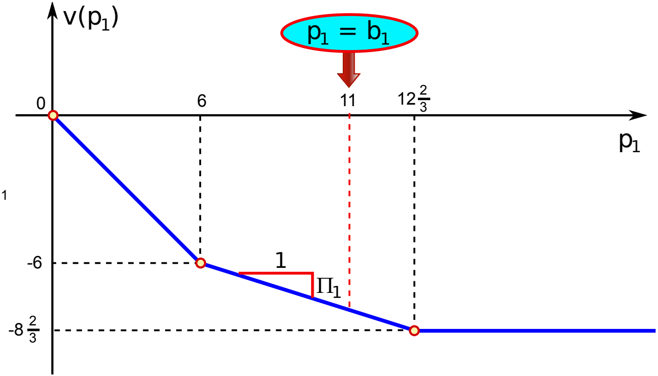
Thus, we find

$$\begin{array}{rcl} v(p) & \geq & \Pi^T p + \min_{x \geq 0} \left\{ (c^T - \Pi^T A) x \right\} \\ & \geq & \Pi^T p \\ & = & \Pi^T b + \Pi^T (p - b) \\ & = & c_B^T B^{-1} b + \Pi^T (p - b) \\ & = & v(b) + \Pi^T (p - b) \end{array}$$

$\square$

Shadow Prices in Example 1

Note: Π_1 is the shadow price for the budget of machine X.



At $p_1 = b_1 = 11$, the optimal costs change by $\Pi_1 = -\frac{2}{5}$ if the availability of X increases by 1.

Interpretation

- Assume the company can buy a "small" additional amount of time on machine X, at price μ_1 per unit.
- Is it worthwhile to buy additional time on X?
  - Yes if $\mu_1 + \Pi_1 < 0$ (overall cost decreases);
  - No if $\mu_1 + \Pi_1 > 0$ (overall cost increases).

$\Rightarrow$ Therefore, $-\Pi_1$ is the maximum price one should pay for one additional unit of time on machine X!

Interpretation

New constraints RHS is therefore given as:

$$p = b + \xi e_t$$

with e_t being a vector with all coordinates 0 except a single 1 at position t:

$$e_t^T = \begin{array}{c} [\,0 \quad \dots \quad 0 \quad \underset{\substack{\uparrow \\ t}}{1} \quad 0 \quad \dots \quad 0\,] \end{array}$$

Interpretation (assuming minimisation)

Accept offer $\Rightarrow$ total production cost:

$$v(b) + \mu_t \xi$$

Extra production $\Rightarrow$ total production cost:

$$\begin{cases} = v(b) + \Pi_t \xi & \text{if } B^{-1}(b + \xi e_t) \geq 0 \\ \geq v(b) + \Pi_t \xi & \text{in general.} \end{cases}$$

ACCEPT offer if $\mu_t + \Pi_t < 0$ and if $B^{-1}(b + \xi e_t) \geq 0$.
REJECT offer if $\mu_t + \Pi_t > 0$ and if $B^{-1}(b + \xi e_t) \geq 0$.
i.e. $-\Pi_t$ is the maximum price one should pay.

Maximisation Problems

For maximisation problems Theorem 8 is unchanged:

Theorem 8' (Local): If $B^{-1}p \geq 0$ then $v(p) = v(b) + \Pi^T(p - b)$.

and inequality is reversed in statement of Theorem 9:

Theorem 9' (Global): $v(p) \leq v(b) + \Pi^T(p - b)$ for all $p \in \mathbb{R}^m$.

Evaluation of Shadow Prices

Q: Can we read shadow prices from final tableau?

Lemma: Suppose row t is initially a "$\leq$-constraint" and a slack variable x_s had been added. Then, $\Pi_t = \beta_s$, where β_s is the objective coefficient of x_s in the final (optimal) tableau.

Proof:

- If x_s is nonbasic in the final tableau, then

$$\begin{aligned}\beta_s &= -r_s = -(c_N - N^T(B^{-1})^T c_b)^T e_s \\ &= -c_s + \Pi^T a_s = 0 + \Pi^T e_t = \Pi_t.\end{aligned}$$

where e_s is a vector of zeros except for a one in the s-th position, $a_s = Ne_s$ is column s of A, and since x_s is the slack for row t we noted that $c_s = 0$ and $a_s = e_t$.

Evaluation of Shadow Prices (cont)

If x_s is basic in the final tableau, then

$$\beta_s = 0 = c_s = e_s^T c_B = e_s^T B^T \Pi = e_t^T \Pi = \Pi_t.$$

Which completes the proof.

Lemma: Suppose row t is initially a "$\geq$-constraint" and a surplus variable x_s had been added. Then, $\Pi_t = -\beta_s$, where β_s is the objective coefficient of x_s in the final tableau.

Example 1 (revisited)

The final tableau for Example 1 is:

BV	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	0	0	$-\frac{2}{5}$	$-\frac{1}{5}$	0	-8
x_2	0	0	1	$-\frac{1}{5}$	$\frac{2}{5}$	0	5
x_5	0	0	0	$-\frac{3}{5}$	$\frac{1}{5}$	1	1
x_1	0	1	0	$\frac{3}{5}$	$-\frac{1}{5}$	0	3

- The constraint on the availability of X was standardised by introducing the slack variable x_3.
- The shadow price Π_1 for that constraint thus coincides with the coefficient of x_3 in the objective row of the above tableau $\Rightarrow \Pi_1 = -\frac{2}{5}$.