60016 OPERATIONS RESEARCH

Basic feasible solutions

Last Lecture

- Taxonomy of LP models
  - Resource allocation & blending models
  - Operations planning models
  - Shift scheduling models
  - Time-phased models
  - ...

This Lecture

- Basic solutions
- Algebra vs. geometry
- Fundamental theorem of linear programming
- Basic representations

Assumptions

- From now on we focus on LPs in standard form

$$\begin{aligned} \min \quad & z = c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned} \tag{$\mathcal{LP}$}$$

  with data $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$ (where $b \geq 0$).
- We assume:
  - $\#$ of variables $= n \geq m = \#$ of equations (otherwise, the system $Ax = b$ is overdetermined);
  - rows of A are linearly independent (otherwise, the constraints are redundant or inconsistent).

  $\Rightarrow$ $\boxed{\text{rank}(A) = m}$

Linear Dependence

Linear dependence of rows in A implies either:

- contradictory constraints
  i.e., no solution to $A\,x = b$, e.g.

$$
\begin{aligned}
x_1 + x_2 &= 1 \\
x_1 + x_2 &= 2
\end{aligned}
$$

- redundant constraints, e.g.

$$
\begin{aligned}
x_1 + x_2 &= 1 \\
2x_1 + 2x_2 &= 2
\end{aligned}
$$

Index Sets

- Consider only the system of linear equations in problem $\mathcal{LP}$.

$$\boxed{A\,x = b}$$

- Let $A = [a_1, \ldots, a_n]$, where $a_i \in \mathbb{R}^m$ is the ith column vector of A.
- Select a subset of m columns a_i that are linearly independent. This is always possible since $m = \operatorname{rank}(A)$ and $n \geq m$.
- Collect in the index set I the indexes for these m columns. I is therefore a subset of $\{1, \ldots, n\}$.

Definition: The matrix $B = B(I) \in \mathbb{R}^{m \times m}$ consisting of the columns $\{a_i\}_{i \in I}$ is called the basis corresponding to the index set I.

Example: Partition of A

$$A = \begin{bmatrix} 2 & 4 & 3 & 3 & 1 & 0 \\ 3 & -3 & 4 & 2 & 0 & 1 \\ -1 & 2 & 1 & 2 & 0 & 0 \end{bmatrix}$$

Choose $I = \{1, 5, 2\}$

$$\Rightarrow B(I) = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 0 & -3 \\ -1 & 0 & 2 \end{bmatrix}$$

Basic Solutions

Definition: A solution x to $Ax = b$ with $x_i = 0$ for all $i \notin I$ is a basic solution (BS) to $Ax = b$ with respect to the index set I.

Definition: A solution x satisfying both $Ax = b$ and $x \geq 0$ is a feasible solution (FS).

Definition: A feasible solution which is also basic is a basic feasible solution (BFS).

Basic Solutions (cont)

Assume for example that $I = \{1, \ldots, m\}$.

$$\begin{array}{ccc}
a_{11}x_1 + \ldots + a_{1m}x_m + a_{1,m+1}x_{m+1} + \ldots + a_{1n}x_n & = & b_1 \\
a_{21}x_1 + \ldots + a_{2m}x_m + a_{2,m+1}x_{m+1} + \ldots + a_{2n}x_n & = & b_2 \\
\vdots \qquad \vdots \qquad \vdots & & \vdots \\
a_{m1}x_1 + \ldots + a_{mm}x_m + a_{m,m+1}x_{m+1} + \ldots + a_{mn}x_n & = & b_m
\end{array}$$

The following system is equivalent to $Bx_B = b$.

$$\begin{array}{ccccc}
a_{11}x_1 + \ldots + a_{1m}x_m & + & a_{1,m+1}0 + \ldots + a_{1n}0 & = & b_1 \\
a_{21}x_1 + \ldots + a_{2m}x_m & + & a_{2,m+1}0 + \ldots + a_{2n}0 & = & b_2 \\
\vdots \qquad \vdots & & \vdots \qquad \vdots & & \vdots \\
a_{m1}x_1 + \ldots + a_{mm}x_m & + & a_{m,m+1}0 + \ldots + a_{mn}0 & = & b_m
\end{array}$$

Basic Solutions (cont)

Assume for example that $I = \{1, \ldots, m\}$.

$$
\begin{array}{rcl}
a_{11}x_1 + \ldots + a_{1m}x_m + a_{1,m+1}x_{m+1} + \ldots + a_{1n}x_n & = & b_1 \\
a_{21}x_1 + \ldots + a_{2m}x_m + a_{2,m+1}x_{m+1} + \ldots + a_{2n}x_n & = & b_2 \\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots & & \vdots \\
a_{m1}x_1 + \ldots + a_{mm}x_m + a_{m,m+1}x_{m+1} + \ldots + a_{mn}x_n & = & b_m
\end{array}
$$

The following system is equivalent to $Bx_B = b$.

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a_{21}x_1 & + & \ldots & + & a_{2m}x_m & = & b_2 \\
\vdots & & & & \vdots & & \vdots \\
a_{m1}x_1 & + & \ldots & + & a_{mm}x_m & = & b_m
\end{array}
$$

Remove non-linear independent columns =>
Linear Independent Column Index Sets

Basic Solutions (cont)

Observation: The basic solution corresponding to I is unique.

As the vectors $\{a_i\}_{i \in I}$ are linearly independent, the basis B is invertible. Thus, the system

$$\boxed{B\, x_B = b}$$

has a unique solution $x_B = B^{-1} b \in \mathbb{R}^m$.

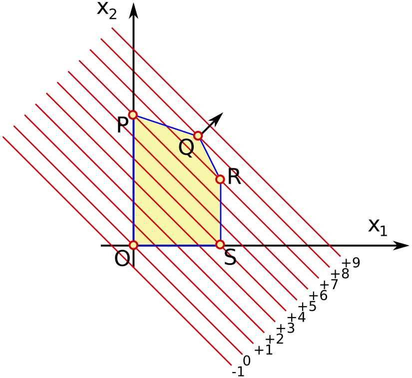
Define $x = (x_1, \ldots, x_n)$ through

$$(x_i)_{i \in I} = x_B \quad \text{and} \quad (x_i)_{i \notin I} = 0.$$

This x is the unique basic solution to $Ax = b$ with respect to. I.

Algebra vs. Geometry

- Geometric intuition: LP solution at corner of feasible set



- Algebra: Corners of feasible set correspond to basic feasible solutions

Example 1 (revisited)

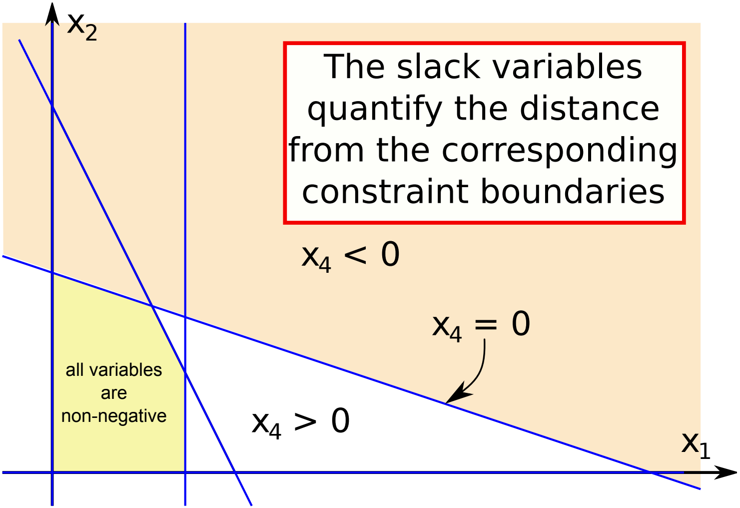
$$\begin{array}{lll} \max & y = x_1 + x_2 & : \text{ objective function} \\ \text{s.t.} & 2x_1 + x_2 \leq 11 & : \text{ constraint on availability of X} \\ & x_1 + 3x_2 \leq 18 & : \text{ constraint on availability of Y} \\ & x_1 \leq 4 & : \text{ constraint on demand of A} \\ & x_1, x_2 \geq 0 & : \text{ non-negativity constraints} \end{array}$$

In standard form: $n = 5$ variables & $m = 3$ constraints

$$\begin{array}{ll} -\min & z = -x_1 - x_2 \\ \text{s.t.} & 2x_1 + x_2 + x_3 = 11 \\ & x_1 + 3x_2 + x_4 = 18 \\ & x_1 + x_5 = 4 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{array}$$

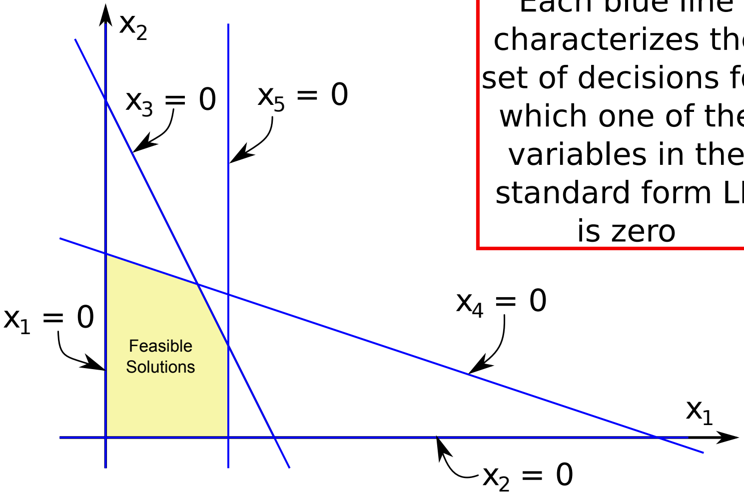
x3, x4, x5 are slack variables

Algebra vs. Geometry (cont)



5 Dimensional Problems as 5 decision variables.

Algebra vs. Geometry (cont)



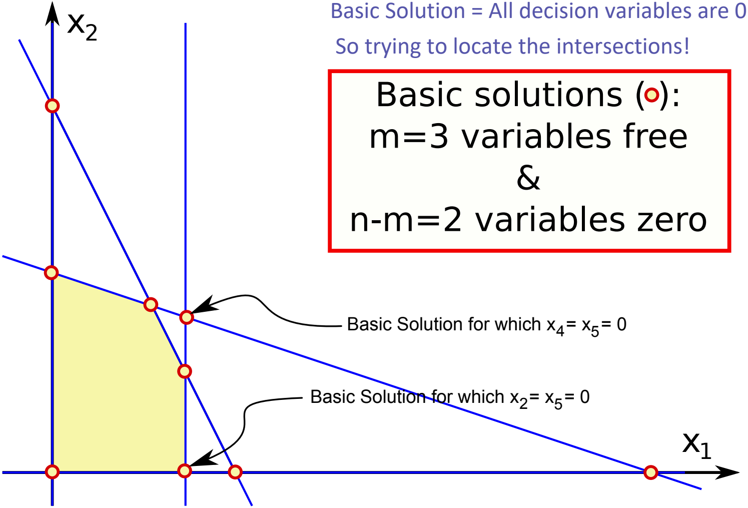
Each blue line characterizes the set of decisions for which one of the variables in the standard form LP is zero

Algebra vs. Geometry (cont)

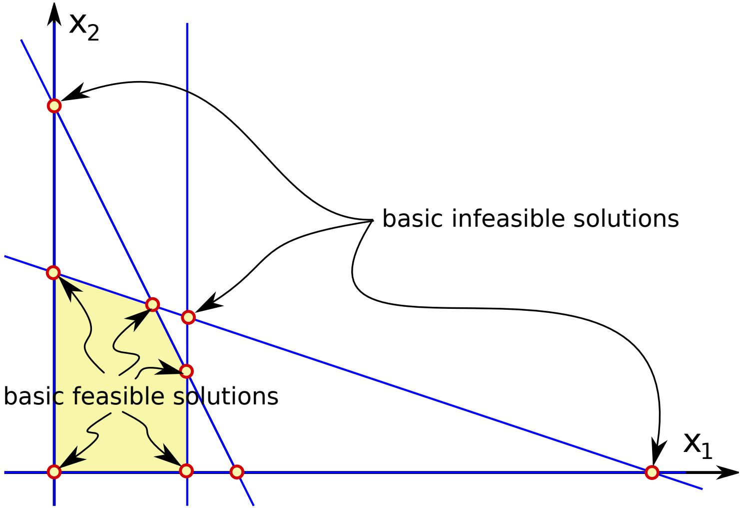
Basic Solution = All decision variables are 0

So trying to locate the intersections!

Basic solutions (o):
m=3 variables free
&
n-m=2 variables zero



Algebra vs. Geometry (cont)



Importance of BFS

$\boxed{\text{Vertices of the feasible set} = \text{basic feasible solutions!}}$

- Geometry: optimum always achieved at a vertex
- Algebra: optimum always achieved at a BFS

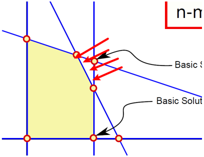
Definition: Given an LP in standard form, a feasible solution to the constraints $\{A\,x = b; x \geq 0\}$ that achieves the optimal value of the objective function is called an optimal feasible solution. If the solution is basic then it is an optimal BFS.

Fundamental Theorem of LP

Theorem 1: For an LP in standard form with $\operatorname{rank}(A) = m \leq n$:

1. $\exists$ a feasible solution $\Rightarrow \exists$ a BFS.
2. $\exists$ an optimal solution $\Rightarrow \exists$ an optimal BFS.

Feasible, but not basic



The reverse is in general not true:

- there may be feasible solutions that are not BFS
- there may be optimal solutions that are not BFS

The naïve statement "an LP has an optimal BFS" is also in general not true as the LP may be infeasible or unbounded.

Searching for Optima

- Theorem 1 reduces solving an LP to searching over BFS's.
- For an LP in standard form with n variables and m constraints, there are

$$\binom{n}{m} = \frac{n!}{m!\,(n-m)!}$$

possibilities of selecting m columns in the A matrix.

$\Rightarrow$ There are at most $\binom{n}{m}$ basic solutions: a finite number of possibilities!

$\Rightarrow$ Theorem 1 offers an obvious but terribly inefficient way of computing the optimum through a finite search.

Number of BFS

Note: There are $\binom{n}{m}$ index sets $I \subseteq \{1, \ldots, n\}$ with $|I| = m$.

$\Rightarrow$ The number of distinct BFS is finite and usually $< \binom{n}{m}$ for the following reasons:

1. $B(I)$ may be singular,
2. the BS corresponding to I may not be feasible.

A "Small" Problem

Let $m = 30$, and $n = 100$.

$$\binom{100}{30} = \frac{100!}{30!\,70!} \approx 2.9 \times 10^{25}.$$

It takes approximately 10^{12} years if we check 10^6 sets/sec.

(The age of the universe is $\approx 14 \times 10^9$ years!)

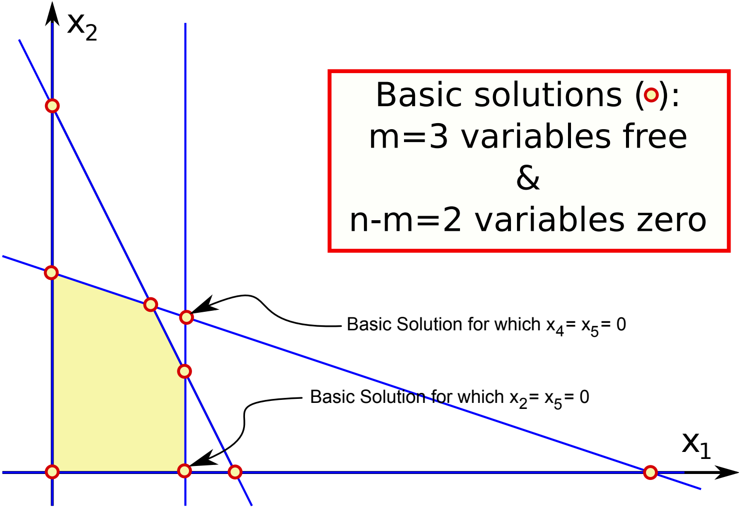
Basic Variables

Fix an index set I with $|I| = m$ and $B(I)$ invertible.

Definition The variables $\{x_i\}_{i \in I}$ are referred to as the basic variables (BV), while the variables $\{x_i\}_{i \notin I}$ are called the nonbasic variables (NBV) corresponding to I.

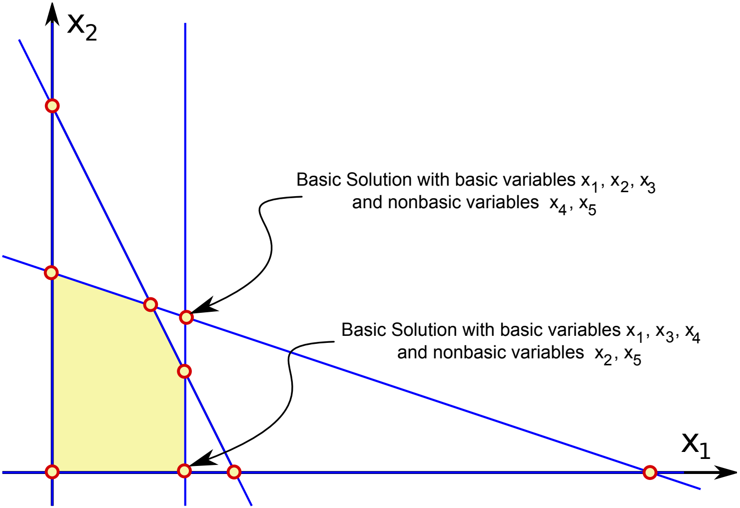
Note: By construction, the nonbasic variables are always zero, but the basic variables can be zero or non-zero.

Example: Algebra vs Geometry



Basic solutions ( ):
m=3 variables free
&
n-m=2 variables zero

Example: Basic vs Nonbasic Variables



Basic Representation

Fix an index set I with $|I| = m$ and $B(I)$ invertible.

Definition: The basic representation corresponding to I is the (unique) reformulation of the system $(z = c^T x, Ax = b)$ which expresses the objective function value z and each BV as a linear function of the NBV's:

$$\begin{bmatrix} z \\ x_B \end{bmatrix} = f(x_N),$$

where

- $x_B = [x_i | i \in I]$ (BV's),
- $x_N = [x_i | i \notin I]$ (NBV's) and
- $f : \mathbb{R}^{n-m} \to \mathbb{R}^{m+1}$ is linear.

Matrix Partition

Let $A = [a_1, \ldots a_n]$, where $a_i \in \mathbb{R}^m$ is the ith column of A. For any index set $I \subseteq \{1, \ldots, n\}$ with $|I| = m$. Define

- $B = B(I) = [a_i | i \in I]$;
- $N = N(I) = [a_i | i \notin I]$;
- $c_B = c_B(I) = [c_i | i \in I]$;
- $c_N = c_N(I) = [c_i | i \notin I]$;
- $x_B = x_B(I) = [x_i | i \in I]$;
- $x_N = x_N(I) = [x_i | i \notin I]$.

This implies

$$Ax = Bx_B + Nx_N \quad \text{and} \quad c^T x = c_B^T x_B + c_N^T x_N .$$

Example: Partition of A

$$A = \begin{bmatrix} 2 & 4 & 3 & 3 & 1 & 0 \\ 3 & -3 & 4 & 2 & 0 & 1 \\ -1 & 2 & 1 & 2 & 0 & 0 \end{bmatrix}$$

Choose $I = \{3, 4, 5\}$

$$\Rightarrow B(I) = \begin{bmatrix} 3 & 3 & 1 \\ 4 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix} \quad \text{and} \quad N(I) = \begin{bmatrix} 2 & 4 & 0 \\ 3 & -3 & 1 \\ -1 & 2 & 0 \end{bmatrix}$$

Basic Representation (cont)

Given this partition, we have:

$$\left.\begin{array}{rcl} z & = & c^T x \\ A\,x & = & b \end{array}\right\} \quad \Longleftrightarrow \quad \left\{\begin{array}{rcl} z & = & c_B^T x_B + c_N^T x_N \\ B\,x_B & = & b - N\,x_N \end{array}\right.$$

Since B is invertible by construction, this implies that

$$x_B = B^{-1}(b - N x_N) = B^{-1} b - B^{-1} N x_N\,.$$

Substituting this formula into the expression for z we find

$$z = c_B^T x_B + c_N^T x_N = c_B^T B^{-1} b + (c_N^T - c_B^T B^{-1} N) x_N\,,$$

which may be equivalently rewritten as

$$z = c_B^T B^{-1} b + (c_N - N^T B^{-T} c_B)^T x_N$$

where we use the shorthand notation $B^{-T} = (B^{-1})^T$.

Basic Representation (cont)

Thus, the original system $z = c^T x$, $Ax = b$ is equivalent to the basic representation

$$\left.\begin{array}{rcl} z & = & c_B^T B^{-1} b + (c_N - N^T B^{-T} c_B)^T x_N \\ x_B & = & B^{-1} b - B^{-1} N x_N, \end{array}\right\} \qquad (*)$$

which expresses z and x_B as linear functions of x_N.

Note: By setting $x_N = 0$ in $(*)$ we obtain the basic solution $x = (x_B, x_N) = (B^{-1}b, 0)$ with objective value $z = c_B^T B^{-1} b$.

Definition: We call $r = c_N - N^T B^{-T} c_B$ the reduced cost vector. This vector characterises the sensitivity of the objective function value z with respect to the nonbasic variables x_N.

Example: Basic Representation

Consider the following LP:

$$\min z = 6x_1 + 3x_2 + 4x_3 + 2x_4 - 3x_5 + 4x_6$$

subject to:

$$\begin{array}{rrrrrrrrl}
2x_1 & -1x_2 & +3x_3 & +2x_4 & +3x_5 & +2x_6 & +x_7 & & = 4 \\
3x_1 & +4x_2 & +2x_3 & +2x_4 & +3x_5 & & & +x_8 & = 2
\end{array}$$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \geq 0$$

Example: Basic Representation (cont)

Thus, we are given the following problem data:

$$A = \begin{bmatrix} 2 & -1 & 3 & 2 & 3 & 2 & 1 & 0 \\ 3 & 4 & 2 & 2 & 3 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \quad c = \begin{bmatrix} 6 \\ 3 \\ 4 \\ 2 \\ -3 \\ 4 \\ 0 \\ 0 \end{bmatrix}$$

Example: Basic Representation (cont)

Choose $I = \{4, 3\}$. Then, we have

$$B = \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix} \quad \Rightarrow \quad B^{-1} = \begin{bmatrix} -1 & \frac{3}{2} \\ 1 & -1 \end{bmatrix},$$

$$N = \begin{bmatrix} 2 & -1 & 3 & 2 & 1 & 0 \\ 3 & 4 & 3 & 0 & 0 & 1 \end{bmatrix},$$

$$c_B^T = \begin{bmatrix} 2 & 4 \end{bmatrix}, \quad c_N^T = \begin{bmatrix} 6 & 3 & -3 & 4 & 0 & 0 \end{bmatrix}.$$

Example: Basic Representation (cont)

Using $(*)$, we find that the original system

$$
\begin{array}{rcccccccccl}
z = & 6x_1 & & +3x_2 & +4x_3 & +2x_4 & -3x_5 & +4x_6 & & & \\
& 2x_1 & -1x_2 & +3x_3 & +2x_4 & +3x_5 & +2x_6 & +x_7 & & = 4 \\
& 3x_1 & +4x_2 & +2x_3 & +2x_4 & +3x_5 & & & +x_8 & = 2
\end{array}
$$

is equivalent to the basic representation

$$
\begin{array}{rcccccccc}
z = & 6 & +5x_1 & +9x_2 & & -6x_5 & & -2x_7 & +x_8 \\
x_4 = & -1 & -\frac{5}{2}x_1 & -7x_2 & & -\frac{3}{2}x_5 & +2x_6 & +x_7 & -\frac{3}{2}x_8 \\
x_3 = & 2 & +x_1 & +5x_2 & & & -2x_6 & -x_7 & +x_8
\end{array}
$$

The corresponding BS is not feasible:

$$(z, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = (6, 0, 0, 2, -1, 0, 0, 0, 0)$$

Importance of Basic Representations

Fix an index set I with $|I| = m$.

- Assume that
  - the basis B is invertible and
  - the corresponding BS with $x_B = B^{-1}b$ and $x_N = 0$ is feasible, i.e., $B^{-1}b \geq 0$.
- The objective value of this BFS is $z = c_B^T B^{-1} b$.
- Any other feasible solution satisfies $x_N \geq 0$.
- The basic representation

$$z = c_B^T B^{-1} b + r^T x_N \quad \text{and} \quad x_B = B^{-1} b - B^{-1} N x_N$$

tells us how z and x_B change when the nonbasic variables increase.

Importance of Basic Representations

In particular, the reduced cost vector r enables us to:

- recognise whether the current BFS is optimal (this is the case iff $r \geq 0$; then, no other feasible solution can have a lower objective value than the current BFS);
- find a new BFS with a lower objective value if the current BFS is not optimal (by increasing a nonbasic variable with a negative reduced cost).