COMPUTATIONAL FINANCE: 422

Fixed-Income Securities

Panos Parpas

(Slides courtesy of Daniel Kuhn)

p.parpas@imperial.ac.uk

Imperial College

London

This Lecture

- Basic terminology
- Examples of fixed-income securities
  - Annuities
  - Bonds
- Valuation of fixed-income securities
- Risk management of fixed-income securities
  - Credit and interest rate risk
  - Yield
  - Duration
  - Immunization

Fri 2h (LEC)
· Tue 2h (LEC)
· Fri 2h (LEC)
Tue 2h (Tut)

Further reading:

- D.G. Luenberger: *Investment Science*, Chapters 3,5

Definitions

- Financial instrument: a legal obligation or claim having monetary value.
- Examples: stocks, bonds, mortgages, futures, insurance, etc.
- Security: a tradable financial instrument satisfying legal and regulatory requirements.
- Fixed-income security: security that promises a fixed (that is, definite) income over a span of time.
- Examples: bonds, mortgages, annuities, etc.

$\Rightarrow$ A fixed-income security represents the ownership of a definite cash flow stream.

Remarks

The issuer of a fixed-income security could default (by, say, going bankrupt).

$\Rightarrow$ There is a (typically small) chance that the promised income is discontinued or delayed.

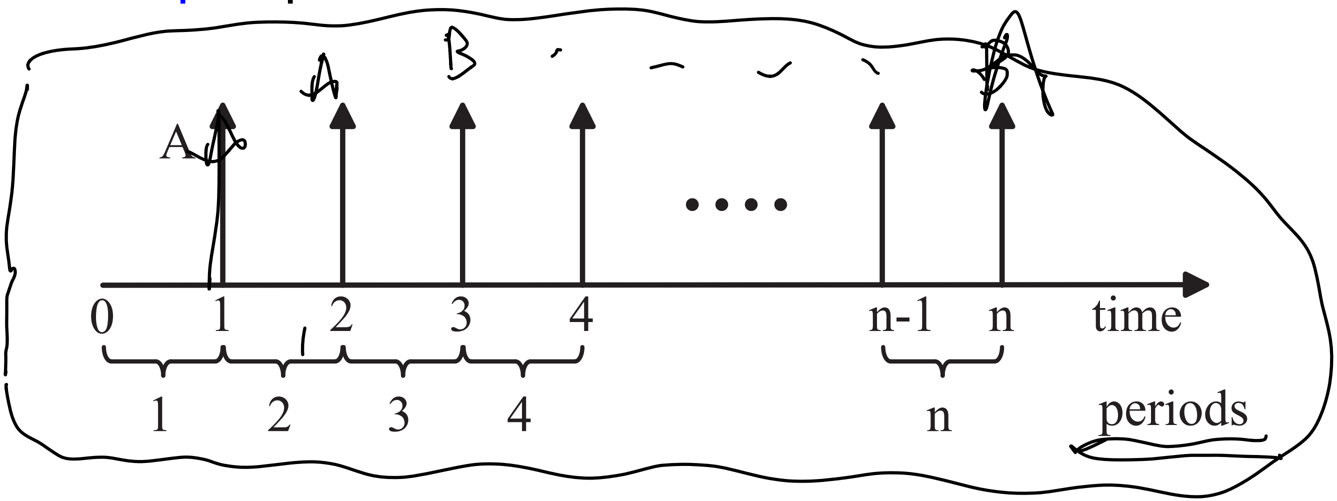
Some securities whose cash flows depend on contingencies or fluctuating indices are also called fixed-income:

- Adjustable-rate mortgages
- Callable bonds
- etc.

General idea: a fixed-income security's cash flow stream is fixed except in well-defined contingent circumstances.

Annuities

- **Annuity**: a contract that pays the holder money periodically according to a fixed schedule.
- **Example**: pension benefits



The present value of an annuity that pays a fixed amount A at the end of each of n equally spaced periods is

$$\boxed{\text{PV} = \sum_{k=1}^{n} d_k A\,.}$$

$$d_k = \frac{1}{(1+r)^k}$$

Geometric Series

$x = \frac{1}{1+r}$

Calculate $\boxed{S_n = \sum_{k=0}^{n} x^k}$ for $n \in \mathbb{N}$. We use the recursions:

$$S_{n+1} = 1 + x\underbrace{(1 + x + x^2 + \cdots + x^n)}_{S_n} = 1 + xS_n \quad (1)$$

$$S_{n+1} = \underbrace{(1 + x + x^2 + \cdots + x^n)}_{S_n} + x^{n+1} = S_n + x^{n+1} \quad (2)$$

$S_{n+1} = 1 + x + x^2 + \ldots x^{n+1}$

By combining (1) and (2), we obtain

(1) ~ (2)

$$\boxed{S_n = \frac{1 - x^{n+1}}{1 - x}.}$$

$x^{n+1} \to 0$

$r > 0$, $0 < x < 1$

$x = 1/1+r$

$n \to \infty$

For $x < 1$, we can calculate $S_\infty = \sum_{k=0}^{\infty} x^k$ to be

$$\boxed{S_\infty = \lim_{n\to\infty} \frac{1 - x^{n+1}}{1 - x} = \frac{1}{1 - x}.}$$

Annuity Valuation

$S_n = \dfrac{1 - x^{n+1}}{1 - x}$

- If the spot rate curve is constant, $s_k = r$ for all $k = 1, 2, \ldots, n$, then the value of the annuity becomes

$x = \dfrac{1}{1+r}$

$$\boxed{\text{PV} = \sum_{k=1}^{n} \frac{A}{(1+r)^k}} = A \left(\frac{1 - \frac{1}{(1+r)^{n+1}}}{1 - \frac{1}{1+r}} - 1 \right)$$

valid from 0 to n

$$\boxed{= \frac{A}{r} \left(1 - \frac{1}{(1+r)^n} \right).}$$

Equivalently, we have

$$\boxed{A = \frac{r(1+r)^n \text{PV}}{(1+r)^n - 1}.}$$

Tables give PV/A as a function of r and n.

Amortization

- You have borrowed $1,000 at 12% interest compounded monthly, and you must repay this loan with equal monthly payments over 5 years.
- How much are the monthly payments?

Given: $\mathrm{PV} = \$1{,}000$

r/m $\quad r = 12\%/12 = 1\%$ per month

$n = 5 \times 12 = 60$

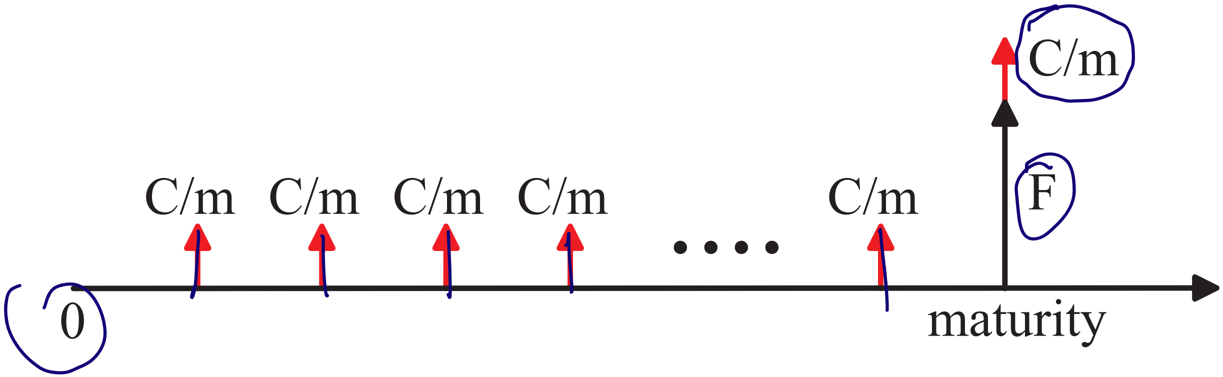
$$\Rightarrow \quad A = \frac{r(1+r)^n \mathrm{PV}}{(1+r)^n - 1} = \boxed{\$22.24} \text{ per month}$$

Replacing a current obligation by periodic payments is called amortization.

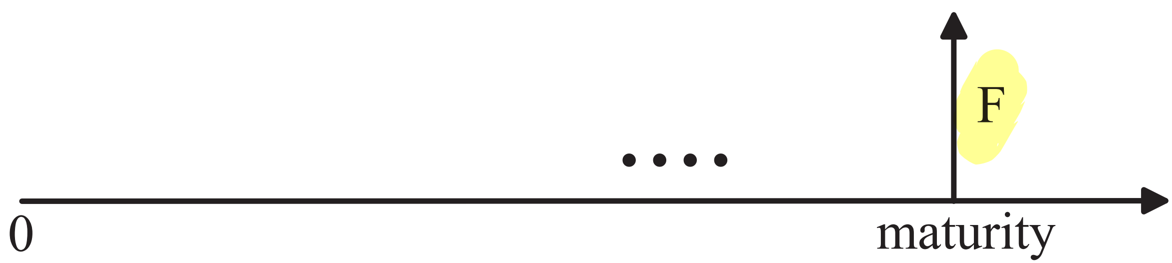
Bonds

- Bond: an agreement to pay money according to the rules of the issue.
- Bond specifications:
  - maturity date: time of the last payment
  - face value or par value: an amount to be paid at the maturity date
  - coupon payment: an amount paid periodically expressed as a percentage of the face value (the last coupon is paid at maturity)
- F = face value, r_C = coupon rate
  $\Rightarrow \boxed{C = r_C F}$ = yearly coupon amount; \$, £
  if there are m coupon payments per year, then each subannual coupon payment amounts to $\boxed{C/m}$.

Payoff Structure of a Bond



A zero coupon bond has no coupon payments; it only pays the face value at maturity.



Example: Price of a 10-year Bond

~ 8%

Consider an 8% bond maturing in 10 years:

- the bond pays \$8 at the end of the years $1, 2, \dots, 9$ and \$108 at the end of year 10.
- the end-of-year discount factors for years $1, 2, \dots, 10$ can be calculated from a given spot rate curve.
- We take the products of the cash flows with the corresponding discount factors and sum.

$\Rightarrow$ The value of the bond is \$97.34.

$46.50 \approx \dfrac{100+8}{(1+0.08793)^{10}}$

$8/(1+0.0571)$

Year	1	2	3	4	5	6	7	8	9	10	Total PV
Spot Rate	5.571	6.088	6.555	6.978	7.361	7.707	8.020	8.304	8.561	8.793	
Discount	0.947	0.889	0.827	0.764	0.701	0.641	0.583	0.528	0.477	0.431	
Cash Flow	8	8	8	8	8	8	8	8	8	108	
PV	7.58	7.11	6.61	6.11	5.61	5.12	4.66	4.23	3.82	46.50	97.34

$100, + 8 \times 10$

Bond Terminology

- corporate bond: issued by a corporation
- municipal bond: issued by a municipality
- treasury bond: issued by government, maturity more than 10 years, 2 coupon payments per year
- treasury note: similar to treasury bond, but issued for 1-10 years
- treasury bill: similar to treasury bond, no coupon payments, matures in 3, 6, or 9 months

Quality Ratings I

- Bonds offer principally a deterministic income stream. However, they are subject to default if the issuer falls into bankruptcy.
- Rating classifications are published by Moody's and Standard & Poor's.
- Treasury securities are not rated, since they are considered to be essentially free of default risk.
- A bond with a low rating will have a lower price than a comparable bond with a higher rating.

Quality Ratings II

	Moody's	S&P	
	Aaa	$\boxed{\text{AAA}}$	Best quality, smallest credit risk ~1%
Investment	Aa	AA	High grade
grade	A	A	High to medium grade
	Baa	BBB	Medium grade
	Ba	BB	Judged to be speculative ~15-20%
Speculative	B	B	Increasingly speculative
grade	Caa	CCC	Danger of default
(junk bonds)	Ca	CC	High chance of default
	C	C	Small chance of no default
	D	D	In default

Bond Quotes (finance.yahoo.com)

100

BOND SCREENER RESULTS

P.V P%

Type	Issue	State	Price	Coupon(%)	Maturity	YTM(%)	Current Yield(%)	Fitch Ratings	Callable
Corp	EXPORT IMPORT BK KOREA	-	102.17	5.125	14-Feb-2011	-52.654	5.016	A	No
Corp	BANK AMER CORP SUB INTNTS BE	-	102.05	6.500	15-Feb-2011	-44.685	6.369	A	No
Corp	FEDERAL NATL MTG ASSN	-	102.22	4.500	15-Feb-2011	-38.219	4.402	AAA	No
Corp	NORTHROP GRUMMAN CORP	-	102.24	7.125	15-Feb-2011	-47.904	6.969	BBB	No
Corp	TYCO INTL GROUP S A	-	102.24	6.750	15-Feb-2011	-48.290	6.602	A	No
Corp	WAL MART STORES INC	-	102.15	4.125	15-Feb-2011	-49.144	4.038	AA	No
Muni	ALABAMA WTR POLLUTION CTL AUTH REF BDS	AL	104.12	4.000	15-Feb-2011	-91.731	3.842	Not Rated	No
Muni	SAN DIEGO CALIF PUB FACS FING REF LEASE REV BDS	CA	104.18	5.000	15-Feb-2011	-91.594	4.799	A	No
Zero	U S TREAS SEC STRIPPED INT PMT 15-Feb-2011	-	100.03	0.000	15-Feb-2011	-0.677	0.000	AAA	No
Treas	T-NOTE 5.000 15-Feb-2011	-	100.26	5.000	15-Feb-2011	0.021	4.987	AAA	No
Corp	FEDERAL NATL MTG ASSN MTN	-	102.21	4.125	17-Feb-2011	-34.320	4.036	AAA	No
Corp	CISCO SYS INC	-	102.29	5.250	22-Feb-2011	-32.493	5.133	A	No
Corp	FEDERAL HOME LOAN BANKS	-	102.00	0.250	25-Feb-2011	-24.964	0.245	AAA	No
Treas	T-NOTE 4.500 28-Feb-2011	-	100.39	4.500	28-Feb-2011	0.041	4.482	AAA	No
Corp	HERTZ CORP	-	102.55	7.400	1-Mar-2011	-21.916	7.216	B	No

Remark: The current yield of a bond is the ratio of the annual coupon payment and the bond's current price.

$P = \sum_{k=1}^{n} d_k P_k$

Yield

~ 2:10 (Break until)

- A bond's yield is the interest rate at which the PV of the stream of payments (coupon payments & face value redemption) is equal to the current price.
- More properly, it is called yield to maturity (YTM).
- Consider a bond with price P and face value F that makes m coupon payments of C/m per year, and there are n remaining periods. The bond's YTM λ is such that

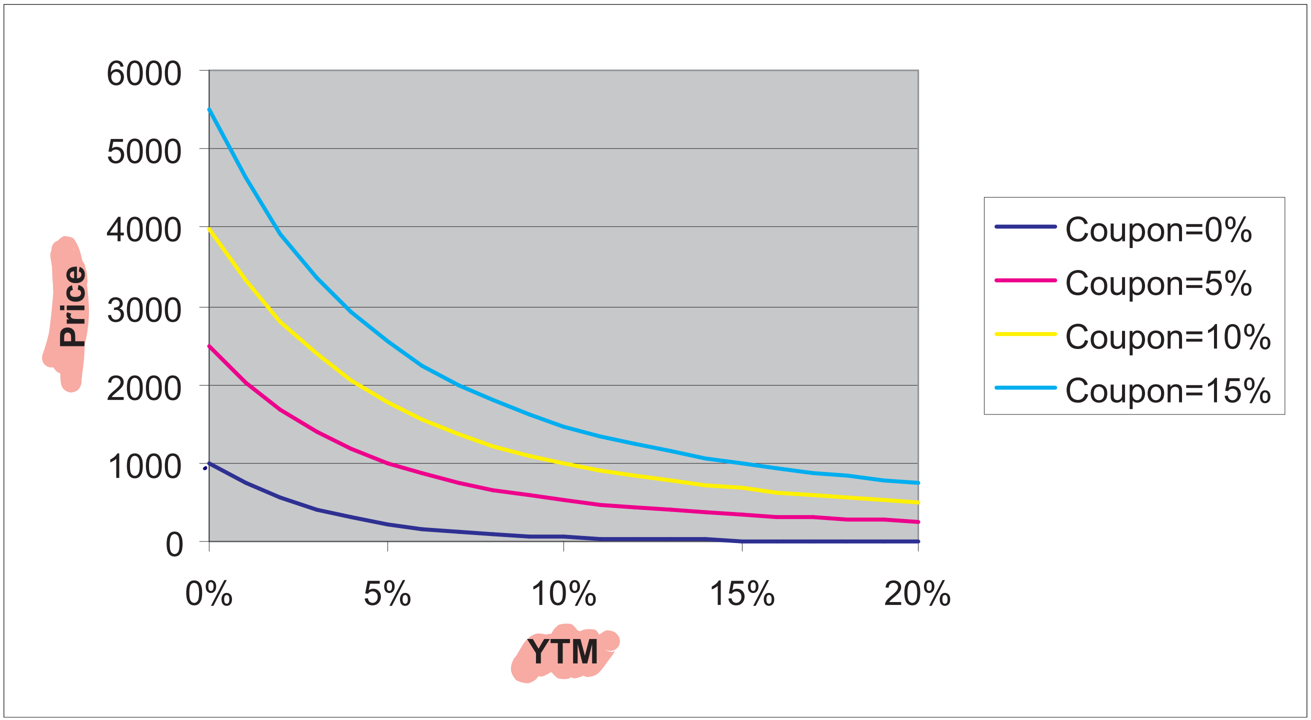
✓100 ≈ 8% × F

$$
\begin{aligned}
P &= \underline{\frac{F}{[1+(\lambda/m)]^n}} + \boxed{\sum_{k=1}^{n} \frac{C/m}{[1+(\lambda/m)]^k}} \\
&= \frac{F}{[1+(\lambda/m)]^n} + \frac{C}{\lambda}\left\{1 - \frac{1}{[1+(\lambda/m)]^n}\right\}.
\end{aligned}
$$

Use Geometric Series

Price-Yield Curve and Coupon Rate

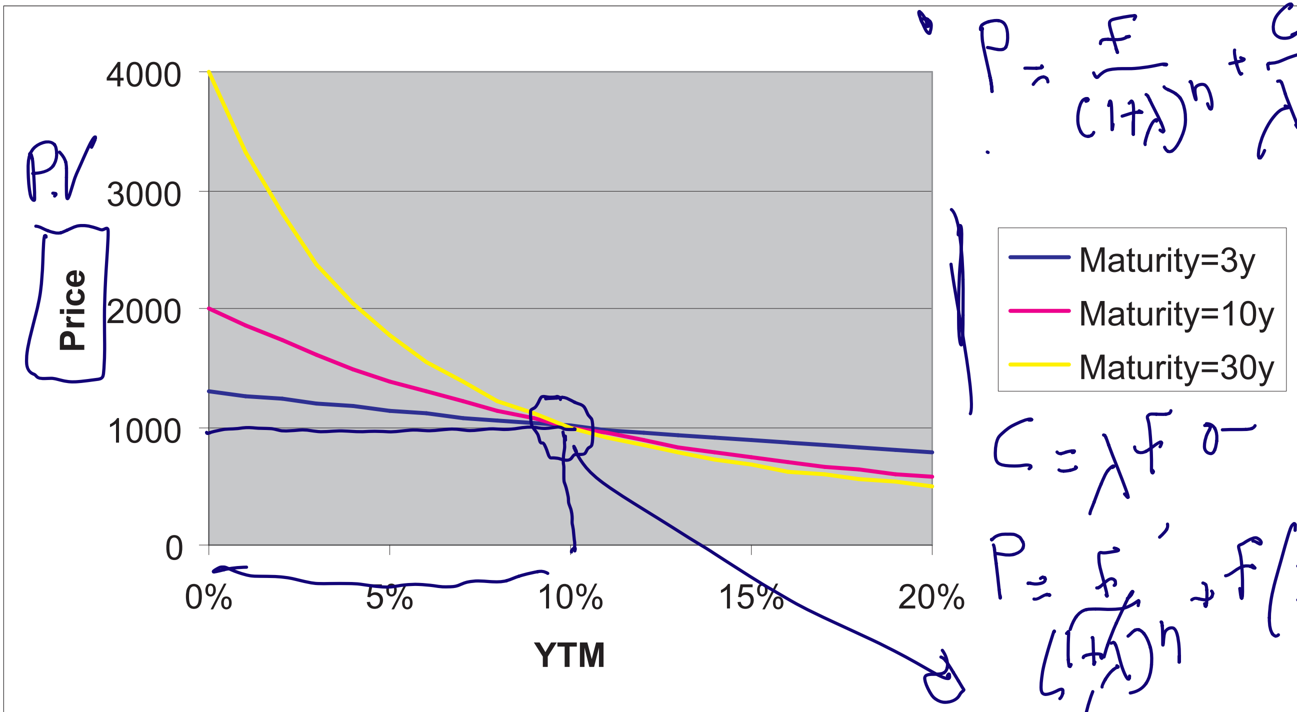
- Bond data: $F = 1000$, $m = 2$, $n = 60$ (i.e., 30 years).



$\Rightarrow$ Inverse dependence between price and yield!

Price-Yield Curve and Maturity

- Bond data: $F = 1000$, $m = 2$, Coupon $= 10\%$.



$P = \frac{F}{(1+\lambda)^n} + \frac{C}{\lambda}\left(1 - \frac{1}{(1+\lambda)^n}\right)$

$C = \lambda F$

$P = \frac{F}{(1+\lambda)^n} + F\left(1 - \frac{1}{(1+\lambda)^n}\right)$

$= F$

$\Rightarrow$ The longer the time to maturity, the more sensitive is the price of the bond to the yield.

Interest Rate Risk

- The price-yield curve describes the interest rate risk associated with a bond.
- The yield can roughly be identified with the market rate for the underlying bond.
- Suppose you bought a 10% bond with $F = 1000$, $m = 2$, $n = 60$, when the yield was $10\% \Rightarrow$ price $= 1000$.
- If the yield rises to 11%, then the price of your bond drops to $913 \Rightarrow$ a 8.72% loss!

Duration

0-coupon

10 y

- Long bonds are more sensitive to interest rate changes than short bonds.

1 year coupon

- When there are coupons, maturity time does not exactly correspond to sensitivity.

- Another measure of time length termed duration does give a quantitative measure of interest rate sensitivity.

Macaulay Duration I

$PV_{tot} = \sum_{k=1}^{n} P_k d_k$

PV(0) PV(1) PV(2) P(t_n)

- Suppose that cash flows are received at times $t_0, t_1, \ldots, t_n$. The Macaulay duration of this stream is

$$\boxed{D = \frac{\mathrm{PV}(t_0)t_0 + \mathrm{PV}(t_1)t_1 + \cdots + \mathrm{PV}(t_n)t_n}{\underbrace{\mathrm{PV}(t_0) + \mathrm{PV}(t_1) + \cdots \mathrm{PV}(t_n)}_{PV_{tot}}},}$$

where $\mathrm{PV}(t_k)$ denotes the present value of the cash flow that occurs at time t_k computed w.r.t. the yield.

$\Rightarrow$ Weighted average of the cash flow times, where the weight of time t_k is given by $\boxed{w_k = \mathrm{PV}(t_k)/\mathrm{PV}_{\mathrm{tot}}}$ with $\mathrm{PV}_{\mathrm{tot}} = \mathrm{PV}(t_0) + \mathrm{PV}(t_1) + \cdots + \mathrm{PV}(t_n)$.

$\sum_{k=0}^{n} w_k t_k = (\text{time}).$

years (time)

dimensionless

Macaulay Duration II

$D = \sum_{i=0}^{n} w_i t_i$ $\quad w_i = 0 \quad i=0, \ldots, n-1$ $\quad w_n = 1$ $\quad D = t_n$

maturity date

- Macaulay duration is a weighted average of times $\Rightarrow$ it is quoted in years.
- In practice, when people refer to duration, they often mean Macaulay duration.
- By definition of D, the following relations hold.
  - Zero-coupon bond:

    Macaulay duration $=$ maturity date
  - Coupon bond:

    t_n

    Macaulay duration $<$ maturity date

$t_0 < t_1 < t_2 < \cdots < t_n$

Macaulay Duration III

- Consider a bond with the following specifications:
  - c - coupon rate per year
  - λ - yield
  - m - periods per year
  - n - remaining periods

$\Rightarrow$ This bond's Macaulay duration is given by

$$\boxed{D = \frac{1+\lambda/m}{\lambda} - \frac{1+\lambda/m+n(c/m-\lambda/m)}{c[(1+\lambda/m)^n - 1]+\lambda}.}$$

Modified Duration I

$P(\lambda) = F/(1+\lambda)^n + \sum_{k=1}^{n} C/(1+\lambda)^k$

- Let $P(\lambda)$ be the price of a bond as a function of yield, and denote by λ_0 the current yield.
- **Def.**: The modified duration D_M of this bond is

$$\boxed{D_M = -\frac{1}{P(\lambda_0)} \left. \frac{\mathrm{d}P(\lambda)}{\mathrm{d}\lambda} \right|_{\lambda=\lambda_0}} .$$

Intuition:

- sensitivity $=$ derivative;
- the minus sign ensures that $D_M \geq 0$;
- dividing by $P(\lambda_0)$ makes D_M a relative sensitivity.

Modified Duration II

- Modified duration determines the percentage change in the price of a bond given a change in yield:

$$D_M = -\frac{1}{P(\lambda_0)}\,\frac{\mathrm{d}P(\lambda)}{\mathrm{d}\lambda}\bigg|_{\lambda=\lambda_0} \approx -\frac{1}{P}\frac{\Delta P}{\Delta\lambda}$$

$P(\lambda_0) = PV_{tot} = \text{price today}.$

$$\Rightarrow \quad \Delta P \approx -D_M P \Delta\lambda\,.$$

- If the yield increases by 1%, what percentage change will occur in a bond price with modified duration of 5?

Answer: The price will decrease by 5%!

Relation between D and D_M I

Theorem 0.1 *If λ is the yield and m the number of compounding periods per year, then*

$$D_M = \frac{D}{1 + \lambda/m}.$$

$D = \sum_{k=0}^{n} w_k t_k$

Next week

4h Lectures

$\Rightarrow$ *In the case of continuous compounding $D_M = D$.*

$m \to \infty$

Proof. For the stream $c_0, c_1, \ldots, c_n$ we have

t_k weight of k

$$\mathrm{PV}_k = \frac{c_k}{(1+\lambda/m)^k}, \quad \mathrm{PV}_{\text{tot}} = \sum_{k=0}^{n} \frac{c_k}{(1+\lambda/m)^k}, \quad \boxed{D = \sum_{k=0}^{n} \frac{k}{m} \frac{\mathrm{PV}_k}{\mathrm{PV}_{\text{tot}}}}.$$

$$\Rightarrow \quad \frac{\mathrm{d}\,\mathrm{PV}_{\text{tot}}}{\mathrm{d}\lambda} = \sum_{k=0}^{n} \frac{-(k/m)c_k}{(1+\lambda/m)^{k+1}} = -\frac{1}{1+\lambda/m} \sum_{k=0}^{n} \frac{(k/m)c_k}{(1+\lambda/m)^k}$$

$$= -\frac{1}{1+\lambda/m} \sum_{k=0}^{n} \frac{k}{m} \mathrm{PV}_k = -\frac{\mathrm{PV}_{\text{tot}} D}{1+\lambda/m}$$

Relation between D and D_M II

Thus, we find

$$D_M = -\frac{1}{\mathrm{PV}_{\mathrm{tot}}} \frac{\mathsf{d}\,\mathrm{PV}_{\mathrm{tot}}}{\mathsf{d}\lambda} = \frac{D}{1+\lambda/m}\,.$$

This observation completes the proof. □

Example: A 10%, 30 year bond with semiannual coupons is selling at par and has Macaulay duration $D = 9.94$.

By how many percent will the bond's price change if yield increases by 1%?

$$\frac{\Delta P}{P} \approx -D_M \Delta\lambda = -\frac{D}{1+\lambda/m}\Delta\lambda = -\frac{9.94}{1+10\%/2} 1\% = -9.47\%$$

$\Rightarrow$ A 1% increase in yield causes a 9.47% drop in price!

Duration of a Portfolio I

- Consider a set of m fixed-income securities with prices P_i and durations D_i for $i = 1, 2, \ldots, m$, all computed at a common yield.
- The portfolio consisting of the aggregate of these securities has price P and duration D given by

$$\begin{aligned} P &= P_1 + P_2 + \cdots + P_m \\ D &= w_1 D_1 + w_2 D_2 + \cdots + w_n D_n \end{aligned}$$

where $w_i = P_i / P$, $i = 1, 2, \ldots, m$.

- The formula for D is true for the Macaulay duration as well as for the modified duration!

Duration of a Portfolio II

Consider two cash flow streams A and B with

$$D^A = \sum_{k=0}^{n} \frac{t_k \mathrm{PV}_k^A}{P^A} \quad \text{and} \quad D^B = \sum_{k=0}^{n} \frac{t_k \mathrm{PV}_k^B}{P^B}.$$

The Macaulay duration of the aggregate stream $A + B$ is ?

$$\begin{aligned}
D^{A+B} &= \sum_{k=0}^{n} \frac{t_k (\mathrm{PV}_k^A + \mathrm{PV}_k^B)}{P^A + P^B} = \sum_{k=0}^{n} \frac{t_k \mathrm{PV}_k^A}{P^A + P^B} + \sum_{k=0}^{n} \frac{t_k \mathrm{PV}_k^B}{P^A + P^B} \\
&= \frac{P^A}{P^A + P^B} \sum_{k=0}^{n} \frac{t_k \mathrm{PV}_k^A}{P^A} + \frac{P^B}{P^A + P^B} \sum_{k=0}^{n} \frac{t_k \mathrm{PV}_k^B}{P^B} \\
&= \frac{P^A}{P^A + P^B} D^A + \frac{P^B}{P^A + P^B} D^B.
\end{aligned}$$

Immunization I

- Problem of major practical value:
  - Construct a portfolio which is protected (immunized) against changes in interest rates.

- Example:
  - You have an obligation to pay £1,000 in 2 years, but you can only buy bonds of maturities 1 or 5 years.
    - Buying 1 year bonds: you face reinvestment risk as you do not know the bond prices in 1 year.
    - Buying 5 year bonds: you may fail to meet your obligation when interest rates change, i.e., you may not be able to sell the bond after 1 year at the desired price.

Immunization II

- Assume that you face a series of cash obligations and you wish to acquire a portfolio that you will use to pay these obligations.
- The stream of obligations has present value P and duration D.
- You can invest in two bonds with present values P_1 and P_2 and durations D_1 and D_2, respectively.

$\Rightarrow$ Construct a portfolio whose present value and duration equal those of your obligations.

$$\left.\begin{array}{l} P = x_1 P_1 + x_2 P_2 \\ D = \frac{x_1 P_1}{P} D_1 + \frac{x_2 P_2}{P} D_2 \end{array}\right\} \quad \text{Solve for } x_1 \text{ and } x_2!$$

Immunization III

- Obligation: £1,000 in 2 years.
- You can invest in a 1 year and 5 year zero coupon bond.

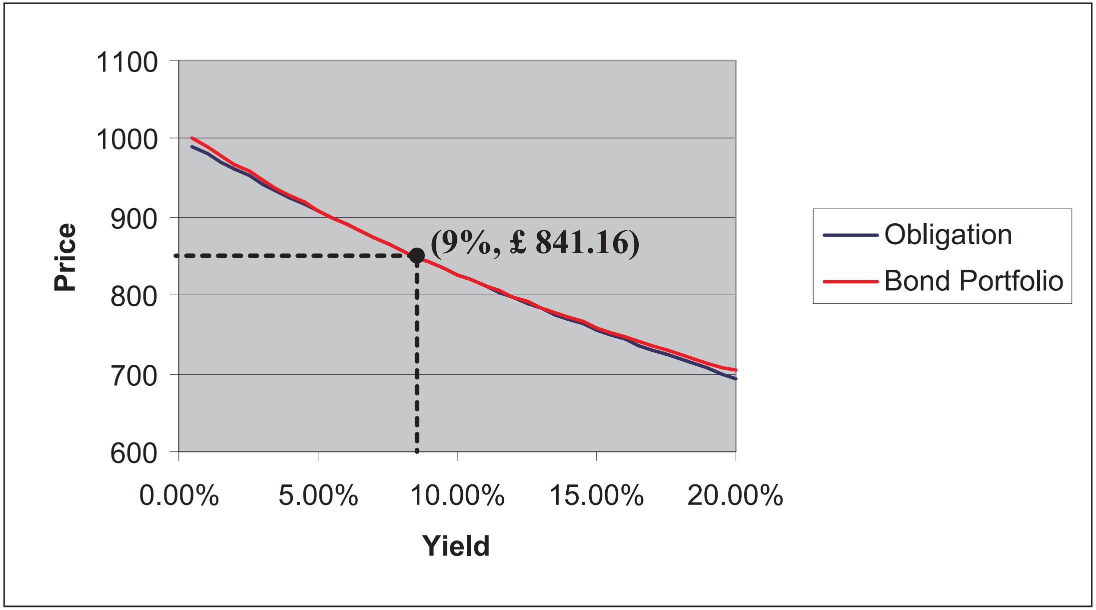
	Obligation	Bond 1	Bond 2
Face value	1000	100	100
Maturity	2	1	5
Coupon	0	0	0
Yield	0.09	0.09	0.09
Mac. Duration	2	1	5
Price	841.68	91.74	64.99

- Form a portfolio with the same price and duration as the obligation.
- Let x_1 and x_2 be the numbers of bond 1 and 2 held in the portfolio, respectively.

Immunization IV

- Match prices: $91.74x_1 + 64.99x_2 = 841.68$
- Match durations: $\frac{91.74x_1}{841.68}1 + \frac{64.99x_2}{841.68}5 = 2$

$\Rightarrow$ $x_1 = 6.88$ and $x_2 = 3.24$



Convexity I

- Idea behind immunization: Taylor expansion!
- λ_0 is the current value of the yield.
- We can expand a fixed-income security's price as a function of yield around λ_0:

$$P(\lambda) = P(\lambda_0) + P'(\lambda_0)(\lambda - \lambda_0) + \frac{1}{2} P''(\lambda_0)(\lambda - \lambda_0)^2 + \cdots$$

  - $P(\lambda_0)$ is the price.
  - $P'(\lambda_0) = -D_M P(\lambda_0)$ is the (unnormalized) duration.
  - $P''(\lambda_0) = P(\lambda_0) C$ is the (unnormalized) convexity.

- We can match as many terms as we like to adjust the Taylor series of our portfolio to that of our obligation.

Convexity II

- The Convexity of a fixed-income security is defined as

$$C = \frac{1}{P(\lambda_0)} \left. \frac{\mathrm{d}^2 P(\lambda)}{\mathrm{d}\lambda^2} \right|_{\lambda=\lambda_0} .$$

- If $\Delta\lambda$ is a small change in yield and ΔP is the corresponding change in price, then

$$\Delta P \approx -D_M P \Delta\lambda + \frac{PC}{2} (\Delta\lambda)^2 .$$

This is a second-order approximation to the price-yield curve.

$\Rightarrow$ Convexity can be used to improve immunization!