60016 OPERATIONS RESEARCH

Game Theory
Mixed Strategies

16 November 2020

Two-Person Zero-Sum Games

Two-person zero-sum games:

- m row strategies and n column strategies
- RP tries to maximise payoff, CP tries to minimise loss
- Dominated strategies are never played
- In a Nash equilibrium, players do not unilatarelly change their strategy when told what the opponent would do
- Equilibrium exists if

$$\max_{i=1,\dots,m} \alpha_i = \min_{j=1,\dots,n} \beta_j$$

α_i and β_j being payoffs for row strategy i, column strategy j.

Example 1: Election Game (with different payoffs)

MIN

CP

NOT

NOT

			L	B	S	α_i
MAX		~~L~~	0	-1	2	-1
	RP	~~B~~	$\boxed{5}$	4	$\boxed{-3}$	-3
		S	2	3	-4	-4
		β_j	5	4	2	

- No Nash equilibrium in pure strategies
- CP would switch to strategy *B* if told RP's strategy

Example 2: Odds-and-Evens

MIN

		CP		
		1	2	$\bar{\alpha}_i$
MAX RP	1	-1	1	-1
	2	1	-1	-1
	β_j	1	1	

Example 2: Odds-and-Evens

$$
\begin{array}{cc|cc|c}
 & & \multicolumn{2}{c}{CP} & \\
 & & 1 & 2 & \alpha_i \\
\hline
\multirow{2}{*}{RP} & 1 & -1 & 1 & -1 \\
 & 2 & 1 & -1 & -1 \\
\hline
 & \beta_j & 1 & 1 &
\end{array}
$$

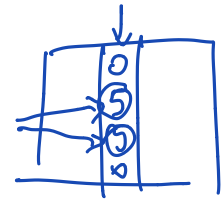
- No Nash equilibrium in pure strategies
- For any strategy pair, the losing player can always improve if told the strategy chosen by the winning player

Example: Odds-and-Evens (towards mixed strategies)

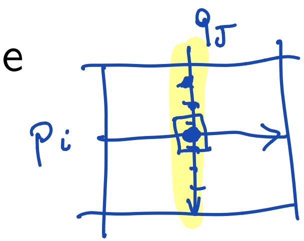
$$
\begin{array}{ccc|cc|}
 & & & \multicolumn{2}{c}{CP} \\
 & & & q_1 & q_2 \\
 & & & 1 & 2 \\
\cline{4-5}
\multirow{2}{*}{RP} & p_1 & 1 & -1 & 1 \\
 & p_2 & 2 & 1 & -1 \\
\cline{4-5}
\end{array}
$$

- Players randomly pick strategy with equal probabilities
- Each strategy pair is played with probability 0.25
- Expected value of the game is 0 for both players
  - No reason to unilaterally change probabilities
  - Example of Nash equilibrium in mixed strategies

Mixed Strategies



- In a mixed strategy $(p_1, \dots, p_m; q_1, \dots, q_n)$:
  - RP plays strategy i with probability $p_i \geq 0$, $\sum_{i=1}^{m} p_i = 1$.
  - CP plays strategy j with probability $q_j \geq 0$, $\sum_{j=1}^{n} q_j = 1$.
- If $p_k = 1$ or $q_k = 1$, then k is a pure strategy
- The payoff of the mixed strategy (p, q) will be

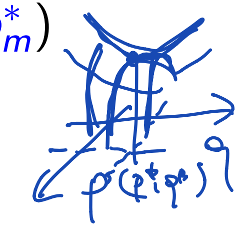


$(q_1, \dots, q_m)$

$$V(p, q) = \underbrace{\boxed{\sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j a_{ij}}}_{\text{EXPECTED PAYOFF/COST}}$$

$(p_1, \dots, p_m)$

- RP seeks probabilities that maximise payoff $(p_1^*, \dots, p_m^*)$
- CP seeks probabilities that minimise loss $(q_1^*, \dots, q_n^*)$



Definition

A mixed Nash equilibrium is a pair of mixed strategies (p^*, q^*) such that $\boxed{V(p, q^*) \leq} \boxed{V(p^*, q^*)} \leq V(p^*, q)$ for all other mixed strategies (p, q) [i.e., no agent has any incentive in unilaterally deviating].

Example: Election Game (revised in mixed strategies)

$$
\begin{array}{cc|ccc|c}
 & & & CP & & \\
 & & L & B & S & \\
\hline
 & L & 0 & -1 & 2 & p_L \\
RP & B & 5 & 4 & -3 & p_B \\
 & S & 2 & 3 & -4 & p_S \\
\hline
 & & q_L & q_B & q_S &
\end{array}
$$

- The strategy pair (S,B) is played with probability $p_S q_B$
- (S,B) contributes $p_S q_B \cdot 3$ to the mixed strategy payoff
- How can player find their optimal probabilities?

Column Player's Perspective

- Remember the Assumption: "Each player chooses a strategy that enables him/her to do best, reasoning in face of the worst-case opponent"
- CP expects RP to respond with optimal p_i's for any choice of q_j's. How should CP choose the q_j's?

$$V_{CP} = \min_{q_1,\dots,q_n} \left[\max_{p_1,\dots,p_m} \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j a_{ij} \right]$$

$$\min_{q} \left[\max_{p} V(p,q) \right]$$

subject to

$$\sum_{j=1}^{n} q_j = 1, \qquad \sum_{i=1}^{m} p_i = 1,$$

$$q_j \geq 0, \qquad p_i \geq 0$$

$$= \min_{q} \max_{p} \sum_{i} p_i \underbrace{\left(\sum_{j} q_j a_{ij} \right)}_{\alpha_i}$$

Column Player's Perspective (inner program)

Goal: reduce optimisation problem to linear program

- Let us rewrite and focus on the inner problem

$$V_{CP} = \min_{q_1,\ldots,q_n} V_{CP}^{in}(q_1, \ldots, q_n)$$

subject to

$$\sum_{j=1}^{n} q_j = 1, \qquad ,$$

$$q_j \geq 0,$$

Column Player's Perspective (inner program is trivial!)

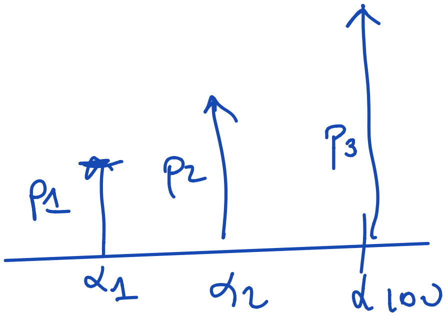
- For any choice of q_j's, let $\alpha_i = \sum_{j=1}^{n} q_j a_{ij}$ be row payoffs
- Then the inner maximisation problem is:

$$V_{CP}^{in}(q_1, \ldots, q_n) = \max_{p_1, \ldots, p_m} \sum_{i=1}^{m} p_i \alpha_i$$

subject to

$$\sum_{i=1}^{m} p_i = 1,$$

$$p_i \geq 0$$



- The solution is $p_i = 1$ for the largest α_i, $p_k = 0$ for $k \neq i$.
- Example: maximise $3p_1 + 2p_2 + 5p_3 \Rightarrow p_3 = 1$.

Example 1: inner program is trivial

- CP evaluates a pure strategy $q_S = 1$

$$
\begin{array}{cc|ccc|c}
 & & & CP & & \\
 & & L & B & S & \\
\hline
 & L & 0 & -1 & 2 & p_L \\
RP & B & 5 & 4 & -3 & p_B \\
 & S & 2 & 3 & -4 & p_S \\
\hline
 & & 0.0 & 0.0 & 1.0 &
\end{array}
$$

- if RP plays L, $\alpha_L = 0.0 \times 0 + 0.0 \times -1 + 1.0 \times \quad 2 = \quad 2$
- if RP plays B, $\alpha_B = 0.0 \times 5 + 0.0 \times \quad 4 + 1.0 \times -3 = -3$
- if RP plays S, $\alpha_S = 0.0 \times 2 + 0.0 \times \quad 3 + 1.0 \times -4 = -4$
- RP optimal response to CP is $p_L = 1 \Rightarrow V_{CP}^{in} = 2$

Example 2: inner program is trivial

- CP changes guess and evaluates a mixed strategy

$$
\begin{array}{cc|ccc|c}
 & & & CP & & \\
 & & L & B & S & \\
\hline
 & L & 0 & -1 & 2 & {\color{blue}p_L} \\
RP & B & 5 & 4 & -3 & {\color{blue}p_B} \\
 & S & 2 & 3 & -4 & {\color{blue}p_S} \\
\hline
 & & {\color{red}0.7} & {\color{red}0.2} & {\color{red}0.1} &
\end{array}
$$

- if RP plays L, $\alpha_L = {\color{red}0.7} \times 0 + {\color{red}0.2} \times -1 + {\color{red}0.1} \times \quad 2 = 0$
- if RP plays B, $\alpha_B = {\color{red}0.7} \times 5 + {\color{red}0.2} \times \quad 4 + {\color{red}0.1} \times -3 = 4$
- if RP plays S, $\alpha_S = {\color{red}0.7} \times 2 + {\color{red}0.2} \times \quad 3 + {\color{red}0.1} \times -4 = 1.6$
- RP optimal response to CP is ${\color{blue}p_B} = 1 \Rightarrow V_{CP}^{in} = 4$

Column Player (substitute inner in outer program)

- The inner maximisation optimal value is thus simply

$$V_{CP}^{in}(q_1, \ldots, q_n) = \underbrace{\max\{\overset{\sum_i \alpha_i p_i}{\alpha_1, \ldots, \alpha_m}\}}$$

- Expanding the definitions of the α_i's, we conclude that CP is in fact solving a min-max problem

$$\left\{\begin{aligned} V_{CP} = \min_{q_1,\ldots,q_n} & \left[\overbrace{\max\left\{\sum_{j=1}^{n} q_j a_{1j}, \ldots, \sum_{j=1}^{n} q_j a_{mj}\right\}}^{\text{inner}}\right] \\ \text{subject to} \quad & \sum_{j=1}^{n} q_j = 1, \\ & q_j \geq 0 \end{aligned}\right.$$

Column Player (final LP)

- The min-max problem is equivalent to a linear program

$$V_{CP} = \min_{\tau, q_1, \ldots, q_n} \tau$$

subject to

$$\tau \geq \sum_{j=1}^{n} q_j a_{ij}, \qquad \forall i = 1, \ldots, m$$

$$\sum_{j=1}^{n} q_j = 1,$$

$$q_j \geq 0,$$

- Election Game: $q_L^* = 0, q_B^* = \frac{1}{2}, q_S^* = \frac{1}{2} \Rightarrow V_{CP}^* = \frac{1}{2}$
- Note: the optimal q_j^*'s are independent of the p_i^*'s

Row Player's Perspective

- A similar reasoning applies to the row player, who instead optimises

$$V_{RP} = \max_{p_1,\ldots,p_m} \min_{q_1,\ldots,q_n} \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j a_{ij}$$

subject to

$$\sum_{i=1}^{m} p_i = 1, \qquad \sum_{j=1}^{n} q_j = 1,$$

$$p_i \geq 0, \qquad q_j \geq 0$$

Row Player's Perspective

- The max-min problem can be shown equivalent to a linear program

$$V_{RP} = \max_{\tau, p_1, \ldots, p_m} \tau$$

subject to

$$\tau \leq \sum_{i=1}^{m} p_i a_{ij}, \qquad \forall j = 1, \ldots, n$$

$$\sum_{i=1}^{m} p_i = 1,$$

$$p_i \geq 0,$$

- Election Game: $p_L^* = \frac{7}{10}, p_B^* = \frac{3}{10}, p_S^* = 0 \Rightarrow V_{RP}^* = \frac{1}{2}$
- Observation: p_i^*'s will be independent of the q_j^*'s.

Minimax Theorem

Theorem (Von Neumann, 1928). For every two-person zero-sum game, the RP and CP linear programs have the same optimal value, i.e.,

$$V_{RP} = \max_{p_1,\ldots,p_m} \min_{q_1,\ldots,q_n} \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j a_{ij} = \min_{q_1,\ldots,q_n} \max_{p_1,\ldots,p_m} \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j a_{ij} = V_{CP}$$

Proof: ideas?

Minimax Theorem

Theorem (Von Neumann, 1928). For every two-person zero-sum game, the RP and CP linear programs have the same optimal value, i.e.,

$$V_{RP} = \max_{p_1,\ldots,p_m} \min_{q_1,\ldots,q_n} \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j a_{ij} = \min_{q_1,\ldots,q_n} \max_{p_1,\ldots,p_m} \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j a_{ij} = V_{CP}$$

Proof: ideas? Result follows by strong duality since the two programs are the duals of each other.

Consequences:

- A Nash Equilibrium in mixed strategies always exists!!!
  - Players expect identical payoffs
  - Neither player has an incentive to change p_i or q_j
- Statement generalises to M players (Nash, 1949).

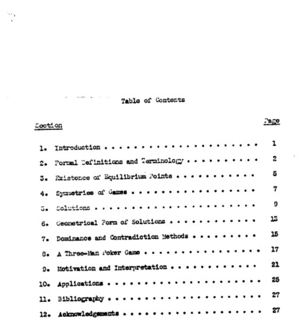
Historical Notes

BACK @ 11:04

- In 1928, Von Neumann first proved the Minmax Theorem for zero-sum games. He later wrote:

  "As far as I can see, there could be no theory of games . . . without that theorem . . . I thought there was nothing worth publishing until the Minimax Theorem was proved."

- In 1949, Nash gave a **one-page** proof (in 27-page thesis) that games with any number of players have a mixed equilibria.



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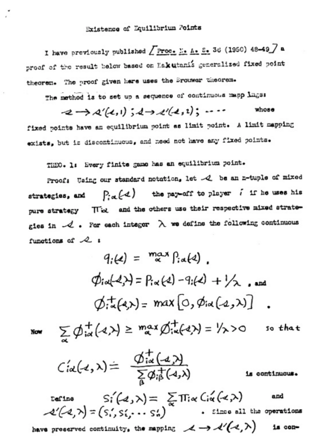


Existence of Equilibrium Points

I have previously published [*Proc. N. A. S.* 36 (1950) 48–49] a proof of the result below based on Kakutani's generalized fixed point theorem. The proof given here uses the Brouwer theorem.

The method is to set up a sequence of continuous mappings: $s \to s'(s,1)$; $s \to s'(s,2)$; $\cdots$ whose fixed points have an equilibrium point as limit point. A limit mapping exists, but is discontinuous, and need not have any fixed points.

THEO. 1: Every finite game has an equilibrium point.

Proof: Using our standard notation, let s be an n-tuple of mixed strategies, and $p_{i\alpha}(s)$ the pay-off to player i if he uses his pure strategy $\pi_{i\alpha}$ and the others use their respective mixed strategies in s. For each integer λ we define the following continuous functions of s:

$$q_i(s) = \max_\alpha p_{i\alpha}(s),$$

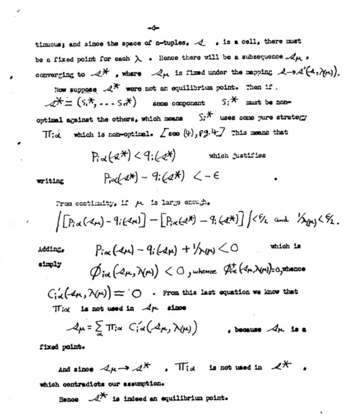
$$\phi_{i\alpha}(s,\lambda) = p_{i\alpha}(s) - q_i(s) + 1/\lambda, \text{ and}$$

$$\phi^+_{i\alpha}(s,\lambda) = \max[0, \phi_{i\alpha}(s,\lambda)].$$

Now $\sum_\alpha \phi^+_{i\alpha}(s,\lambda) \geq \max_\alpha \phi^+_{i\alpha}(s,\lambda) = 1/\lambda > 0$ so that

$$C'_{i\alpha}(s,\lambda) = \frac{\phi^+_{i\alpha}(s,\lambda)}{\sum_\beta \phi^+_{i\beta}(s,\lambda)}$$ is continuous.

Define $s'_i(s,\lambda) = \sum_\alpha \pi_{i\alpha} C'_{i\alpha}(s,\lambda)$ and $s'(s,\lambda) = (s'_1, s'_2, \ldots s'_n)$. Since all the operations have preserved continuity, the mapping $s \to s'(s,\lambda)$ is con-



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tinuous; and since the space of n-tuples, s, is a cell, there must be a fixed point for each λ. Hence there will be a subsequence s_μ, converging to s^*, where s_μ is fixed under the mapping $s \to s'(s, \lambda(\mu))$.

Now suppose s^* were not an equilibrium point. Then if $s^* = (s_1^*, \ldots s_n^*)$ some component s_i^* must be non-optimal against the others, which means s_i^* uses some pure strategy $\pi_{i\alpha}$ which is non-optimal. [see (4), pg. 4] This means that

$$p_{i\alpha}(s^*) < q_i(s^*)$$ which justifies writing $$p_{i\alpha}(s^*) - q_i(s^*) < -\epsilon.$$

From continuity, if μ is large enough,

$$\left| [p_{i\alpha}(s_\mu) - q_i(s_\mu)] - [p_{i\alpha}(s^*) - q_i(s^*)] \right| < \epsilon/2 \text{ and } 1/\lambda(\mu) < \epsilon/2.$$

Adding, $p_{i\alpha}(s_\mu) - q_i(s_\mu) + 1/\lambda(\mu) < 0$ which is simply $\phi_{i\alpha}(s_\mu, \lambda(\mu)) < 0$, whence $\phi^+_{i\alpha}(s_\mu,\lambda(\mu)) = 0$, whence $C'_{i\alpha}(s_\mu, \lambda(\mu)) = 0$. From this last equation we know that $\pi_{i\alpha}$ is not used in s_μ since

$$s_\mu = \sum_\alpha \pi_{i\alpha} C'_{i\alpha}(s_\mu, \lambda(\mu))$$, because s_μ is a fixed point.

And since $s_\mu \to s^*$, $\pi_{i\alpha}$ is not used in s^*, which contradicts our assumption.

Hence s^* is indeed an equilibrium point.

- In 1994, Nash was awarded the Nobel Prize for this work