IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2016

MEng Honours Degree in Electronic and Information Engineering Part IV
MEng Honours Degree in Mathematics and Computer Science Part IV
MEng Honours Degrees in Computing Part IV
MSc in Advanced Computing
MSc in Computing Science
MSc in Computing Science (Specialist)
MRes in Advanced Computing
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute

PAPER C422

COMPUTATIONAL FINANCE

Wednesday 23 March 2016, 10:00
Duration: 120 minutes

Answer THREE questions

Paper contains 4 questions
Calculators required

1 a Describe Macaulay duration, modified duration, and convexity for fixed income securities.

b Show that under continuous compounding the Macaulay duration is given by

$$D = \frac{\sum_{k=0}^{n} t_k c_k e^{-\lambda t_k}}{P}$$

where P is the present value of the bond, λ is the yield, c_k denotes the k^{th} coupon paid at time t_k. Find the derivative $(dP/d\lambda)$ of the present value (P) with respect to λ and write it in terms of D and P. What is the financial meaning of $dP/d\lambda$?

c Consider a bond that has a coupon rate of c paid m times a year and a yield of $\lambda > 0$. Show that under discrete compounding the limiting value of duration, as maturity is increased to infinity, is given by,

$$\frac{1 + \lambda/m}{\lambda},$$

where λ is the yield of the bond, and m is the number of coupon payments per year.

d Find the convexity of a zero-coupon bond maturing at time T under continuous compounding.

e Show that when the yield of a bond is equal to its coupon rate then its present value is equal to its face value. Prove this for a bond with price P and face value F that makes m coupon payments of C/m per year, and with n remaining periods.

The five parts carry, respectively, 20%, 20%, 10%, 25%, and 25% of the marks.

2a Suppose that an investor is interested in computing the weights of a portfolio of n stocks such that it tracks a specified portfolio as closely as possible – in the sense of minimising the variance of the difference in returns. In particular, suppose that the target portfolio has a (random) rate of return denoted by r_m.The n assets have random rates of return given by $r_1, r_2, \ldots, r_n$. We wish to the find the portfolio rate of return,

$$r = \sum_{i=1}^{n} \alpha_i r_i$$

with $\sum_{i=1}^{n} \alpha_i = 1$ such that $\text{var}(r - r_m)$ is minimised. State (but do not solve) the necessary conditions that α_i, $i = 1, \ldots, n$ must satisfy.

b Suppose that the investor has the additional requirement of obtaining an average return of $\bar{r}_p$. State (but do not solve) the necessary conditions that α_i, $i = 1, \ldots, n$ must satisfy in this case.

c Suppose that an investor constructs a portfolio of n assets, where the weight of each asset is $1/n$. Suppose that all assets have mean μ, variance σ^2 and that covariance between asset i and j is given by $\text{cov}(r_i, r_j) = 0.3\sigma^2$ for $i \neq j$. Show that no matter how large n is, the variance of this portfolio can never be lower than $0.3\sigma^2$.

d Consider a European call option on a non-dividend paying stock. The strike price is K, the time to expiration is T, and the price of one unit of a zero-coupon bond maturing at time T is $B(T)$. Denote the price of the call by $C(S, T)$ show that

$$C(S, T) \geq \max(0, S - KB(T)).$$

Hint: Consider a portfolio of one call, a share of stock, financed by selling K bonds.

e A perpetual call option is one that never expires. Such an option must be of American style. Use part (d) to show that the value of a perpetual call on a non-dividend paying stock is $C = S$, where S is the current price of the stock.

The five parts carry, respectively, 15%, 10%, 15%, 25%, and 35% of the marks.

3 Your company is considering to lease its computer servers over a period of three years. The servers in total provide 100 units of CPU time per year, and each unit costs your company £85 in costs (electricity, staff costs etc). The current price per unit of CPU time is £100, but the future prices are subject to uncertainty. These prices are represented by a binomial lattice. Each year the price either increases by a factor of 1.2 (with probability $p = 0.75$) or decreases by a factor of 0.9 (with probability $1 - p = 0.25$). The term structure of interest rates is assumed to be flat at 10%. We assume that the price obtained from leasing the computer servers in a given year is the price that was observed at the beginning of the year, but all cash flows occur at the end of the year. You may assume that there exists a liquid market for buying and selling CPU time, just like any other financial asset.

a How does the value of the servers change qualitatively when the probability p increases? Justify your answer (no calculation is required).

b Determine the arbitrage free price of the lease. Fully explain the steps in your solution.

The two parts carry, respectively, 25% and 75% of the marks.

4a Utility functions in finance are required to satisfy the following conditions

$$\frac{dU(x)}{dx} > 0,$$
$$\frac{d^2U(x)}{dx^2} \leq 0,$$

(i) Explain why these two conditions are desirable.
(ii) Under what conditions are two utility functions equivalent?
(iii) Describe three different ways to obtain a utility function for a particular investor.

b Consider a simple market consisting of only two risky assets and a risk-free asset. Relevant data about the risky assets is provided in the following table.

	Price/Share	Mean Return	No. of Shares	StDev. of Return
Asset 1	£20	15%	1,000	2.5%
Asset 2	£60	55%	150	32.5%

The covariance between the assets is 50%. Assume that all investors share the indicated predictions of means, variances, and covariances of the asset returns. If the market is in equilibrium in the sense of the Capital Asset Pricing Model, what is the one fund of efficient risky assets?

c Consider the following three propositions:

(i) : I flip a coin. If it is heads, you are paid £3; if it is tails you are paid £0. It will cost you £1 to participate in this proposition. You may enter this proposition repeatedly, and at any level, and the payoffs will be scaled accordingly.

(ii) : You may keep your money in your pocket and earn no interest.

(iii) : I flip a coin three times. If at least two of the flips are heads, you are paid £27; otherwise 0.

How much is proposition (iii) worth?

The three parts carry, respectively, 25%, 25%, and 50% of the marks.