IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2014-2015

BEng Honours Degree in Computing Part III
BEng Honours Degree in Electronic and Information Engineering Part III
MEng Honours Degree in Electronic and Information Engineering Part III
MEng Honours Degree in Mathematics and Computer Science Part IV
BEng Honours Degree in Mathematics and Computer Science Part III
MEng Honours Degree in Mathematics and Computer Science Part III
MEng Honours Degrees in Computing Part III
MSc in Computing Science (Specialist)
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute

PAPER C343

OPERATIONS RESEARCH

Wednesday 17 December 2014, 10:00
Duration: 120 minutes

Answer THREE questions

Paper contains 4 questions
Calculators required

1 a Consider the following linear programming (LP) problem.

$$\begin{array}{ll} \text{minimise} & z = 2x_1 + 3x_2 \\ \text{subject to} & \frac{1}{2}x_1 + \frac{1}{4}x_2 \leq 4 \\ & x_1 + 3x_2 \geq 20 \\ & x_1 + x_2 = 10 \\ & x_1, x_2 \geq 0 \end{array}$$

Solve the LP using the two-phase simplex method, justifying at each step the choice of the pivot. Indicate if the optimal solution found is unique.

b Consider a set $\mathcal{S} = \{x_1, x_2, \ldots, x_V\}$ of V points in $\mathbb{R}^n$. We say that a point $x_i \in \mathcal{S}$ is a *convex combination* of the other points if there exist weights $\lambda_j \geq 0$, $j \neq i$, such that

$$x_i = \sum_{\substack{j=1..V \\ j \neq i}} \lambda_j x_j, \qquad \sum_{\substack{j=1..V \\ j \neq i}} \lambda_j = 1.$$

A point $x_i \in \mathcal{S}$ that is **not** a convex combination of the other points is called an *extreme point*.

i) Assume that you know all the points in $\mathcal{S}$. Write a linear program to decide if a given point $x_i \in \mathcal{S}$ is an extreme point or not, explaining how to interpret the optimal solution. (**Note**: you cannot use binary or integer variables.)

The two parts carry equal marks.

2a Consider a min-max problem of the form:

$$\begin{array}{ll} \text{minimise} & \phi(x) \\ \text{subject to} & Ax = b, \\ & x \geq 0 \end{array}$$

where $\phi(x) = \max_{i=1,\ldots,I} \left\{ c(i)^T x \right\}$, $c(i) \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$.

i) Write a linear program (LP) to solve the min-max problem when $I = 2$ and

$$A = \begin{bmatrix} 10 & 5 \\ 5 & 9 \end{bmatrix}, \qquad b = \begin{bmatrix} -2 \\ 5 \end{bmatrix}, \qquad c(i) = [5 - i, 3 - i]$$

ii) Prove that the optimal solution of the linear program is also optimal for the min-max problem.

b Consider the following mixed integer linear program

$$\begin{array}{ll} \text{minimise } z = & -8x_1 - 5x_2 \\ \text{subject to} & x_1 + x_2 + x_3 = 2p \\ & 9x_1 + 5x_2 + x_4 = 15p \\ & x_1, x_2, x_3, x_4 \geq 0 \\ & x_1, x_2 \in \mathbb{N}_0 \end{array}$$

where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and $p \in \mathbb{R}_0^+$ is a parameter.

i) Assume $p = 3$. The LP relaxation of this program has optimal tableau

BV	z	x_1	x_2	x_3	x_4	RHS
z	1	0	0	1.25	0.75	−41.25
x_1	0	1	0	−1.25	0.25	3.75
x_2	0	0	1	2.25	−0.25	2.25

Specify a Gomory cut for the variable x_1.

ii) For $p = 3$, determine the shadow prices for rows 1 and 2 in the LP relaxation. How would you use the shadow prices to determine the value function of this LP?

The two parts carry equal marks.

3 a Eve is testing MM, a memory manager, on a tablet with 8 memory banks of 1024 MB each. Eve runs on her test 5 applets of type A, 7 of type B, and 1 of type C.

Each A applet allocates 256MB on a single memory bank. Each B applet allocates 64MB on a single memory bank. Applet C uses 768MB of a single memory bank and also allocates an object cache, distributed on one or more of the other banks. The object cache size is decided by MM and its global size is between 128MB and 512MB. Applets of the same type can either use the same bank or different banks. Also, B applets cannot use a bank used by the C applet or its object cache.

MM has to decide: (i) the bank that each applet should use; (ii) the amount of memory and banks used by the object cache. Unused banks are switched *off* by MM. In case there are multiple possible solutions, MM prefers to minimize memory fragmentation: thus if a bank is *on*, it prefers it to be fully utilised.

Formulate the decision of MM on Eve's test scenario as a mixed integer linear program. Treat memory size as a continuous quantity. Do *not* solve the program.

b i) Discuss the maximum number of iterations for the standard simplex algorithm for a general problem with m rows and n columns. Include in your answer a comment on the significance of the Klee-Minty cube.

ii) Prove finite termination of the standard simplex algorithm for linear programs without degenerate basic feasible solutions. Then, give an example of Bland's rule.

The two parts carry, respectively, 60% and 40% of the marks.

4a i) Consider the following (primal) linear program (P):

$$\begin{aligned} \max z = \quad & x_4 \\ \text{subject to} \quad & x_4 \leq x_2 - x_3 \\ & x_4 \leq -x_1 + x_3 \\ & x_4 \leq x_1 - x_2 \\ & x_1 + x_2 + x_3 = 1 \\ & x_1, x_2, x_3 \geq 0 \\ & x_4 \text{ free} \end{aligned}$$

Derive the dual linear program (D) describing all steps.

ii) Consider now a zero-sum game in mixed strategy and call it G. If D is the linear program for the column player and P is the linear program for the row player, could you give a payoff matrix for the game G? Is this matrix unique?

iii) Does game G have a Nash equilibrium in pure strategies? If so, determine it. Otherwise, explain why it does not exist.

iv) Does game G have a Nash equilibrium in mixed strategies? Do *not* determine any equilibrium, but justify in details your answer.

b Solve the following problem using the branch and bound technique:

$$\begin{aligned} \text{maximise } y = \quad & 2x_1 + x_2 \\ \text{subject to} \quad & 5x_1 + 2x_2 \leq 8 \\ & x_1 + x_2 \leq 3 \\ & x_1, x_2 \geq 0 \\ & x_1 \in \{0, 1\} \end{aligned} \qquad \text{(LP)}$$

Provide in your answer the details of the calculations.

The two parts carry equal marks.