Tutorial 2 - 60016 Operations Research

Basic Solutions and Basic Representations

Exercise 1 *Find the basic solutions (BS) of the system of equations:*

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & 3 \\ 3x_1 & + & 2x_2 & + & 4x_3 & = & 10 \end{array}$$

Solution Every pair of columns of the matrix of coefficients forms a basis for $\mathbb{R}^2$, thus there are three basic solutions. Set $x_1 = 0$, obtain as basic solution:

$$(0, 1, 2),$$

For $x_2 = 0$, we obtain

$$(2, 0, 1)$$

Set $x_3 = 0$, we obtain the following basic solution:

$$(4, -1, 0)$$

Exercise 2. Consider the following optimisation problem:

$$\max y = x_1 + 3x_2 \tag{1a}$$

subject to

$$2x_1 + x_2 \leq 4 \tag{1b}$$
$$x_1 + 2x_2 \leq 4 \tag{1c}$$
$$x_1, x_2 \geq 0. \tag{1d}$$

(a) Bring the problem into standard form by introducing slack variables s_1 and s_2.

(b) For the problem in standard form, determine all basic solutions. Which of these solutions are feasible, and what are their objective values?

(c) Draw the feasible region of problem (1) in the (x_1, x_2)-plane. Where are the basic solutions from part (b)? Which feasible solutions satisfy $s_1 = 0$? Which feasible solutions satisfy $s_2 = 0$?

Solution

(a) The standard form is given by

$$-\min\ z = -x_1 - 3x_2$$

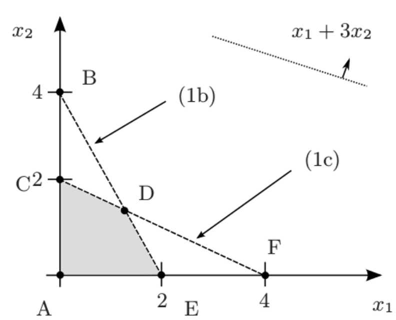
subject to

$$\begin{aligned} 2x_1 + x_2 + s_1 &= 4 \\ x_1 + 2x_2 + s_2 &= 4 \\ x_1, x_2, s_1, s_2 &\geq 0. \end{aligned}$$

(b) Since the constraint matrix A contains two linearly independent rows, we need to consider all pairs of linearly independent columns of A. The variables that correspond to these columns become the basic variables (BVs), whereas the remaining variables are non-basic variables (NBVs). We set the values of the non-basic variables to zero and solve the resulting 2×2-system of equations in the basic variables. This gives us a basic solution, which is feasible if and only if it is non-negative:

BVs	NBVs	(x_1, x_2, s_1, s_2)	feasible?	objective value
$\{x_1, x_2\}$	$\{s_1, s_2\}$	$(4/3, 4/3, 0, 0)$	yes	$-16/3$
$\{x_1, s_1\}$	$\{x_2, s_2\}$	$(4, 0, -4, 0)$	no	-4
$\{x_1, s_2\}$	$\{x_2, s_1\}$	$(2, 0, 0, 2)$	yes	-2
$\{x_2, s_1\}$	$\{x_1, s_2\}$	$(0, 2, 2, 0)$	yes	-6
$\{x_2, s_2\}$	$\{x_1, s_1\}$	$(0, 4, 0, -4)$	no	-12
$\{s_1, s_2\}$	$\{x_1, x_2\}$	$(0, 0, 4, 4)$	yes	0

(c) The feasible region looks as follows.



The basic solution correspond to the letters in the graph:

$\{x_1, x_2\}$	$\{x_1, s_1\}$	$\{x_1, s_2\}$	$\{x_2, s_1\}$	$\{x_2, s_2\}$	$\{s_1, s_2\}$
D	F	E	C	B	A

All solutions on line segment DE are feasible and satisfy $s_1 = 0$. Similarly, all solutions on line segment DC are feasible and satisfy $s_2 = 0$.

Exercise 3. Consider the basic solution from Exercise 1 (b) that has x_1 and x_2 as basic variables.

(a) Determine the basic representation for this basic solution.

(b) Is this basic solution optimal? Justify your answer both graphically (see Exercise 1 (c)) and from the basic representation!

(c) Find a non-basic variable such that increasing its value improves the objective value. How much can we increase the value of this variable without leaving the feasible region? Which is the resulting basic solution? Is this solution optimal?

Solution

(a) The basic representation is:

$$\begin{array}{ccccccccc} z & & & + & 1/3s_1 & - & 5/3s_2 & = & -16/3 \\ & & x_2 & - & 1/3s_1 & + & 2/3s_2 & = & 4/3 \\ & x_1 & & + & 2/3s_1 & - & 1/3s_2 & = & 4/3 \end{array}$$

(b) The first line is equivalent to $z = -1/3s_1 + 5/3s_2$ - 16/3. Since we want to minimise z, the solution is *not* optimal: by increasing s_1, we can decrease z. The solution corresponds to point D in the graph; for the chosen objective function, point C results in a better objective value.

(c) As already mentioned, increasing s_1 improves the objective value. By which amount λ can we increase s_1 while keeping $s_2 = 0$ constant? We need to guarantee that all variables (apart from z) remain non-negative:

constraint	allowed increase	BV that becomes zero
$x_2 = 4/3 + 1/3\lambda$	any	none
$x_1 = 4/3 - 2/3\lambda$	$\lambda \leq 2$	x_1

Hence, we set $s_1 = 2$. This causes x_1 to become a non-basic variable. The new basic representation is:

$$\begin{array}{cccccccccc} z & - & 1/2x_1 & & & & & - & 3/2s_2 & = & -6 \\ & & 1/2x_1 & + & x_2 & & & + & 1/2s_2 & = & 2 \\ & & 3/2x_1 & & & + & s_1 & - & 1/2s_2 & = & 2 \end{array}$$

In the graph, this new solution is represented by point C. Both the graph and the basic representation indicate that this solution is optimal.

Exercise 4. Entrepreneur S. McDuck runs a 24h supermarket. Since the number of customers per hour varies with the time of the day, the number of required staff does so, too. McDuck decides to establish three shifts that cover different time periods: midnight to noon, 6.00am to 6.00pm and noon to midnight. His estimates for the staff required for each shift are as follows.

	shift			
time period	1	2	3	required staff
midnight – 6.00am	X			2
6.00am – noon	X	X		4
noon – 6.00pm		X	X	5
6.00pm – midnight			X	4

McDuck is planning to pay £48 per day and employee for shift 1, while he has to pay £72 per day and employee for the other two shifts.

McDuck wants to determine a personnel schedule (i.e., how many staff to hire for which shift) that satisfies the stated minimum staff requirements at the lowest daily cost. Formulate the corresponding linear program! For simplicity, assume that employees are 'continuously divisible.'

Solution Let x_i denote the number of staff hired for shift i, $i = 1, 2, 3$. Then the optimisation problem can be formulated as follows.

$$\min z = 48x_1 + 72x_2 + 72x_3$$

subject to

$$\begin{aligned} x_1 &\geq 2 \\ x_1 + x_2 &\geq 4 \\ x_2 + x_3 &\geq 5 \\ x_3 &\geq 4 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

The constraints ensure satisfaction of the staff requirements in the four different time periods.

Exercise 5 *(Exam 2015, Q1b).* Consider a set $\mathcal{S} = \{x_1, x_2, \ldots, x_V\}$ of V points in $\mathbb{R}^n$. We say that a point $x_i \in \mathcal{S}$ is a *convex combination* of the other points if there exist weights $\lambda_j \geq 0$, $j \neq i$, such that

$$x_i = \sum_{\substack{j=1..V \\ j\neq i}} \lambda_j x_j, \qquad \sum_{\substack{j=1..V \\ j\neq i}} \lambda_j = 1.$$

A point $x_i \in \mathcal{S}$ that is **not** a convex combination of the other points is called an *extreme point.*

Assume that you know all the points in $\mathcal{S}$. Write a linear program to decide if a given point $x_i \in \mathcal{S}$ is an extreme point or not, explaining how to interpret the optimal solution. (You cannot use binary or integer variables.)

Solution

$$\begin{array}{ll} \min & \lambda_i \\ \text{s.t.} & x_i = \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_V x_V, \\ & \sum_{v=1}^{V} \lambda_v = 1 \\ & \lambda_v \geq 0 \qquad v = 1, \ldots, V \end{array} \tag{LP}$$

Note that the variables are λ_i, and the x_i are vectors of constant coefficients.

Let λ_i^* be the i-th coordinate of the optimal solution of (LP). If $\lambda_i^* = 0$ at optimality, then x_i can be expressed as a convex combination of the other points, by the definition given in the problem statement.

If $\lambda_i^* = 1$ at optimality, then the constraints imply the identity $x_i = x_i$ and x_i cannot be expressed as a convex combination. This is true because otherwise we would have some $\lambda_j^* > 0$, $j \neq i$, that define the convex combination, which by the constraints would imply $\lambda_i^* < 1$ against the assumption.

Consider now the case $0 < \lambda_i^* < 1$. We show that this case cannot occur as it leads to a contradiction. In this case, since $\lambda_i^* \neq 1$ we can rewrite the constraints evaluated at the optimal solution as

$$x_i = \sum_{j \neq i} \lambda'_j x_j, \qquad \sum_{j \neq i} \lambda'_j = 1 \tag{2}$$

where $\lambda'_j = \lambda_j^*/(1 - \lambda_i^*) \geq 0$. Note that (2) may be seen as a particular instance of the constraints of the LP in which we have set $\lambda'_i = 0$. This means that the point at coordinates $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_V)$, with $\lambda'_i = 0$, by (2) will satisfy the constraints of (LP). Therefore, it is feasible for (LP).

If λ' is feasible for (LP), then it must also be optimal since $\lambda'_i = 0$ and we are minizing on the i-th coordinate of the solution vector. Thus it is impossible that the considered point with $0 < \lambda_i^* < 1$ is optimal. We have thus shown a contradiction, implying that the case $0 < \lambda_i^* < 1$ never occurs at optimality.