Interest Rate Risk Modeling

The Fixed Income Valuation Course

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- **Interest Rate Risk Modeling : The Fixed Income Valuation Course.** Sanjay K. Nawalkha, Gloria M. Soto, Natalia K. Beliaeva, 2005, Wiley Finance.
  - **Chapter 2: Bond Price, Duration, and Convexity**

- Goals:
  - Introduce the concepts of duration and convexity.
  - Show how the relation between interest rate and bond price can be described by duration and convexity.
  - Discuss common fallacies concerning duration and convexity.
  - Provide closed-form formulas for duration and convexity.

Chapter 2: Bond Price, Duration, and Convexity

- Introduction
- Bond Price under Continuous Compounding
- Duration
- Convexity
- Common Fallacies Concerning Duration and Convexity
- Formulas for Duration and Convexity

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Risk of Fixed Income Securities

- Maturity: approximates the interest rate risk of a default-free bond.

- Duration: a more appropriate measure of risk of a default-free bond; it's the weighted average maturity of the bond.

- Convexity: captures the *non-linear* relationship between bond returns and the changes in interest rates.

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The Future Value of a Single Sum Formula

- The future value of a single sum formula is given as:

$$FV_t = PV\left(1+\frac{APR_k}{k}\right)^{t\times k} \qquad (2.1)$$

- Federal regulations require that all quotes of interest rates be given by using an APR.
- APR_k is the Annual Percentage Rate with *k* compounding periods over one year.

An example: power of compounding

- Consider a student who in a desperate moment borrows \$1,000 for one year from a pawnbroker at an APR_{12} of 300%.

- The money that she is going to return is:

$$FV_t = 1000\left(1+\frac{300\%}{12}\right)^{1\times 12} = 1000\left(1+0.25\right)^{12} = \$14,551.92 \qquad (2.2)$$

- In contrast, if she is borrowing at an APR_1 of 300%, the return amount will be just \$4000.

Continuous compounding, where interest on interest is paid out continuously, allows mathematical tractability

- The most common compounding frequencies in the fixed income markets are annual, semi-annual, monthly, and daily.

- Let *y* represent the *APR* assuming continuous compounding. Then by using the compounding rule, equation (2.1) can be rewritten as:

$$FV_t = \lim_{k \to \infty} PV \left(1 + \frac{y}{k}\right)^{t \times k} \qquad (2.3)$$

Continuous Compounding Formulas

- When k goes to infinity, applying the exponential constant e, (2.3) is written as:

$$FV_t = PV \cdot e^{yt} \tag{2.4}$$

- If an *APR* is quoted with a compounding frequency k, then an equivalent *APR* under continuous compounding can be given as:

$$y = \ln\left(1 + \frac{APR_k}{k}\right) \times k \tag{2.5}$$

Continuous Compounding Formulas

- So given APR_{12} (i.e., with monthly compounding), the continuously-compounded APR is given as:

$$y = \ln\left(1 + \frac{APR_{12}}{12}\right) \times 12 \qquad (2.6)$$

- By (2.4), the present value of a single sum is:

$$PV = \frac{FV_t}{e^{yt}} \qquad (2.7)$$

Bond Price under Continuous Compounding

- By applying the present value rule to every cash flow of a bond, the price of a bond with a periodic coupon *C* paid *k* times a year, and face value *F*, is:

$$P = \frac{C}{e^{yt_1}} + \frac{C}{e^{yt_2}} + \frac{C}{e^{yt_3}} + \ldots + \frac{C}{e^{yt_N}} + \frac{F}{e^{yt_N}} \qquad (2.8)$$

where t_1, t_2, t_3, …,t_N are the *N* cash flow payment dates of the bond.

Bond Price under Continuous Compounding

- Assuming the bond matures at time $tN = T$, and the time intervals between all cash flow payments are equal, then $N = Tk$, and $t_1 = 1/k$, $t_2 = 2/k$, $t_3 = 3/k$, …, $t_N = N/k$, the bond price can be express by:

$$P = \frac{C}{e^{i} - 1}\left[1 - \frac{1}{e^{Ni}}\right] + \frac{F}{e^{Ni}} \qquad (2.9)$$

where $i = y/k$ is the continuously compounded *APR* divided by k.

Price Formulas for Annuities

- Annuities like the mortgage loans do not make a lump-sum payment at the maturity date, then the annuity formula (F=0) is given as:

$$P = \frac{C}{e^{i} - 1}\left[1 - \frac{1}{e^{Ni}}\right] \tag{2.10}$$

- Perpetuities are annuities with infinite life. The perpetuity formula (N=infinity) is given as:

$$P = \frac{C}{e^{i} - 1} \tag{2.11}$$

Bond Price under Continuous Compounding: An example

- Example 2.1: Consider a 30-year home-equity loan with 360 monthly payments (i.e., 30 X 12 = 360) of $100. Suppose that the quoted *APR* with monthly compounding for the loan is 6% and we wish to calculate *y*, the continuously-compounded *APR*.

  Using (2.5), this yield is calculated as:

$$y = \ln(1 + APR_k / k) \times k = \ln(1 + 0.06 / 12) \times 12 = 0.0598505$$

Bond Price under Continuous Compounding: An example

- The present value of the loan can be computed in two different ways.
- (1) Using the discrete monthly rate = *APR*/12 = 0.06/12 = 0.005, the loan's present value is given as:

$$P = \frac{100}{(1+0.005)} + \frac{100}{(1+0.005)^2} + ... + \frac{100}{(1+0.005)^{360}}$$

$$= \$16,679.16$$

Bond Price under Continuous Compounding: An example

- (2) Using the continuously-compounded yield, y = 0.0598505, the loan's present value is given by (2.8) as follows:

$$P = \frac{100}{e^{(0.0598505)\times(1/12)}} + \frac{100}{e^{(0.0598505)\times(2/12)}} + \ldots + \frac{100}{e^{(0.0598505)\times(360/12)}}$$
$$= \$16{,}679.16$$

- Since both approaches give identical answers, we can use the second approach based upon continuous-compounding, which turns to be more tractable mathematically.

Bond Price under Continuous Compounding: An example

- We do not have to do a summation of the 360 terms as shown above. The present value of the mortgage loan can be computed directly by using (2.10), with C = \$100, $i = y/12 = 0.0598505/12 = 0.00498754$, and $N = 360$, as follows:

$$P = \frac{\$100}{e^{0.00498754} - 1}\left[1 - \frac{1}{e^{360 \times 0.00498754}}\right] = \$16{,}679.16$$

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Duration

- Duration is the weighted-average maturity of a bond, where weights are the present values of the bond's cash flows, given as proportions of bond's price:

$$D = \sum_{t=t_1}^{t=t_N} t \; w_t \qquad (2.13)$$

$$w_t = \left\{ \frac{C_t}{e^{ty}} \right\} / P \qquad (2.14)$$

Duration

- Under continuous compounding, the duration measure gives the negative of the percentage price change of a bond, divided by an infinitesimally small change in the yield of a bond:

$$D = -\lim_{\Delta y \to 0} \frac{\Delta P / P}{\Delta y} = -\frac{\partial P / \partial y}{P} \qquad (2.15)$$

- The duration measure defined by (2.13) and (2.15) increases with maturity, decreases with coupon rate, and decreases with the yield.

Duration

- By using an approximation of (2.15), we have the following expression for the percentage change in the bond price:

$$\frac{\Delta P}{P} \cong -D\Delta y \qquad (2.16)$$

- (2.16) assumes a parallel and infinitesimal shift in the yield curve. Non-parallel shifts in the yield curve result in unequal changes in the yields for different bonds, which invalidates using the above definition of duration.

Duration and Maturity

- By definition, the magnitude of duration is always less than or equal to the maturity of the bond. However, this is true only for securities such as bonds that have non-negative cash flows.

- If one or more of the cash flows are negative (when computing the duration of fixed income derivatives), then duration may exceed the maturity of the underlying security, or may even be negative..

Computing Duration: Example 2.2

- Example 2.2: Consider a 5-year bond with $1,000 face value. The bond makes annual coupon payments at a 10% coupon rate. Assume that the continuously-compounded annualized yield of this bond equals 5% (i.e., $y = \ln(1+APR) = 5\%$).

- The following table shows how to compute the price and duration of this bond.

Computing Duration: Example 2.2

Table 2.1

Maturity	Cash flows	Present value of the cash flow	Weight of the cash flow	Product of weight and maturity	Product of present value and maturity
t	C	$PV = C/e^{t \cdot y}$	$w = PV/P$	$w \cdot t$	$PV \cdot t$
1	$100	$95.12	0.079	0.079	95.12
2	$100	$90.48	0.075	0.149	180.97
3	$100	$86.07	0.071	0.213	258.21
4	$100	$81.87	0.068	0.271	327.49
5	$100	$77.88	0.064	0.322	389.40
5	$1,000	$778.80	0.643	3.218	3,894.00
Total		P = $1,210.23	1.000	D = 4.251	5,145.20

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Computing Duration: Example 2.2

- The sum of the present values of the cash flows gives the bond price. The duration of the bond is given by (2.13) as:

$$D = \sum_{t=1}^{t=N} t \; w_t$$

and shown at the bottom of the fifth column as D = 4.251 years.

Computing Duration: Example 2.2

- The last column of the table allows us to compute duration in a different way. Substituting (2.14) into (2.13) gives:

$$D = \frac{1}{P}\sum_{t=1}^{t=N}\frac{tC_t}{e^{ty}}$$

- Therefore, duration can be also computed by dividing the sum of the products of the present value and the maturity of each cash flow by the bond price. This gives again:

$$D = \frac{5145.20}{1210.23} = 4.251 \text{ years}$$

Computing Duration: Duration of a Portfolio

- The duration of a bond portfolio is a weighted average of the durations of the bonds in the portfolio, where the weights are defined as the proportions of investments in the bonds.

- Consider another bond B with a maturity of 10 years and a coupon rate of 10%. Using the same yield, the price of bond B is $1,373.96 and its duration is 7.257 years. The duration of bond B is longer since it has a longer maturity.

Computing Duration: Duration of a Portfolio

- Now consider a bond portfolio including one bond A and two bonds B. The portfolio value is thus 1210.23 + 2 (1373.96) = $3,958.15. The portfolio duration is computed as follows:

$$D_{PORT} = \frac{1210.23}{3958.15} \times 4.251 + \frac{2 \times 1373.96}{3958.15} \times 7.257 = 6.338 \text{ years}$$

Durations for different bonds

- In general, higher coupon rate gives a higher bond price, and thus a lower duration (ceteris paribus), since the weight of the earlier cash flows (i.e., coupons) gives more weight to lower maturities.

Price change and Duration: Example 2.3

- Example 2.3: Reconsider the $1,000 face value, 10% coupon rate, 5-year bond introduced in the previous example. As shown above this bond has a price of $1,210.23 and a duration equal to 4.251 years. Now suppose that interest rates increase (due to unexpected news on inflation) so that the yield of the bond rises up to 6%.

Price change and Duration: Example 2.3

- The new bond price consistent with the new yield can be computed using (2.8) as follows:

$$P_{new} = \frac{100}{e^{0.06}} + \frac{100}{e^{0.06\times 2}} + \frac{100}{e^{0.06\times 3}} + \frac{100}{e^{0.06\times 4}} + \frac{1100}{e^{0.06\times 5}} = \$1{,}159.96$$

- The percentage bond price change is given as:

$$\frac{\Delta P}{P} = \frac{P_{new} - P_{old}}{P_{old}} = \frac{-50.27}{1210.23} = -0.04154 = -4.154\%$$

Price change and Duration: Example 2.3

- Using the duration risk measure, we can approximate the percentage change in the bond price using (2.16), as follows:

$$\frac{\Delta P}{P} \cong -D\Delta y = -4.251 \times 0.01 = -0.04251 = -4.251\%$$

- The percentage price change approximated by duration is very close to the true percentage price change. The difference between actual and approximated percentage price change is -4.154% - (-4.251%) = 0.097%

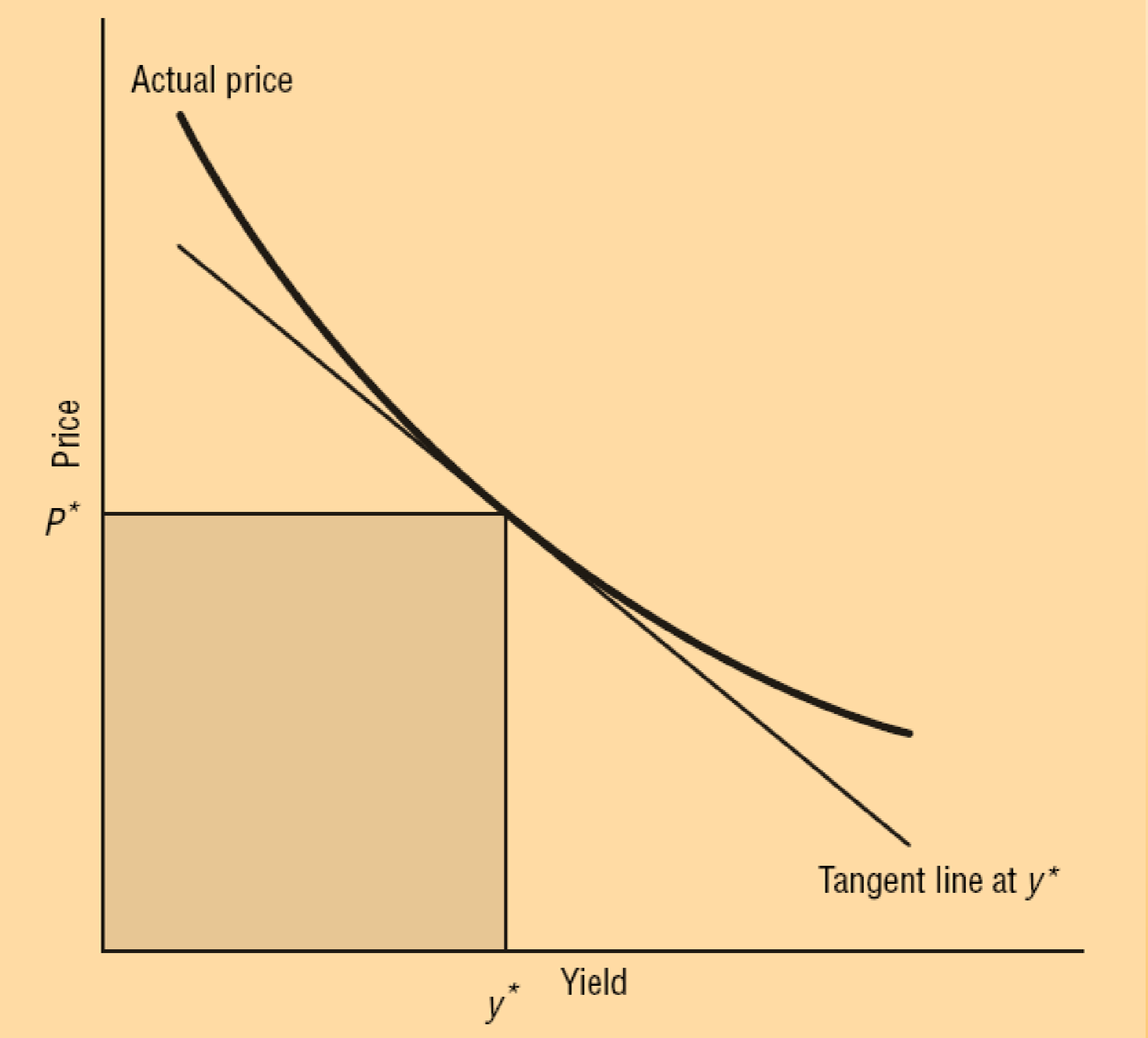
Price change and Duration: Example 2.3

- In the above example, if the yield decreased to 4%, then the bond price would have increased to $1,262.9, giving a percentage bond price change equal to 4.352%.

- In this case, the difference between the actual and estimated percentage change would have been 4.352% - 4.251% = 0.101%.

- Whether the yield increases or decreases, the actual minus the estimated percentage price change is *always positive.* This is due to the so-called “convexity” of the bond, which is related to the curvature of the bond price-yield relationship shown in Figure 2.1.

Figure 2.1 Bond price/yield relationship with tangent line



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Convexity

- Convexity is given as the weighted-average of maturity-squares of a bond, where weights are the present values of the bond's cash flows, given as proportions of bond's price. Convexity can be mathematically expressed as follows:

$$CON = \sum_{t=t_1}^{t=t_N} t^2 w_t \qquad (2.17)$$

Convexity

- Under continuous compounding, the convexity measure is obtained as the second derivative of the bond price with respect to the yield of the bond, divided by the bond price:

$$CON = \frac{\partial^2 P / \partial y^2}{P} \qquad (2.18)$$

Convexity

- For large changes in the interest rates, the definitions of duration and convexity in (2.15) and (2.18), respectively, are used to derive a two-term Taylor series expansion for approximating the percentage change in the bond price as follows:

$$\frac{\Delta P}{P} \cong -D\Delta y + \frac{1}{2} CON(\Delta y)^2 \qquad (2.19)$$

Convexity

- (2.19) suggests that for bonds with identical durations, higher convexity is always preferable.

- However, the above result is based upon the assumption of a large and parallel shift in the yield curve. Not only are large and parallel shifts in the yield curve inconsistent with arbitrage-free term structure dynamics, such shifts occur rarely in the bond markets.

Convexity

- Bond convexity increases with maturity, decreases with coupon rate, and decreases with yield.
- By definition, the magnitude of convexity is always less than the square of the maturity of the bond. If the underlying security has one or more negative cash flows then convexity may exceed maturity-square, or may even be negative.

Convexity: Example 2.4

- Example 2.4: The convexity of a bond is computed identically to the duration of a bond, except that the longevity of each cash flow is replaced by the longevity squared. The following table gives the convexities of the three bonds considered in Example 2.2.

Convexity: Example 2.4

- Table 2.2 Bonds’ characteristics

Bond	Maturity (years)	Annual coupon rate (%)	Yield to maturity (%)	Price ($)	Duration (years)	Convexity (years squared)
A	5	10	5	1,210.23	4.251	19.797
B	10	10	5	1,373.96	7.257	63.162
C	5	12	5	1,296.52	4.161	19.172

- As can be seen from Table 2.3, convexity increases with maturity and decreases with coupon rate.

Convexity: Convexity of Portfolios

- The convexity of a bond portfolio can be calculated as the weighted average of the convexities of the individual bonds. For the portfolio of bonds A and B in Example 2.2, the convexity of the portfolio is given as the weighted-average of the individual bond convexities:

$$CON_{PORT} = \frac{1210.23}{3958.15} \times 19.797 + \frac{2 \times 1373.96}{3958.15} \times 63.162 = 49.903$$

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Common Fallacies Concerning Duration and Convexity

1. Duration measures the sensitivity of a bond's price to changes in its yield, and is thus given by the (negative of the) slope of the plot of bond price versus bond yield.

2. Duration decreases (increases) as bond yield increases (decreases) – this property holds for all option-free bonds.

3. Duration is the steepness of the tangent line in Figure 2.1. The steeper the tangent line, the greater the duration; the flatter the tangent line, the lower the duration.

Common Fallacies Concerning Duration and Convexity

4. Yield-induced changes in duration accelerate (decelerate) changes in prices as yields decrease (increase). This is why absolute and percentage price changes are greater when yields decline than when they increase by the same number of basis points.

5. Bond convexity is a second order measure of the sensitivity of a bond's price to changes in its yield, and is thus given by the curvature (i.e., rate of change of slope) of the plot of bond price versus bond yield.

Common Fallacies Concerning Duration and Convexity

6. Bond convexity is the rate of change of duration as yields change.

7. Bond convexity decreases (increases) as bond yield increases (decreases) – this property holds for all option-free bonds.

- Each and every one of the above statements is false unless accompanied by additional assumptions or restrictions. If the (flawed) intuition behind these statements is applied to bonds with embedded options it creates substantial confusion.

Common Fallacies Concerning Duration and Convexity

- Simple Counter Examples
- Explanation of the Fallacies
- Applications to Callable Bonds
- A New Graph

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Simple Counter Examples

- Consider a five-year zero-coupon bond (i.e., a “zero”). The plot of bond price versus yield to maturity for the zero looks like that in Figure 2.1.

- A zero has duration equal to its maturity, so a five-year zero has *D* equal to five – regardless of its yield. It follows immediately that in the case of the zero, the changing slope of the plot in Figure 2.1 cannot be equal to the negative of the zero's Macaulay duration because the zero's *D* is fixed at five regardless of yield.

Simple Counter Examples

- Also, the convexity of a zero equals its maturity squared regardless of yield. However, the rate of change of the zero's duration with respect to its yield is zero (because the zero's duration does not change with yield). Thus bond convexity cannot be simply the sensitivity of duration to changes in yield.

- We also conclude that convexity (fixed at *CON*=25 for our five-year zero) need not change with changing yield (even though the curvature of bond price in Figure 2.1 decreases with increasing yield).

Simple Counter Examples

- This is not simply a matter of the given statements failing to hold for zeroes or failing to hold when using continuous yields.

- The seven casual statements mentioned before overlook the following interrelated facts:

  (1) The slope of Figure 2.1 is not (the negative of) duration.

  (2) The curvature (i.e., rate of change of slope) in Figure 2.1 does not illustrate changing duration.

  (3) The curvature (i.e., rate of change of slope) in Figure 2.1 is not bond convexity.

  (4) Changing curvature in Figure 2.1 does not illustrate changing convexity.

Common Fallacies Concerning Duration and Convexity

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Explanation of the Fallacies

- Based on the two-term Taylor series expansion of the bond return given in (2.19), we get:

$$\Delta P \cong -DP\Delta y + \frac{1}{2} CON \cdot P(\Delta y)^2 \qquad (2.20)$$

- It follows from (2.19) duration and convexity are directly related to the first two coefficients in a second order approximation of "instantaneous bond return" (i.e., $\Delta P/P$) with respect to change in the yield.

Explanation of the Fallacies

- Looking at (2.20), we find that the roles of duration and convexity are "contaminated" by price level. Indeed, this can be shown by the first and second order derivatives of bond price with respect to the yield, given as:

$$\frac{\partial P}{\partial y} = -DP \qquad (2.21)$$

$$\frac{\partial^2 P}{\partial y^2} = CON \cdot P \qquad (2.22)$$

Explanation of the Fallacies

- These equation say that the slope of the plot in Figure 2.1 is not *-D*, but *-DP* (-"dollar duration") and the curvature (i.e., rate of change of the slope) of the plot in Figure 2.1 is not *CON*, but $CON \cdot P$ ("dollar convexity").

- That is, when you relate price to yield, price contaminates the roles of duration and convexity.

- We suspect that many of the earlier statements drawn from the literature refer to dollar duration and dollar convexity respectively.

Explanation of the Fallacies

- The definition in (2.21) also yields the following:

$$\partial D / dy = D^2 - CON \qquad (2.23)$$

- Thus the sensitivity of duration to changes in yield is not convexity, but the difference between duration-squared and convexity.

- Changing slope in Figure 2.1 illustrates changing dollar duration and this does not necessarily tell us anything about duration *D*. Similarly, the curvature in Figure 2.1 illustrates dollar convexity (i.e., $CON{\cdot}P$) and this differs substantially from convexity, *CON*. See Table 2.3.

Explanation of the Fallacies

Table 2.3 Numerical Examples

Panel A - Five Year Zero-Coupon Bond with Face 100							
Yield	y	0.00	0.05	0.10	0.15	0.20	0.25
Price	P	100.00	77.88	60.65	47.24	36.79	28.65
Duration	D	5.00	5.00	5.00	5.00	5.00	5.00
Convexity	CON	25.00	25.00	25.00	25.00	25.00	25.00
Slope	$-DP$	-500.00	-389.40	-303.27	-236.18	-183.94	-143.25
Curvature	$CON \cdot P$	2500.00	1947.00	1516.33	1180.92	919.70	716.26
Panel B – Five Year 15% Annual-Coupon Bond with Face $100							
Yield	y	0.00	0.05	0.10	0.15	0.20	0.25
Price	P	175.00	142.59	116.77	96.14	79.61	66.33
Duration	D	4.14	4.05	3.94	3.83	3.71	3.59
Convexity	CON	19.00	18.38	17.70	16.99	16.23	15.44
Slope	$-DP$	-725.00	-577.08	-460.45	-368.37	-295.58	-237.95
Curvature	$CON \cdot P$	3325.00	2620.30	2067.36	1633.21	1292.08	1023.84

Explanation of the Fallacies

- It can be seen that duration does not change with yield for a zero-coupon bond, but duration does decrease slowly with increasing yield for a coupon-bearing bond;

- Dollar duration decreases rapidly with increasing yield regardless of coupon rate; convexity does not change with yield for a zero-coupon bond, but convexity does decrease slowly with increasing yield for a coupon-bearing bond;

Explanation of the Fallacies

- Dollar convexity decreases rapidly with increasing yield regardless of coupon rate;

- And finally, other things being equal duration and convexity decrease with increasing coupon level, but dollar duration and dollar convexity increase with increasing coupon level.

Common Fallacies Concerning Duration and Convexity

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- Explanation of the Fallacies
- Applications to Callable Bonds
- A New Graph

Applications to Callable Bonds

- Let us now take the more complicated example of a security with an embedded option – a callable zero-coupon bond – to illustrate how misleading these statements can be if applied more generally.

- Consider a $100 face value 10-year zero-coupon bond that is callable (European-style) in one year at 80% of it face value. Figure 2.2 plots the bond's price, duration, and dollar duration as a function of yield.

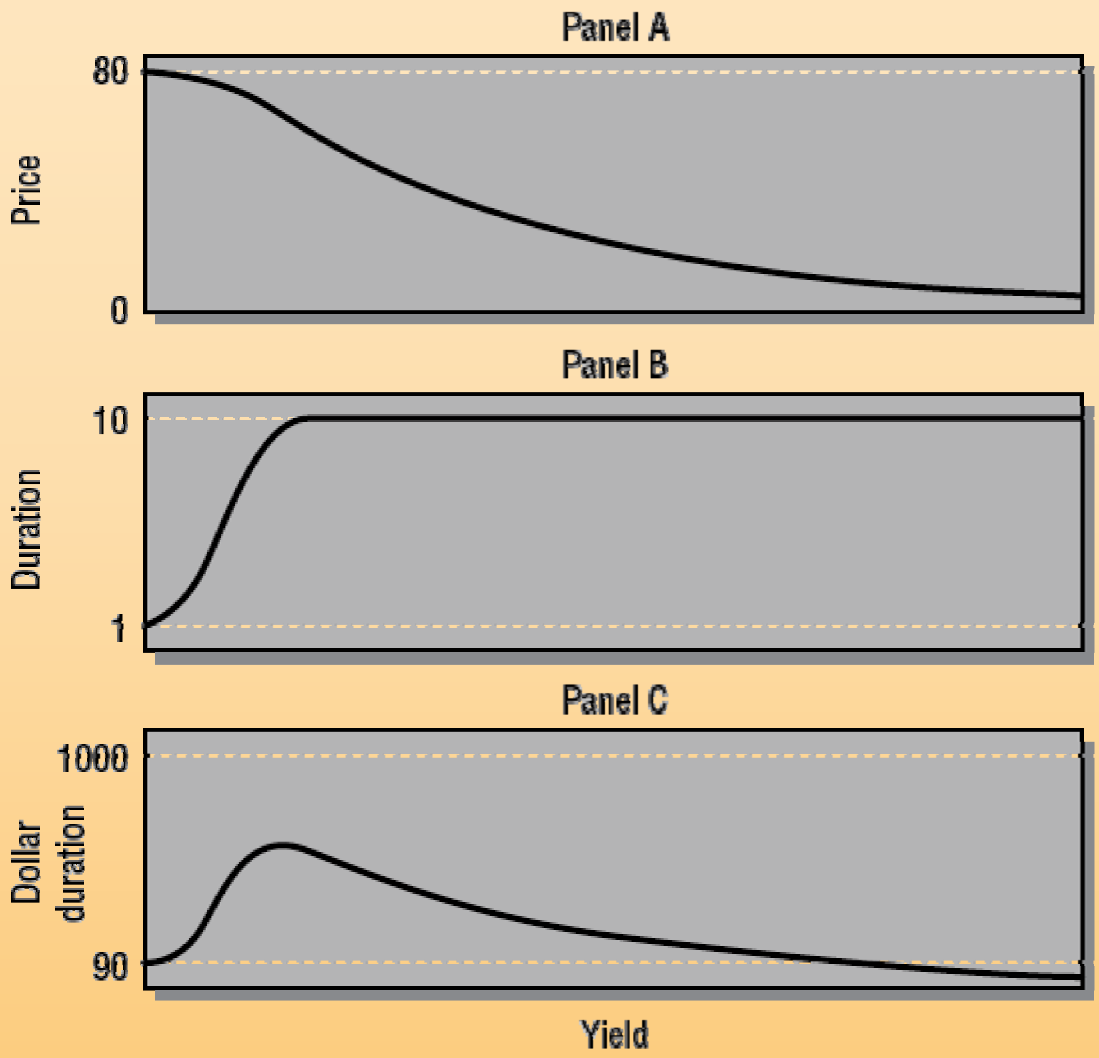
Applications to Callable Bonds

- The bond price as a function of yield first steepens, and then flattens as yield increases (see Panel A of Figure 2.2).

- Inferring duration from the slope in Panel A of Figure 2.2 implies *incorrectly* that duration first increases and then decreases as yield rises – whereas, the duration of the callable bond is monotonically increasing in yield (see Panel B of Figure 2.2). The correct inference is that it is dollar duration that first increases and then decreases with increasing yield (see Panel C of Figure 2.2).

- We conclude that inferring duration from the slope of the price-yield relationship causes substantial confusion in the case of a callable bond.

Figure 2.2 Callable zero-coupon bond



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A New Graph

- Two ways to reduce these problems come immediately to mind: either plot log price against yield, or plot instantaneous return (i.e., $\Delta P/P$) against yield.

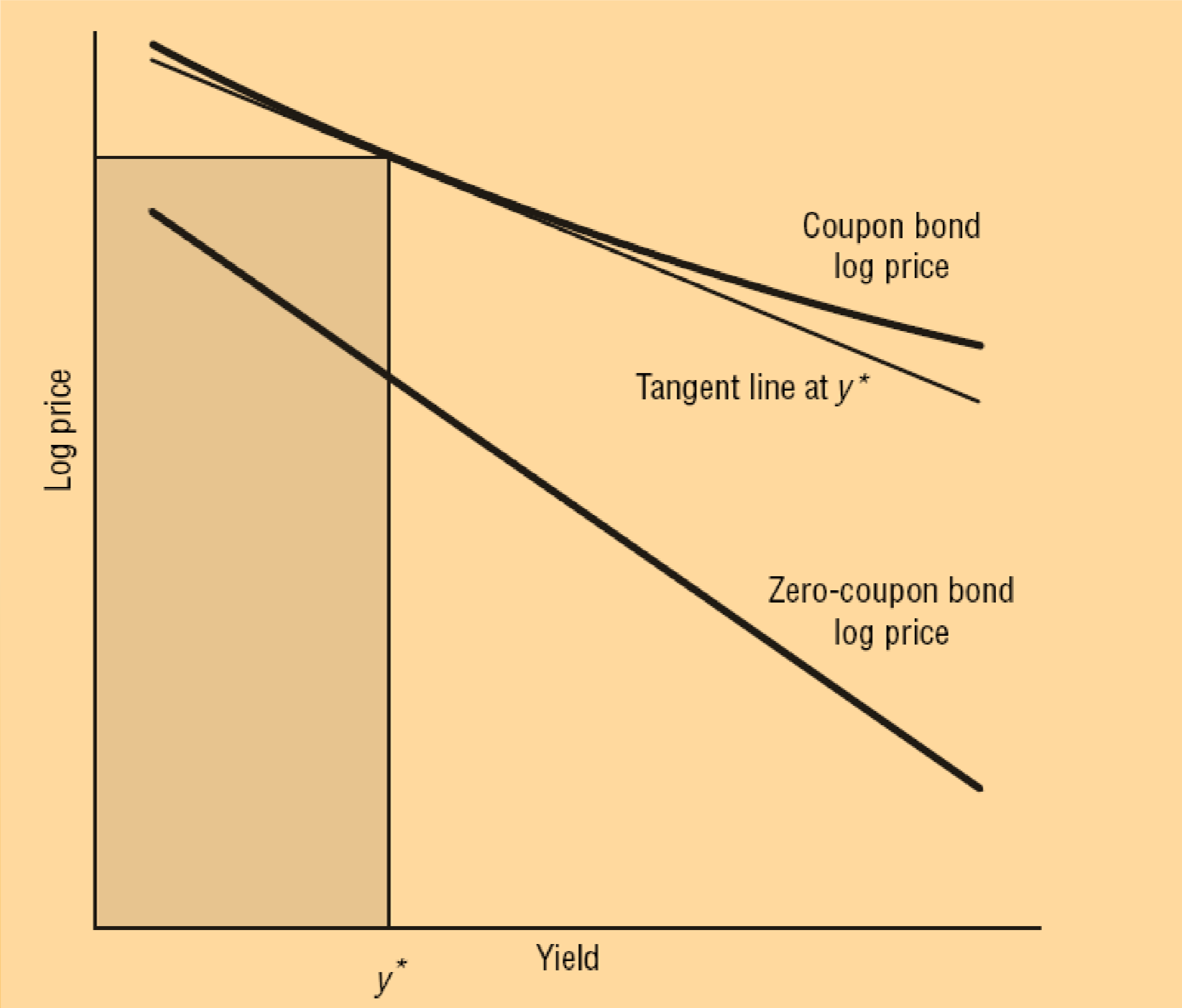
- Suppose we follow the first approach and plot log price against yield as shown in Figure 2.3. In this case the slope of the plot is easily shown to be $-D$, where D is the duration of the bond. The change in slope can be shown to be $-(D^2 - CON)$.

A New Graph

- The vertical change along the plot from the initial point is the log of 1 plus the instantaneous return on the bond. The plot is unbounded below (because log of the price is unbounded below as $P \rightarrow 0$).

- In the special case of a zero-coupon bond (where $CON = D^2$ so the change in slope is zero), the plot of log price versus yield is a straight line (see Figure 2.3).

Figure 2.3 Log price versus yield



A New Graph

- The absolute value of the slope in Figure 2.3 is duration, and duration does not change very much with changing yield. Contrast this with the traditional plot (Figure 2.1) where the slope is $-DP$, which does vary a lot with changing yield because P varies a lot.

- Unfortunately, even though the slope of the log price versus yield graph is equal to $-D$, the curvature (i.e., change in slope) of the graph does not give convexity. To address this shortcoming, we recommend a second approach that plots instantaneous return against yield (see Figure 2.4).

A New Graph

- The instantaneous bond return is the instantaneous price change divided by initial price, $(P - P^*)/P^*$. The slope of the tangent line *at the initial yield* is easily shown to be -*D*. The rate of change of slope *at the initial yield* is easily shown to be *CON*.

- Given that the value of the slope and its rate of change are -*D* and *CON* respectively, this plot is more appropriate than plotting log price versus yield (it also avoids the potential confusion arising from log price being unbounded below).

Figure 2.4 Instantaneous return versus yield

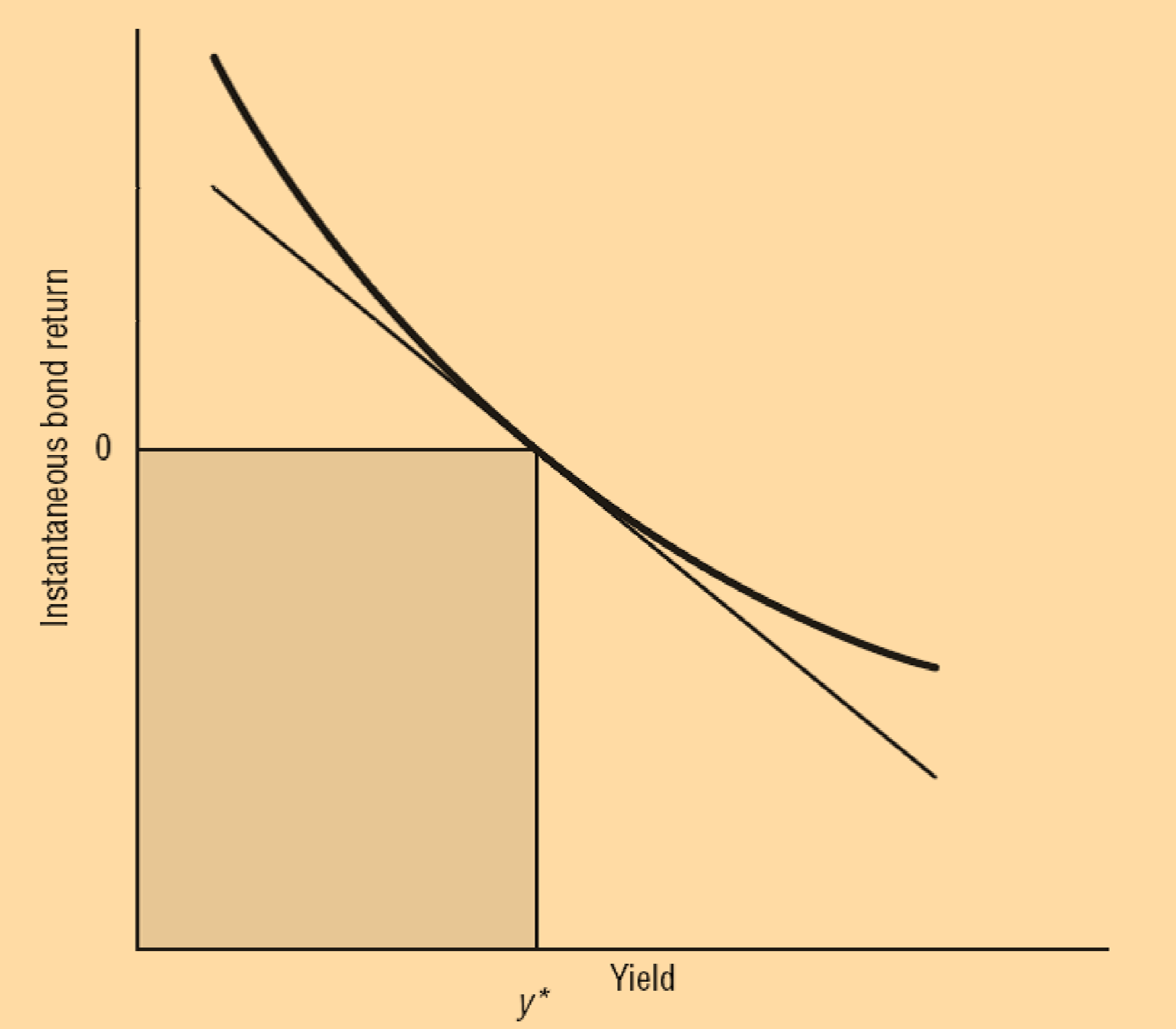


Figure 2.5 Instantaneous return versus yield for different initial yields

For different initial yields, the curve in Figure 2.4 “slides” sideways as shown in Figure 2.5:

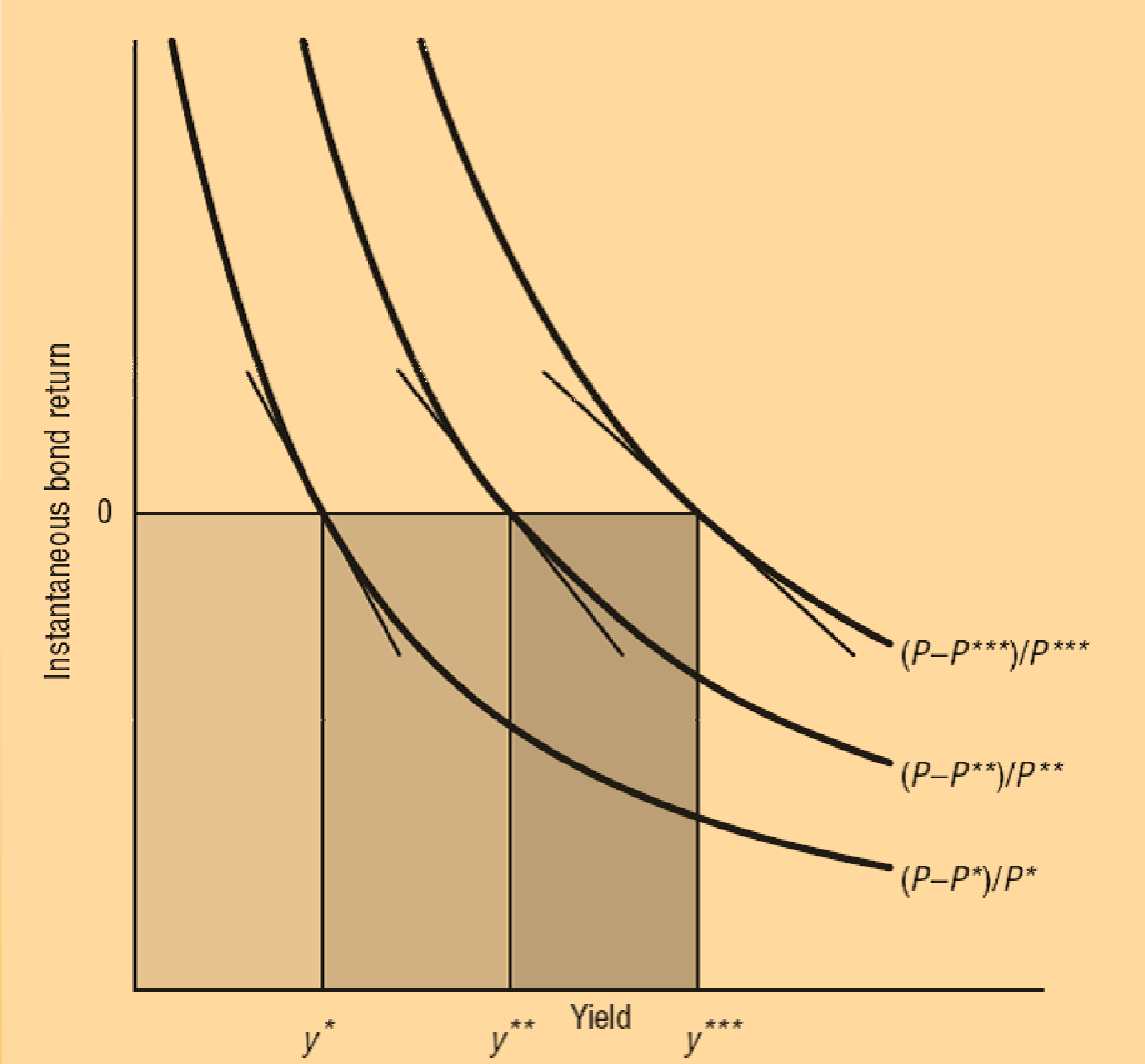
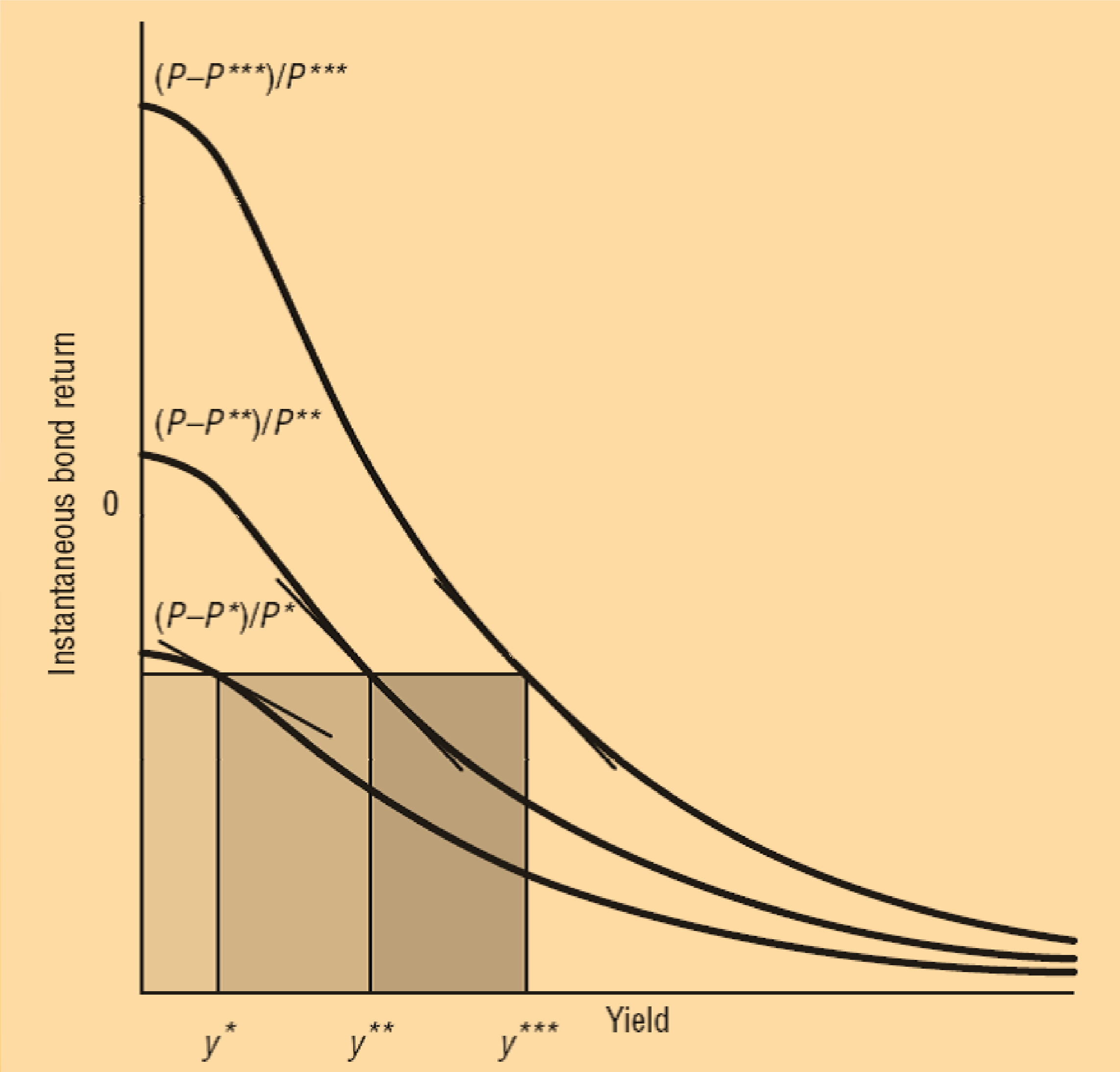


Figure 2.6 Instantaneous return versus yield for callable zero



A New Graph

- Figure 2.6 presents the plot of instantaneous return versus yield for the callable zero-coupon bond that we discussed earlier. It can be seen that the duration (absolute value of slope at initial yield) increases for increasing initial yield – as in Panel B of Figure 2.2 – and the convexity (change in slope at initial yield) goes from negative to zero to positive as initial yield increases.

- The monotonically increasing relationship between the duration of a callable zero-coupon bond and its yield cannot be inferred directly from the traditional price-yield graph such as Panel A of Figure 2.2 – thus illustrating the importance of our new plot.

Chapter 2: Bond Price, Duration, and Convexity

- Introduction
- Bond Price under Continuous Compounding
- Duration
- Convexity
- Common Fallacies Concerning Duration and Convexity
- Formulas for Duration and Convexity

Formulas for Duration and Convexity

- Duration and Convexity Formulas for Regular Bonds

- Duration and Convexity Formulas for Annuities and Perpetuities

Formulas for Duration and Convexity

- Duration and Convexity Formulas for Regular Bonds

- Duration and Convexity Formulas for Annuities and Perpetuities

Duration and Convexity Formulas for Regular Bonds

- Consider a bond which makes a periodic coupon payment of C dollars k times a year. The face value of the bond is F dollars, and its continuously-compounded annualized yield equals y.
- The bond has a total of N cash flows remaining. Let s define the time elapsed since the last coupon payment date *in the units of the time interval between coupon payments*. At the coupon payment dates, $s = 0$.
- Further, define $c = C/F$ as the periodic coupon rate, and $i = y/k$, the continuously compounded annualized yield divided by k.

Duration and Convexity Formulas for Regular Bonds

- The formula for the duration of a regular bond between coupon payment dates is given as follows:

$$D = \frac{c\left[e^{Ni}\left(e^{i}(1-s)+s\right)-e^{i}\right]+\left(e^{i}-1\right)(N-s)\left(e^{i}-1-c\right)}{k\left[c\left(e^{i}-1\right)\left(e^{Ni}-1\right)+\left(e^{i}-1\right)^{2}\right]}$$

(2.24)

- For the special case s=0, the above formula simplifies to:

$$D_{s=0} = \frac{ce^{i}\left(e^{Ni}-1\right)+N\left(e^{i}-1\right)\left(e^{i}-1-c\right)}{k\left[c\left(e^{i}-1\right)\left(e^{Ni}-1\right)+\left(e^{i}-1\right)^{2}\right]} \qquad (2.25)$$

Duration and Convexity Formulas for Regular Bonds

- The formula for the convexity of a regular bond between coupon payment dates is given as follows:

$$CON = \frac{c\left\{e^{Ni}\left[\left(e^{i}(1-s)+s\right)^{2}+e^{i}\right]-e^{i}\left[2\left(1+(N-s)\left(e^{i}-1\right)\right)+e^{i}-1\right]\right\}}{k^{2}\left[c\left(e^{i}-1\right)^{2}\left(e^{Ni}-1\right)+\left(e^{i}-1\right)^{3}\right]}$$

$$+\frac{\left(e^{i}-1\right)^{2}(N-s)^{2}\left(e^{i}-1-c\right)}{k^{2}\left[c\left(e^{i}-1\right)^{2}\left(e^{Ni}-1\right)+\left(e^{i}-1\right)^{3}\right]} \qquad (2.26)$$

$$CON_{s=0} =$$

$$\frac{c\left\{e^{i}\left(1+e^{i}\right)\left(e^{Ni}-1\right)-2\cdot N\cdot e^{i}\left(e^{i}-1\right)\right\}+N^{2}\left(e^{i}-1\right)^{2}\left(e^{i}-1-c\right)}{k^{2}\left[c\left(e^{i}-1\right)^{2}\left(e^{Ni}-1\right)+\left(e^{i}-1\right)^{3}\right]}$$

(2.27)

Duration and Convexity Formulas for Regular Bonds

- The duration formulas given above adjust for compounding frequency, and are given in annualized units (i.e., number of years). Hence, they can be directly used for approximating the percentage price change in (2.16), given a change Δy in the continuously compounded annualized yield.

- The convexity formulas given above also adjust for compounding frequency, and are given in annualized units (i.e., number of years squared). Hence, they can be directly used for approximating the percentage price change in (2.19), given a squared change $(\Delta y)2$ in the continuously-compounded annualized yield.

Formulas for Duration and Convexity

- Duration and Convexity Formulas for Regular Bonds
- Duration and Convexity Formulas for Annuities and Perpetuities

Duration and Convexity Formulas for Annuities and Perpetuities

- Consider an annuity with a periodic cash flow of C dollars made k times a year, a continuously compounded annualized yield equal to y, and N cash flows remaining.

- Let s define the time elapsed since the last annuity payment date *in the units of the time interval between the annuity payments*. So, at the annuity payment dates, $s = 0$.

- Let $i = y/k$, the continuously-compounded annualized yield divided by k.

Duration and Convexity Formulas for Annuities and Perpetuities

- The formula for the duration of an annuity between the payment dates is given as follows:

$$D = \left[\frac{e^i}{e^i - 1} - \frac{N}{e^{Ni} - 1} - s \right] / k \qquad (2.28)$$

$$D_{s=0} = \left[\frac{e^i}{e^i - 1} - \frac{N}{e^{Ni} - 1} \right] / k \qquad (2.29)$$

Duration and Convexity Formulas for Annuities and Perpetuities

- The formula for the convexity of an annuity between coupon payment dates is given as follows:

$$CON = \left[\frac{(e^i(1-s)+s)^2 + e^i}{(e^i-1)^2} + \frac{(e^i-1)(s^2-(N-s)^2)-2Ne^i}{(e^i-1)(e^{Ni}-1)} \right] / k^2 \qquad (2.30)$$

$$CON_{s=0} = \left[\frac{e^i(1+e^i)}{(e^i-1)^2} + \frac{-Ne^i(N+2)+N}{(e^i-1)(e^{Ni}-1)} \right] / k^2 \qquad (2.31)$$

- The duration and convexity formulas given in this section all adjust for compounding frequency, and are given in annualized units.

Duration and Convexity Formulas for Annuities and Perpetuities

- A perpetuity is an annuity with an infinite number of cash flows. The duration and convexity formulas for perpetuity at and between coupon payment dates can be obtained by a simple inspection of the above equations, and substitution of N = infinity as follows:

$$D = \left[\frac{e^i}{e^i - 1} - s \right] / k, \quad D_{s=0} = \left[\frac{e^i}{e^i - 1} \right] / k \qquad (2.32), (2.33)$$

$$CON = \left[\frac{(e^i(1-s)+s)^2 + e^i}{(e^i - 1)^2} \right] / k^2 \qquad (2.34)$$

$$CON_{s=0} = \left[\frac{e^i(1+e^i)}{(e^i - 1)^2} \right] / k^2 \qquad (2.35)$$

Duration and Convexity Formulas for Annuities and Perpetuities: Example 2.5

- Example 2.5: Reconsider the 5-year, 10% annual coupon bond A with a continuously-compounded yield of 5% given in Example 2.2. In that example, we obtained the bond's duration as 4.251 years. In Example 2.4, we obtained the same bond's convexity as 19.797. These values can be computed using the closed-form formulas given above. Since the bond matures in *exactly* 5 years, the closed-form formulas with s=0 is applied.

Duration and Convexity Formulas for Annuities and Perpetuities: Example 2.5

- Using (2.25) and (2.27), the bond A's duration and convexity are calculated as:

$$D_{s=0} = \frac{0.1 \times e^{0.05}\left(e^{0.05\times 5} - 1\right) + 5\left(e^{0.05} - 1\right)\left(e^{0.05} - 1.1\right)}{0.1\left(e^{0.05} - 1\right)\left(e^{0.05\times 5} - 1\right) + \left(e^{0.05} - 1\right)^2} = 4.251 \text{ years}$$

$$CON_{s=0} =$$

$$\frac{0.1\left\{e^{0.05}\left(1 + e^{0.05}\right)\left(e^{0.05\times 5} - 1\right) - 10 \times e^{0.05}\left(e^{0.05} - 1\right)\right\} + 5^2\left(e^{0.05} - 1\right)^2\left(e^{0.05} - 1.1\right)}{0.1\left(e^{0.05} - 1\right)^2\left(e^{0.05\times 5} - 1\right) + \left(e^{0.05} - 1\right)^3} = 19.797$$

- As expected, the two formulas give the same values obtained in earlier examples.

Duration and Convexity Formulas for Annuities and Perpetuities: Example 2.5

- Now, consider the duration and convexity of this bond after 9 months. Assume that the yield to maturity is still 5%. Since the bond has not paid any coupons, the number of coupons before maturity remains 5 (N=5), and the first coupon is due in 3 months. The time elapsed since the date of the last coupon relative to time between two coupon payments is s = 9 months / 12 months = 0.75 years.

Duration and Convexity Formulas for Annuities and Perpetuities: Example 2.5

- To calculate the duration and convexity we use the formulas given in (2.24) and (2.26) respectively as follows:

$$D = \frac{0.1\left[e^{0.05\times 5}\left(0.25\times e^{0.05}+0.75\right)-e^{0.05}\right]+4.25\times\left(e^{0.05}-1\right)\left(e^{0.05}-1.1\right)}{0.1\left(e^{0.05}-1\right)\left(e^{0.05\times 5}-1\right)+\left(e^{0.05}-1\right)^{2}} = 3.501\text{ years}$$

$$CON = \frac{0.1\left\{e^{0.05\times 5}\left[\left(0.25\times e^{0.05}+0.75\right)^{2}+e^{0.05}\right]-e^{0.05}\left[2\left(1+4.25\left(e^{0.05}-1\right)\right)+e^{0.05}-1\right]\right\}}{0.1\left(e^{0.05}-1\right)^{2}\left(e^{0.05\times 5}-1\right)+\left(e^{0.05}-1\right)^{3}}$$

$$+\frac{\left(e^{0.05}-1\right)^{2}4.25^{2}\left(e^{0.05}-1.1\right)}{0.1\left(e^{0.05}-1\right)^{2}\left(e^{0.05\times 5}-1\right)+\left(e^{0.05}-1\right)^{3}} = 13.982$$

Interest Rate Risk Modeling

The Fixed Income Valuation Course

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