Tutorial 5 – 60016 Operations Research

Duality & Shadow Prices

Exercise 1. Consider the following optimisation problem.

$$
\begin{array}{ll}
\underset{\mathbf{x},\mathbf{y},\mathbf{z}}{\text{minimise}} & \mathbf{c_1}^\top \mathbf{x} + \mathbf{c_2}^\top \mathbf{y} + \mathbf{c_3}^\top \mathbf{z} \\
\text{subject to} & \mathrm{A}_1\mathbf{x} + \mathrm{B}_1\mathbf{y} + \mathrm{C}_1\mathbf{z} \leq \mathbf{b_1} \\
& \mathrm{A}_2\mathbf{x} + \mathrm{B}_2\mathbf{y} + \mathrm{C}_2\mathbf{z} = \mathbf{b_2} \\
& \mathrm{A}_3\mathbf{x} + \mathrm{B}_3\mathbf{y} + \mathrm{C}_3\mathbf{z} \geq \mathbf{b_3} \\
& \mathbf{x}_1, \ldots, \mathbf{x}_k \geq 0 \\
& \mathbf{y}_1, \ldots, \mathbf{y}_k \in \mathbb{R} \\
& \mathbf{z}_1, \ldots, \mathbf{z}_k \leq 0
\end{array}
$$

Where capital letters denote matrices and bold lowercase letters denote vectors and

$$\mathbf{x} \in \mathbb{R}^k, \mathbf{y} \in \mathbb{R}^k, \mathbf{z} \in \mathbb{R}^k.$$

You can think about this whole problem as having $3k$ variables. Also

$$\mathbf{b_1} \in \mathbb{R}^l, \mathbf{b_2} \in \mathbb{R}^l, \mathbf{b_3} \in \mathbb{R}^l.$$

You may assume all other vectors and matrices are of the appropriate dimensions. Dualise this problem.

Exercise 2. Write the dual formulation of the following (primal) linear program:

@ 11:30

(OR)

$$
\begin{array}{ll}
\text{minimise} & -x_1 + x_2 \\
\text{subject to} \ (1) & 2x_1 + x_2 \leq 10 \\
& 3x_1 + 7x_2 \geq 20 \\
& x_1, x_2 \geq 0
\end{array}
$$

Now write the dual formulation of the following (primal) linear program. What is the objective value for the new primal and dual problem? How are the primal solutions related? How are the dual solutions related?

$$
\begin{array}{ll}
\text{minimise} & -x_1 + x_2 \\
\text{subject to} & 4x_1 + 2x_2 \leq 20 \quad \leftarrow \text{this is } 2\times(1) \\
& 3x_1 + 7x_2 \geq 20 \\
& x_1, x_2 \geq 0
\end{array}
$$

Generalising, what is the dual formulation of the following (primal) linear program if $\mu > 0$? What is the new objective value and how are the primal and dual solutions related?

(M)

$$
\begin{array}{ll}
\text{minimise} & -x_1 + x_2 \\
\text{subject to} & 2 \cdot \mu x_1 + \mu x_2 \leq \mu \cdot 10 \\
& 3x_1 + 7x_2 \geq 20 \\
& x_1, x_2 \geq 0
\end{array}
$$

$$\begin{cases} \min\ -x_1 + x_2 \\ \text{s.t. } \mu 2x_1 + x_2\mu \le 10\mu \\ 3x_1 + 7x_2 \ge 20 \\ x \ge 0 \end{cases} \Rightarrow \begin{cases} \max\ x_1 - x_2 \\ \text{s.t: } \mu 2x_1 + \mu x_2 \le 10\mu \\ -3x_1 - 7x_2 \le -20 \\ x \ge 0 \end{cases}$$

(P) $\max\{c^T x \text{ s.t. } Ax \le b,\ x \ge 0\}$

(D) $\min\{b^T y \text{ s.t } A^T y \ge c,\ y \ge 0\}$

$c = (1, -1)$ $\quad A = \begin{pmatrix} 2\mu & 1\mu \\ -3 & -7 \end{pmatrix}$ $\quad b = \begin{pmatrix} 10\mu \\ -20 \end{pmatrix}$

$$\begin{cases} -\min\ \mu 10 y_1 - 20 y_2 \\ \text{s.t.: } \mu 2 y_1 - 3y_2 \ge 1 \\ \mu y_1 - 7y_2 \ge -1 \\ y_1, y_2 \ge 0 \end{cases}$$

A:- primal solution of (OR) is same of (M)

- let $\mu y_1 = \bar{y}_1$, dual program with $\bar{y}_1$ has same solution of (OR), so that $y_1 = \dfrac{\bar{y}_1}{\mu}$ ← solut original (OR)

↑ new solution (M)

The dual solution y_1 of (M) is equal $\frac{1}{\mu}$ dual solution y_1 of (OR)

y_2 remains unchanged

Exercise 3. Consider the following (primal) linear program (P):

$$\begin{array}{ll} \max z = & x_4 \\ \text{subject to} & x_4 \leq x_2 - x_3 \\ & x_4 \leq -x_1 + x_3 \\ & x_4 \leq x_1 - x_2 \\ & x_1 + x_2 + x_3 = 1 \\ & x_1, x_2, x_3 \geq 0 \\ & x_4 \text{ free} \end{array}$$

Derive the dual linear program (D).

Exercise 4. Let:

$$v(b) = \min\left\{c^T x \mid Ax = b,\, x \geq 0\right\},$$

with $c,\, x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$. If the vector b is perturbed by a vector $\epsilon \in \mathbb{R}^m$, we consider the perturbed problem:

$$v(b+\epsilon) \min\left\{c^T x \mid Ax = b + \epsilon,\, x \geq 0\right\}.$$

In general, the solution of the latter problem does not have the same basic representation, and hence the optimal basis matrix, as the former. Using the optimal basis matrix of the former, define its shadow prices and use the shadow prices to establish a relationship between $v(b)$ and $v(b+\epsilon)$.

Exercise 5. Construct an example of a primal problem that has no feasible solutions and whose corresponding dual also has no feasible solutions.

Exercise 6. Let $A \subseteq B \subseteq \mathbb{R}^n$, and $c, x \in \mathbb{R}^n$. Prove that

$$\min_{x \in B} c^T x \leq \min_{x \in A} c^T x.$$

Hint: using the definition of minimum and the properties of subsets is sufficient.