Tutorial 6 - 60016 Operations Research

Game theory

Exercise 1. Consider the following setting were a man and a woman try to decide if they will go to the opera, to the football match or stay home. This game is known as the "battle of the sexes" game.

		Man: Opera	Man: football	Man: home
Woman	Opera	5	-10	3
	football	9	4	5
	home	4	2	3

For the game,

1. is there a dominant strategy equilibrium? If so, find it.
2. is there a pure Nash equilibrium? If so, find it.
3. write down the linear program that the row player (woman) has to solve in order to find her best mixed strategy Nash equilibrium. Do not solve the linear program.

Solution.

1. Try to find the dominant equilibrium by sequentially eliminating dominated strategies.

Step 1: The man will never choose to go to the opera as the opera column is dominated by football column. Eliminate opera column for the man.

		Man: Opera	Man: football	Man: home
Woman	Opera	5	-10	3
	football	9	4	5
	home	4	2	3

		Man: football	Man: home
Woman	Opera	-10	3
	football	4	5
	home	2	3

Step 2: The woman will never choose to stay home as the home row is dominated by the football row. Eliminate home row for the woman.

		Man: football	Man: home
Woman	Opera	-10	3
	football	4	5
	home	2	3

		Man: football	Man: home
Woman	Opera	-10	3
	football	4	5

Step 3: The man will not choose to stay home as the home column is dominated by football column. Eliminate home column for the man.

		Man football	Man home
Woman	Opera	-10	3
	football	4	5

		Man football
Woman	Opera	-10
	football	4

Step 4: The woman will not choose to go to the opera as the opera row is dominated by the football row. Eliminate opera row for the woman.

		Man football
Woman	Opera	-10
	football	4

		Man football
Woman	football	4

Therefore, the dominant strategy is for both the man and woman to go the football match.

2. The Nash equilibrium in pure strategies can be found by the woman taking the max-min over the strategies and the man taking the min-max over the strategies. Therefore,

		Man Opera	Man football	Man home	
Woman	Opera	5	-10	3	minimum payoff -10
	football	9	4	5	minimum payoff 4
	home	4	2	3	minimum payoff 2

The max-min payoff for the woman is to go to the football match.

		Man Opera	Man football	Man home	
Woman	Opera	5	-10	3	minimum -10
	football	9	4	5	minimum 4
	home	4	2	3	minimum 2

For the man,

		Man Opera	Man football	Man home
Woman	Opera	5	-10	3
	football	9	4	5
	home	4	2	3
		maximum 9	maximum 4	maximum 5

The min-max payment for the man is to go to the football match.

		Man		
		Opera	football	home
Woman	Opera	5	-10	3
	football	9	4	5
	home	4	2	3
		maximum 9	maximum 4	maximum 5

Therefore, the pure Nash equilibrium is for both the man and the woman to go to the football match. Notice that the dominant strategy equilibrium and the Nash equilibrium is the same, i.e. if there is a dominant strategy equilibrium, then it is a Nash equilibrium as well.

3. For the mixed strategy, let the woman play her strategies with probability w_o, w_f, w_h and the man with probability m_o, m_f, m_h for the opera, football and home respectively. The woman needs to solve the following optimisation problem,

$$\begin{array}{lll} \underset{w_o,w_f,w_h}{\text{maximise}} & \underset{(m_o,m_f,m_h)\in\Delta}{\text{minimise}} & \sum_{i\in\{\text{o,f,h}\}} \sum_{j\in\{\text{o,f,h}\}} w_i m_j a_{ij} \\ & \text{subject to} & (w_o, w_f, w_h) \in \Delta \end{array}$$

where $\Delta = \{(w_o, w_f, w_h) : w_o + w_f + w_h = 1, w_o, w_f, w_h \geq 0\}$. By noticing that there is always an optimal solution (m_o, m_f, m_h) to the inner minimisation problem with $(m_o, m_f, m_h) \in \{0, 1\}$ (see lecture notes), we can rewrite the above max-min problem as the following linear program.

$$\begin{array}{ll} \underset{w_o,w_f,w_h}{\text{maximise}} & \tau \\ \text{subject to} & \tau \leq \sum_{i\in\{\text{o,f,h}\}} w_i a_{i,o} \\ & \tau \leq \sum_{i\in\{\text{o,f,h}\}} w_i a_{i,f} \\ & \tau \leq \sum_{i\in\{\text{o,f,h}\}} w_i a_{i,h} \\ & w_o + w_f + w_h = 1 \\ & w_o, w_f, w_h \geq 0 \end{array}$$

Exercise 2. Consider the following two-person zero-sum game

1. Does this game have a dominant strategy equilibrium? If yes, find it. If not, explain why.

2. Does this game have a pure Nash equilibrium? If yes, find it. If not, explain why.

3. Formulate the linear program that the column player must solve in order to find his best mixed strategy (you are not required to solve this problem).

		Column Player		
		A	B	C
Row Player	A	1	-1	17
	B	-1	15	-3
	C	5	9	9

Solution.

1. From the definition of dominance (row and column players), there are no dominated strategies in this problem. Thus, there is no dominant strategy equilibrium.

2. In order to determine whether this game has a pure Nash equilibrium, we investigate the worst-case (minimum) payoff of the row player for each of his strategy choices and the worst-case (maximum) payment of the column player for each of his strategy choices.

   For the row player, we have:

   - If the row player selects strategy A, his worst-case payoff is -1 (strategy B of the column player)
   - If the row player selects strategy B, his worst-case payoff is -3 (strategy C of the column player)
   - If the row player selects strategy C, his worst-case payoff is 5 (strategy A of the column player)

   For the column player, we have:

   - If the column player selects strategy A, his worst-case payment is 5 (strategy C of the row player)
   - If the column player selects strategy B, his worst-case payment is 15 (strategy B of the row player)
   - If the column player selects strategy C, his worst-case payment is 17 (strategy A of the row player)

   Thus, by playing strategy C, the row player wins at least 5. The column player therefore cannot expect to lose less than -5. By playing strategy A, the column player loses at most 5. The row player therefore cannot expect to win more than 5.

   Since neither of the players can expect a better outcome by deviating from these strategies (cell "C", "A"), the only rational choice is for the row player to play strategy "C" and for the column player to play strategy "A".

3. In order to find his best mixed strategy, the column player must find the probabilities q_A, q_B and q_C associated with strategies A, B and C respectively, so as to minimize the worst-case (maximum) expected payment to the row player, under the assumption that the row player also uses a mixed (randomized) strategy.

   Let Δ denote the probability simplex over the three strategies A, B and C:
   $$\Delta = \{(p_1, p_2, p_3) : p_1 + p_2 + p_3 = 1,\ p_1, p_2, p_3 \geq 0\}.$$
   Then, the column player's problem can be formulated as follows:
   $$\begin{array}{ll} \text{minimize}_{q_A,q_B,q_C} & \displaystyle\max_{(p_A,p_B,p_C)\in\Delta} \sum_{i\in\{A,B,C\}} \sum_{j\in\{A,B,C\}} p_i q_j a_{ij} \\ \text{subject to} & (q_A, q_B, q_C) \in \Delta, \end{array}$$
   where a_{ij} = payment (from payoff table) when the row player chooses strategy i and the column player chooses strategy j.

   Consider the inner maximization problem:
   $$\begin{array}{ll} \text{maximize}_{p_A,p_B,p_C} & \displaystyle\sum_{i\in\{A,B,C\}} p_i \left\{ \sum_{j\in\{A,B,C\}} q_j a_{ij} \right\} \\ \text{subject to} & p_A + p_B + p_C = 1 \\ & p_A, p_B, p_C \geq 0. \end{array} \tag{1}$$
   Since there always exists an optimal solution (p_A, p_B, p_C) to (1) with $p_A, p_B, p_C \in \{0, 1\}$, its optimal objective value is given by
   $$\max \left\{ \sum_{j\in\{A,B,C\}} q_j a_{Aj}, \sum_{j\in\{A,B,C\}} q_j a_{Bj}, \sum_{j\in\{A,B,C\}} q_j a_{Cj} \right\}.$$

Plugging this into the overall optimization problem, we obtain
$$\begin{array}{ll} \text{minimize}_{q_A,q_B,q_C} & \displaystyle\max_{i\in\{A,B,C\}} \sum_{j\in\{A,B,C\}} q_j a_{ij} \\ \text{subject to} & q_A + q_B + q_C = 1 \\ & q_A, q_B, q_C \geq 0. \end{array}$$

This is a min-max problem, which can be readily reformulated as an LP.

The optimization problem that the column player must solve in order to find his best mixed strategy is thus

$$
\begin{aligned}
\text{minimize}_{q_A, q_B, q_C} \quad & \tau \\
\text{subject to} \quad & \tau \geq \textstyle\sum_{j \in \{A,B,C\}} q_j a_{Aj} \\
& \tau \geq \textstyle\sum_{j \in \{A,B,C\}} q_j a_{Bj} \\
& \tau \geq \textstyle\sum_{j \in \{A,B,C\}} q_j a_{Cj} \\
& q_A + q_B + q_C = 1 \\
& q_A, q_B, q_C \geq 0.
\end{aligned}
$$

Exercise 3. Consider the following two-player zero-sum game.

Column Player

Row Player	c_1	c_2	c_3	c_4	c_5
r_1	-4	0	5	-1	2
r_2	4	9	-5	1	-5
r_3	3	-3	0	-7	5
r_4	7	2	6	0	5
r_5	-8	-4	8	-5	9

1. Is there a dominant strategy equilibrium? If so, find it. If not, can we at least remove dominated strategies from the problem?
2. Is there a pure Nash equilibrium? If so, find it.
3. Write down the linear program that the row player has to solve. Do not solve the problem.
4. Write down the linear program that the column player has to solve. Do not solve the problem.

Solution. The payoff matrix is

Column Player

		c_1	c_2	c_3	c_4	c_5
Row Player	r_1	-4	0	5	-1	2
	r_2	4	9	-5	1	-5
	r_3	3	-3	0	-7	5
	r_4	7	2	6	0	5
	r_5	-8	-4	8	-5	9

Investigate Dominant Strategies

We first begin by removing dominated strategies.
For Row Player, strategy r_4 dominates strategy r_1 and therefore we can remove strategy r_1.

Column Player

		c_1	c_2	c_3	c_4	c_5
Row Player	~~r_1~~	~~-4~~	~~0~~	~~5~~	~~-1~~	~~2~~
	r_2	4	9	-5	1	-5
	r_3	3	-3	0	-7	5
	r_4	7	2	6	0	5
	r_5	-8	-4	8	-5	9

For Row Player, strategy r_4 dominates strategy r_3 and therefore we can remove strategy r_3.

Column Player

Row Player	c_1	c_2	c_3	c_4	c_5
r_2	4	9	-5	1	-5
~~r_3~~	~~3~~	~~-3~~	~~0~~	~~-7~~	~~5~~
r_4	7	2	6	0	5
r_5	-8	-4	8	-5	9

For Column Player, strategy c_4 dominates strategy c_2 and therefore we can remove strategy c_2.

Column Player

Row Player	c_1	~~c_2~~	c_3	c_4	c_5
r_2	4	~~9~~	-5	1	-5
r_4	7	~~2~~	6	0	5
r_5	-8	~~-4~~	8	-5	9

We have removed all dominated strategies and the payoff matrix is not 1x1, so we do not have a dominant strategy equilibrium.

Column Player

Row Player	c_1	c_3	c_4	c_5
r_2	4	-5	1	-5
r_4	7	6	0	5
r_5	-8	8	-5	9

We now try to find a nash equilibrium in pure strategies.

Investigate Nash Equilibrium in Pure Strategies

For each strategy of Row Player, the red number indicates the minimum payoff. The red rectangle indicates which strategy achieves the maximum over all minimum payoffs. For each strategy of Column Player, the blue number indicates the maximum payoff. The blue rectangle indicates which strategy achieves the minimum over all maximum payoffs.

Column Player

Row Player	c_1	c_3	c_4	c_5	
r_2	4	-5	1	-5	-5
r_4	7	6	0	5	0
r_5	-8	8	-5	9	-8
	7	8	1	9	

Suppose Row Player and Column Player play strategies r_4 and c_4 respectively. Row Player cannot expect to win more than 0. Column Player cannot expect to lose more than 1. Therefore there is no saddlepoint and thus no pure Nash Equilibrium.

Investigate Nash Equilibrium in Mixed Strategies

We try to find a nash equilibrium in mixed strategies. Consider the optimisation problem for Row Player

Row Player Optimisation Problem

$$\begin{aligned} &\underset{p_{r_2},p_{r_4},p_{r_5}}{\text{maximise}} \quad \min_{(p_{c_1},p_{c_3},p_{c_4},p_{c_5})\in\Delta} \textstyle\sum_{i\in\{r_2,r_4,r_5\}} \sum_{j\in\{c_1,c_3,c_4,c_5\}} p_i p_j a_{i,j} \\ &\text{subject to} \quad (p_{r_2},p_{r_4},p_{r_5}) \in \Delta \end{aligned}$$

This is equivalent to

$$\begin{aligned} &\underset{p_{r_2},p_{r_4},p_{r_5}}{\text{maximise}} \quad \min\left\{\textstyle\sum_{i\in\{r_2,r_4,r_5\}} p_i a_{i,c_1} \ , \ \sum_{i\in\{r_2,r_4,r_5\}} p_i a_{i,c_3} \ , \ \sum_{i\in\{r_2,r_4,r_5\}} p_i a_{i,c_4} \ , \ \sum_{i\in\{r_2,r_4,r_5\}} p_i a_{i,c_5}\right\} \\ &\text{subject to} \quad (p_{r_2},p_{r_4},p_{r_5}) \in \Delta \end{aligned}$$

This is equivalent (applying the hypograph transformation) to

$$\begin{aligned} \underset{\tau, p_{r_2}, p_{r_4}, p_{r_5}}{\text{maximise}} \quad & \tau \\ \text{subject to} \quad & (p_{r_2}, p_{r_4}, p_{r_5}) \in \Delta \\ & \tau \leq \textstyle\sum_{i \in \{r_2, r_4, r_5\}} p_i a_{i,c_1} \\ & \tau \leq \textstyle\sum_{i \in \{r_2, r_4, r_5\}} p_i a_{i,c_3} \\ & \tau \leq \textstyle\sum_{i \in \{r_2, r_4, r_5\}} p_i a_{i,c_4} \\ & \tau \leq \textstyle\sum_{i \in \{r_2, r_4, r_5\}} p_i a_{i,c_5} \end{aligned}$$

Substituting payoffs

$$\begin{aligned} \underset{\tau, p_{r_2}, p_{r_4}, p_{r_5}}{\text{maximise}} \quad & \tau \\ \text{subject to} \quad & \tau - 4p_{r_2} - 7p_{r_4} + 8p_{r_5} \leq 0 \\ & \tau + 5p_{r_2} - 6p_{r_4} - 8p_{r_5} \leq 0 \\ & \tau - p_{r_2} + 5p_{r_5} \leq 0 \\ & \tau + 5p_{r_2} - 5p_{r_4} - 9p_{r_5} \leq 0 \\ & p_{r_2} + p_{r_4} + p_{r_5} = 1 \\ & p_{r_2}, p_{r_4}, p_{r_5} \geq 0 \end{aligned}$$

Column Player Optimisation Problem

$$\begin{aligned} \underset{p_{c_1}, p_{c_3}, p_{c_4}, p_{c_5}}{\text{minimise}} \quad & \max_{(p_{r_2}, p_{r_4}, p_{r_5}) \in \Delta} \textstyle\sum_{i \in \{c_1, c_3, c_4, c_5\}} \sum_{j \in \{r_2, r_4, r_5\}} p_i p_j a_{j,i} \\ \text{subject to} \quad & (p_{c_1}, p_{c_3}, p_{c_4}, p_{c_5}) \in \Delta \end{aligned}$$

Following the same reasoning of the row player, this is equivalent to

$$\begin{aligned} \underset{p_{c_1}, p_{c_3}, p_{c_4}, p_{c_5}}{\text{minimise}} \quad & \max \left\{ \textstyle\sum_{i \in \{c_1, c_3, c_4, c_5\}} p_i a_{r_2,i} \ , \ \sum_{i \in \{c_1, c_3, c_4, c_5\}} p_i a_{r_4,i} \ , \ \sum_{i \in \{c_1, c_3, c_4, c_5\}} p_i a_{r_5,i} \right\} \\ \text{subject to} \quad & (p_{c_1}, p_{c_3}, p_{c_4}, p_{c_5}) \in \Delta \end{aligned}$$

Which can be rewritten as

$$\begin{aligned}
\underset{\tau, p_{c_1}, p_{c_3}, p_{c_4}, p_{c_5}}{\text{minimise}} \quad & \tau \\
\text{subject to} \quad & (p_{c_1}, p_{c_3}, p_{c_4}, p_{c_5}) \in \Delta \\
& \tau \geq \textstyle\sum_{i \in \{c_1, c_3, c_4, c_5\}} p_i a_{r_2, i} \\
& \tau \geq \textstyle\sum_{i \in \{c_1, c_3, c_4, c_5\}} p_i a_{r_4, i} \\
& \tau \geq \textstyle\sum_{i \in \{c_1, c_3, c_4, c_5\}} p_i a_{r_5, i}
\end{aligned}$$

Substituting payoffs

$$\begin{aligned}
\underset{\tau, p_{c_1}, p_{c_3}, p_{c_4}, p_{c_5}}{\text{minimise}} \quad & \tau \\
\text{subject to} \quad & -\tau + 4p_{c_1} - 5p_{c_3} + p_{c_4} - 5p_{c_5} \leq 0 \\
& -\tau + 7p_{c_1} + 6p_{c_3} + 5p_{c_5} \leq 0 \\
& -\tau - 8p_{c_1} + 8p_{c_3} - 5p_{c_4} + 9p_{c_5} \leq 0 \\
& p_{c_1} + p_{c_3} + p_{c_4} + p_{c_5} = 1 \\
& p_{c_1}, p_{c_3}, p_{c_4}, p_{c_5} \geq 0
\end{aligned}$$

A Word about the Solution

If we solve the optimisation problems, then we see that Row Player should play strategies r_1, r_2, r_3, r_4, r_5 with probabilities $0, \frac{5}{11}, 0, \frac{6}{11}, 0$ respectively, giving an expected payoff of $\frac{5}{11}$. Similarly, the Column Player should play strategies c_1, c_2, c_3, c_4, c_5 with probabilities $0, 0, 0, \frac{10}{11}, \frac{1}{11}$ respectively, also giving an expected payoff of $\frac{5}{11}$. It is no coincidence that the expected payoffs of both players are the same, as will be seen later in the course.

Bonus exercise. During Monday's lecture we have discussed the minimax theorem. A consequence of this result is that every two-players zero-sum game with finite actions admits a mixed Nash equilibrium. Prove this statement.

[**Hint:** Let A be the matrix of payoffs/costs. To prove the claim it is instrumental to consider a pair of probability distributions $(p_1^*, \ldots, p_m^*, q_1^*, \ldots, q_n^*) = (p^*, q^*)$ constructed as follows

$$p^* \in \arg\max_{p \in \Delta_m} \left(\min_{q \in \Delta_n} V(p, q) \right), \qquad q^* \in \arg\min_{q \in \Delta_n} \left(\max_{p \in \Delta_m} V(p, q) \right),$$

where $\Delta_m = \{(p_1, \ldots, p_m)$ s.t. $\sum_{i=1}^m p_i = 1,$ and $p_i \geq 0\ \forall i = 1, \ldots, m\}$ (similarly for Δ_n), and where $V(p,q) = \sum_{i=1}^m \sum_{j=1}^n p_i q_j a_{ij}$. The statement is shown, if you can prove that (p^*, q^*) is a mixed Nash equilibrium, i.e., if you prove that $V(p, q^*) \leq V(p^*, q^*) \leq V(p^*, q)$ for all alternative distributions p, q.]

Solution. Thank to the definition of q^*, it is

$$\min_{q \in \Delta_n} \max_{p \in \Delta_m} V(p,q) = \max_{p \in \Delta_m} V(p, q^*) \geq V(p, q^*) \quad \forall p \in \Delta_m,$$

where the inequality holds by definition of max. The minimax theorem allows us to <u>exchange</u> min and max and thus obtain

$$\min_{q \in \Delta_n} \max_{p \in \Delta_m} V(p,q) = \max_{p \in \Delta_m} \min_{q \in \Delta_n} V(p,q) = \min_{q \in \Delta_n} V(p^*, q) \leq V(p^*, q^*),$$

where the second equality holds by definition of p^* and the third by definition of min. Putting together the chain of inequalities, starting from the latter, gives

$$V(p^*, q^*) \geq \max_{p \in \Delta_m} \min_{q \in \Delta_n} V(p,q) = \min_{q \in \Delta_n} \max_{p \in \Delta_m} V(p,q) \geq V(p, q^*) \quad \forall p$$

Repeating the reasoning symmetrically, one concludes.