Tutorial 4 – 60016 Operations Research

Two-Phase Simplex Method

Exercise 1 *Solve the following LP problem*[1]*:*

$$\max\ y = 4x_1 + 3x_2$$

subject to:

$$\begin{array}{rcrcr} 3x_1 & + & 4x_2 & \leq & 12 \\ 3x_1 & + & 3x_2 & \leq & 10 \\ 4x_1 & + & 2x_2 & \leq & 8 \end{array}$$

and

$$x_1 \geq 0, x_2 \geq 0.$$

Solution Standard Form:

$$\min\ z = -4x_1 - 3x_2$$

subject to:

$$\begin{array}{rcrcrcrcrcr} 3x_1 & + & 4x_2 & + & x_3 & & & & & = & 12 \\ 3x_1 & + & 3x_2 & & & + & x_4 & & & = & 10 \\ 4x_1 & + & 2x_2 & & & & & + & x_5 & = & 8 \end{array}$$

and

$$x_1 \geq 0, x_2 \geq 0 x_3 \geq 0, x_4 \geq 0, x_5 \geq 0.$$

[1] This is a problem that can be solved also with the standard simplex method, we will later compare in Exercise 2 the solution against the one obtained with the two-phase algorithm.

Simplex Algorithm:

BV	x_1	x_2	x_3	x_4	x_5	RHS
z	4	3	0	0	0	0
x_3	3	4	1	0	0	12
x_4	3	3	0	1	0	10
x_5	$\boxed{4}$	2	0	0	1	8
z	0	1	0	0	-1	-8
x_3	0	$\boxed{\frac{5}{2}}$	1	0	$-\frac{3}{4}$	6
x_4	0	$\frac{3}{2}$	0	1	$-\frac{3}{4}$	4
x_1	1	$\frac{1}{2}$	0	0	$\frac{1}{4}$	2
z	0	0	$-\frac{2}{5}$	0	$-\frac{7}{10}$	$-10\frac{2}{5}$
x_2	0	1	$\frac{2}{5}$	0	$-\frac{3}{10}$	$2\frac{2}{5}$
x_4	0	0	$-\frac{3}{5}$	1	$-\frac{3}{10}$	$\frac{2}{5}$
x_1	1	0	$-\frac{1}{5}$	0	$\frac{2}{5}$	$\frac{4}{5}$

The optimal BS is thus given by:

$$(x_1, x_2, x_3, x_4, x_5) = \left(\frac{4}{5}, \frac{12}{5}, 0, \frac{2}{5}, 0\right).$$

and the optimal value (for the max problem) is 10.4.

Exercise 2 *Consider the following LP similar to the problem in Exercise 1:*

$$\max\ y = 4x_1 + 3x_2$$

subject to:

$$\begin{array}{rcrcr} 3x_1 & + & 4x_2 & \leq & 12 \\ 3x_1 & + & 3x_2 & \leq & 10 \\ 4x_1 & + & 2x_2 & \leq & 8 \\ x_1 & + & x_2 & \geq & 1 \end{array}$$

and

$$x_1 \geq 0, x_2 \geq 0.$$

Solution Standard Form:

$$\min\ z = -4x_1 - 3x_2$$

subject to:

$$\begin{array}{rcrcrcrcrcrcr} 3x_1 & + & 4x_2 & + & x_3 & & & & & & & = & 12 \\ 3x_1 & + & 3x_2 & & & + & x_4 & & & & & = & 10 \\ 4x_1 & + & 2x_2 & & & & & + & x_5 & & & = & 8 \\ x_1 & + & x_2 & & & & & & & - & x_6 & = & 1 \end{array}$$

and

$$x_1 \geq 0, x_2 \geq 0 x_3 \geq 0, x_4 \geq 0, x_5 \geq 0, x_6 \geq 0.$$

Introducing artificial variables:

$$\min\ -4x_1 - 3x_2$$

subject to:

$$\begin{array}{rcrcrcrcrcrcrcl} 3x_1 & + & 4x_2 & + & x_3 & & & & & & & & & = & 12 \\ 3x_1 & + & 3x_2 & & & + & x_4 & & & & & & & = & 10 \\ 4x_1 & + & 2x_2 & & & & & + & x_5 & & & & & = & 8 \\ x_1 & + & x_2 & & & & & & & - & x_6 & + & \xi & = & 1 \end{array}$$

and

$$x_1 \geq 0, x_2 \geq 0 x_3 \geq 0, x_4 \geq 0, x_5 \geq 0, x_6 \geq 0, \xi \geq 0.$$

Phase I: Introduce ζ

$$\begin{array}{ccccccccccc} \zeta & & & & & & & - & \xi & = & 0 \\ & & x_1 & + & x_2 & - & x_6 & + & \xi & = & 1 \\ \hline \zeta & + & x_1 & + & x_2 & - & x_6 & & & = & 1 \end{array}$$

$$\min\ \zeta = 1 - x_1 - x_2 + x_6.$$

Simplex Algorithm:

BV	x_1	x_2	x_3	x_4	x_5	x_6	ξ	RHS
ζ	1	1	0	0	0	−1	0	1
x_3	3	4	1	0	0	0	0	12
x_4	3	3	0	1	0	0	0	10
x_5	4	2	0	0	1	0	0	8
ξ	$\boxed{1}$	1	0	0	0	−1	1	1
ζ	0	0	0	0	0	0	−1	0
x_3	0	1	1	0	0	3	−3	9
x_4	0	0	0	1	0	3	−3	7
x_5	0	−2	0	0	1	4	−4	4
x_1	1	1	0	0	0	−1	1	1

Found $\min \zeta = 0$.

Phase II: Drop column for ξ and replace row for ζ by original objective function.

We cannot take simply:

BV	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	4	3	0	0	0	0	0
x_3	0	1	1	0	0	3	9
x_4	0	0	0	1	0	3	7
x_5	0	−2	0	0	1	4	4
x_1	1	1	0	0	0	−1	1

as this is not a basic representation! Eliminate x_1 (as it is basis for Phase I) from z:

$$\begin{array}{rcrcrcrcrl} z & + & 4x_1 & + & 3x_2 & & & = & 0 & \\ & & x_1 & + & x_2 & - & x_6 & = & 1 & \times -4 \end{array}$$

or better:

$$\begin{array}{rcrcrcrcr} z & + & 4x_1 & + & 3x_2 & & & = & 0 \\ & - & 4x_1 & - & 4x_2 & + & 4x_6 & = & -4 \\ \hline z & & & - & x_2 & + & 4x_6 & = & -4 \end{array}$$

Proceed with Simplex Algorithm:

BV	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	0	-1	0	0	0	4	-4
x_3	0	1	1	0	0	3	9
x_4	0	0	0	1	0	3	7
x_5	0	-2	0	0	1	$\boxed{4}$	4
x_1	1	1	0	0	0	-1	1
z	0	1	0	0	-1	0	-8
x_3	0	$\boxed{\frac{5}{2}}$	1	0	$-\frac{3}{4}$	0	6
x_4	0	$\frac{3}{2}$	0	1	$-\frac{3}{4}$	0	4
x_6	0	$-\frac{1}{2}$	0	0	$\frac{1}{4}$	1	1
x_1	1	$\frac{1}{2}$	0	0	$\frac{1}{4}$	0	2
z	0	0	$-\frac{2}{5}$	0	$-\frac{7}{10}$	0	$-10\frac{2}{5}$
x_2	0	1	$\frac{2}{5}$	0	$-\frac{3}{10}$	0	$2\frac{2}{5}$
x_4	0	0	$-\frac{3}{5}$	1	$-\frac{3}{10}$	0	$\frac{2}{5}$
x_6	0	0	$\frac{1}{5}$	0	$\frac{1}{10}$	1	$2\frac{1}{5}$
x_1	1	0	$-\frac{1}{5}$	0	$\frac{2}{5}$	0	$\frac{4}{5}$

As expected: The optimal BS is the same as before:

$$(x_1, x_2, x_3, x_4, x_5) = \left(\frac{4}{5}, \frac{12}{5}, 0, \frac{2}{5}, 0\right).$$

and the optimal value (for the max problem) is 10.4.

Exercise 3 *Solve the following LP problem:*

$$\max\ y = 4x_1 + 3x_2$$

subject to:

$$\begin{array}{rcrcr} 3x_1 & + & 4x_2 & \leq & 12 \\ 5x_1 & + & 2x_2 & \leq & 8 \\ x_1 & + & x_2 & \geq & 5 \end{array}$$

and

$$x_1 \geq 0, x_2 \geq 0.$$

Solution

$$\min\ z = -4x_1 - 3x_2$$

subject to:

$$\begin{array}{rcrcrcrcrcrcl} 3x_1 & + & 4x_2 & + & x_3 & & & & & & & = & 12 \\ 5x_1 & + & 2x_2 & & & + & x_4 & & & & & = & 8 \\ x_1 & + & x_2 & & & & & - & x_5 & + & \xi & = & 5 \end{array}$$

and

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0, \xi \geq 0.$$

Two-Phase Simplex Algorithm:

$$\zeta = \xi = -x_1 - x_2 + x_5 + 5$$

BV	x_1	x_2	x_3	x_4	x_5	ξ	RHS
ζ	1	1	0	0	-1	0	5
x_3	3	4	1	0	0	0	12
x_4	$\boxed{5}$	2	0	1	0	0	8
ξ	1	1	0	0	-1	1	5
ζ	0	$\frac{3}{5}$	0	$-\frac{1}{5}$	-1	0	$\frac{17}{5}$
x_3	0	$\boxed{\frac{14}{5}}$	1	$-\frac{3}{5}$	0	0	$\frac{36}{5}$
x_1	1	$\frac{2}{5}$	0	$\frac{1}{5}$	0	0	$\frac{8}{5}$
ξ	0	$\frac{3}{5}$	0	$\frac{1}{5}$	-1	1	$\frac{17}{5}$
ζ	0	0	$-\frac{3}{14}$	$-\frac{1}{14}$	1	0	$\frac{13}{7}$
x_2	0	1	$\frac{5}{14}$	$-\frac{3}{14}$	0	0	$\frac{18}{7}$
x_1	1	0	$-\frac{1}{7}$	$\frac{2}{7}$	0	0	$\frac{4}{7}$
ξ	0	0	$-\frac{3}{14}$	$-\frac{1}{14}$	-1	1	$\frac{13}{7}$

After Phase I, we see: The minimum of ζ reached is non-zero. Therefore there is no feasible solution!

Exercise 4 (Exam 2018, Q1a) *Consider the following linear programming (LP) problem.*

$$\max\ y = 4x_1 + 8x_2$$

subject to

$$x_1 + 2x_2 \leq 3$$
$$3x_1 + x_2 \leq 8$$
$$x_1 + x_2 \geq 2$$

and

$$x_1 \geq 0, x_2 \geq 0$$

Solve the LP using the two-phase simplex method, justifying at each step the choice of the variable that leaves the basis.

Solution i) After standardisation this LP becomes:

$$\begin{array}{ll} \text{minimise} & z = -4x_1 - 8x_2 \\ \text{subject to} & x_1 + 2x_2 + x_3 = 3 \\ & 3x_1 + x_2 + x_4 = 8 \\ & x_1 + x_2 - x_5 = 2 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{array}$$

We now introduce an artificial in the last equality and in the phase-1 objective

$$\begin{array}{ll} \text{minimise} & \zeta = \xi_1 \\ \text{subject to} & x_1 + 2x_2 + x_3 = 3 \\ & 3x_1 + x_2 + x_4 = 8 \\ & x_1 + x_2 - x_5 + \xi_1 = 2 \\ & x_1, x_2, x_3, x_4, x_5, \xi_1 \geq 0 \end{array}$$

We now select x_3, x_4, ξ_1 to be in the initial basis and use the last equality to express ζ as a function of the non-basic variables. This gives the initial phase-1 tableau:

BV	x_1	x_2	x_3	x_4	x_5	ξ_1	RHS
ζ	1	1	0	0	-1	0	2
x_3	1	2	1	0	0	0	3
x_4	3	1	0	1	0	0	8
ξ_1	$\boxed{1}$	1	0	0	-1	1	2
ζ	0	0	0	0	0	-1	0
x_3	0	1	1	0	1	-1	1
x_4	0	-2	0	1	3	-3	2
x_1	1	1	0	0	-1	1	2

Therefore the index set $I = \{3, 4, 1\}$ can be used to initialize phase-2. The associated basic representation has objective row

$$z = -4x_1 - 8x_2 = -8 - 4x_2 - 4x_5$$

Thus in phase-2

BV	x_1	x_2	x_3	x_4	x_5	RHS
z	0	4	0	0	4	-8
x_3	0	$\boxed{1}$	1	0	1	1
x_4	0	-2	0	1	3	2
x_1	1	1	0	0	-1	2
z	0	0	-4	0	0	-12
x_2	0	1	1	0	1	1
x_4	0	0	2	1	5	4
x_1	1	0	-1	0	-2	1

The optimal value is thus $y^* = 12$ and the optimal solution is $x^* = (1, 1, 0, 4, 0)$.

Exercise 5 (Exam 2017, Q1) *You are given the following linear programming (LP) problem:*

$$\min z = 3x_1 + x_2$$

subject to

$$\begin{aligned} 5x_1 + 5x_2 &\geq 15 \\ -2x_1 - x_2 &\geq -3 \\ x_1 + x_2 &= 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Using Phase 1 of the simplex algorithm, determine an initial basic feasible solution for this LP. The write the initial tableau for Phase-2 and state if the initial basic feasible solution found is degenerate or non-degenerate.

Solution Standard form:

$$\min z = 3x_1 + x_2$$

subject to

$$\begin{aligned} x_1 + x_2 - x_3 + \xi_1 &= 3 \\ 2x_1 + x_2 + x_4 &= 3 \\ x_1 + x_2 + \xi_2 &= 3 \\ x_1, x_2, x_3, x_4, \xi_1, \xi_2 &\geq 0 \end{aligned}$$

The phase-1 objective is therefore

$$\zeta = \xi_1 + \xi_2 = 6 - 2x_1 - 2x_2 + x_3$$

and thus

BV	x_1	x_2	x_3	x_4	ξ_1	ξ_2	RHS	$ratio$
ζ	2	2	-1	0	0	0	6	
ξ_1	1	1	-1	0	1	0	3	3
x_4	$\boxed{2}$	1	0	1	0	0	3	3/2
ξ_2	1	1	0	0	0	1	3	3
ζ	0	1	-1	-1	0	0	3	
ξ_1	0	$\boxed{1/2}$	-1	$-1/2$	1	0	3/2	3
x_1	1	1/2	0	1/2	0	0	3/2	3
ξ_2	0	1/2	0	$-1/2$	0	1	3/2	3
ζ	0	0	1	0	-2	0	0	
x_2	0	1	-2	-1	2	0	3	
x_1	1	0	1	1	-1	0	0	
ξ_2	0	0	$\boxed{1}$	0	-1	1	0	

Therefore we have found that the original LP is feasible. However, we are now in the special case where $\zeta^* = 0$ and an artificial variable ξ_2 is basic. We therefore need to further pivot to remove it from the basis:

BV	x_1	x_2	x_3	x_4	ξ_1	ξ_2	RHS	$ratio$
ζ	0	0	1	0	-2	0	0	
x_2	0	1	-2	-1	2	0	3	
x_1	1	0	1	1	-1	0	0	
ξ_2	0	0	$\boxed{1}$	0	-1	1	0	$deg.$
ζ	0	0	0	0	-1	-1	0	
x_2	0	1	0	-1	0	2	3	
x_1	1	0	0	1	0	-1	0	
x_3	0	0	1	0	-1	1	0	

Note that even though a negative reduce cost is positive, this basis is already optimal and we can stop.

The initial tableau for Phase 2 is determined as follows. We first use the final tableau in Phase-1 to write z as a function of the non-basic variables:

$$z = 3x_1 + x_2 = 3(-x_4)$$

We are then ready to write the initial tableau for Phase-2:

BV	x_1	x_2	x_3	x_4	RHS
z	0	0	0	2	3
x_2	0	1	0	-1	3
x_1	1	0	0	1	0
x_3	0	0	1	0	0

The initial BFS found has more than $n - m$ zeros, being n the number of rows and m the number of columns in the LP, hence it is degenerate. This is due to the fact that the third constraint in the original LP intersects the first constraint at the points where $x_1 + x_3 = 3$.