Computational Finance
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Tutorial 1

Exercise 1

1. Let x and y be two dependent random variables, and let α and β be real numbers. Prove that

$$\text{var}(\alpha x + \beta y) = \alpha^2 \text{var}(x) + 2\alpha\beta \text{cov}(x, y) + \beta^2 \text{var}(y).$$

2. Suppose that there are two stocks. Let x and y denote the random values of the first and second stock, respectively, after one year. Furthermore, we know that $\text{std}(x) = 0.20$, $\text{std}(y) = 0.18$, and $\text{cov}(x, y) = 0.01$. A *portfolio* is composed out of $\alpha = 2$ units of stock 1 and $\beta = 3$ units of stock 2. Calculate the variance of the *portfolio value* in one year, that is, $\text{var}(\alpha x + \beta y)$.

Exercise 2

Find the mean and the variance of a random variable described by the probability density function

$$p(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Exercise 3

Write the second order Taylor series expansion of

1. $f(x) = e^x$, around $x = 1$.

2. $f(x) = e^{x^2}$, around $x = 1$.
3. $f(x_1, x_2) = e^{x_1 x_2}$, around $x_1 = x_2 = 0$.

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Tutorial 2

Exercise 1

Write down the relation between the nominal and the effective interest rate. Using this relation, calculate the effective rates form the given nominal interest rates:

1. 3% compounded monthly.
2. 18% compounded monthly.
3. 18% compounded quarterly.

Exercise 2

You are considering the purchase of a nice home. It is in every way perfect for you and in excellent condition, except for the roof. The roof has only 5 years of life remaining. A new roof would last 20 years, but would cost £20,000. The house is expected to last forever. Assuming that costs will remain constant and that the interest rate is 5%, what value would you assign to the existing roof? *Note: It is assumed that the roof will be replaced every 20 years, forever.*

Exercise 3

1. Given the yearly spot rate curve $\mathbf{s} = (0.05, 0.053, 0.056, 0.058, 0.06, 0.061)$, find the spot rate curve for next year under expectation dynamics.
2. Consider two 5-year bonds : one has a 9% coupon and sells for £101.00; the other has a 7% coupon and sells for £93.20. Assuming that both

bonds have a face value of £100.00, find the price of a 5-year zero-coupon bond. Note that all bonds are priced under the same spot rate curve. However, the spot rates are not given.

Exercise 4

The *Have A Nice Day* Corporation has just paid a dividend of \$1.37 per share. The company is expected to grow at 10% for the foreseeable future, and hence most analysts project a similar growth in dividends. The discount rate used for this type of company is 15%. The value of a share is given by the net present value of all future dividend payments. Calculate the share price.

Computational Finance Questions

Tutorial 3

Exercise 1

Consider four bonds having annual payments as shown in the table. They are traded to produce a 15% yield.

1. Determine the price of each bond.
2. Determine the Macaulay duration of each bond.
3. Which bond is most sensitive to a change in the yield?
4. Suppose you owe 2000 at the end of 2 years. Concern about interest rate risk suggests that a portfolio consisting of the bonds and the obligation should be immunized. If V_A, V_B, V_C and V_D are the total values of bonds purchased of types A,B,C and D, respectively, what are the necessary constraints to implement the immunization? (Hint: there are two equations. Just state them, don't solve them.)
5. In order to immunize the portfolio, you decide to use bond C and one other bond. Which other bond should you choose? Find the amounts (in total value) of each of these to purchase.

End of year payments	Bond A	Bond B	Bond C	Bond D
Year 1	100	50	0	1000
Year 2	100	50	0	0
Year 3	1100	1050	1000	0

Exercise 2

We roll two dice. Let z be the product of their values. What are the expected value and the variance of the random variable z? (Hint: Use the independence of the two separate dice.)

Exercise 3

Calculate the modified duration D_M for an infinite maturity bond by

1. using the definition of D_M.

2. using the formula

$$D = \frac{1 + \lambda/m}{\lambda} - \frac{1 + \lambda/m + n(c/m - \lambda/m)}{c[(1 + \lambda/m)^n - 1] + \lambda}$$

   and the relationship between D and D_M.

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Tutorial 4

Exercise 1
Consider a simple market consisting of two assets. Relevant data about the asset returns is provided in the following table.

	$\bar{r}$	σ
Asset 1	10%	15%
Asset 2	18%	30%

Furthermore, the correlation between the assets' rates of return amounts to $\rho = 0.1$. Suppose that we hold a portfolio with weights $w_1 = 25\%$ and $w_2 = 75\%$. Calculate the portfolio's expected return and standard deviation. (*Hint: recall that* $\rho = \sigma_{12}/(\sigma_1\sigma_2)$.)

Exercise 2
Kate is planning to invest £1 million in a rock concert to be held one year from now. She figures that she will obtain £3 million revenues from the investment – unless, my goodness, it rains at the time of the concert. If it rains, she will lose her entire investment. As there is a 50% chance that it will rain, she thinks of buying rain insurance. She can buy one unit of insurance for £0.50, and this unit pays £1.00 if it rains and nothing if it does not. She may purchase as many units as she wishes, up to 3 million.

1. What is the expected rate of return of Kate's investment when she buys u units of insurance? (*Hint: the cost of the insurance is in addition to her £1 million investment.*)

2. How many units of insurance should Kate buy to minimise the variance of the investment's rate of return? Calculate this minimum variance and the corresponding expected rate of return. (*Hint: is it possible to achieve a variance of zero?*)

Exercise 3
Consider two assets with corresponding expected rates of return $\bar{r}_1$ and $\bar{r}_2$; the corresponding variances and covariances are σ_1^2, σ_2^2 and σ_{12}.

1. What percentages of total investment should be invested in each of the two stocks to minimise the variance of the portfolio?

2. What is the variance of the rate of return of the resulting portfolio?

3. What is the expected rate of return of this portfolio?

Exercise 4
Assume that there are n assets. All the assets have the same expected rate of return $\bar{r}$, and variance σ^2. Furthermore, the correlation between the rates of return of any two assets is equal to some constant ρ. Find the variance of the portfolio defined by taking equal weights in each of the assets. What is the limit of the portfolio variance when $n \to \infty$?

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Tutorial 5

Exercise 1

(Bounds on returns) Consider a universe of just three securities. They have expected rates of return of 10%, 20% and 10%, respectively. Two portfolios are known to lie on the minimum-variance set. They are defined by the portfolio weights

$$w = [0.6, 0.2, 0.2]^T, v = [0.8, -0.2, 0.4]^T.$$

It is also known that the market portfolio is efficient. Furthermore, short-selling is allowed.

1. Given this information, what are the minimum and maximum possible values for the expected rate of return on the market portfolio? Hint: The market cannot contain assets in negative amounts.

2. Now suppose you are told that w represents the minimum-variance portfolio. Does this change your answer to part (1)?

Exercise 2

(Capital market line) Assume that the expected rate of return on the market portfolio is 23% and the rate of return on T-bills (the risk-free rate) is 7%. The standard deviation of the market is 32%. Assume that the market portfolio is efficient.

1. What is the equation of the capital market line?

2. (a) If an expected return of 39% is desired, what is the standard deviation of this position?

(b) If you have \$1000 to invest, how should you allocate it to achieve the above position?

3. If you invest \$300 in the risk-free asset and \$700 in the market portfolio, how much money should you expect to have at the end of the year?

Exercise 3

(A small world) Consider a world in which there are only two risky assets, A and B, and a risk free asset F. The two risky assets are in equal supply in the market; that is, the market portfolio M can be represented as $M = \frac{1}{2}(A+B)$. The following information is known: $r_f = 0.1$, $\sigma_A^2 = 0.04$, $\sigma_{AB} = 0.01$, $\sigma_B^2 = 0.02$ and $\bar{r}_M = 0.18$.

1. Find a general expression (without substituting values) for σ_M^2, β_A, β_B.

2. According to CAPM, what are the numerical values of $\bar{r}_A$, $\bar{r}_B$?

Exercise 4

(Simpleland) In Simpleland there are only two risky stocks, A and B, whose details are listed in the following table.

	Number of shares outstanding	Price per share	Expected rate of return	Standard deviation of return
Stock A	100	\$1.5	15%	15%
Stock B	150	\$2	12%	9%

Furthermore, the correlation coefficient between the returns of stocks A and B is $\rho_{AB} = \frac{1}{3}$. There is also a risk-free asset, and Simpleland satisfies the CAPM exactly.

1. What is the expected rate of return of the market portfolio?

2. What is the standard deviation of the market portfolio?

3. What is the beta of stock A?

4. What is the risk-free rate in Simpleland?

Computational Finance
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Tutorial 6

Exercise 1
The utility function is $U(x) = x - 0.04x^2$. Two investment possibilities are given. The first one is risk free and has a £5 payoff, the second one is based on a toss of a (fair) coin. When the outcome of the coin toss is heads, the payoff is £10, and when the outcome is tails, you earn nothing.

a Evaluate the expected utility of the two alternatives. Which one is to be preferred?

b Find the certainty equivalent to the risky investment.

Exercise 2
Jérôme has the utility function $U(x) = \sqrt[4]{x}$. Having worked for a well-known investment bank, he now has an offer from a reputable hedge-fund which pays £80,000 as a basic salary. The bonus will be £0, £10,000, £20,000, £30,000, £40,000, £50,000, or £60,000, each with equal probability. What is the certainty equivalent of this offer?

Exercise 3
$U(x)$ is a utility function with Arrow-Pratt *absolute* risk aversion coefficient $a(x)$. Let $V(x) = c + bU(x)$. What is the risk aversion coefficient of $V(x)$?

Exercise 4
The Arrow-Pratt *relative* risk aversion coefficient is defined by

$$\mu(x) := -\frac{xU''(x)}{U'(x)}$$

Show that this coefficient is constant for the utility functions $U(x) = \ln(x)$ and $U(x) = \gamma x^{\gamma}$.

Exercise 5
In this exercise we want to derive a useful approximation to the certainty equivalent. A second order expansion of the utility function $U(x)$ near $\bar{x} = E(x)$ is given by

$$U(x) \approx U(\bar{x}) + U'(\bar{x})(x - \bar{x}) + \frac{1}{2}U''(\bar{x})(x - \bar{x})^2.$$

Using the linearity of the expected value we have

$$E(U(x)) \approx U(\bar{x}) + \frac{1}{2}U''(\bar{x})\text{var}(x). \tag{1}$$

If we denote the certainty equivalent by C and assume it is close to $\bar{x}$, we can also use a first order approximation for $U(C)$:

$$U(C) \approx U(\bar{x}) + U'(\bar{x})(C - \bar{x}). \tag{2}$$

a Using the approximations (1) and (2), show that

$$C \approx \bar{x} + \frac{1}{2}\frac{U''(\bar{x})}{U'(\bar{x})}\text{var}(x).$$

b Using the previous approximation, calculate the certainty equivalent of the risky investment in exercise 1 and compare it to the exact value found in 1(b).

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Tutorial 7

Exercise 1

[Residual rights]

Table 1

	Film Venture	Risk Free	Film Residuals	Probability
Hight Success	3.0	1.2	6.0	0.3
Moderate Success	1.0	1.2	0.0	0.4
Failure	0.0	1.2	0.0	0.3

Consider the film venture example on page 30 of the notes. The investor now has the option of investing in three types of securities: the film venture, film residuals and a risk free asset. The total returns of the securities in the three scenarios are shown in Table 1. All securities have price £1. Find the optimal asset allocation for an investor with initial capital W and a logarithmic utility function.

Exercise 2

At the horse races one Saturday afternoon Joe Blog studies the racing form and concludes that the horse No Arbitrage has a 25% chance to win and is posted at 4 to 1 odds (for every pound Joe bets, he receives £5 if the horse wins and nothing if it loses). He can either bet on this horse or keep his money in his pocket. Joe decides that he has a square-root utility function for money.

1. What fraction of his money should Joe bet on No Arbitrage?
2. Consider the following finite state securities:
   (a) Keep the money in the pocket
   (b) Bet on the horse
   (c) Bet against the horse No Arbitrage, (if No Arbitrage wins, Joe gets no money, otherwise he gets an amount of, say, £y)

Write down the payoff vectors and the prices of the securities.

3. Find the value of y of the last security (bet against No Arbitrage) that makes Joe's market arbitrage free.

   [Hint : Use linear pricing]

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Tutorial 8

Exercise 1 (**Stock lattice**)

A stock with current value $S(0) = 100$ has an expected growth rate of its logarithm of $\nu = 12\%$ and a volatility of that growth rate of $\sigma = 20\%$. Find suitable parameters of a binomial lattice representing this stock with basic period length 3 months. Draw the lattice for a period of 1 year (thus comprising four basic periods) and calculate the stock prices in all the nodes. Also, calculate the probabilities of attaining the various end nodes of the lattice.

Exercise 2 (**Time scaling**)

A stock price S is governed by the model

$$\ln S(k+1) = \ln S(k) + w(k)$$

where the period length is 1 month. Let $\nu = \mathrm{E}[w(k)]$ and $\sigma^2 = \mathrm{var}[w(k)]$ for all k. Now suppose the basic period length is changed to 1 year. Then the model is

$$\ln S(K+1) = \ln S(K) + W(K)$$

where each movement in K corresponds to 1 year. What is the natural definition of $W(K)$? Show that $\mathrm{E}[W(K)] = 12\nu$ and $\mathrm{var}[W(K)] = 12\sigma^2$. Hence parameters scale in proportion to time.

Exercise 3 (**Expectations**)

A stock price is governed by a geometric Brownian motion with $\nu = 0.12$ and $\sigma = 0.40$. The initial price is $S(0) = 1$. Evaluate the four quantities

$$\begin{array}{rl} \mathrm{E}[\ln S(1)], & \mathrm{stdev}[\ln S(1)] \\ \mathrm{E}[S(1)], & \mathrm{stdev}[S(1)]. \end{array}$$

Exercise 4 (**Lognormal random variables**)

Find the expected value of a lognormal random variable.

[*Hint: Justify the steps on slide 13 in your notes*]

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Tutorial 9

Exercise 1 (**Monotonicity of the price of a call**)
Two call options on a stock are identical apart from the strike price. The value of a call option with strike price K is denoted by $C(K)$. Use an arbitrage argument to show that

$$K_1 < K_2 \Rightarrow C(K_1) \geq C(K_2).$$

Exercise 2 (**A Bull spread**)
An investor who is bullish about a stock (believing that it will rise) may wish to construct a *bull spread* for that stock. One way to construct such a spread is to buy a call with strike price K_1 and sell another call with the same expiration date but with a strike price of $K_2 > K_1$.

a Draw the payoff curve of the spread.

b Is the initial cost of the spread positive or negative?

Exercise 3 (**My coin**)
There are two propositions:

a I flip a coin. If it is heads, you are paid £3; if it is tails, you are paid £0. To participate in this bet you must pay £1.

b You may keep your money in your pocket (earning no interest).

You may participate in these propositions at any level, and the payoff scales accordingly. Now we introduce a third proposition: *I flip the coin three times, if at least two flips are heads, you are paid £27; otherwise nothing.*
What is the fair price for this third proposition?
Hint: Draw three binomial trees representing the three propositions. Then, try to replicate the third proposition in terms of the first and the second one.

Exercise 4 (**A "happy call"**)
A firm in the city is offering a new financial instrument called a "happy call"[1] It has a payoff function at time T equal to $\max\{0.5 \cdot S, S - K\}$, where S is the price of the stock and K is the strike price. Let P be the price of the stock at time $t = 0$, and let C_1 and C_2 be the prices of ordinary calls with strike prices K and $2K$, respectively. The fair price of the happy call is of the form

$$C_H = \alpha P + \beta C_1 + \gamma C_2$$

Determine α, β, and γ.

Exercise 5 (**Forward price formula**)
A *forward contract* is a contract to purchase a specific amount of a commodity at a specific price and at a specific expiration time in the future.[2] The price that is paid at expiry is called the *forward price*. At expiration, the buyer of the forward contract receives the agreed amount of the commodity from the seller. In return, the buyer pays the agreed forward price. This price is negotiated so that the initial payment is zero, that is, the *value of the contract* is zero when it is initiated. We now want to find the *forward price* at time $t = 0$ for a forward commitment to deliver one unit of the commodity at time T. Suppose that the commodity can be stored at zero cost and also sold short. The current unit price (at $t = 0$) of the commodity is S. The discount factor for the time between 0 and T is d. Show that the *forward price* F is

$$F = \frac{S}{d}\ .$$

Hint: Construct an arbitrage argument.

[1] The name is derived from the fact that you always receive a positive payoff with such an instrument.

[2] A typical forward contract might be to purchase 100,000 kilos of sugar at 12 pennies per kilo on the 15th of March next year.