IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2018-2019

BEng Honours Degree in Computing Part III
BEng Honours Degree in Electronic and Information Engineering Part III
MEng Honours Degree in Electronic and Information Engineering Part III
MEng Honours Degree in Mathematics and Computer Science Part IV
BEng Honours Degree in Mathematics and Computer Science Part III
MEng Honours Degree in Mathematics and Computer Science Part III
MEng Honours Degrees in Computing Part III
MSc in Computing Science (Specialist)
for Internal Students of the Imperial College of Science, Technology and Medicine

*This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute*

# PAPER C343

# OPERATIONS RESEARCH

Wednesday 12th December 2018, 14:00
Duration: 120 minutes

*Answer THREE questions*

Paper contains 4 questions
Calculators required

1 a Consider the following linear programming (LP) problem.

$$\begin{array}{ll} \max & y = 4x_1 + 8x_2 \\ \text{subject to} & x_1 + 2x_2 \leq 3 \\ & 3x_1 + x_2 \leq 8 \\ & x_1 + x_2 \geq 2 \\ & x_1, x_2 \geq 0 \end{array}$$

Solve the LP using the two-phase simplex method, justifying at each step the choice of the variable that leaves the basis.

b Formulate the following optimization problem as a linear program in standard form (you are *not* required to solve it):

$$\min \frac{10x_1 + 20x_2 - 10}{5 + x_1 + x_2}$$

subject to

$$\begin{aligned} x_1 + 2x_2 &\geq 3 \\ 4x_1 - 2x_2 &\geq -3 \\ x_1 + x_2 &\leq 8 \\ x_1 \text{ free}, x_2 &\geq 0 \end{aligned}$$

c Given an index $i$, such that $1 \leq i \leq V$, consider the following linear program

$$\begin{array}{ll} \min & \lambda_i \\ \text{s.t.} & a_i = \lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_V a_V, \\ & \sum_{v=1}^{V} \lambda_v = 1 \\ & \lambda_v \geq 0 \qquad v = 1, \ldots, V \end{array} \qquad (\text{LP}_i)$$

where $a_v \in \mathbb{R}^n$ are constant vectors and $\lambda_v$ are decision variables. Assume that (LP$_i$) is feasible. Show that the optimal value for (LP$_i$) cannot be $0 < \lambda_i^* < 1$.

*The three parts carry, respectively, 40%, 30%, and 30% of the marks.*

2 a Consider the following min-min problem:

$$\begin{array}{ll} \min & \phi(x) \\ \text{subject to} & Ax = b,\ x \geq 0 \end{array} \tag{MM}$$

where

$$\phi(x) = \min_{i=1,\dots,I} \left| c(i)^T x + d(i) \right|$$

with $c(i) \in \mathbb{R}^n$, $d(i) \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$.

i) Explain why (MM) can be solved exactly using linear programming.

ii) Find the optimal value for (MM) when $I = 2$ and

$$A = \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad c(i) = [2i, 3], \qquad d(i) = i$$

(Note: you do *not* need to use the simplex algorithm.)

b An oil shipment company wants to plan operations for the next two months. The company ships oil from two sites, $A$ and $B$, to a final destination $F$. Oil shipped from $A$ visits in a sequence two intermediate sites $I$ and $J$ prior to reaching $F$, while oil shipped from $B$ visits only site $I$ before reaching $F$. The transportation cost between any two directly connected sites is £ 7 per oil barrel.

At both sites $I$ and $J$, the company can store oil in a storage unit. The oil can then be retrieved from storage in the following month if needed to meet the demand at $F$.

The following production and demand data is forecasted (in million barrels):

| *Month* | *Demand at F* | *Production at A* | *Production at B* |
|---|---|---|---|
| 1 | 10 | 12 | 5 |
| 2 | 20 | 9 | 4 |

Write a linear program to minimize the company costs in month 1, while meeting the demand at $F$ in both months. Assume that storage units are initially empty and have an unlimited capacity, and that the time for the oil to travel between connected sites is immediate.

c Explain what a Klee-Minty cube is. Then discuss its importance in linear programming.

*The three parts carry, respectively, 45%, 45%, and 10% of the marks.*

3 **Duality**

a Consider the possibilities for a primal/dual pair:

| | | Primal | | |
|---|---|---|---|---|
| | | Finite optimal | Infeasible | Unbounded |
| | Finite optimal | | | |
| Dual | Infeasible | | | |
| | Unbounded | | | |

i) Please replicate this table in the answer sheet to show which of the 9 possible primal/dual pairs may arise in linear optimization. Just denoting yes ($\checkmark$) or no ($\times$) is sufficient for this question subpart.

ii) For each *possible* primal/dual pair noted in (i) by $\checkmark$, give a corresponding example of a primal linear program and its dual.

b Let:

$$v(\ b) = \min \left\{ c^T x \mid Ax = \ b,\, x \geq 0 \right\},$$

with $c,\, x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$. If each element in the vector $b$ is multiplied by 2, we consider the perturbed problem:

$$v(2b) = \min \left\{ c^T x \mid Ax = 2b,\, x \geq 0 \right\}.$$

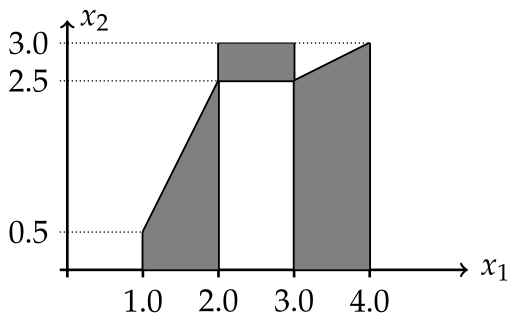
i) In general, the solution of the latter problem does not have the same basic representation, and hence the optimal basis matrix, as the former. Using the optimal basis matrix of the former, define its shadow prices $\Pi \in \mathbb{R}^m$ and use the shadow prices to establish a relationship between $v(b)$ and $v(2b)$. Please justify each step.

ii) Under what condition(s) could it be that $2v(b) = v(2b)$?

*The two parts carry, respectively, 45% and 55% of the marks.*

4 **Integer Programming**

a Develop constraints restricting the feasible set to the shaded region shown in the figure. In words, variable $x_2$ is restricted on $x_1 \in [1.0, 2.0]$ between $x_2 \geq 0$ and $x_2 \leq 2x_1 - 1.5$, on $x_1 \in [2.0, 3.0]$ between $2.5 \leq x_2 \leq 3.0$, and on $x_1 \in [3.0, 4.0]$ to $x_2 \geq 0$ and $x_2 \leq 0.5x_1 + 1$.




Please formulate the mixed-integer linear optimization constraints but do not solve it. Remember to justify the values of any big-M parameters.

b Consider the integer linear programming problem:

$$
\begin{aligned}
\max_{x_1, x_2} \quad & 2x_1 + x_2 \\
\text{s.t.} \quad & 2x_1 + 5x_2 \leq 17 \\
& 3x_1 + 2x_2 \leq 10 \\
& x_1, x_2 \geq 0 \\
& x_1, x_2 \in \mathbb{N}_0
\end{aligned}
$$

Ignoring the integer requirement and applying the simplex algorithm to the resulting linear program (LP), we obtain the following optimal tableau:

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | RHS |
|---|---|---|---|---|---|
| $z$ | 0 | -1/3 | 0 | -2/3 | -20/3 |
| $x_3$ | 0 | 11/3 | 1 | -2/3 | 31/3 |
| $x_1$ | 1 | 2/3 | 0 | 1/3 | 10/3 |

Derive any Gomory cut(s) possible based on the above tableau. Justify why these cut(s) do or do not exclude the LP optimal solution. Formulate the next linear program that incorporates all possible Gomory cut(s).

c What are all the minimal cover cuts for the following constraint?

$$
\begin{aligned}
3x_1 + 5x_2 + 4x_3 + 3x_4 + 7x_5 &\leq 9 \\
x_1, x_2, x_3, x_4, x_5 &\in \{0, 1\}.
\end{aligned}
$$

*The three parts carry, respectively, 40%, 30%, and 30% of the marks.*