60016 OPERATIONS RESEARCH

Linear Programs in Standard Form

Last Lecture

LP as a tool for optimal decision making:

- Maximise/minimise a linear objective function
- Linear constraints (equalities and/or inequalities)
- The feasible region is a convex polyhedron
- The vertices of the feasible region contain a solution to the LP problem (if the LP is well-defined)

$\Rightarrow$ An LP can be solved by examining all vertices, but this approach is computationally prohibitive!

This Lecture

- How to formulate an LP in a standard way

LPs in Standard Form

We want to use computers to solve LP problems

$\Rightarrow$ We need a standardised specification of LP problems

Definition: An LP is in standard form if:

- The aim is to minimise a linear objective function;
- All constraints are linear equality constraints;
- All constraint right hand sides are non-negative;
- All decision variables are non-negative.

LPs in Standard Form

An LP in standard form looks as follows:

$$
\begin{array}{lccccccccc}
\text{minimise} & z = c_1x_1 & + & c_2x_2 & + & \dots & + & c_nx_n & & \\
\text{subject to} & a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\
 & a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\
 & \vdots & & \vdots & & & & \vdots & & \vdots \\
 & a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m
\end{array}
$$

$$x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0$$

The input parameters b_i, c_j, and a_{ij} are fixed real constants that encode the LP problem. We require $b_i \geq 0$, $\forall i = 1, \dots, m$. (The decision variables x_i, $i = 1, \dots, n$, are yet to be found.)

Compact Notation

Collect the input parameters in vectors and matrices:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$$c^T = [c_1, c_2, \dots, c_n]$$

LPs in Standard Form (cont)

- With matrix notation, the LP in standard form reduces to

$$\begin{array}{ll} \text{minimise} & z = c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0, \end{array}$$

  where $b \geq 0$.

- Inequalities of the type $x \geq 0$ are understood to hold component-wise, i.e., $x_i \geq 0$, $\forall x_i \in x$.

Standardising General LPs

General LP problems can

- be maximisation (instead of minimisation) problems;
- have inequality (instead of equality) constraints;
- have equality constraints with negative (instead of non-negative) right hand sides;
- have free (instead of non-negative) decision variables.

These general LPs can be transformed to standard LPs in a systematic way.

$$\min \ z = -2x_1 - x_2$$

$$\text{s.t.} \quad x_1 - 4x_2 + s_1 = 1$$

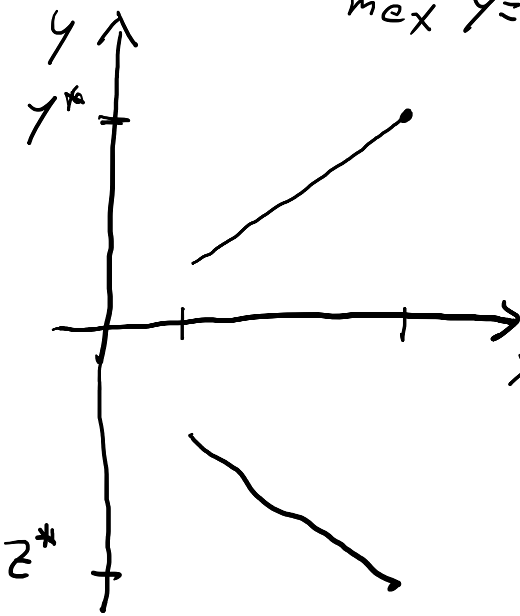
$$+x_1 + 5x_2 - s_2 = 3$$

$$x_1, x_2 \geq 0 \quad s_1 \geq 0 \quad s_2 \geq 0$$

$$-z^*$$

$$y^*$$

$$\max \ y = x$$



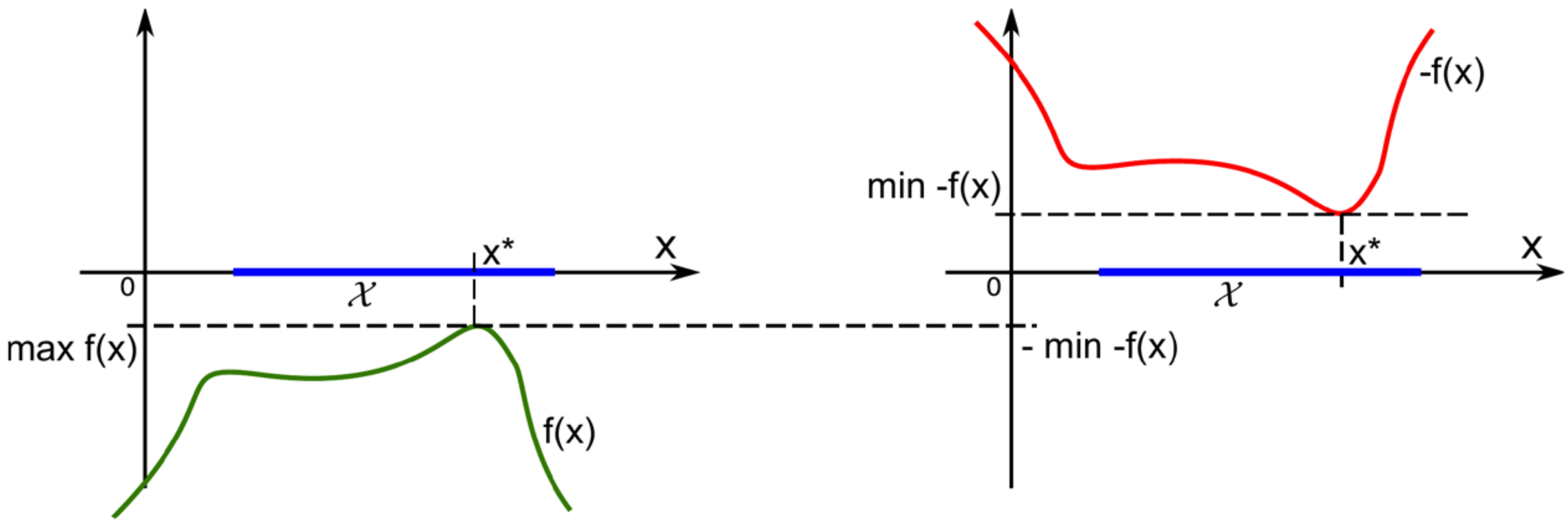
$$-y^* = z^*$$

$$\min z = -x$$

Maximisation $\rightarrow$ Minimisation

$$\left.\begin{array}{ll} \max & y = f(x) \\ \text{s.t.} & x \in \mathcal{X} \end{array}\right\} \quad = \quad \left\{\begin{array}{ll} -\min & z = -f(x) \\ \text{s.t.} & x \in \mathcal{X} \end{array}\right.$$

Inverting the objective preserves the optimal solution x^*



Optimal value of the objective is $y^* = -z^*$.

$\leq$ Inequalities $\rightarrow$ Equalities

$$\text{minimise} \quad z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

subject to:

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & \leq & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & \leq & b_2 \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & \leq & b_m
\end{array}$$

and

$$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$$

$\leq$ Inequalities $\rightarrow$ Equalities

$$\text{minimise} \quad z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

subject to:

$$\begin{array}{ccccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & + & s_1 & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & + & s_2 & = & b_2 \\
\vdots & & \vdots & & & & \vdots & & & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & + & s_m & = & b_m
\end{array}$$

and

$$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0, \quad s_1 \geq 0, s_2 \geq 0, \ldots, s_m \geq 0$$

Slack Variables

- To reformulate $\leq$ inequalities as equalities, we introduced m slack variables
  - Original variables: $x_1, x_2, \ldots, x_n$
  - Slack variables: $s_1, s_2, \ldots, s_m$

  $\Rightarrow$ After transformation, LP has $n + m$ variables!
- With matrix notation we can write

$$\left.\begin{array}{ll} \min & z = c^T x \\ \text{s.t.} & Ax \leq b \\ & x \geq 0 \end{array}\right\} = \left\{\begin{array}{ll} \min & z = c^T x \\ \text{s.t.} & Ax + s = b \\ & x \geq 0, s \geq 0, \end{array}\right.$$

  where $s = (s_1, \ldots, s_m)^T$.
- Slack variables take the value of the difference $b - Ax$

$\geq$ Inequalities $\rightarrow$ Equalities

$$\text{minimise} \quad z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

subject to:

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & \geq & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & \geq & b_2 \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & \geq & b_m
\end{array}$$

and

$$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$$

$\geq$ Inequalities $\to$ Equalities

$$\text{minimise} \quad z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

subject to:

$$\begin{array}{ccccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & - & s_1 & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & - & s_2 & = & b_2 \\
\vdots & & \vdots & & & & \vdots & & & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & - & s_m & = & b_m
\end{array}$$

and

$$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0, \quad s_1 \geq 0, s_2 \geq 0, \ldots, s_m \geq 0$$

Excess Variables

- To reformulate $\geq$ inequalities as equalities, we introduced m excess variables (a.k.a. surplus variables)
  - Original variables: $x_1, x_2, \ldots, x_n$
  - Excess variables: $s_1, s_2, \ldots, s_m$

  $\Rightarrow$ After transformation, LP has $n+m$ variables!
- With matrix notation we can write

$$\left.\begin{array}{ll} \min & z = c^T x \\ \text{s.t.} & Ax \geq b \\ & x \geq 0 \end{array}\right\} = \left\{\begin{array}{ll} \min & z = c^T x \\ \text{s.t.} & Ax - s = b \\ & x \geq 0, s \geq 0, \end{array}\right.$$

  where $s = (s_1, \ldots, s_m)^T$.
- Excess variables take the value of the difference $Ax - b$

Equivalence

- Assume initial problem not in standard form
- x: feasible solution to the initial problem
- (x, s): feasible solution to the standardised problem
- x can be associated with one and only one (x, s) using the reformulations we have defined.
- In particular, the optimal solutions will be x^* and (x^*, s^*)
  - x^* will be the same in both formulations

Negative Right Hand Sides

- If the right hand side of the ith constraint is negative, i.e., if $b_i < 0$ in
$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n = b_i,$$
then this constraint should be multiplied by -1.
- This yields
$$(-a_{i1})x_1 + (-a_{i2})x_2 + \ldots + (-a_{in})x_n = -b_i.$$
- The new constraint has a non-negative right hand side, i.e., we have $-b_i \geq 0$.

Free Variables (1st Approach)

Free variables:

- Suppose there is no constraint $x_j \geq 0$,i.e., x_j can be positive or negative.
- Substitute $\underline{x_j = x_j^+ - x_j^-}$ with $x_j^+, x_j^- \geq 0$.
- The LP has now $(n+1)$ variables:

$$x_1, \ldots, x_{j-1}, \underline{x_j^+, x_j^-}, x_{j+1}, \ldots, x_n$$

ORIGINAL

$(\ldots, x_j, \ldots)$

$\underline{x_j} = 7.5$

-7.5

$(x_j^+ - x_j^-)$

$10 - 2.5 = 7.5$

$11 - 3.5 = 7.5$

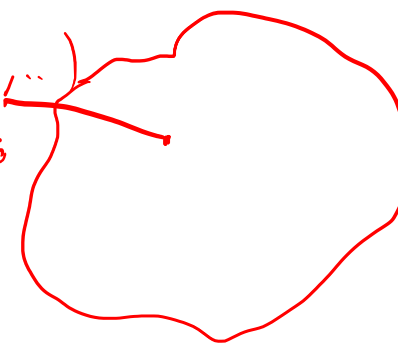
$2.5 - 10 = -7.5$

$\vdots$

NEW

$(\ldots, x_j^+, x_j^-, \ldots)$

10 2.5



$x_j + x_1 = 3$

Free Variables (2nd Approach)

Free variables:

- Suppose there is no constraint $x_j \geq 0$,i.e., x_j can be positive or negative.
- Any equality constraint involving x_j can be used to eliminate x_j.
- Example: x_1 is free

$$\left.\begin{array}{ll} \min & z = x_1 + 3x_2 + 4x_3 \\ \text{s.t.} & x_1 + 2x_2 + x_3 = 5 \ (*) \\ & 2x_1 + 3x_2 + x_3 = 6 \\ & x_2, x_3 \geq 0 \end{array}\right\} = \left\{\begin{array}{ll} \min & z = x_2 + 3x_3 + 5 \\ \text{s.t.} & x_2 + x_3 = 4 \\ & x_2, x_3 \geq 0 \end{array}\right.$$

Use $(*)$ to substitute $x_1 = 5 - 2x_2 - x_3$.