# Functional Dependencies

# A Mathematical Model

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## Overview

A discussion of functional dependencies occurs in the context of a set of *attributes* referenced by the functional dependencies. As presented here, the attributes are indexed as a sequence with each attribute having a unique index value.

Corresponding to n attributes is an n-dimensional *attribute space.* Each attribute in the sequence of attributes corresponds to a dimension in the attribute space. A *point* in the attribute space is an n-tuple coordinate where the coordinate value for each attribute is 0 or 1. For an n-dimensional attribute space there are $2^n$ points.

A *predicate* is a description of a subset of the points in an attribute space. An *atomic predicate* is an attribute or its negation. A *compound predicate* is a conjunction of atomic predicates.

A *functional dependency* is a predicate on the attribute space together with the designation that the set of points described by the predicate is *"eliminated from use"*.

# Attribute Spaces and Predicates

Informally, an *attribute space* represents all possible combinations of a set of attributes and their negations.

**Definition 1** $\mathbb{A}^n \stackrel{\text{def}}{=} (\alpha_1, \ldots, \alpha_n)$ is a sequence of n attributes. Each $\alpha \in \mathbb{A}^n$ has a unique index in $\mathbb{A}^n$, designated by idx($\alpha$).

Per Definition 1 , note: For $\alpha \in \mathbb{A}^n$ , $\mathbb{A}^n[\text{idx}(\alpha)] = \alpha$.

# Attribute Spaces

Corresponding to a sequence $\mathbb{A}^n$ of attributes is an n-dimensional *attribute space.*

**Definition 2** An *attribute space* $\mathfrak{A}^n \stackrel{\text{def}}{=} (p_1, \ldots, p_{2^n})$ is a sequence of $2^n$ *points,* where for each point $p \in \mathfrak{A}^n$, $p \in \{0, 1\}^n$ . Each $p \in \mathfrak{A}^n$ has a unique position index in the sequence $\mathfrak{A}^n$, designated by pos(p). Hence, $\mathfrak{A}^n[\text{pos}(p)] = p$.

# Atomic Predicates

**Definition 3** An *atomic predicate* on attribute space $\mathfrak{A}^n$ is an attribute in $\mathbb{A}^n$ or its negation. If $\alpha$ is an attribute, designate its negation by $\sim\alpha$.

**Definition 4** For atomic predicate $\alpha \in \mathbb{A}^n$, $\alpha$: $\mathfrak{A}^n \rightarrow \{0, 1\}$ a function such that for point $p \in \mathfrak{A}^n$,

$$\alpha(p) \stackrel{\text{def}}{=} p[\text{idx}(\alpha)] .$$

**Definition 5** For atomic predicate $\alpha \in \mathbb{A}^n$, $\sim\alpha$: $\mathfrak{A}^n \rightarrow \{0, 1\}$ a function such that for point $p \in \mathfrak{A}^n$,

$$\sim\alpha(p) \stackrel{\text{def}}{=} \alpha(p) + 1 \bmod 2 .$$

**Definition 6** $\sim\mathbb{A}^n \stackrel{\text{def}}{=} \{ \sim\alpha : \alpha \in \mathbb{A}^n \}$

**Notation 1** $^{\oplus}\mathbb{A}^n \stackrel{\text{def}}{=} \mathbb{A}^n \cup \sim\mathbb{A}^n$, the set of all atomic predicates on $\mathfrak{A}^n$.

**Definition 7** For atomic predicates $A \subseteq {}^{\oplus}\mathbb{A}^n$, $\wedge A \stackrel{\text{def}}{=} (\beta_i \wedge \ldots \wedge \beta_j)$, for all $\beta_i .. \beta_j \in A$ .

**Definition 8** For $A, B \subseteq {}^{\oplus}\mathbb{A}^n$, $(\wedge A) \wedge (\wedge B) = \wedge(A \cup B)$ .

Note that $|\alpha(\mathfrak{A}^n)| = ½\ |\mathfrak{A}^n|$.[1] That is, for any $\alpha \in \mathbb{A}^n$, for *half* of the the $p \in \mathfrak{A}^n$, $p(\alpha) = 1$.

**Exercise 1** For $\alpha, \beta \in \mathbb{A}^n$ , $\alpha \neq \beta$, how many members are in $\alpha(\mathfrak{A}^n) \cap \beta(\mathfrak{A}^n)$?
How many members in $\bigcap \{ \alpha(\mathfrak{A}^n) : \alpha \in \mathbb{A}^n \}$ ? [Answer in terms of n]. [2]

# Predicates

**Definition 9** If $\mathfrak{P}$ is an atomic predicate on $\mathfrak{A}^n$ then $\mathfrak{P}$ is a *predicate* on $\mathfrak{A}^n$.

**Definition 10** If $\mathfrak{P}_1$, $\mathfrak{P}_2$ are predicates on $\mathfrak{A}^n$, then $\mathfrak{P}_1 \wedge \mathfrak{P}_2$ is a predicate on $\mathfrak{A}^n$.

**Assumption 1** $\mathfrak{P}$ a predicate on $\mathfrak{A}^n$ only if $\mathfrak{P}$ a predicate per Definition 9 and Definition 10.

**Definition 11** For predicates $\mathfrak{P}_1$ and $\mathfrak{P}_2$ on $\mathfrak{A}^n$, define:

$(\mathfrak{P}_1 \wedge \mathfrak{P}_2) : \mathfrak{A}^n \to \{0, 1\}$ a function
such that
$\forall\ p \in \mathfrak{A}^n,\ (\mathfrak{P}_1 \wedge \mathfrak{P}_2)(p) = \min(\mathfrak{P}_1(p), \mathfrak{P}_2(p))$ .

**Lemma 1** If $\mathfrak{P}$ a predicate on $\mathfrak{A}^n$ and $\mathfrak{P}$ not an atomic predicate, then there exist predicates $\mathfrak{P}_1$, $\mathfrak{P}_2$ on $\mathfrak{A}^n$, such that $\mathfrak{P} = \mathfrak{P}_1 \wedge \mathfrak{P}_2$ .

*Proof* If $\mathfrak{P}$ is not an atomic predicate then $\mathfrak{P}$ is a predicate only per Definition 10, which implies there exist $\mathfrak{P}_1$ and $\mathfrak{P}_2$ , predicates on $\mathfrak{A}^n$, such that $\mathfrak{P} = \mathfrak{P}_1 \wedge \mathfrak{P}_2$ . □

**Definition 12** $\mathbb{P}^n \stackrel{\text{def}}{=} \{ \mathfrak{P} : \mathfrak{P} \text{ a predicate on } \mathfrak{A}^n \}$

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[1] For a set S, the notation |S| refers to the *order* of S, which is the number of members of S.
[2] For $\mathfrak{S}$ a set of sets, $\bigcap \mathfrak{S} \stackrel{\text{def}}{=} \bigcap \{ S : S \in \mathfrak{S} \}$, the intersection of all sets S in $\mathfrak{S}$.

**Definition 13** For predicate $\mathfrak{P}$ on $\mathfrak{A}^n$, $\mathfrak{P}(\mathfrak{A}^n) \stackrel{\text{def}}{=} \{ p \in \mathfrak{A}^n : \mathfrak{P}(p) = 1 \}$

**Theorem 1** For predicates $\mathfrak{P}_1$ and $\mathfrak{P}_2$ on $\mathfrak{A}^n$, $(\mathfrak{P}_1 \wedge \mathfrak{P}_2)(\mathfrak{A}^n) = \mathfrak{P}_1(\mathfrak{A}^n) \cap \mathfrak{P}_2(\mathfrak{A}^n)$.

*Proof*

a) $(\mathfrak{P}_1 \wedge \mathfrak{P}_2)(\mathfrak{A}^n) =$

b) $\{ p \in \mathfrak{A}^n : (\mathfrak{P}_1 \wedge \mathfrak{P}_2)(p) = 1 \} =$ [Definition 11]

c) $\{ p \in \mathfrak{A}^n : \min( \mathfrak{P}_1(p), \mathfrak{P}_2(p) ) = 1 \} =$

d) $\{ p \in \mathfrak{A}^n : \mathfrak{P}_1(p) = 1$ *and* $\mathfrak{P}_2(p) = 1 \} =$

e) $\mathfrak{P}_1(\mathfrak{A}^n) \cap \mathfrak{P}_2(\mathfrak{A}^n)$ □

**Theorem 2** For predicates $\mathfrak{P}_1$, $\mathfrak{P}_2$, $\mathfrak{P}_3$ on $\mathfrak{A}^n$, $(\mathfrak{P}_1 \wedge \mathfrak{P}_2) \wedge \mathfrak{P}_3 = \mathfrak{P}_1 \wedge (\mathfrak{P}_2 \wedge \mathfrak{P}_3)$

*Proof*

a) $(\mathfrak{P}_1 \wedge \mathfrak{P}_2) \wedge \mathfrak{P}_3 = \mathfrak{P}_1 \wedge (\mathfrak{P}_2 \wedge \mathfrak{P}_3)$ if and only if
For any $p \in \mathfrak{A}^n$, $((\mathfrak{P}_1 \wedge \mathfrak{P}_2) \wedge \mathfrak{P}_3)(p) = (\mathfrak{P}_1 \wedge (\mathfrak{P}_2 \wedge \mathfrak{P}_3))(p)$

b) $((\mathfrak{P}_1 \wedge \mathfrak{P}_2) \wedge \mathfrak{P}_3)(p) =$ [Definition 11]

c) $\min((\mathfrak{P}_1 \wedge \mathfrak{P}_2)(p), \mathfrak{P}_3(p)) =$

d) $\min( \min(\mathfrak{P}_1(p), \mathfrak{P}_2(p)), \mathfrak{P}_3(p)) =$

e) $\min(\mathfrak{P}_1(p), \mathfrak{P}_2(p), \mathfrak{P}_3(p)) =$

f) $\min( \mathfrak{P}_1(p), \min(\mathfrak{P}_2(p), \mathfrak{P}_3(p)) ) =$

g) $\min( \mathfrak{P}_1(p), (\mathfrak{P}_2 \wedge \mathfrak{P}_3)(p)) ) =$

h) $(\mathfrak{P}_1 \wedge (\mathfrak{P}_2 \wedge \mathfrak{P}_3))(p)$ □

# Functional Dependencies

We have established a representation for the attribute space of a set of attributes, and we have established a notion of predication on the attribute space. We are now ready to establish a notion of functional dependency based on the foregoing notion of predication. Central to our notion of functional dependency is that a *functional dependency is a constraint on the attribute space*. A functional dependency "eliminates a subset of the attribute space from use."

**Definition 14** A *functional dependency* is a set of attributes $A \subset \mathbb{A}^n$, and an attribute $\beta \in \mathbb{A}^n$, commonly written $A \rightarrow \beta$, together with the *designation* that the subset of $\mathfrak{A}^n$, $(\bigwedge A \wedge {\sim}\beta)(\mathfrak{A}^n)$, is *eliminated from use*.

Motivation: $A \rightarrow \beta$ can be read as: "The attributes A together *imply* the attribute $\beta$."[3] From propositional calculus, (X implies Y) is the same as (not (X and not-Y)). Hence our definition. Note that the outermost "not" in "not (X and not-Y)" corresponds to our notion of "eliminated from use." The assertion is that particular combination of attributes *cannot occur*. That is, the combination of attributes represented by any $p \in (\bigwedge A \wedge {\sim}\beta)(\mathfrak{A}^n)$ cannot occur. In context of database tuples, this means no tuple can conform to the combination of attributes designated by p.[4,5]

**Definition 15** A subset of $\mathfrak{A}^n$ is *eliminated from use* if and only if a fuctional dependency designates it as eliminated from use.

The notion of "eliminated from use" as used here is purely nominal.

**Definition 16** For functional dependency $f = A \rightarrow \beta$, $A \subseteq \mathbb{A}^n$, $\beta \in \mathbb{A}^n$, $\mathrm{Pred}(f) \stackrel{\text{def}}{=} \bigwedge A \wedge {\sim}\beta$.

**Lemma 2** f a functional dependency if and only if $\mathrm{Pred}(f)(\mathfrak{A}^n)$ is designated eliminated from use.

*Proof* Definition 14, Definition 15, and Definition 16.

[3] More precisely, if the attributes A are uniquely given, then the attribute $\beta$ is uniquely given
[4] What does it mean for a tuple to conform to a combination of attributes?
[5] Note that per our definition a functional dependency can only have a single atomic attribute to the right of the arrow.

**Definition 17** Define $\mathfrak{E} \subseteq \mathfrak{A}^n$ such that $\mathfrak{E} \stackrel{\text{def}}{=} \{ p \in \mathfrak{A}^n : p \text{ is designated eliminated from use} \}$.

**Definition 18** $\mathfrak{F}^n \stackrel{\text{def}}{=} \{ f : f \text{ a functional dependency on } \mathfrak{A}^n \}$

**Definition 19** For $F \subseteq \mathfrak{F}^n$, $\text{Pred}(F)(\mathfrak{A}^n) \stackrel{\text{def}}{=} \cup \{ \text{Pred}(f)(\mathfrak{A}^n) : f \in F \}$ [6]

**Definition 20** For $F \subseteq \mathfrak{F}^n$, $F \Leftrightarrow \forall f \in F, f$ .

**Theorem 3** For $F \subset \mathfrak{F}^n$, $F \Leftrightarrow \text{Pred}(F)(\mathfrak{A}^n) \subseteq \mathfrak{E}$ .

*Proof*

a) For $F \subset \mathfrak{F}^n$, $F \Leftrightarrow$ [Definition 20]
b) $\forall f \in F, f \Leftrightarrow$
c) $\forall f \in F, \text{Pred}(f)(\mathfrak{A}^n) \subseteq \mathfrak{E} \Leftrightarrow$ [Definition 14,Definition 17]
d) $\cup \{ \text{Pred}(f)(\mathfrak{A}^n): f \in F \} \subseteq \mathfrak{E} \Leftrightarrow$ [Definition 19]
e) $\text{Pred}(F)(\mathfrak{A}^n) \subseteq \mathfrak{E}$. □

# Implication

Here we develop the notion of implication with respect to functional dependencies and their predicates.

**Theorem 4** For $F \subset \mathfrak{F}^n$, $f \in \mathfrak{F}^n$, $\text{Pred}(f)(\mathfrak{A}^n) \subseteq \text{Pred}(F)(\mathfrak{A}^n) \Longrightarrow F$ implies f

*Proof*

a) $F \Longrightarrow \text{Pred}(F)(\mathfrak{A}^n) \subseteq \mathfrak{E}$
b) $\text{Pred}(f)(\mathfrak{A}^n) \subseteq \text{Pred}(F)(\mathfrak{A}^n) \Longrightarrow \text{Pred}(f)(\mathfrak{A}^n) \subseteq \mathfrak{E} \Longrightarrow f$ [Given and a)]
c) Therefore if $\text{Pred}(f)(\mathfrak{A}^n) \subseteq \text{Pred}(F)(\mathfrak{A}^n)$ then F implies f □

[6] In general, if $\diamond$ is a binary operation on set S, and $A \subseteq S$, then $\diamond A \stackrel{\text{def}}{=} (a_i \diamond \ldots \diamond a_j)$, for all $a_i .. a_j \in A$

**Theorem 5** For $F \subset \mathfrak{F}^n$, $f \in \mathfrak{F}^n$, F implies f $\Longrightarrow$ $Pred(f)(\mathfrak{A}^n) \subseteq Pred(F)(\mathfrak{A}^n)$ .

*Proof*

a) F implies f $\Longleftrightarrow$ [Theorem 3 ]

b) $Pred(F)(\mathfrak{A}^n) \subseteq \mathfrak{E}$ implies f $\Longleftrightarrow$ [Lemma 2 ]

c) $Pred(F)(\mathfrak{A}^n) \subseteq \mathfrak{E}$ implies $Pred(f)(\mathfrak{A}^n) \subseteq \mathfrak{E}$, hence,

d) F implies f $\Longleftrightarrow$ $Pred(F)(\mathfrak{A}^n) \subseteq \mathfrak{E}$ implies $Pred(f)(\mathfrak{A}^n) \subseteq \mathfrak{E}$

e) Let $p \in Pred(f)(\mathfrak{A}^n)$ and assume $p \notin Pred(F)(\mathfrak{A}^n)$

f) Then $Pred(F)(\mathfrak{A}^n) \subseteq \mathfrak{E}$ does not imply $p \subseteq \mathfrak{E}$ $\rightarrow\leftarrow$

g) Therefore, if $Pred(F)(\mathfrak{A}^n) \subseteq \mathfrak{E}$ implies $Pred(f)(\mathfrak{A}^n) \subseteq \mathfrak{E}$,
then $p \in Pred(f)(\mathfrak{A}^n)$ implies $p \in Pred(F)(\mathfrak{A}^n)$ , or

h) If $Pred(F)(\mathfrak{A}^n) \subseteq \mathfrak{E} \Longrightarrow Pred(f)(\mathfrak{A}^n) \subseteq \mathfrak{E}$
then $Pred(f)(\mathfrak{A}^n) \subseteq Pred(F)(\mathfrak{A}^n)$

Combining d) and h) we get,

i) F implies f $\Longrightarrow$ $Pred(f) \subseteq Pred(F)$ □

**Theorem 6** For $F_1, F_2 \subseteq \mathfrak{F}^n$, $Pred(F_1)(\mathfrak{A}^n) \subseteq Pred(F_2)(\mathfrak{A}^n) \Longleftrightarrow F_2$ implies $F_1$ .

*Proof*

a) $Pred(F_1)(\mathfrak{A}^n) \subseteq Pred(F_2)(\mathfrak{A}^n)$ $\Longleftrightarrow$

b) $\forall\ f \in F_1$, $Pred(f)(\mathfrak{A}^n) \subseteq Pred(F_2)(\mathfrak{A}^n)$ $\Longleftrightarrow$ [Theorem 4 ]

c) $\forall\ f \in F_1$, $F_2$ implies f $\Longleftrightarrow$

d) $F_2$ implies $F_1$ □

**Theorem 7** For $F, F' \subseteq \mathfrak{F}^n$, If $F' \subseteq F$ then F implies F´

*Proof*

a) $F' \subseteq F$ $\Rightarrow$

b) $\forall\ f \in F'$, $f \in F$ $\Rightarrow$

c) $\forall\ f \in F'$, $Pred(f)(\mathfrak{A}^n) \subseteq Pred(F)(\mathfrak{A}^n)$ $\Rightarrow$

d) $Pred(F')(\mathfrak{A}^n) \subseteq Pred(F)(\mathfrak{A}^n)$ □

Note the converse, *If F implies F´ then F´ ⊆ F*, is not generally true.

**Notation 2** For $F \subset \mathfrak{F}^n$, $f \in \mathfrak{F}^n$, if F implies f, then write $F \vDash f$ .

# Covers

Since per our definition a functional dependency corresponds to a subset of the attribute space, the notion of "cover" falls into place as simply that of set containment.

**Definition 21** For predicates $\mathfrak{P}_1$, $\mathfrak{P}_2$ on $\mathfrak{A}^n$, $\mathfrak{P}_1(\mathfrak{A}^n) \subseteq \mathfrak{P}_2(\mathfrak{A}^n) \iff \mathfrak{P}_2$ *covers* $\mathfrak{P}_1$ .

**Definition 22** For $F_1, F_2 \subset \mathfrak{F}^n$, Pred($F_1$) covers Pred($F_2$) $\iff F_1$ covers $F_2$ .

# Closure

Closure of a set of functional dependencies under implication.

**Definition 23** For $F \subseteq F^n$, $F^+ \stackrel{\text{def}}{=} \{ f \in F^n \ni F \text{ implies } f \}$

**Definition 24** For $A \subseteq \mathbb{A}^n$, $A^+ \stackrel{\text{def}}{=} \{ \beta \in \mathbb{A}^n \text{ such that } \exists\ (A \rightarrow \beta) \in F^+ \}$

# The Armstrong Axioms

## Introduction

Conventional approaches to functional dependency are based on the three Armstrong Axioms: *Reflexivity, Transitivity, and Augmentation*. In such treatments the notions of cover and minimum cover are based on the axioms. As seen above, we have developed the notions of cover and minimum cover without use of the Armstrong Axioms. Since we have defined a functional dependency in terms of a subset of the attribute space, the notions of cover and minimum cover are naturally expressed in terms of set containment.

Here we prove the Armstrong Axioms as theorems. This shows that the semantics of our treatment implies the usual semantics based on Armstrong's Axioms.

Note that in our proofs of the Armstrong Axioms none of the Armstrong Axioms are employed. Hence we show that the Armstrong Axioms are natural consequences of the attribute space model for functional dependencies.

# Proofs of Armstrong's Axioms

## Reflexivity

**Theorem 8** For attribute $\alpha \in \mathbb{A}^n$, $\alpha \rightarrow \alpha$ .

*Proof*

a) $\alpha \rightarrow \alpha \Leftrightarrow$

b) Pred$(\alpha \wedge {\sim}\alpha)(\mathfrak{A}^n) \subseteq \mathfrak{E}$

c) Since Pred$(\alpha \wedge {\sim}\alpha)(\mathfrak{A}^n) = \varnothing$, Pred$(\alpha \wedge {\sim}\alpha)(\mathfrak{A}^n) \subseteq \mathfrak{E}$ □

## Transitivity

**Theorem 9** For attributes $\alpha, \beta, \gamma \in \mathfrak{A}^n$, if $\alpha \rightarrow \beta$, and $\beta \rightarrow \gamma$ then $\alpha \rightarrow \gamma$.

*Proof*

a) $\{ \alpha \rightarrow \beta, \beta \rightarrow \gamma \}$ implies $\alpha \rightarrow \gamma$ $\Rightarrow$

b) Pred(f)$(\mathfrak{A}^n) \subseteq$ Pred(F)$(\mathfrak{A}^n)$ $\Rightarrow$ [Theorem 5 ]

c) Pred$(\alpha \rightarrow \gamma)(\mathfrak{A}^n) \subseteq$ Pred$(\{ \alpha \rightarrow \beta, \beta \rightarrow \gamma \})(\mathfrak{A}^n)$ $\Leftrightarrow$

d) For $p \in \mathfrak{A}^n$, $p \in$ Pred$(\alpha \wedge {\sim}\gamma)(\mathfrak{A}^n)$ implies

$p \in$ Pred$(\alpha \wedge {\sim}\beta) \cup$ Pred$(\beta \wedge {\sim}\gamma)(\mathfrak{A}^n)$

e) $p \in$ Pred$(\alpha \wedge {\sim}\gamma)(\mathfrak{A}^n) \Leftrightarrow p(\alpha) = 1$ and $p(\gamma) = 0$

f) $p \in$ Pred$(\alpha \wedge {\sim}\beta) \cup$ Pred$(\beta \wedge {\sim}\gamma)(\mathfrak{A}^n) \Leftrightarrow$

[$p(\alpha)$=1 and $p(\beta)$=0] or [$p(\beta)$=1 and $p(\gamma)$=0]

Case $p(\beta) = 1$:

g) Then $p \in$ Pred$(\alpha \wedge {\sim}\gamma)(\mathfrak{A}^n) \Rightarrow p \in$ (Pred$(\beta \wedge {\sim}\gamma)(\mathfrak{A}^n)$ [e) f) ]

Case $p(\beta) = 0$:

h) Then $p \in$ Pred$(\alpha \wedge {\sim}\gamma)(\mathfrak{A}^n) \Rightarrow p \in$ Pred$(\alpha \wedge {\sim}\beta)(\mathfrak{A}^n)$ [e) f) ]

i) Therefore $p \in$ Pred$(\alpha \wedge {\sim}\gamma)(\mathfrak{A}^n)$ implies [g) h) ]

$p \in$ Pred$(\alpha \wedge {\sim}\beta) \cup$ Pred$(\beta \wedge {\sim}\gamma)(\mathfrak{A}^n)$ $\Rightarrow$

j) Pred$(\alpha \rightarrow \gamma)(\mathfrak{A}^n) \subseteq$ Pred$(\{ \alpha \rightarrow \beta, \beta \rightarrow \gamma \})(\mathfrak{A}^n)$ $\Rightarrow$ [Theorem 4 ]

k) $\alpha \rightarrow \beta$ and $\beta \rightarrow \gamma$ implies $\alpha \rightarrow \beta$ □

## Augmentation

**Theorem 10** For attributes $\alpha, \beta, \delta \in \mathbb{A}^n$, if $\alpha \rightarrow \beta$ then $\alpha,\delta \rightarrow \beta$

*Proof*

a) $\alpha \rightarrow \beta$ implies $\alpha, \delta \rightarrow \beta$ $\Leftrightarrow$

b) $\mathrm{Pred}(\alpha, \delta \rightarrow \beta) \subseteq \mathrm{Pred}(\alpha \rightarrow \beta)$ $\Leftrightarrow$

c) For any $p \in \mathfrak{A}^n$, $(\alpha \wedge \gamma \wedge {\sim}\beta)(p) = 1$ implies $(\alpha \wedge {\sim}\beta)(p) = 1$

d) Since $(\alpha \wedge \gamma \wedge {\sim}\beta)(p) = 1$ only if $\alpha(p) = 1$ and ${\sim}\beta(p) = 1$

e) $p \in \mathrm{Pred}(\alpha \wedge \gamma \wedge {\sim}\beta)$ implies $p \in \mathrm{Pred}(\alpha \wedge {\sim}\beta)$ $\Rightarrow$

f) $\mathrm{Pred}(\alpha \wedge \gamma \wedge {\sim}\beta) \subseteq \mathrm{Pred}(\alpha \wedge {\sim}\beta)$ $\Rightarrow$ [Theorem 4 ]

g) $\alpha \rightarrow \beta$ implies $\alpha,\delta \rightarrow \beta$ □

# Comparison to Attribute Space Model

Does the attribute space model imply the same functional dependencies as Armstrong's Axioms?

**Definition 25** For $F \subseteq \mathfrak{F}^n$, the *Armstrong closure* of F, here written F*, is the smallest set containing F such that Armstrong's Axioms cannot be applied to the set to yield a functional dependency not in the set.

**Theorem 11** For $F \subset \mathfrak{F}^n$, $F^+ = F^*$.

*Proof*

a) Per transitivity we have shown that for $\{\alpha \rightarrow \beta, \beta \rightarrow \gamma\} \subset \mathfrak{F}^n$, $\{\alpha \rightarrow \beta, \beta \rightarrow \gamma\} \vDash \alpha \rightarrow \gamma$, therefore if f is derived from $F^+$ by transitivity, then $f \in F^+$.

b) Per augmentation we have shown that for $\{\alpha, \beta \rightarrow \gamma, \delta \rightarrow \beta\} \subset \mathfrak{F}^n$, $\{\alpha, \beta \rightarrow \gamma, \delta \rightarrow \beta\} \vDash \alpha, \delta \rightarrow \gamma$, therefore if $\alpha, \delta \rightarrow \gamma$ is derived from $F^+$ by augmentation, then $\alpha, \delta \rightarrow \gamma \in F^+$.

c) Per reflexivity, since for any $F \subset \mathfrak{F}^n$, for any $\alpha \in \mathbb{A}^n$, $F \vDash \alpha \rightarrow \alpha$ , [Corrolary 1 ] for any $f \in \mathfrak{F}^n$ such that f is derived from $F^+$ via reflexivity, $f \in F^+$.

d) We have shown that $(F^+)^* \subseteq F^+$. But since $F \subseteq F^+$, $F^* \subseteq (F^+)^*$, and we have shown that $F^* \subseteq F^+$ □

# Subsets of Attribute Space $\mathfrak{A}^n$ as Set Maps

We have represented functional dependencies as predicates on $\mathfrak{A}^n$. To facilitate operations on such predicates, or more precisely the subsets of $\mathfrak{A}^n$ designated by the predicates, we now introduce a mapping of subsets of $\mathfrak{A}^n$ as designated by predicates to a simplified set representaion on which set operations can be easily performed.

Note that a predicate describes a set of points in $\mathfrak{A}^n$, and that such sets are integers in the range 1 to $2^n$. Such sets can be represented by $2^n$-tupes of values in {0, 1}. [7]

**Definition 26** $\mathfrak{M}^n \stackrel{\text{def}}{=} \{0, 1\}^{2^n}$.

$\mathfrak{M}^n$ consists of $2^n$-tuples of the values {0, 1}.

**Definition 27** For $m_1, m_2 \in \mathfrak{M}^n$, $m_1 \sqcap m_2 \stackrel{\text{def}}{=} m' \in \mathfrak{M}^n \ni \forall\ i \in 1..\ 2^n$, $m'[i] = \min(m_1[i], m_2[i])$

**Definition 28** For $m_1, m_2 \in \mathfrak{M}^n$, $m_1 \sqcup m_2 \stackrel{\text{def}}{=} m' \in \mathfrak{M}^n \ni \forall\ i \in 1..\ 2^n$, $m'[i] = \max(m_1[i], m_2[i])$
Considering members of $\mathfrak{M}^n$ as representing sets, $\sqcup$ and $\sqcap$ correspond to union and intersection operations on members of $\mathfrak{M}^n$.

**Definition 29** For $M \subseteq \mathfrak{M}^n$, $\sqcap M \stackrel{\text{def}}{=} \sqcap \{ m_i \in M \} = m_i \sqcap \ldots \sqcap m_j$ , for $m_i, \ldots, m_j \in M$.

**Definition 30** For $m_1, m_2 \in \mathfrak{M}^n$, define $m_1 = m_2$ if and only if $\forall\ i \in 1..\ 2^n$, $m_1[i] = m_2[i]$ .

**Definition 31** For $m_1, m_2 \in \mathfrak{M}^n$, $m_1 \sqsubseteq m_2 \iff \forall\ i \in 1..2^n$, $m_1[i] = 1 \implies m_2[i] = 1$ .

Here we map predicates on $\mathfrak{A}^n$ to members of $\mathfrak{M}^n$.

**Definition 32** Map:$\mathfrak{P}^n \rightarrow \mathfrak{M}^n$ a function such that for $\mathcal{P} \in \mathfrak{P}^n$, $\text{Map}(\mathcal{P}) \stackrel{\text{def}}{=} m \in \mathfrak{M}^n$ such that for $i \in 1..2^n$, $p_i \in \mathfrak{A}^n$, $m[i] = \mathcal{P}(p_i)$ .

Note that for predicate $\mathcal{P}$, $\mathcal{P}(\mathfrak{A}^n) = \{ p \in \mathfrak{A}^n : \mathcal{P}(p) = 1 \}$

For any predicate $\mathcal{P}$, Map($\mathcal{P}$) is a representation of the set of points $p \in \mathcal{P}(\mathfrak{A}^n)$. Since a functional dependency f corresponds to a subset of $\mathfrak{A}^n$, Pred(f)($\mathfrak{A}^n$), we can represent the set designated by a functional dependency with the map Map(Pred(f)). A set of functional dependencies F also designates a subset of $\mathfrak{A}^n$, Pred(F)($\mathfrak{A}^n$), with the map representation Map(Pred(F)). Hence we define,

[7] That is, tuples of $2^n$ terms.

**Definition 33** For a set of functional dependencies F, define $\mathrm{Map}(F) \stackrel{\text{def}}{=} \mathrm{Map}(\mathrm{Pred}(F))$.

**Theorem 12** For predicates $\mathfrak{P}_1, \mathfrak{P}_2 \in \mathfrak{P}^n$, if $\mathfrak{P}_1(\mathfrak{A}^n) \subseteq \mathfrak{P}_2(\mathfrak{A}^n)$ then $\mathrm{Map}(\mathfrak{P}_1) \sqsubseteq \mathrm{Map}(\mathfrak{P}_2)$ .

*Proof*

a) For $i \in 1..2^n$, $\mathrm{Map}(\mathfrak{P}_1)[i] = 1 \Leftrightarrow p_i \in \mathfrak{P}_1(\mathfrak{A}^n)$

b) For $i \in 1..2^n$, $\mathrm{Map}(\mathfrak{P}_2)[i] = 1 \Leftrightarrow p_i \in \mathfrak{P}_2(\mathfrak{A}^n)$

c) Since for $i \in 1..2^n$, $p_i \in \mathfrak{P}_1(\mathfrak{A}^n)$ only if $p_i \in \mathfrak{P}_2(\mathfrak{A}^n)$, [$\mathfrak{P}_1(\mathfrak{A}^n) \subseteq \mathfrak{P}_2(\mathfrak{A}^n)$]
$\mathrm{Map}(\mathfrak{P}_1)[i] = 1$ only if $\mathrm{Map}(\mathfrak{P}_2)[i] = 1$

d) Since $\forall\ i \in 1..2^n$, $\mathrm{Map}(\mathfrak{P}_1)[i] = 1 \Rightarrow \mathrm{Map}(\mathfrak{P}_2)[i] = 1$, [Definition 31]
$\mathrm{Map}(\mathfrak{P}_1) \sqsubseteq \mathrm{Map}(\mathfrak{P}_2)$ □

**Theorem 13** For $F_1, F_2 \subset \mathfrak{F}^n$, $F_1$ implies $F_2$, $\Leftrightarrow \mathrm{Map}(F_2) \sqsubseteq \mathrm{Map}(F_1)$ .

*Proof*

a) $F_1$ implies $F_2 \Leftrightarrow$

b) $\mathrm{Pred}(F_1)(\mathfrak{A}^n) \subseteq \mathrm{Pred}(F_2)(\mathfrak{A}^n) \Leftrightarrow$ [Theorem 12 ]

c) $\mathrm{Map}(\mathrm{Pred}(F_1)) \sqsubseteq \mathrm{Map}(\mathrm{Pred}(F_2)) \Leftrightarrow$

d) $\mathrm{Map}(F_1) \sqsubseteq \mathrm{Map}(F_2)$ □

**Theorem 14** For $F_1, F_2 \subset \mathfrak{F}^n$, $F_1$ covers $F_2$, $\Leftrightarrow \mathrm{Map}(F_2) \sqsubseteq \mathrm{Map}(F_1)$ .

*Proof*

a) $F_1$ covers $F_2 \Leftrightarrow$

b) $\mathrm{Pred}(F_1)(\mathfrak{A}^n)$ covers $\mathrm{Pred}(F_2)(\mathfrak{A}^n) \Leftrightarrow$

c) $\mathrm{Pred}(F_2)(\mathfrak{A}^n) \subseteq \mathrm{Pred}(F_1)(\mathfrak{A}^n) \Leftrightarrow$

d) $\mathrm{Map}(\mathrm{Pred}(F_2)) \sqsubseteq \mathrm{Map}(\mathrm{Pred}(F_1)) \Leftrightarrow$

e) $\mathrm{Map}(F_2) \sqsubseteq \mathrm{Map}(F_1)$ □

Theorem 13 and Theorem 14 show that we can represent the fundamental relations between sets of functional dependencies of implication and coverage in terms of their maps.

Given the obvious interpretation of the maps $\mathfrak{M}^n$ as bitmaps in a computer program, in the next article we present a Python program for determining minimum covers for a set of functional dependencies, as well as the sets of functional dependences that imply a given functional dependency.

Definition 32 is a key definition. $\mathfrak{A}^n$ is a sequence of $2^n$ terms, where each term is itself a sequence of n terms. $\phi(\mathbb{A}^n)$ is a sequence of n terms, where for i∈1..n, $\phi(\mathbb{A}^n)[i] = \phi(\alpha_i) = \mathrm{m}_{\alpha_i}$.

Hence $\phi(\mathbb{A}^n)$ is a sequence of n terms where each term is a sequence of $2^n$ terms. If $\mathfrak{A}^n$ is regarded as a two-dimensional array of $2^n$ columns and n rows, $\phi(\mathbb{A}^n)$ is a two-dimensional array of n rows and $2^n$ columns. The $i^{th}$ row in $\phi(\mathbb{A}^n)$ is a "slice" across the columns of $\mathfrak{A}^n$, taking the $i^{th}$ term in each column of $\mathfrak{A}^n$. Note the $i^{th}$ row in the "virtual" 2-dimensional array $\mathfrak{A}^n$ refers to the single attribute $\alpha_i$, and the $i^{th}$ row $\mathrm{m}_{\alpha_i}$ in $\phi(\mathbb{A}^n)$ refers to the single attribute $\alpha_i$.

An attribute $\alpha_i$ specifies a subset of the terms in $p \in \mathfrak{A}^n$ where $p(\alpha_i) = p[i] = 1$.

The conjunction of two attributes $\alpha_i$ and $\alpha_j$ specifies the subset of terms $p \in \mathfrak{A}^n$ where $p(\alpha_i) = 1$ *and* $p(\alpha_j) = 1$. In other words, $(\alpha_i \wedge \alpha_j)(\mathfrak{A}^n) = \alpha_i(\mathfrak{A}^n) \cap \alpha_j(\mathfrak{A}^n)$.

Any conjunction of atomic attributes or their negations specifies a subset of $\mathfrak{A}^n$.

**Theorem 15** For atomic predicates α and β, $\phi(\alpha(\mathfrak{A}^n)) = \phi(\beta(\mathfrak{A}^n)) \Leftrightarrow \alpha = \beta$

*Proof* *CLEAN UP*

$\mathrm{idx}(\alpha) \neq \mathrm{idx}(\beta) \Leftrightarrow \exists\, p \in \mathfrak{A}^n$ such that $p[\mathrm{idx}(\alpha)] \neq p[\mathrm{idx}[\beta]] \Leftrightarrow \phi(\alpha) \neq \phi(\beta)$ .

**Theorem 16** For predicates $\mathcal{P}_1$ and $\mathcal{P}_2$ on $\mathfrak{A}^n$, $\phi(\mathcal{P}_1 \wedge \mathcal{P}_2)(\mathfrak{A}^n) = \phi(\mathcal{P}_1)(\mathfrak{A}^n) \sqcap \phi(\mathcal{P}_2)(\mathfrak{A}^n)$

*Proof*

a)

**Definition 34** For $f \in \mathfrak{F}^n$, $\phi(f) \stackrel{\text{def}}{=} \phi(\mathrm{Pred}(f))$.

**Definition 35** For F a set of functional dependencies on attributes $\mathbb{A}^n$,

$\phi(F) \stackrel{\text{def}}{=} \sqcup \{ \phi(f) : f \in F \}$

Example

If $F = \{f_1, f_3, f_4\}$ then $\phi(F) = \phi(f_1) \sqcup \phi(f_3) \sqcup \phi(f_4)$ .

**Definition 36** For $A \subset \mathbb{A}^n$, $\beta \in \mathbb{A}^n$, and functional dependency $A \rightarrow \beta$,

$$\phi(A \rightarrow \beta)(\mathfrak{A}^n) \stackrel{\text{def}}{=} \phi(\bigwedge A \wedge {\sim}\beta)(\mathfrak{A}^n)$$ [8]

**Definition 37** For F a set of functional dependencies, $F' \subseteq F$,

If $F'$ is a *cover* for F then $\phi(F) \sqsubseteq \phi(F')$ .

**Theorem 17** For $F_1$ and $F_2$, sets of functional dependencies defined on attributes $\mathbb{A}^n$,

If $F_1$ implies $F_2$ then $\phi(F_2)(\mathfrak{A}^n) \sqsubseteq \phi(F_1)(\mathfrak{A}^n)$ .

*Proof*

a) If $F_1$ implies $F_2$ then $\mathrm{Pred}(F_2)(\mathfrak{A}^n) \subseteq \mathrm{Pred}(F_1)(\mathfrak{A}^n)$ [

b)

[8] **CAN MAKE A PROPOSITION?**

 

# Appendix A

Before proceeding to a proof of Definition 40some groundwork to be laid.

**Definition 38** Define $\multimap$ a binary relation on $\mathfrak{P}^n$ such that,

c) For $\mathrm{P}_1$ , $\mathrm{P}_2 \in \mathfrak{P}^n$, $\mathrm{P}_1 \multimap (\mathrm{P}_1 \wedge \mathrm{P}_2)$ and $\mathrm{P}_2 \multimap (\mathrm{P}_1 \wedge \mathrm{P}_2)$

d) For $\mathrm{P}$, $\mathrm{P}_1$, $\mathrm{P}_2 \in \mathfrak{P}^n$, $\mathrm{P} \neq \mathrm{P}_1, \mathrm{P}_2$, $\mathrm{P} \multimap (\mathrm{P}_1 \wedge \mathrm{P}_2)$ implies $\mathrm{P} \multimap \mathrm{P}_1$ or $\mathrm{P} \multimap \mathrm{P}_2$

Informally, $\multimap$ may be interpreted as indicating the relationship *is a component of.*

**Definition 39** Define $\bullet\!\!\multimap$ a binary relation on $\mathfrak{P}^n$ such that for $\mathrm{P}_1$, $\mathrm{P}_2 \in \mathfrak{P}^n$,

If $\mathrm{P}_1 \multimap \mathrm{P}_2$ and $\mathrm{P}_1 \in {}^{\oplus}\mathbb{A}^n$ then $\mathrm{P}_1 \bullet\!\!\multimap \mathrm{P}_2$ .

Informally, $\bullet\!\!\multimap$ may be interpreted as indicating the relationship: *is an atomic component of.*

**Definition 40** For $\mathrm{P} \in \mathfrak{P}^n$, $\mathrm{Base}(\mathrm{P}) \stackrel{\text{def}}{=} \{ \beta \in {}^{\oplus}\mathbb{A}^n : \beta \bullet\!\!\multimap \mathrm{P} \}$ )

**Exercise 2** If Base($\mathrm{P}$) includes both $\alpha \in \mathbb{A}^n$, as well as $\sim\alpha$,
what can be said about the set $\mathrm{P}(\mathfrak{A}^n)$?

**Exercise 3** If Base($\mathrm{P}$) = $\mathbb{A}^n$, what can be said about the set $\mathrm{P}(\mathfrak{A}^n)$ ?

**Theorem 18** For predicate $\mathrm{P}$ on $\mathfrak{A}^n$, for $p \in \mathfrak{A}^n$, $\mathrm{P}(p) = \min(\{ \beta(p) : \beta \in \mathrm{Base}(\mathrm{P}) \})$ .

*Proof* See Appendix A for a proof of Definition 40.

Informally, $\mathcal{L}$ can be regarded as assigning a "level" to a predicate $\mathrm{P}$. For an atomic predicate $\beta \in {}^{\oplus}\mathbb{A}^n$ , $\mathcal{L}(\beta) = 1$. For a compound predicate $\alpha \wedge \beta$ , $\mathcal{L}(\alpha \wedge \beta) = 2$. $\mathcal{L}( \alpha \wedge (\beta \wedge \delta) ) = 3$.

Formally,

**Definition 1** $\mathcal{L} : \mathfrak{P}^n \to \mathbb{N}$ a function such that:

a) For $\beta \in {}^{\oplus}\mathbb{A}^n$ , $\mathcal{L}(\beta) = 1$

b) For $\mathrm{P}_1$, $\mathrm{P}_2 \in \mathfrak{P}^n$, $\mathcal{L}(\mathrm{P}_1 \wedge \mathrm{P}_2) = \max(\mathcal{L}(\mathrm{P}_1), \mathcal{L}(\mathrm{P}_2)) + 1$

**Example** For $\alpha, \beta \in \mathbb{A}^n$, $\mathcal{L}(\alpha \wedge \beta)$ = max($\mathcal{L}$(

For $\alpha, \beta, \delta \in \mathbb{A}^n$, $\mathcal{L}(\alpha \wedge (\beta \wedge \delta)) = \max( \mathcal{L}(\alpha), \mathcal{L}(\beta \wedge \delta) ) + 1 =$

$\max( \mathcal{L}(\alpha), \max(\mathcal{L}(\beta), \mathcal{L}(\delta))+1) + 1 = \max(1, \max(1, 1)+1) +1 = 3$

## Proof of the Theorem

Here the function $\mathcal{L}$ is used to assign a "level" to the components of $\mathrm{P}$ ( $\{ \mathrm{P}_i \in \mathfrak{P}^n : \mathrm{P}_i \multimap \mathrm{P} \}$ ) to create a hierarchical structure for applying an inductive proof. First a demonstration of the proposition to be proved for k=2. Next a demonstration that if the proposition is assumed true for $\mathcal{L}$ = k-1, the proposition is true for $\mathcal{L}$ = k. Finally the induction principle is invoked to prove the proposition for all k ≥ 2.

**Theorem 1** For any $\mathrm{P} \in \mathfrak{P}^n$, $p \in \mathfrak{A}^n$, $\mathrm{P}(p) = \min(\{\beta(p) : \beta \multimapdot \mathrm{P}\})$

*Proof*

Case 1: Assume $\mathcal{L}(\mathrm{P}) = 2$

a) There exist $\alpha, \beta \in {}^{\oplus}\mathbb{A}^n$ such that $\mathrm{P} = (\alpha \wedge \beta) \Rightarrow$ [ ]

b) $\mathrm{P}(p) = (\alpha \wedge \beta)(p) = \min(\alpha(p), \beta(p)) \Rightarrow$

c) $\mathrm{P}(p) = \min(\{\beta(p) : \beta \multimapdot \mathrm{P}\})$ [because $\alpha, \beta$ atomic, $\alpha, \beta \multimapdot \mathrm{P}$]

Case 2: Assume $\mathcal{L}(\mathrm{P}) = k$, $\mathrm{P} = \mathrm{P}_1 \wedge \mathrm{P}_2$

a) For $\mathrm{P}_0 \in \mathfrak{P}^n$, assume $\mathcal{L}(\mathrm{P}_0) \le k-1$ implies $\mathrm{P}_0(p) = \min(\{\beta(p) : \beta \multimapdot \mathrm{P}_0\})$ [Def. of $\mathcal{L}$ ]

b) $\mathcal{L}(\mathrm{P}_1)$ , $\mathcal{L}(\mathrm{P}_2) < \mathcal{L}(\mathrm{P}) \Rightarrow$ [Case 2 Assumption]

c) $\mathrm{P}_1(p) = \min(\{\beta(p) : \beta \multimapdot \mathrm{P}_1\})$ and $\mathrm{P}_2(p) = \min(\{\beta(p) : \beta \multimapdot \mathrm{P}_2\})$

d) $\mathrm{P}(p) = \min(\mathrm{P}_1(p), \mathrm{P}_2(p)) = \min(\min(\{\beta(p): \beta \multimapdot \mathrm{P}_1\}), \min(\{\beta(p) : \beta \multimapdot \mathrm{P}_2\}))$

e) $\{\beta : \beta \multimapdot \mathrm{P}\} = \{\beta : \beta \multimapdot \mathrm{P}_1\} \cup \{\beta : \beta \multimapdot \mathrm{P}_2\} \Rightarrow$ [JUSTIFICATION]
$\beta \multimapdot \mathrm{P}_1 \Rightarrow \beta \multimapdot \mathrm{P}$

f) $\min(\min\{\beta(p): \beta \multimapdot \mathrm{P}_1\}), \min(\beta(p) : \beta \multimapdot \mathrm{P}_2\}) = \min(\{\beta(p) : \beta \multimapdot \mathrm{P}\}) \Rightarrow$

g) $\mathrm{P}(p) = \min(\{\beta(p) : \beta \multimapdot \mathrm{P}\})$

We have shown the proposition is true for $\mathrm{P}$ is $\mathcal{L}(\mathrm{P}) = 2$,
We have shown that if the propostion is true for $\mathrm{P}$ such that $\mathcal{L}(\mathrm{P})$ = k-1, then it is true for $\mathrm{P}$ such that $\mathcal{L}(\mathrm{P})$ = k.
Hence by induction we have proven the proposition for all $\mathrm{P}$ such that $\mathcal{L}(\mathrm{P}) \ge 2$. □

**Theorem 2** For $\mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_3 \in \mathfrak{P}^n$, if $\mathrm{P}_1 \bullet\!\!\multimap \mathrm{P}_2$ and $\mathrm{P}_2 \multimap \mathrm{P}_3$ , then $\mathrm{P}_1 \bullet\!\!\multimap \mathrm{P}_3$ .

*Proof*

a) $\mathrm{P}_1 \bullet\!\!\multimap \mathrm{P}_2 \Longrightarrow \mathrm{P}_1 \multimap \mathrm{P}_2$

b) $\mathrm{P}_1 \multimap \mathrm{P}_2$ and $\mathrm{P}_2 \multimap \mathrm{P}_3 \Longrightarrow \mathrm{P}_1 \multimap \mathrm{P}_3$ [transitivity]

c) But $\mathrm{P}_1 \bullet\!\!\multimap \mathrm{P}_2 \Longrightarrow \mathrm{P}_1$ is atomic.

d) $\mathrm{P}_1$ atomic and $\mathrm{P}_1 \multimap \mathrm{P}_3 \Longrightarrow \mathrm{P}_1 \bullet\!\!\multimap \mathrm{P}_3$ $\square$

**Theorem 3** For any $\mathrm{P} \in \mathfrak{P}^n$, there exists $k \in \mathbb{N}$ such that $\mathcal{L}(\mathrm{P}) = k$ .

**Theorem 4** For any $\mathrm{P}, \mathrm{P}' \in \mathfrak{P}^n$, $\mathrm{P}' \multimap \mathrm{P} \Longrightarrow \mathcal{L}(\mathrm{P}') < \mathcal{L}(\mathrm{P})$ .

**Theorem 5** For $\mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_3 \in \mathfrak{P}^n$, $\mathrm{P}_1 \bullet\!\!\multimap \mathrm{P}_2$ and $\mathrm{P}_2 \multimap \mathrm{P}_3$ implies $\mathrm{P}_1 \bullet\!\!\multimap \mathrm{P}_3$ .

*Proof*

a)

**Corrolary 1** For any $F \subset \mathfrak{F}^n$, $F \vDash \alpha \to \alpha$

*Proof*

a) From Theorem 8 $\text{Pred}(\alpha \to \alpha)(\mathfrak{A}^n) = \varnothing \implies$

b) For any $F \subset \mathfrak{F}^n$, $\text{Pred}(\alpha \to \alpha)(\mathfrak{A}^n) \subseteq \text{Pred}(F)(\mathfrak{A}^n) \implies$

c) $F \vDash \alpha \to \alpha$ □

## Pseudo-Transitivity

**Theorem 6** For attributes $\alpha, \beta, \gamma, \delta \in \mathbb{A}^n$, $\alpha \to \beta$, and $\beta, \gamma \to \delta$ imply $\alpha, \gamma \to \delta$

*Proof*

a) $\{ (\alpha \to \beta), (\beta, \gamma \to \delta)\}$ imply $(\alpha, \gamma \to \delta) \iff$ [Theorem 5 ]

b) $\text{Pred}(\alpha, \gamma \to \delta) \subseteq \text{Pred}(\{ (\alpha \to \beta), (\beta, \gamma \to \delta)\}) \iff$

c) For any $p \in \text{Pred}(\alpha, \gamma \to \delta)$, $p \in \text{Pred}(\alpha \to \beta) \cup \text{Pred}(\beta, \gamma \to \delta) \iff$

d) $p \in \text{Pred}(\alpha, \gamma \to \delta) \iff p(\alpha) = 1$, $p(\gamma) = 1$ and $p(\delta) = 0$

e) $p \in \text{Pred}(\alpha \to \beta) \cup \text{Pred}(\beta, \gamma \to \delta) \iff$

$p(\alpha) = 1$ and $p(\beta) = 0$ or $p(\beta) = 1$, $p(\gamma) = 1$ and $p(\delta) = 0$

Case $p(\beta) = 0$:

f) $p \in \text{Pred}(\alpha, \gamma \to \delta) \implies p(\alpha) = 1$ and $p \in \text{Pred}(\alpha \to \beta)$ [d) ]

Case $p(\beta) = 1$:

g) $p \in \text{Pred}(\alpha, \gamma \to \delta) \implies p(\gamma) = 1$, $p(\delta) = 0$, and $p \in \text{Pred}(\alpha \to \beta)$

h) Hence $p \in \text{Pred}(\alpha, \gamma \to \delta) \implies p \in \text{Pred}(\alpha \to \beta)$ or $p \in \text{Pred}(\alpha \to \beta)$, or

$\text{Pred}(\alpha, \gamma \to \delta) \subseteq \text{Pred}(\alpha \to \beta) \cup \text{Pred}(\alpha \to \beta)$ □

i)