

©2022-2023 William Emerson Six, licensed under the MIT License (at end of this document)

Problem 1. Given vectors \vec{a} and \vec{b} in \mathbb{R}^3 , find a vector \vec{c} in \mathbb{R}^3 that is perpendicular both to \vec{a} and to \vec{b} that follows the right-hand rule. Also, $\|\vec{c}\| = \|\vec{a}\| \|\vec{b}\| \sin(\theta)$

Proof. **Rotate \vec{a} and \vec{b} so that \vec{a} is on the x axis:**

$$\text{Let } \vec{a} \triangleq \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \text{ and let } \vec{b} \triangleq \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

$$\text{let } k_1 \triangleq \sqrt{a_1^2 + a_2^2}.$$

Define $\vec{f}_a^{zx}(\vec{v})$ such that $\vec{f}_a^{zx}(\vec{a})$ is rotated onto the xz plane.

$$\vec{a'} \triangleq \vec{f}_a^{zx} \left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \right) \triangleq \begin{bmatrix} \frac{a_1}{k_1} \\ \frac{-a_2}{k_1} \\ 0 \end{bmatrix} * v_1 + \begin{bmatrix} \frac{a_2}{k_1} \\ \frac{a_1}{k_1} \\ 0 \end{bmatrix} * v_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} * v_3 \quad (1)$$

Define $\vec{f}_{a'}^x(\vec{v})$ such that $(\vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{a})$ is rotated onto the x axis.

$$\vec{f}_{a'}^x(\vec{v}) \triangleq \begin{bmatrix} \frac{k_1}{\|\vec{a}\|} \\ 0 \\ \frac{-a_3}{\|\vec{a}\|} \end{bmatrix} * v_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} * v_2 + \begin{bmatrix} \frac{a_3}{\|\vec{a}\|} \\ 0 \\ \frac{k_1}{\|\vec{a}\|} \end{bmatrix} * v_3 \quad (2)$$

$$\vec{a}'' \triangleq (\vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{v}) = \begin{bmatrix} \frac{k_1}{\|\vec{a}\|} & 0 & \frac{a_3}{\|\vec{a}\|} \\ 0 & 1 & 0 \\ \frac{-a_3}{\|\vec{a}\|} & 0 & \frac{k_1}{\|\vec{a}\|} \end{bmatrix} * \left(\begin{bmatrix} \frac{a_1}{k_1} & \frac{a_2}{k_1} & 0 \\ \frac{-a_2}{k_1} & \frac{a_1}{k_1} & 0 \\ 0 & 0 & 1 \end{bmatrix} * \vec{v} \right) \quad (3)$$

$$= \begin{bmatrix} \frac{a_1}{\|\vec{a}\|} & \frac{a_2}{\|\vec{a}\|} & \frac{a_3}{\|\vec{a}\|} \\ \frac{-a_2}{k_1} & \frac{a_1}{k_1} & 0 \\ \frac{-a_1 a_3}{k_1 \|\vec{a}\|} & \frac{-a_2 a_3}{k_1 \|\vec{a}\|} & \frac{k_1}{\|\vec{a}\|} \end{bmatrix} * \vec{v} \quad (4)$$

$$(\vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{a}) = \begin{bmatrix} \frac{a_1^2}{\|\vec{a}\|} + \frac{a_2^2}{\|\vec{a}\|} + \frac{a_3^2}{\|\vec{a}\|} \\ \frac{-a_2 a_1}{k_1} + \frac{a_1 a_2}{k_1} \\ \frac{-a_1^2 a_3}{k_1 \|\vec{a}\|} + \frac{-a_2^2 a_3}{k_1 \|\vec{a}\|} + \frac{k_1 a_3}{\|\vec{a}\|} \end{bmatrix} \quad (5)$$

$$= \begin{bmatrix} \frac{a_1^2 + a_2^2 + a_3^2}{\|\vec{a}\|} \\ 0 \\ \frac{1}{\|\vec{a}\|} * \left(\frac{-a_1^2 a_3}{k_1} + \frac{-a_2^2 a_3}{k_1} + k_1 a_3 \right) \end{bmatrix} \quad (6)$$

$$= \begin{bmatrix} \frac{\|\vec{a}\|^2}{\|\vec{a}\|} \\ 0 \\ \frac{1}{\|\vec{a}\|} * \left(\frac{-a_1^2 a_3}{k_1} + \frac{-a_2^2 a_3}{k_1} + \frac{k_1^2 a_3}{k_1} \right) \end{bmatrix} \quad (7)$$

$$= \begin{bmatrix} \|\vec{a}\| \\ 0 \\ \frac{1}{\|\vec{a}\|} * \left(\frac{-a_1^2 a_3}{k_1} + \frac{-a_2^2 a_3}{k_1} + \frac{((a_1^2 + a_2^2)) a_3}{k_1} \right) \end{bmatrix} \quad (8)$$

$$= \begin{bmatrix} \|\vec{a}\| \\ 0 \\ 0 \end{bmatrix} \quad (9)$$

$$\vec{b}'' \triangleq (\vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b}) = \begin{bmatrix} \frac{a_1 b_1}{\|\vec{a}\|} + \frac{a_2 b_2}{\|\vec{a}\|} + \frac{a_3 b_3}{\|\vec{a}\|} \\ \frac{-a_2 b_1}{k_1} + \frac{a_1 b_2}{k_1} \\ \frac{-a_1 a_3 b_1}{k_1 \|\vec{a}\|} + \frac{-a_2 a_3 b_2}{k_1 \|\vec{a}\|} + \frac{k_1 b_3}{\|\vec{a}\|} \end{bmatrix} \quad (10)$$

Rotate $\vec{b}'' \triangleq (\vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b})$ onto the xy plane

$$\text{let } k_2 \triangleq \sqrt{b_2''^2 + b_3''^2}.$$

$$\vec{f}_{b''}^{xy}(\vec{v}) \triangleq \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} * v_1 + \begin{bmatrix} 0 \\ \frac{b_2''}{k_2} \\ \frac{-b_3''}{k_2} \end{bmatrix} * v_2 + \begin{bmatrix} 0 \\ \frac{b_3''}{k_2} \\ \frac{b_2''}{k_2} \end{bmatrix} * v_3 \quad (11)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{b_2''}{k_2} & \frac{b_3''}{k_2} \\ 0 & \frac{-b_3''}{k_2} & \frac{b_2''}{k_2} \end{bmatrix} * \vec{v} \quad (12)$$

$$\vec{b}''' \triangleq (\vec{f}_{b''}^{xy} \circ \vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{b_2''}{k_2} & \frac{b_3''}{k_2} \\ 0 & \frac{-b_3''}{k_2} & \frac{b_2''}{k_2} \end{bmatrix} * (\vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b}) \quad (13)$$

Project $(\vec{f}_{b''}^{xy} \circ \vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b})$ **onto the** yz **plane**

Define $\vec{f}_{b'''}^y(\vec{v})$ to project any vector v onto the y axis.

$$\vec{f}_{b'''}^y(\vec{v}) \triangleq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} * v_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} * v_2 + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} * v_3 \quad (14)$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} * \vec{v} \quad (15)$$

$$(\vec{f}_{b'''}^y \circ \vec{f}_{b''}^{xy} \circ \vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{b_2''}{k_2} & \frac{b_3''}{k_2} \\ 0 & \frac{-b_3''}{k_2} & \frac{b_2''}{k_2} \end{bmatrix} * (\vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b}) \quad (16)$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{b_2''}{k_2} & \frac{b_3''}{k_2} \\ 0 & 0 & 0 \end{bmatrix} * (\vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b}) \quad (17)$$

Rotate $(\vec{f}_{b'''}^y \circ \vec{f}_{b''}^{xy} \circ \vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b})$ **90 degrees on the yz plane.**

Define $\vec{f}_y^z(\vec{v})$ rotate any vector v around the yz plane.

$$\vec{f}_y^z(\vec{v}) \triangleq \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} * v_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} * v_2 + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} * v_3 \quad (18)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} * \vec{v} \quad (19)$$

$$(\vec{f}_y^z \circ \vec{f}_{b'''}^y \circ \vec{f}_{b''}^{xy} \circ \vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} * \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{b_2''}{k_2} & \frac{b_3''}{k_2} \\ 0 & 0 & 0 \end{bmatrix} * (\vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b}) \quad (20)$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{b_2''}{k_2} & \frac{b_3''}{k_2} \end{bmatrix} * (\vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b}) \quad (21)$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{b_2''}{k_2} & \frac{b_3''}{k_2} \end{bmatrix} * \begin{bmatrix} b_1'' \\ b_2'' \\ b_3'' \end{bmatrix} \quad (22)$$

$$= \begin{bmatrix} 0 \\ 0 \\ \frac{b_2''^2 + b_3''^2}{\sqrt{b_2''^2 + b_3''^2}} \end{bmatrix} \quad (23)$$

$$= \begin{bmatrix} 0 \\ 0 \\ k_2 \end{bmatrix} \quad (24)$$

Apply inverse of $\vec{f}_{b''}^{xy}$ to $(\vec{f}_y^z \circ \vec{f}_{b'''}^y \circ \vec{f}_{b''}^{xy} \circ \vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b})$

$$((\vec{f}_{b''}^{xy})^{-1} \circ \vec{f}_y^z \circ \vec{f}_{b'''}^y \circ \vec{f}_{b''}^{xy} \circ \vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{b_2''}{k_2} & \frac{-b_3''}{k_2} \\ 0 & \frac{b_3''}{k_2} & \frac{b_2''}{k_2} \end{bmatrix} * \begin{bmatrix} 0 \\ 0 \\ k_2 \end{bmatrix} \quad (25)$$

$$= \begin{bmatrix} 0 \\ -b_3'' \\ b_2'' \end{bmatrix} \quad (26)$$

$$= \begin{bmatrix} 0 \\ \frac{a_1 a_3 b_1}{k_1 \|\vec{a}\|} + \frac{a_2 a_3 b_2}{k_1 \|\vec{a}\|} + \frac{-k_1 b_3}{\|\vec{a}\|} \\ \frac{-a_2 b_1}{k_1} + \frac{a_1 b_2}{k_1} \end{bmatrix} \quad (27)$$

$$= \frac{1}{k_1 \|\vec{a}\|} * \begin{bmatrix} 0 \\ a_1 a_3 b_1 + a_2 a_3 b_2 + -k_1^2 b_3 \\ -a_2 b_1 \|\vec{a}\| + a_1 b_2 \|\vec{a}\| \end{bmatrix} \quad (28)$$

Rotate the x axis back to \vec{a}

$$f \triangleq ((\vec{f}_a^{zx})^{-1} \circ (\vec{f}_{a'}^x)^{-1} \circ (\vec{f}_{b''}^{xy})^{-1} \circ \vec{f}_y^z \circ \vec{f}_{b'''}^y \circ \vec{f}_{b''}^{xy} \circ \vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b}) \quad (29)$$

$$= \begin{bmatrix} \frac{a_1}{\|\vec{a}\|} & \frac{-a_2}{k_1} & \frac{-a_1 a_3}{k_1 \|\vec{a}\|} \\ \frac{a_2}{\|\vec{a}\|} & \frac{a_1}{k_1} & \frac{-a_2 a_3}{k_1 \|\vec{a}\|} \\ \frac{a_3}{\|\vec{a}\|} & 0 & \frac{k_1}{\|\vec{a}\|} \end{bmatrix} * \begin{bmatrix} 0 \\ \frac{a_1 a_3 b_1}{k_1 \|\vec{a}\|} + \frac{a_2 a_3 b_2}{k_1 \|\vec{a}\|} + \frac{-k_1 b_3}{\|\vec{a}\|} \\ \frac{-a_2 b_1}{k_1} + \frac{a_1 b_2}{k_1} \end{bmatrix} \quad (30)$$

$$= \frac{1}{k_1^2 \|\vec{a}\|^2} * \begin{bmatrix} a_1 k_1 & -a_2 \|\vec{a}\| & -a_1 a_3 \\ a_2 k_1 & a_1 \|\vec{a}\| & -a_2 a_3 \\ a_3 k_1 & 0 & k_1^2 \end{bmatrix} * \begin{bmatrix} 0 \\ a_1 a_3 b_1 + a_2 a_3 b_2 + -k_1^2 b_3 \\ -a_2 b_1 \|\vec{a}\| + a_1 b_2 \|\vec{a}\| \end{bmatrix} \quad (31)$$

$$= \frac{1}{k_1^2 \|\vec{a}\|^2} * \begin{bmatrix} \|\vec{a}\| * (-a_1 a_2 a_3 b_1 + -a_2^2 a_3 b_2 + k_1^2 a_2 b_3 + a_1 a_2 a_3 b_1 + -a_1^2 a_3 b_2) \\ \|\vec{a}\| * (a_1^2 a_3 b_1 + a_1 a_2 a_3 b_2 + -k_1^2 a_1 b_3 + a_2^2 a_3 b_1 + -a_1 a_2 a_3 b_2) \\ k_1^2 \|\vec{a}\| * (-a_2 b_1 + a_1 b_2) \end{bmatrix} \quad (32)$$

$$= \frac{1}{k_1^2 \|\vec{a}\|} * \begin{bmatrix} -a_2^2 a_3 b_2 + k_1^2 a_2 b_3 + -a_1^2 a_3 b_2 \\ a_1^2 a_3 b_1 + -k_1^2 a_1 b_3 + a_2^2 a_3 b_1 \\ k_1^2 \|\vec{a}\| * (-a_2 b_1 + a_1 b_2) \end{bmatrix} \quad (33)$$

$$= \frac{1}{\|\vec{a}\|} * \begin{bmatrix} a_2 b_3 + \frac{-a_2^2 a_3 b_2 - a_1^2 a_3 b_2}{k_1^2} \\ -a_1 b_3 + \frac{a_1^2 a_3 b_1 + a_2^2 a_3 b_1}{k_1^2} \\ -a_2 b_1 + a_1 b_2 \end{bmatrix} \quad (34)$$

$$= \frac{1}{\|\vec{a}\|} * \begin{bmatrix} a_2 b_3 + \frac{-(a_1^2 + a_2^2) a_3 b_2}{k_1^2} \\ -a_1 b_3 + \frac{(a_1^2 + a_2^2) a_3 b_1}{k_1^2} \\ -a_2 b_1 + a_1 b_2 \end{bmatrix} \quad (35)$$

$$= \frac{1}{\|\vec{a}\|} * \begin{bmatrix} a_2 b_3 + \frac{-k_1^2 a_3 b_2}{k_1^2} \\ -a_1 b_3 + \frac{k_1^2 a_3 b_1}{k_1^2} \\ -a_2 b_1 + a_1 b_2 \end{bmatrix} \quad (36)$$

Scale $\vec{f}(\vec{b})$ by $\|\vec{a}\|$

$$\|\vec{a}\| * \vec{f}(\vec{b}) = \|\vec{a}\| * \frac{1}{\|\vec{a}\|} * \begin{bmatrix} a_2b_3 - a_3b_2 \\ -a_1b_3 + a_3b_1 \\ a_1b_2 - a_2b_1 \end{bmatrix} \quad (39)$$

$$= \begin{bmatrix} a_2b_3 - a_3b_2 \\ -a_1b_3 + a_3b_1 \\ a_1b_2 - a_2b_1 \end{bmatrix} \quad (40)$$

$$= a \times b \quad (41)$$

■

License of proof

©2022-2023 William Emerson Six

Permission is hereby granted, free of charge, to any person obtaining a copy of this software and associated documentation files (the "Software"), to deal in the Software without restriction, including without limitation the rights to use, copy, modify, merge, publish, distribute, sublicense, and/or sell copies of the Software, and to permit persons to whom the Software is furnished to do so, subject to the following conditions:

The above copyright notice and this permission notice shall be included in all copies or substantial portions of the Software.

THE SOFTWARE IS PROVIDED "AS IS", WITHOUT WARRANTY OF ANY KIND, EXPRESS OR IMPLIED, INCLUDING BUT NOT LIMITED TO THE WARRANTIES OF MERCHANTABILITY, FITNESS FOR A PARTICULAR PURPOSE AND NONINFRINGEMENT. IN NO EVENT SHALL THE AUTHORS OR COPYRIGHT HOLDERS BE LIABLE FOR ANY CLAIM, DAMAGES OR OTHER LIABILITY, WHETHER IN AN ACTION OF CONTRACT, TORT OR OTHERWISE, ARISING FROM, OUT OF OR IN CONNECTION WITH THE SOFTWARE OR THE USE OR OTHER DEALINGS IN THE SOFTWARE.