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**Problem 1.** Given vectors  $\vec{a}$  and  $\vec{b}$  in  $\mathbb{R}^3$ , verify that a quaternion can be made to rotate the plane spanned by  $\vec{a}$  and  $\vec{b}$  to rotate  $\vec{a}$  to  $\vec{b}$ .

**Define quaternion, and it's multiplication**

A quaternion is defined as an unevaluated addition of a scalar and a three dimensional vector ( $q + \vec{q}$ ).

Quaternion multiplication is defined as being distributive, where the product of two vectors “multiplied” is the cross product added to negative dot product of the vectors.

$$(p + \vec{p})(q + \vec{q}) \triangleq (\underbrace{pq}_{s1} - \underbrace{\vec{p} \cdot \vec{q}}_{s2}) + (\underbrace{p\vec{q}}_{v1} + \underbrace{q\vec{p}}_{v2} + \underbrace{\vec{p} \times \vec{q}}_{v3}) \quad (1)$$

**Calculate the angle of rotation between  $\vec{a}$  and  $\vec{b}$  :**

Let  $\vec{c} \triangleq \vec{a} \times \vec{b}$ .

$$\|\vec{c}\| = \|\vec{a}\| \|\vec{b}\| \sin(\theta_{ab})$$

$$\vec{d} \triangleq \vec{d}_{\parallel} + \vec{d}_{\perp}$$

**Create quaternion:**

$$\text{Let } q \triangleq \underbrace{\cos(\theta_{ab})}_p + \underbrace{\frac{\vec{c}}{\|\vec{c}\|}}_{\vec{p}}.$$

$$\text{Proof. } \vec{r}_a^b(\vec{d}) = \vec{r}_a^b(\vec{d}_{\parallel}) + \vec{d}_{\perp}$$

**Components of vector parallel to plane of rotation:**

$$\vec{r}_a^b(\vec{d}_{\parallel}) \triangleq (\underbrace{\cos(\theta_{ab})}_p + \underbrace{\frac{\vec{c}}{\|\vec{c}\|}}_{\vec{p}})(\underbrace{0}_q + \underbrace{\vec{d}_{\parallel}}_{\vec{q}})(\cos(\theta_{ab}) - \frac{\vec{c}}{\|\vec{c}\|}) \quad (2)$$

$$= ((\underbrace{\cos(\theta_{ab})0}_{s1} - \underbrace{\frac{\vec{c}}{\|\vec{c}\|} \cdot \vec{d}_{\parallel}}_{s2}) + (\underbrace{\cos(\theta_{ab})\vec{d}_{\parallel}}_{v1} + \underbrace{0\frac{\vec{c}}{\|\vec{c}\|}}_{v2} + \underbrace{\frac{\vec{c}}{\|\vec{c}\|} \times \vec{d}_{\parallel}}_{v3}))(\cos(\theta_{ab}) - \frac{\vec{c}}{\|\vec{c}\|}) \quad (3)$$

$$= ((\underbrace{0}_{s1} - \underbrace{0}_{s2}) + (\underbrace{\cos(\theta_{ab})\vec{d}_{\parallel}}_{v1} + \underbrace{0}_{v2} + \underbrace{\frac{\vec{c}}{\|\vec{c}\|} \times \vec{d}_{\parallel}}_{v3}))(\cos(\theta_{ab}) - \frac{\vec{c}}{\|\vec{c}\|}) \quad (4)$$

$$= (\underbrace{0}_{s1-s2} + (\underbrace{\cos(\theta_{ab})\vec{d}_{\parallel}}_{v1} + \underbrace{\frac{\vec{c}}{\|\vec{c}\|} \times \vec{d}_{\parallel}}_{v3}))(\cos(\theta_{ab}) - \frac{\vec{c}}{\|\vec{c}\|}) \quad (5)$$

$$= \vec{e}_{\parallel} \triangleq \cos(\theta_{ab})\vec{d}_{\parallel} + \frac{\vec{c}}{\|\vec{c}\|} \times \vec{d}_{\parallel} \quad (6)$$

$$= (0 + \vec{e}_{\parallel})(\cos(\theta_{ab}) - \frac{\vec{c}}{\|\vec{c}\|}) \quad (7)$$

**License of proof**

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