Problem 1. Given vectors $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^3$, find a vector in $\mathbb{R}^3$ that is perpendicular both to $\vec{a}$ and to $\vec{b}$.

Proof. **Rotate $\vec{a}$ and $\vec{b}$ so that $\vec{a}$ is on the x axis:**

Let $\vec{a} \triangleq \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and let $\vec{b} \triangleq \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.
let $k_1 \triangleq \sqrt{a_1^2 + a_2^2}$.
Define $\vec{f_a^{zx}}(\vec{v})$ such that $\vec{f_a^{zx}}(\vec{a})$ is rotated onto the xz plane.

$$\vec{a'} \triangleq \vec{f_a^{zx}}(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}) \triangleq \begin{bmatrix} \frac{a_1}{k_1} \\ \frac{-a_2}{k_1} \\ 0 \end{bmatrix} * v_1 + \begin{bmatrix} \frac{a_2}{k_1} \\ \frac{a_1}{k_1} \\ 0 \end{bmatrix} * v_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} * v_3 \tag{1}$$

Define $\vec{f_{a'}^{x}}(\vec{v})$ such that $(\vec{f_{a'}^{x}} \circ \vec{f_a^{zx}})(\vec{a})$ is rotated onto the x axis.

$$\vec{f_{a'}^{x}}(\vec{v}) \triangleq \begin{bmatrix} \frac{k_1}{\|a\|} \\ 0 \\ \frac{-a_3}{\|a\|} \end{bmatrix} * v_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} * v_2 + \begin{bmatrix} \frac{a_3}{\|a\|} \\ 0 \\ \frac{k_1}{\|a\|} \end{bmatrix} * v_3 \tag{2}$$

$$\vec{a''} \triangleq (\vec{f_{a'}^{x}} \circ \vec{f_a^{zx}})(\vec{v}) = \begin{bmatrix} \frac{k_1}{\|a\|} & 0 & \frac{a_3}{\|a\|} \\ 0 & 1 & 0 \\ \frac{-a_3}{\|a\|} & 0 & \frac{k_1}{\|a\|} \end{bmatrix} * (\begin{bmatrix} \frac{a_1}{k_1} & \frac{a_2}{k_1} & 0 \\ \frac{-a_2}{k_1} & \frac{a_1}{k_1} & 0 \\ 0 & 0 & 1 \end{bmatrix} * \vec{v}) \tag{3}$$

$$= \begin{bmatrix} \frac{a_1}{\|a\|} & \frac{a_2}{\|a\|} & \frac{a_3}{\|a\|} \\ \frac{-a_2}{k_1} & \frac{a_1}{k_1} & 0 \\ \frac{-a_1 a_3}{k_1\|a\|} & \frac{-a_2 a_3}{k_1\|a\|} & \frac{k_1}{\|a\|} \end{bmatrix} * \vec{v} \tag{4}$$

$$
\begin{aligned}
(\vec{f}^{x}_{a'} \circ \vec{f}^{zx}_{a})(\vec{a}) &= \begin{bmatrix} \frac{a_1^2}{\|a\|} + \frac{a_2^2}{\|a\|} + \frac{a_3^2}{\|a\|} \\ \frac{-a_2 a_1}{k_1} + \frac{a_1 a_2}{k_1} \\ \frac{-a_1^2 a_3}{k_1\|a\|} + \frac{-a_2^2 a_3}{k_1\|a\|} + \frac{k_1 a_3}{\|a\|} \end{bmatrix} && (5) \\
&= \begin{bmatrix} \frac{a_1^2+a_2^2+a_3^2}{\|a\|} \\ 0 \\ \frac{1}{\|a\|} * (\frac{-a_1^2 a_3}{k_1} + \frac{-a_2^2 a_3}{k_1} + k_1 a_3) \end{bmatrix} && (6) \\
&= \begin{bmatrix} \frac{\|a\|^2}{\|a\|} \\ 0 \\ \frac{1}{\|a\|} * (\frac{-a_1^2 a_3}{k_1} + \frac{-a_2^2 a_3}{k_1} + \frac{k_1^2 a_3}{k_1}) \end{bmatrix} && (7) \\
&= \begin{bmatrix} \|a\| \\ 0 \\ \frac{1}{\|a\|} * (\frac{-a_1^2 a_3}{k_1} + \frac{-a_2^2 a_3}{k_1} + \frac{((a_1^2+a_2^2))a_3}{k_1}) \end{bmatrix} && (8) \\
&= \begin{bmatrix} \|a\| \\ 0 \\ 0 \end{bmatrix} && (9)
\end{aligned}
$$

$$
\vec{b''} \triangleq (\vec{f}^{x}_{a'} \circ \vec{f}^{zx}_{a})(\vec{b}) = \begin{bmatrix} \frac{a_1 b_1}{\|a\|} + \frac{a_2 b_2}{\|a\|} + \frac{a_3 b_3}{\|a\|} \\ \frac{-a_2 b_1}{k_1} + \frac{a_1 b_2}{k_1} \\ \frac{-a_1 a_3 b_1}{k_1\|a\|} + \frac{-a_2 a_3 b_2}{k_1\|a\|} + \frac{k_1 b_3}{\|a\|} \end{bmatrix} \tag{10}
$$

Rotate $\vec{b''} \triangleq (\vec{f}^{x}_{a'} \circ \vec{f}^{zx}_{a})(\vec{b})$ onto the xy plane
let $k_2 \triangleq \sqrt{{b''_2}^2 + {b''_3}^2}$.

$$
\begin{aligned}
\vec{f}^{xy}_{b''}(\vec{v}) &\triangleq \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} * v_1 + \begin{bmatrix} 0 \\ \frac{b''_2}{k_2} \\ \frac{-b''_3}{k_2} \end{bmatrix} * v_2 + \begin{bmatrix} 0 \\ \frac{b''_3}{k_2} \\ \frac{b''_2}{k_2} \end{bmatrix} * v_3 && (11) \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{b''_2}{k_2} & \frac{b''_3}{k_2} \\ 0 & \frac{-b''_3}{k_2} & \frac{b''_2}{k_2} \end{bmatrix} * \vec{v} && (12)
\end{aligned}
$$

$$
\vec{b'''} \triangleq (\vec{f}^{xy}_{b''} \circ \vec{f}^{x}_{a'} \circ \vec{f}^{zx}_{a})(\vec{b}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{b''_2}{k_2} & \frac{b''_3}{k_2} \\ 0 & \frac{-b''_3}{k_2} & \frac{b''_2}{k_2} \end{bmatrix} * (\vec{f}^{x}_{a'} \circ \vec{f}^{zx}_{a})(\vec{b}) \tag{13}
$$

Project $(\vec{f}_{b''}^{xy} \circ \vec{f}_{a'}^{x} \circ \vec{f}_{a}^{zx})(\vec{b})$ onto the yz plane
Define $\vec{f}_{b'''}^{y}(\vec{v})$ to project any vector v onto the y axis.

$$\vec{f}_{b'''}^{y}(\vec{v}) \triangleq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} * v_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} * v_2 + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} * v_3 \tag{14}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} * \vec{v} \tag{15}$$

$$(\vec{f}_{b'''}^{y} \circ \vec{f}_{b''}^{xy} \circ \vec{f}_{a'}^{x} \circ \vec{f}_{a}^{zx})(\vec{b}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{b_2''}{k_2} & \frac{b_3''}{k_2} \\ 0 & \frac{-b_3''}{k_2} & \frac{b_2''}{k_2} \end{bmatrix} * (\vec{f}_{a'}^{x} \circ \vec{f}_{a}^{zx})(\vec{b}) \tag{16}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{b_2''}{k_2} & \frac{b_3''}{k_2} \\ 0 & 0 & 0 \end{bmatrix} * (\vec{f}_{a'}^{x} \circ \vec{f}_{a}^{zx})(\vec{b}) \tag{17}$$

Rotate $(\vec{f}_{b'''}^{y} \circ \vec{f}_{b''}^{xy} \circ \vec{f}_{a'}^{x} \circ \vec{f}_{a}^{zx})(\vec{b})$ 90 degrees on the yz plane.
Define $\vec{f}_{y}^{z}(\vec{v})$ rotate any vector v around the yz plane.

$$\vec{f}_{y}^{z}(\vec{v}) \triangleq \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} * v_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} * v_2 + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} * v_3 \tag{18}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} * \vec{v} \tag{19}$$

$$(\vec{f}_y^z \circ \vec{f}_{b'''}^y \circ \vec{f}_{b''}^{xy} \circ \vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} * \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{b_2''}{k_2} & \frac{b_3''}{k_2} \\ 0 & 0 & 0 \end{bmatrix} * (\vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b}) \tag{20}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{b_2''}{k_2} & \frac{b_3''}{k_2} \end{bmatrix} * (\vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b}) \tag{21}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{b_2''}{k_2} & \frac{b_3''}{k_2} \end{bmatrix} * \begin{bmatrix} b_1'' \\ b_2'' \\ b_3'' \end{bmatrix} \tag{22}$$

$$= \begin{bmatrix} 0 \\ 0 \\ \frac{b_2''^2 + b_3''^2}{\sqrt{b_2''^2 + b_3''^2}} \end{bmatrix} \tag{23}$$

$$= \begin{bmatrix} 0 \\ 0 \\ k_2 \end{bmatrix} \tag{24}$$

Apply inverse of $\vec{f}_{b''}^{xy}$ to $(\vec{f}_y^z \circ \vec{f}_{b'''}^y \circ \vec{f}_{b''}^{xy} \circ \vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b})$

$$((\vec{f}_{b''}^{xy})^{-1} \circ \vec{f}_y^z \circ \vec{f}_{b'''}^y \circ \vec{f}_{b''}^{xy} \circ \vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{b_2''}{k_2} & \frac{-b_3''}{k_2} \\ 0 & \frac{b_3''}{k_2} & \frac{b_2''}{k_2} \end{bmatrix} * \begin{bmatrix} 0 \\ 0 \\ k_2 \end{bmatrix} \tag{25}$$

$$= \begin{bmatrix} 0 \\ -b_3'' \\ b_2'' \end{bmatrix} \tag{26}$$

$$= \begin{bmatrix} 0 \\ \frac{a_1 a_3 b_1}{k_1 \|a\|} + \frac{a_2 a_3 b_2}{k_1 \|a\|} + \frac{-k_1 b_3}{\|a\|} \\ \frac{-a_2 b_1}{k_1} + \frac{a_1 b_2}{k_1} \end{bmatrix} \tag{27}$$

$$= \frac{1}{k_1 \|a\|} * \begin{bmatrix} 0 \\ a_1 a_3 b_1 + a_2 a_3 b_2 + -k_1^2 b_3 \\ -a_2 b_1 \|a\| + a_1 b_2 \|a\| \end{bmatrix} \tag{28}$$

Rotate the x axis back to $\vec{a}$

$$f \triangleq ((\vec{f}_a^{zx})^{-1} \circ (\vec{f}_{a'}^{x})^{-1} \circ (\vec{f}_{b''}^{xy})^{-1} \circ \vec{f}_y^{z} \circ \vec{f}_{b'''}^{y} \circ \vec{f}_{b''}^{xy} \circ \vec{f}_{a'}^{x} \circ \vec{f}_a^{zx})(\vec{b}) \tag{29}$$

$$= \begin{bmatrix} \frac{a_1}{\|a\|} & \frac{-a_2}{k_1} & \frac{-a_1a_3}{k_1\|a\|} \\ \frac{a_2}{\|a\|} & \frac{a_1}{k_1} & \frac{-a_2a_3}{k_1\|a\|} \\ \frac{a_3}{\|a\|} & 0 & \frac{k_1}{\|a\|} \end{bmatrix} * \begin{bmatrix} 0 \\ \frac{a_1a_3b_1}{k_1\|a\|} + \frac{a_2a_3b_2}{k_1\|a\|} + \frac{-k_1b_3}{\|a\|} \\ \frac{-a_2b_1}{k_1} + \frac{a_1b_2}{k_1} \end{bmatrix} \tag{30}$$

$$= \frac{1}{k_1^2\|a\|^2} * \begin{bmatrix} a_1k_1 & -a_2\|a\| & -a_1a_3 \\ a_2k_1 & a_1\|a\| & -a_2a_3 \\ a_3k_1 & 0 & k_1^2 \end{bmatrix} * \begin{bmatrix} 0 \\ a_1a_3b_1 + a_2a_3b_2 + -k_1^2b_3 \\ -a_2b_1\|a\| + a_1b_2\|a\| \end{bmatrix} \tag{31}$$

$$= \frac{1}{k_1^2\|a\|^2} * \begin{bmatrix} \|a\| * (-a_1a_2a_3b_1 + -a_2^2a_3b_2 + k_1^2a_2b_3 + a_1a_2a_3b_1 + -a_1^2a_3b_2) \\ \|a\| * (a_1^2a_3b_1 + a_1a_2a_3b_2 + -k_1^2a_1b_3 + a_2^2a_3b_1 + -a_1a_2a_3b_2) \\ k_1^2\|a\| * (-a_2b_1 + a_1b_2) \end{bmatrix} \tag{32}$$

$$= \frac{1}{k_1^2\|a\|} * \begin{bmatrix} -a_2^2a_3b_2 + k_1^2a_2b_3 + -a_1^2a_3b_2 \\ a_1^2a_3b_1 + -k_1^2a_1b_3 + a_2^2a_3b_1 \\ k_1^2\|a\| * (-a_2b_1 + a_1b_2) \end{bmatrix} \tag{33}$$

$$= \frac{1}{\|a\|} * \begin{bmatrix} a_2b_3 + \frac{-a_2^2a_3b_2 - a_1^2a_3b_2}{k_1^2} \\ -a_1b_3 + \frac{a_1^2a_3b_1 + a_2^2a_3b_1}{k_1^2} \\ -a_2b_1 + a_1b_2 \end{bmatrix} \tag{34}$$

$$= \frac{1}{\|a\|} * \begin{bmatrix} a_2b_3 + \frac{-(a_1^2+a_2^2)a_3b_2}{k_1^2} \\ -a_1b_3 + \frac{(a_1^2+a_2^2)a_3b_1}{k_1^2} \\ -a_2b_1 + a_1b_2 \end{bmatrix} \tag{35}$$

$$= \frac{1}{\|a\|} * \begin{bmatrix} a_2b_3 + \frac{-k_1^2a_3b_2}{k_1^2} \\ -a_1b_3 + \frac{k_1^2a_3b_1}{k_1^2} \\ -a_2b_1 + a_1b_2 \end{bmatrix} \tag{36}$$

$$= \frac{1}{\|a\|} * \begin{bmatrix} a_2b_3 - a_3b_2 \\ -a_1b_3 + a_3b_1 \\ a_1b_2 - a_2b_1 \end{bmatrix} \tag{37}$$

$$\tag{38}$$

Scale $\vec{f}(\vec{b})$ by $\|a\|$

$$\|a\| * \vec{f}(\vec{b}) = \|a\| * \frac{1}{\|a\|} * \begin{bmatrix} a_2b_3 - a_3b_2 \\ -a_1b_3 + a_3b_1 \\ a_1b_2 - a_2b_1 \end{bmatrix} \tag{39}$$

$$= \begin{bmatrix} a_2b_3 - a_3b_2 \\ -a_1b_3 + a_3b_1 \\ a_1b_2 - a_2b_1 \end{bmatrix} \tag{40}$$

$$= a \times b \tag{41}$$

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