Problem 1. Given vectors $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^3$, verify that a quaternion can be made to rotate the plane spanned by $\vec{a}$ and $\vec{b}$ to rotate $\vec{a}$ to $\vec{b}$.

Define quaternion, and it's multiplication

A quarternion is defined as an unevaluated addition of a scalar and a three dimensional vection $(q + \vec{q})$.

Quaternion multiplication is defined as being distributive, where the product of two vectors "multiplied" is the cross product added to negative dot product of the vectors.

$$(p + \vec{p})(q + \vec{q}) \triangleq (\underbrace{pq}_{s1} - \underbrace{\vec{p} \cdot \vec{q}}_{s2}) + (\underbrace{p\vec{q}}_{v1} + \underbrace{q\vec{p}}_{v2} + \underbrace{\vec{p} \times \vec{q}}_{v3}) \tag{1}$$

Calculate the angle of rotation between $\vec{a}$ and $\vec{b}$:

Let $\vec{c} \triangleq \vec{a} \times \vec{b}$.

$\|\vec{c}\| = \left\|\vec{a}\right\| \left\|\vec{b}\right\| \sin(\theta_{ab})$

$\vec{d} \triangleq \vec{d}_{\|} + \vec{d}_{\perp}$

Create quaternion:

Let $q \triangleq \underbrace{\cos(\theta_{ab})}_{p} + \underbrace{\frac{\vec{c}}{\|\vec{c}\|}}_{\vec{p}}$.

Proof. $\vec{r}_a^{\,b}(\vec{d}) = \vec{r}_a^{\,b}(\vec{d}_{\|}) + \vec{d}_{\perp}$

Components of vector parallel to plane of rotation:

$$\vec{r}_a^{\,b}(\vec{d}_{\|}) \triangleq (\underbrace{\cos(\theta_{ab})}_{p} + \underbrace{\frac{\vec{c}}{\|\vec{c}\|}}_{\vec{p}})(\underbrace{0}_{q} + \underbrace{\vec{d}_{\|}}_{\vec{q}})(\cos(\theta_{ab}) - \frac{\vec{c}}{\|\vec{c}\|}) \tag{2}$$

$$= ((\underbrace{\cos(\theta_{ab})0}_{s1} - \underbrace{\frac{\vec{c}}{\|\vec{c}\|} \cdot \vec{d}_{\|}}_{s2}) + (\underbrace{\cos(\theta_{ab})\vec{d}_{\|}}_{v1} + \underbrace{0\frac{\vec{c}}{\|\vec{c}\|}}_{v2} + \underbrace{\frac{\vec{c}}{\|\vec{c}\|} \times \vec{d}_{\|}}_{v3}))(\cos(\theta_{ab}) - \frac{\vec{c}}{\|\vec{c}\|}) \tag{3}$$

$$= ((\underbrace{0}_{s1} - \underbrace{0}_{s2}) + (\underbrace{\cos(\theta_{ab})\vec{d}_{\|}}_{v1} + \underbrace{0}_{v2} + \underbrace{\frac{\vec{c}}{\|\vec{c}\|} \times \vec{d}_{\|}}_{v3}))(\cos(\theta_{ab}) - \frac{\vec{c}}{\|\vec{c}\|}) \tag{4}$$

$$= (\underbrace{0}_{s1-s2} + (\underbrace{\cos(\theta_{ab})\vec{d}_{\|}}_{v1} + \underbrace{\frac{\vec{c}}{\|\vec{c}\|} \times \vec{d}_{\|}}_{v3}))(\cos(\theta_{ab}) - \frac{\vec{c}}{\|\vec{c}\|}) \tag{5}$$

$$= \vec{e}_{\|} \triangleq \cos(\theta_{ab})\vec{d}_{\|} + \frac{\vec{c}}{\|\vec{c}\|} \times \vec{d}_{\|} \tag{6}$$

$$= (0 + \vec{e}_{\|})(\cos(\theta_{ab}) - \frac{\vec{c}}{\|\vec{c}\|}) \tag{7}$$

■