Problem 1. Given vectors $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^3$, find a vector $\vec{c}$ in $\mathbb{R}^3$ that is perpendicular both to $\vec{a}$ and to $\vec{b}$ that follows the right-hand rule. Also, $\|\vec{c}\| = \|\vec{a}\| \left\|\vec{b}\right\| \sin(\theta)$

Proof. **Rotate $\vec{a}$ and $\vec{b}$ so that $\vec{a}$ is on the x axis:**

Let $\vec{a} \triangleq \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and let $\vec{b} \triangleq \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.
let $k_1 \triangleq \sqrt{a_1^2 + a_2^2}$.
Define $\vec{f}_a^{zx}(\vec{v})$ such that $\vec{f}_a^{zx}(\vec{a})$ is rotated onto the xz plane.

$$\vec{a'} \triangleq \vec{f}_a^{zx}(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}) \triangleq \begin{bmatrix} \frac{a_1}{k_1} \\ \frac{-a_2}{k_1} \\ 0 \end{bmatrix} * v_1 + \begin{bmatrix} \frac{a_2}{k_1} \\ \frac{a_1}{k_1} \\ 0 \end{bmatrix} * v_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} * v_3 \tag{1}$$

Define $\vec{f}_{a'}^{x}(\vec{v})$ such that $(\vec{f}_{a'}^{x} \circ \vec{f}_a^{zx})(\vec{a})$ is rotated onto the x axis.

$$\vec{f}_{a'}^{x}(\vec{v}) \triangleq \begin{bmatrix} \frac{k_1}{\|\vec{a}\|} \\ 0 \\ \frac{-a_3}{\|\vec{a}\|} \end{bmatrix} * v_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} * v_2 + \begin{bmatrix} \frac{a_3}{\|\vec{a}\|} \\ 0 \\ \frac{k_1}{\|\vec{a}\|} \end{bmatrix} * v_3 \tag{2}$$

$$\vec{a''} \triangleq (\vec{f}^{x}_{a'} \circ \vec{f}^{zx}_{a})(\vec{v}) = \begin{bmatrix} \frac{k_1}{\|\vec{a}\|} & 0 & \frac{a_3}{\|\vec{a}\|} \\ 0 & 1 & 0 \\ \frac{-a_3}{\|\vec{a}\|} & 0 & \frac{k_1}{\|\vec{a}\|} \end{bmatrix} * \left(\begin{bmatrix} \frac{a_1}{k_1} & \frac{a_2}{k_1} & 0 \\ \frac{-a_2}{k_1} & \frac{a_1}{k_1} & 0 \\ 0 & 0 & 1 \end{bmatrix} * \vec{v} \right) \tag{3}$$

$$= \begin{bmatrix} \frac{a_1}{\|\vec{a}\|} & \frac{a_2}{\|\vec{a}\|} & \frac{a_3}{\|\vec{a}\|} \\ \frac{-a_2}{k_1} & \frac{a_1}{k_1} & 0 \\ \frac{-a_1 a_3}{k_1\|\vec{a}\|} & \frac{-a_2 a_3}{k_1\|\vec{a}\|} & \frac{k_1}{\|\vec{a}\|} \end{bmatrix} * \vec{v} \tag{4}$$

$$(\vec{f_{a'}^{x}} \circ \vec{f_{a}^{zx}})(\vec{a}) = \begin{bmatrix} \frac{a_1^2}{\|\vec{a}\|} + \frac{a_2^2}{\|\vec{a}\|} + \frac{a_3^2}{\|\vec{a}\|} \\ \\ \frac{-a_2 a_1}{k_1} + \frac{a_1 a_2}{k_1} \\ \\ \frac{-a_1^2 a_3}{k_1\|\vec{a}\|} + \frac{-a_2^2 a_3}{k_1\|\vec{a}\|} + \frac{k_1 a_3}{\|\vec{a}\|} \end{bmatrix} \tag{5}$$

$$= \begin{bmatrix} \frac{a_1^2+a_2^2+a_3^2}{\|\vec{a}\|} \\ \\ 0 \\ \\ \frac{1}{\|\vec{a}\|} * (\frac{-a_1^2 a_3}{k_1} + \frac{-a_2^2 a_3}{k_1} + k_1 a_3) \end{bmatrix} \tag{6}$$

$$= \begin{bmatrix} \frac{\|\vec{a}\|^2}{\|\vec{a}\|} \\ \\ 0 \\ \\ \frac{1}{\|\vec{a}\|} * (\frac{-a_1^2 a_3}{k_1} + \frac{-a_2^2 a_3}{k_1} + \frac{k_1^2 a_3}{k_1}) \end{bmatrix} \tag{7}$$

$$= \begin{bmatrix} \|\vec{a}\| \\ \\ 0 \\ \\ \frac{1}{\|\vec{a}\|} * (\frac{-a_1^2 a_3}{k_1} + \frac{-a_2^2 a_3}{k_1} + \frac{((a_1^2+a_2^2))a_3}{k_1}) \end{bmatrix} \tag{8}$$

$$= \begin{bmatrix} \|\vec{a}\| \\ \\ 0 \\ \\ 0 \end{bmatrix} \tag{9}$$

$$\vec{b''} \triangleq (\vec{f_{a'}^{x}} \circ \vec{f_{a}^{zx}})(\vec{b}) = \begin{bmatrix} \frac{a_1 b_1}{\|\vec{a}\|} + \frac{a_2 b_2}{\|\vec{a}\|} + \frac{a_3 b_3}{\|\vec{a}\|} \\ \\ \frac{-a_2 b_1}{k_1} + \frac{a_1 b_2}{k_1} \\ \\ \frac{-a_1 a_3 b_1}{k_1\|\vec{a}\|} + \frac{-a_2 a_3 b_2}{k_1\|\vec{a}\|} + \frac{k_1 b_3}{\|\vec{a}\|} \end{bmatrix} \tag{10}$$

Rotate $\vec{b''} \triangleq (\vec{f_{a'}^{x}} \circ \vec{f_{a}^{zx}})(\vec{b})$ onto the xy plane

let $k_2 \triangleq \sqrt{b_2''^2 + b_3''^2}$.

$$\vec{f}_{b''}^{xy}(\vec{v}) \triangleq \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} * v_1 + \begin{bmatrix} 0 \\ \frac{b_2''}{k_2} \\ \frac{-b_3''}{k_2} \end{bmatrix} * v_2 + \begin{bmatrix} 0 \\ \frac{b_3''}{k_2} \\ \frac{b_2''}{k_2} \end{bmatrix} * v_3 \tag{11}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{b_2''}{k_2} & \frac{b_3''}{k_2} \\ 0 & \frac{-b_3''}{k_2} & \frac{b_2''}{k_2} \end{bmatrix} * \vec{v} \tag{12}$$

$$\vec{b'''} \triangleq (\vec{f}_{b''}^{xy} \circ \vec{f}_{a'}^{x} \circ \vec{f}_{a}^{zx})(\vec{b}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{b_2''}{k_2} & \frac{b_3''}{k_2} \\ 0 & \frac{-b_3''}{k_2} & \frac{b_2''}{k_2} \end{bmatrix} * (\vec{f}_{a'}^{x} \circ \vec{f}_{a}^{zx})(\vec{b}) \tag{13}$$

Project $(\vec{f}_{b''}^{xy} \circ \vec{f}_{a'}^{x} \circ \vec{f}_{a}^{zx})(\vec{b})$ onto the yz plane
Define $\vec{f}_{b'''}^{y}(\vec{v})$ to project any vector v onto the y axis.

$$\vec{f}_{b'''}^{y}(\vec{v}) \triangleq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} * v_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} * v_2 + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} * v_3 \tag{14}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} * \vec{v} \tag{15}$$

$$(\vec{f}^{y}_{b'''} \circ \vec{f}^{xy}_{b''} \circ \vec{f}^{x}_{a'} \circ \vec{f}^{zx}_{a})(\vec{b}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{b''_2}{k_2} & \frac{b''_3}{k_2} \\ 0 & \frac{-b''_3}{k_2} & \frac{b''_2}{k_2} \end{bmatrix} * (\vec{f}^{x}_{a'} \circ \vec{f}^{zx}_{a})(\vec{b}) \tag{16}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{b''_2}{k_2} & \frac{b''_3}{k_2} \\ 0 & 0 & 0 \end{bmatrix} * (\vec{f}^{x}_{a'} \circ \vec{f}^{zx}_{a})(\vec{b}) \tag{17}$$

Rotate $(\vec{f}^{y}_{b'''} \circ \vec{f}^{xy}_{b''} \circ \vec{f}^{x}_{a'} \circ \vec{f}^{zx}_{a})(\vec{b})$ 90 degrees on the yz plane.
Define $\vec{f}^{z}_{y}(\vec{v})$ rotate any vector v around the yz plane.

$$\vec{f}^{z}_{y}(\vec{v}) \triangleq \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} * v_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} * v_2 + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} * v_3 \tag{18}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} * \vec{v} \tag{19}$$

$$(\vec{f_y^z} \circ \vec{f_{b'''}^y} \circ \vec{f_{b''}^{xy}} \circ \vec{f_{a'}^x} \circ \vec{f_a^{zx}})(\vec{b}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} * \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{b''_2}{k_2} & \frac{b''_3}{k_2} \\ 0 & 0 & 0 \end{bmatrix} * (\vec{f_{a'}^x} \circ \vec{f_a^{zx}})(\vec{b}) \tag{20}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{b''_2}{k_2} & \frac{b''_3}{k_2} \end{bmatrix} * (\vec{f_{a'}^x} \circ \vec{f_a^{zx}})(\vec{b}) \tag{21}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{b''_2}{k_2} & \frac{b''_3}{k_2} \end{bmatrix} * \begin{bmatrix} b''_1 \\ b''_2 \\ b''_3 \end{bmatrix} \tag{22}$$

$$= \begin{bmatrix} 0 \\ 0 \\ \frac{{b''_2}^2 + {b''_3}^2}{\sqrt{{b''_2}^2 + {b''_3}^2}} \end{bmatrix} \tag{23}$$

$$= \begin{bmatrix} 0 \\ 0 \\ k_2 \end{bmatrix} \tag{24}$$

Apply inverse of $\vec{f_{b''}^{xy}}$ to $(\vec{f_y^z} \circ \vec{f_{b'''}^y} \circ \vec{f_{b''}^{xy}} \circ \vec{f_{a'}^x} \circ \vec{f_a^{zx}})(\vec{b})$

$$((\vec{f}_{b''}^{xy})^{-1} \circ \vec{f}_{y}^{z} \circ \vec{f}_{b'''}^{y} \circ \vec{f}_{b''}^{xy} \circ \vec{f}_{a'}^{x} \circ \vec{f}_{a}^{zx})(\vec{b}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{b_2''}{k_2} & \frac{-b_3''}{k_2} \\ 0 & \frac{b_3''}{k_2} & \frac{b_2''}{k_2} \end{bmatrix} * \begin{bmatrix} 0 \\ 0 \\ k_2 \end{bmatrix} \tag{25}$$

$$= \begin{bmatrix} 0 \\ -b_3'' \\ b_2'' \end{bmatrix} \tag{26}$$

$$= \begin{bmatrix} 0 \\ \frac{a_1 a_3 b_1}{k_1 \|\vec{a}\|} + \frac{a_2 a_3 b_2}{k_1 \|\vec{a}\|} + \frac{-k_1 b_3}{\|\vec{a}\|} \\ \frac{-a_2 b_1}{k_1} + \frac{a_1 b_2}{k_1} \end{bmatrix} \tag{27}$$

$$= \frac{1}{k_1 \|\vec{a}\|} * \begin{bmatrix} 0 \\ a_1 a_3 b_1 + a_2 a_3 b_2 + -k_1^2 b_3 \\ -a_2 b_1 \|\vec{a}\| + a_1 b_2 \|\vec{a}\| \end{bmatrix} \tag{28}$$

Rotate the x axis back to $\vec{a}$

$$f \triangleq ((\vec{f}^{zx}_{a})^{-1} \circ (\vec{f}^{x}_{a'})^{-1} \circ (\vec{f}^{xy}_{b''})^{-1} \circ \vec{f}^{z}_{y} \circ \vec{f}^{y}_{b'''} \circ \vec{f}^{xy}_{b''} \circ \vec{f}^{x}_{a'} \circ \vec{f}^{zx}_{a})(\vec{b}) \tag{29}$$

$$= \begin{bmatrix} \frac{a_1}{\|\vec{a}\|} & \frac{-a_2}{k_1} & \frac{-a_1a_3}{k_1\|\vec{a}\|} \\ \frac{a_2}{\|\vec{a}\|} & \frac{a_1}{k_1} & \frac{-a_2a_3}{k_1\|\vec{a}\|} \\ \frac{a_3}{\|\vec{a}\|} & 0 & \frac{k_1}{\|\vec{a}\|} \end{bmatrix} * \begin{bmatrix} 0 \\ \frac{a_1a_3b_1}{k_1\|\vec{a}\|} + \frac{a_2a_3b_2}{k_1\|\vec{a}\|} + \frac{-k_1b_3}{\|\vec{a}\|} \\ \frac{-a_2b_1}{k_1} + \frac{a_1b_2}{k_1} \end{bmatrix} \tag{30}$$

$$= \frac{1}{k_1^2\|\vec{a}\|^2} * \begin{bmatrix} a_1k_1 & -a_2\|\vec{a}\| & -a_1a_3 \\ a_2k_1 & a_1\|\vec{a}\| & -a_2a_3 \\ a_3k_1 & 0 & k_1^2 \end{bmatrix} * \begin{bmatrix} 0 \\ a_1a_3b_1 + a_2a_3b_2 + -k_1^2b_3 \\ -a_2b_1\|\vec{a}\| + a_1b_2\|\vec{a}\| \end{bmatrix} \tag{31}$$

$$= \frac{1}{k_1^2\|\vec{a}\|^2} * \begin{bmatrix} \|\vec{a}\| * (-a_1a_2a_3b_1 + -a_2^2a_3b_2 + k_1^2a_2b_3 + a_1a_2a_3b_1 + -a_1^2a_3b_2) \\ \|\vec{a}\| * (a_1^2a_3b_1 + a_1a_2a_3b_2 + -k_1^2a_1b_3 + a_2^2a_3b_1 + -a_1a_2a_3b_2) \\ k_1^2\|\vec{a}\| * (-a_2b_1 + a_1b_2) \end{bmatrix} \tag{32}$$

$$= \frac{1}{k_1^2\|\vec{a}\|} * \begin{bmatrix} -a_2^2a_3b_2 + k_1^2a_2b_3 + -a_1^2a_3b_2 \\ a_1^2a_3b_1 + -k_1^2a_1b_3 + a_2^2a_3b_1 \\ k_1^2\|\vec{a}\| * (-a_2b_1 + a_1b_2) \end{bmatrix} \tag{33}$$

$$= \frac{1}{\|\vec{a}\|} * \begin{bmatrix} a_2b_3 + \frac{-a_2^2a_3b_2 - a_1^2a_3b_2}{k_1^2} \\ -a_1b_3 + \frac{a_1^2a_3b_1 + a_2^2a_3b_1}{k_1^2} \\ -a_2b_1 + a_1b_2 \end{bmatrix} \tag{34}$$

$$= \frac{1}{\|\vec{a}\|} * \begin{bmatrix} a_2b_3 + \frac{-(a_1^2 + a_2^2)a_3b_2}{k_1^2} \\ -a_1b_3 + \frac{(a_1^2 + a_2^2)a_3b_1}{k_1^2} \\ -a_2b_1 + a_1b_2 \end{bmatrix} \tag{35}$$

$$= \frac{1}{\|\vec{a}\|} * \begin{bmatrix} a_2b_3 + \frac{-k_1^2a_3b_2}{k_1^2} \\ -a_1b_3 + \frac{k_1^2a_3b_1}{k_1^2} \\ -a_2b_1 + a_1b_2 \end{bmatrix} \tag{36}$$

Scale $\vec{f}(\vec{b})$ by $\|\vec{a}\|$

$$\|\vec{a}\| * \vec{f}(\vec{b}) = \|\vec{a}\| * \frac{1}{\|\vec{a}\|} * \begin{bmatrix} a_2b_3 - a_3b_2 \\ -a_1b_3 + a_3b_1 \\ a_1b_2 - a_2b_1 \end{bmatrix} \tag{39}$$

$$= \begin{bmatrix} a_2b_3 - a_3b_2 \\ -a_1b_3 + a_3b_1 \\ a_1b_2 - a_2b_1 \end{bmatrix} \tag{40}$$

$$= a \times b \tag{41}$$

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