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Problem 1. Given vectors $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^3$, find a vector $\vec{c}$ in $\mathbb{R}^3$ that 1) is perpendicular both to $\vec{a}$ and to $\vec{b}$ 2) follows the right-hand rule. Also, $\|\vec{c}\| = \|\vec{a}\| \left\|\vec{b}\right\| \sin(\theta)$

Proof. **Rotate $\vec{a}$ and $\vec{b}$ so that $\vec{a}$ is on the x axis:**

Let $\vec{a} \triangleq \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$ and let $\vec{b} \triangleq \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$.

let $k \triangleq \sqrt{a_x^2 + a_y^2}$.

Define $\vec{f}_a^{zx}(\vec{v})$ such that $\vec{f}_a^{zx}(\vec{a})$ is rotated onto the xz plane.

$$\vec{a}\,' \triangleq \vec{f}_a^{zx}(\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}) \triangleq \begin{bmatrix} \frac{a_x}{k} \\ \frac{-a_y}{k} \\ 0 \end{bmatrix} * v_x + \begin{bmatrix} \frac{a_y}{k} \\ \frac{a_x}{k} \\ 0 \end{bmatrix} * v_y + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} * v_z \tag{1}$$

Define $\vec{f}_{a'}^{x}(\vec{v})$ such that $(\vec{f}_{a'}^{x} \circ \vec{f}_a^{zx})(\vec{a})$ is rotated onto the x axis.

$$\vec{f}_{a'}^{x}(\vec{v}) \triangleq \begin{bmatrix} \frac{k}{\|\vec{a}\|} \\ 0 \\ \frac{-a_z}{\|\vec{a}\|} \end{bmatrix} * v_x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} * v_y + \begin{bmatrix} \frac{a_z}{\|\vec{a}\|} \\ 0 \\ \frac{k}{\|\vec{a}\|} \end{bmatrix} * v_z \tag{2}$$

$$\vec{a}'' \triangleq (\vec{f}^{x}_{a'} \circ \vec{f}^{zx}_{a})(\vec{v}) = \begin{bmatrix} \frac{k}{\|\vec{a}\|} & 0 & \frac{a_z}{\|\vec{a}\|} \\ 0 & 1 & 0 \\ \frac{-a_z}{\|\vec{a}\|} & 0 & \frac{k}{\|\vec{a}\|} \end{bmatrix} * (\begin{bmatrix} \frac{a_x}{k} & \frac{a_y}{k} & 0 \\ \frac{-a_y}{k} & \frac{a_x}{k} & 0 \\ 0 & 0 & 1 \end{bmatrix} * \vec{v}) \tag{3}$$

$$= \begin{bmatrix} \frac{a_x}{\|\vec{a}\|} & \frac{a_y}{\|\vec{a}\|} & \frac{a_z}{\|\vec{a}\|} \\ \frac{-a_y}{k} & \frac{a_x}{k} & 0 \\ \frac{-a_x a_z}{k\|\vec{a}\|} & \frac{-a_y a_z}{k\|\vec{a}\|} & \frac{k}{\|\vec{a}\|} \end{bmatrix} * \vec{v} \tag{4}$$

$$(\vec{f}_{a'}^{x} \circ \vec{f}_{a}^{zx})(\vec{a}) = \begin{bmatrix} \frac{a_x^2}{\|\vec{a}\|} + \frac{a_y^2}{\|\vec{a}\|} + \frac{a_z^2}{\|\vec{a}\|} \\ \\ \frac{-a_y a_x}{k} + \frac{a_x a_y}{k} \\ \\ \frac{-a_x^2 a_z}{k\|\vec{a}\|} + \frac{-a_y^2 a_z}{k\|\vec{a}\|} + \frac{k a_z}{\|\vec{a}\|} \end{bmatrix} \tag{5}$$

$$= \begin{bmatrix} \frac{a_x^2+a_y^2+a_z^2}{\|\vec{a}\|} \\ \\ 0 \\ \\ \frac{1}{\|\vec{a}\|} * \left(\frac{-a_x^2 a_z}{k} + \frac{-a_y^2 a_z}{k} + k a_z\right) \end{bmatrix} \tag{6}$$

$$= \begin{bmatrix} \frac{\|\vec{a}\|^2}{\|\vec{a}\|} \\ \\ 0 \\ \\ \frac{1}{\|\vec{a}\|} * \left(\frac{-a_x^2 a_z}{k} + \frac{-a_y^2 a_z}{k} + \frac{k^2 a_z}{k}\right) \end{bmatrix} \tag{7}$$

$$= \begin{bmatrix} \|\vec{a}\| \\ \\ 0 \\ \\ \frac{1}{\|\vec{a}\|} * \left(\frac{-a_x^2 a_z}{k} + \frac{-a_y^2 a_z}{k} + \frac{((a_x^2+a_y^2))a_z}{k}\right) \end{bmatrix} \tag{8}$$

$$= \begin{bmatrix} \|\vec{a}\| \\ \\ 0 \\ \\ 0 \end{bmatrix} \tag{9}$$

$$\vec{b}'' \triangleq (\vec{f}_{a'}^{x} \circ \vec{f}_{a}^{zx})(\vec{b}) = \begin{bmatrix} \frac{a_x b_x}{\|\vec{a}\|} + \frac{a_y b_y}{\|\vec{a}\|} + \frac{a_z b_z}{\|\vec{a}\|} \\ \\ \frac{-a_y b_x}{k} + \frac{a_x b_y}{k} \\ \\ \frac{-a_x a_z b_x}{k\|\vec{a}\|} + \frac{-a_y a_z b_y}{k\|\vec{a}\|} + \frac{k b_z}{\|\vec{a}\|} \end{bmatrix} \tag{10}$$

Rotate $\vec{b}'' \triangleq (\vec{f}_{a'}^{x} \circ \vec{f}_{a}^{zx})(\vec{b})$ onto the xy plane

let $c \triangleq \sqrt{b_y''^2 + b_z''^2}$.

$$\vec{f}_{b''}^{xy}(\vec{v}) \triangleq \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} * v_x + \begin{bmatrix} 0 \\ \frac{b_y''}{c} \\ \frac{-b_z''}{c} \end{bmatrix} * v_y + \begin{bmatrix} 0 \\ \frac{b_z''}{c} \\ \frac{b_y''}{c} \end{bmatrix} * v_z \tag{11}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{b_y''}{c} & \frac{b_z''}{c} \\ 0 & \frac{-b_z''}{c} & \frac{b_y''}{c} \end{bmatrix} * \vec{v} \tag{12}$$

$$\vec{b}''' \triangleq (\vec{f}_{b''}^{xy} \circ \vec{f}_{a'}^{x} \circ \vec{f}_{a}^{zx})(\vec{b}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{b_y''}{c} & \frac{b_z''}{c} \\ 0 & \frac{-b_z''}{c} & \frac{b_y''}{c} \end{bmatrix} * (\vec{f}_{a'}^{x} \circ \vec{f}_{a}^{zx})(\vec{b}) \tag{13}$$

Project $(\vec{f}_{b''}^{xy} \circ \vec{f}_{a'}^{x} \circ \vec{f}_{a}^{zx})(\vec{b})$ onto the yz plane
Define $\vec{f}_{b'''}^{y}(\vec{v})$ to project any vector v onto the y axis.

$$\vec{f}_{b'''}^{y}(\vec{v}) \triangleq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} * v_x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} * v_y + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} * v_z \tag{14}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} * \vec{v} \tag{15}$$

$$(\vec{f}^{y}_{b'''} \circ \vec{f}^{xy}_{b''} \circ \vec{f}^{x}_{a'} \circ \vec{f}^{zx}_{a})(\vec{b}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{b''_y}{c} & \frac{b''_z}{c} \\ 0 & \frac{-b''_z}{c} & \frac{b''_y}{c} \end{bmatrix} * (\vec{f}^{x}_{a'} \circ \vec{f}^{zx}_{a})(\vec{b}) \quad (16)$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{b''_y}{c} & \frac{b''_z}{c} \\ 0 & 0 & 0 \end{bmatrix} * (\vec{f}^{x}_{a'} \circ \vec{f}^{zx}_{a})(\vec{b}) \quad (17)$$

Rotate $(\vec{f}^{y}_{b'''} \circ \vec{f}^{xy}_{b''} \circ \vec{f}^{x}_{a'} \circ \vec{f}^{zx}_{a})(\vec{b})$ 90 degrees on the yz plane.
Define $\vec{f}^{z}_{y}(\vec{v})$ rotate any vector v around the yz plane.

$$\vec{f}^{z}_{y}(\vec{v}) \triangleq \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} * v_x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} * v_y + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} * v_z \quad (18)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} * \vec{v} \quad (19)$$

$$(\vec{f}_{y}^{z} \circ \vec{f}_{b'''}^{y} \circ \vec{f}_{b''}^{xy} \circ \vec{f}_{a'}^{x} \circ \vec{f}_{a}^{zx})(\vec{b}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} * \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{b''_y}{c} & \frac{b''_z}{c} \\ 0 & 0 & 0 \end{bmatrix} * (\vec{f}_{a'}^{x} \circ \vec{f}_{a}^{zx})(\vec{b}) \tag{20}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{b''_y}{c} & \frac{b''_z}{c} \end{bmatrix} * (\vec{f}_{a'}^{x} \circ \vec{f}_{a}^{zx})(\vec{b}) \tag{21}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{b''_y}{c} & \frac{b''_z}{c} \end{bmatrix} * \begin{bmatrix} b''_x \\ b''_y \\ b''_z \end{bmatrix} \tag{22}$$

$$= \begin{bmatrix} 0 \\ 0 \\ \frac{{b''_y}^2 + {b''_z}^2}{\sqrt{{b''_y}^2 + {b''_z}^2}} \end{bmatrix} \tag{23}$$

$$= \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix} \tag{24}$$

Apply inverse of $\vec{f}_{b''}^{xy}$ to $(\vec{f}_{y}^{z} \circ \vec{f}_{b'''}^{y} \circ \vec{f}_{b''}^{xy} \circ \vec{f}_{a'}^{x} \circ \vec{f}_{a}^{zx})(\vec{b})$

$$((\vec{f}_{b''}^{xy})^{-1} \circ \vec{f}_{y}^{z} \circ \vec{f}_{b'''}^{y} \circ \vec{f}_{b''}^{xy} \circ \vec{f}_{a'}^{x} \circ \vec{f}_{a}^{zx})(\vec{b}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{b''_y}{c} & \frac{-b''_z}{c} \\ 0 & \frac{b''_z}{c} & \frac{b''_y}{c} \end{bmatrix} * \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix} \tag{25}$$

$$= \begin{bmatrix} 0 \\ -b''_z \\ b''_y \end{bmatrix} \tag{26}$$

$$= \begin{bmatrix} 0 \\ \frac{a_x a_z b_x}{k\|\vec{a}\|} + \frac{a_y a_z b_y}{k\|\vec{a}\|} + \frac{-k b_z}{\|\vec{a}\|} \\ \frac{-a_y b_x}{k} + \frac{a_x b_y}{k} \end{bmatrix} \tag{27}$$

$$= \frac{1}{k\|\vec{a}\|} * \begin{bmatrix} 0 \\ a_x a_z b_x + a_y a_z b_y + -k^2 b_z \\ -a_y b_x\|\vec{a}\| + a_x b_y\|\vec{a}\| \end{bmatrix} \tag{28}$$

Rotate the x axis back to $\vec{a}$

$$f \triangleq ((\vec{f}^{zx}_{a})^{-1} \circ (\vec{f}^{x}_{a'})^{-1} \circ (\vec{f}^{xy}_{b''})^{-1} \circ \vec{f}^{z}_{y} \circ \vec{f}^{y}_{b'''} \circ \vec{f}^{xy}_{b''} \circ \vec{f}^{x}_{a'} \circ \vec{f}^{zx}_{a})(\vec{b}) \tag{29}$$

$$= \begin{bmatrix} \frac{a_x}{\|\vec{a}\|} & \frac{-a_y}{k} & \frac{-a_x a_z}{k\|\vec{a}\|} \\ \frac{a_y}{\|\vec{a}\|} & \frac{a_x}{k} & \frac{-a_y a_z}{k\|\vec{a}\|} \\ \frac{a_z}{\|\vec{a}\|} & 0 & \frac{k}{\|\vec{a}\|} \end{bmatrix} * \begin{bmatrix} 0 \\ \frac{a_x a_z b_x}{k\|\vec{a}\|} + \frac{a_y a_z b_y}{k\|\vec{a}\|} + \frac{-k b_z}{\|\vec{a}\|} \\ \frac{-a_y b_x}{k} + \frac{a_x b_y}{k} \end{bmatrix} \tag{30}$$

$$= \frac{1}{k^2\|\vec{a}\|^2} * \begin{bmatrix} a_x k & -a_y\|\vec{a}\| & -a_x a_z \\ a_y k & a_x\|\vec{a}\| & -a_y a_z \\ a_z k & 0 & k^2 \end{bmatrix} * \begin{bmatrix} 0 \\ a_x a_z b_x + a_y a_z b_y + -k^2 b_z \\ -a_y b_x\|\vec{a}\| + a_x b_y\|\vec{a}\| \end{bmatrix} \tag{31}$$

$$= \frac{1}{k^2\|\vec{a}\|^2} * \begin{bmatrix} \|\vec{a}\| * (-a_x a_y a_z b_x + -a_y^2 a_z b_y + k^2 a_y b_z + a_x a_y a_z b_x + -a_x^2 a_z b_y) \\ \|\vec{a}\| * (a_x^2 a_z b_x + a_x a_y a_z b_y + -k^2 a_x b_z + a_y^2 a_z b_x + -a_x a_y a_z b_y) \\ k^2\|\vec{a}\| * (-a_y b_x + a_x b_y) \end{bmatrix} \tag{32}$$

$$= \frac{1}{k^2\|\vec{a}\|} * \begin{bmatrix} -a_y^2 a_z b_y + k^2 a_y b_z + -a_x^2 a_z b_y \\ a_x^2 a_z b_x + -k^2 a_x b_z + a_y^2 a_z b_x \\ k^2\|\vec{a}\| * (-a_y b_x + a_x b_y) \end{bmatrix} \tag{33}$$

$$= \frac{1}{\|\vec{a}\|} * \begin{bmatrix} a_y b_z + \frac{-a_y^2 a_z b_y - a_x^2 a_z b_y}{k^2} \\ -a_x b_z + \frac{a_x^2 a_z b_x + a_y^2 a_z b_x}{k^2} \\ -a_y b_x + a_x b_y \end{bmatrix} \tag{34}$$

$$= \frac{1}{\|\vec{a}\|} * \begin{bmatrix} a_y b_z + \frac{-(a_x^2 + a_y^2) a_z b_y}{k^2} \\ -a_x b_z + \frac{(a_x^2 + a_y^2) a_z b_x}{k^2} \\ -a_y b_x + a_x b_y \end{bmatrix} \tag{35}$$

$$= \frac{1}{\|\vec{a}\|} * \begin{bmatrix} a_y b_z + \frac{-k^2 a_z b_y}{k^2} \\ -a_x b_z + \frac{k^2 a_z b_x}{k^2} \\ -a_y b_x + a_x b_y \end{bmatrix} \tag{36}$$

Scale $\vec{f}(\vec{b})$ by $\|\vec{a}\|$

$$\|\vec{a}\| * \vec{f}(\vec{b}) = \|\vec{a}\| * \frac{1}{\|\vec{a}\|} * \begin{bmatrix} a_y b_z - a_z b_y \\ -a_x b_z + a_z b_x \\ a_x b_y - a_y b_x \end{bmatrix} \tag{39}$$

$$= \begin{bmatrix} a_y b_z - a_z b_y \\ -a_x b_z + a_z b_x \\ a_x b_y - a_y b_x \end{bmatrix} \tag{40}$$

$$= a \times b \tag{41}$$

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License of proof

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