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Problem 1. Given vectors \vec{a} and \vec{b} in \mathbb{R}^3 , find a vector \vec{c} in \mathbb{R}^3 that 1) is perpendicular both to \vec{a} and to \vec{b} 2) follows the right-hand rule. Also, $\|\vec{c}\| = \|\vec{a}\| \|\vec{b}\| \sin(\theta)$

Proof. **Rotate \vec{a} and \vec{b} so that \vec{a} is on the x axis:**

$$\text{Let } \vec{a} \triangleq \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \text{ and let } \vec{b} \triangleq \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}.$$

$$\text{let } k \triangleq \sqrt{a_x^2 + a_y^2}.$$

Define $\vec{f}_a^{zx}(\vec{v})$ such that $\vec{f}_a^{zx}(\vec{a})$ is rotated onto the xz plane.

$$\vec{a}' \triangleq \vec{f}_a^{zx} \left(\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \right) \triangleq \begin{bmatrix} \frac{a_x}{k} \\ \frac{-a_y}{k} \\ 0 \end{bmatrix} * v_x + \begin{bmatrix} \frac{a_y}{k} \\ \frac{a_x}{k} \\ 0 \end{bmatrix} * v_y + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} * v_z \quad (1)$$

Define $\vec{f}_{a'}^x(\vec{v})$ such that $(\vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{a})$ is rotated onto the x axis.

$$\vec{f}_{a'}^x(\vec{v}) \triangleq \begin{bmatrix} \frac{k}{\|\vec{a}\|} \\ 0 \\ \frac{-a_z}{\|\vec{a}\|} \end{bmatrix} * v_x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} * v_y + \begin{bmatrix} \frac{a_z}{\|\vec{a}\|} \\ 0 \\ \frac{k}{\|\vec{a}\|} \end{bmatrix} * v_z \quad (2)$$

$$\vec{a}'' \triangleq (\vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{v}) = \begin{bmatrix} \frac{k}{\|\vec{a}\|} & 0 & \frac{a_z}{\|\vec{a}\|} \\ 0 & 1 & 0 \\ \frac{-a_z}{\|\vec{a}\|} & 0 & \frac{k}{\|\vec{a}\|} \end{bmatrix} * \left(\begin{bmatrix} \frac{a_x}{k} & \frac{a_y}{k} & 0 \\ \frac{-a_y}{k} & \frac{a_x}{k} & 0 \\ 0 & 0 & 1 \end{bmatrix} * \vec{v} \right) \quad (3)$$

$$= \begin{bmatrix} \frac{a_x}{\|\vec{a}\|} & \frac{a_y}{\|\vec{a}\|} & \frac{a_z}{\|\vec{a}\|} \\ \frac{-a_y}{k} & \frac{a_x}{k} & 0 \\ \frac{-a_x a_z}{k\|\vec{a}\|} & \frac{-a_y a_z}{k\|\vec{a}\|} & \frac{k}{\|\vec{a}\|} \end{bmatrix} * \vec{v} \quad (4)$$

$$(\vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{a}) = \begin{bmatrix} \frac{a_x^2}{\|\vec{a}\|} + \frac{a_y^2}{\|\vec{a}\|} + \frac{a_z^2}{\|\vec{a}\|} \\ \frac{-a_y a_x}{k} + \frac{a_x a_y}{k} \\ \frac{-a_x^2 a_z}{k\|\vec{a}\|} + \frac{-a_y^2 a_z}{k\|\vec{a}\|} + \frac{k a_z}{\|\vec{a}\|} \end{bmatrix} \quad (5)$$

$$= \begin{bmatrix} \frac{a_x^2 + a_y^2 + a_z^2}{\|\vec{a}\|} \\ 0 \\ \frac{1}{\|\vec{a}\|} * \left(\frac{-a_x^2 a_z}{k} + \frac{-a_y^2 a_z}{k} + k a_z \right) \end{bmatrix} \quad (6)$$

$$= \begin{bmatrix} \frac{\|\vec{a}\|^2}{\|\vec{a}\|} \\ 0 \\ \frac{1}{\|\vec{a}\|} * \left(\frac{-a_x^2 a_z}{k} + \frac{-a_y^2 a_z}{k} + \frac{k^2 a_z}{k} \right) \end{bmatrix} \quad (7)$$

$$= \begin{bmatrix} \|\vec{a}\| \\ 0 \\ \frac{1}{\|\vec{a}\|} * \left(\frac{-a_x^2 a_z}{k} + \frac{-a_y^2 a_z}{k} + \frac{((a_x^2 + a_y^2)) a_z}{k} \right) \end{bmatrix} \quad (8)$$

$$= \begin{bmatrix} \|\vec{a}\| \\ 0 \\ 0 \end{bmatrix} \quad (9)$$

$$\vec{b}'' \triangleq (\vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b}) = \begin{bmatrix} \frac{a_x b_x}{\|\vec{a}\|} + \frac{a_y b_y}{\|\vec{a}\|} + \frac{a_z b_z}{\|\vec{a}\|} \\ \frac{-a_y b_x}{k} + \frac{a_x b_y}{k} \\ \frac{-a_x a_z b_x}{k\|\vec{a}\|} + \frac{-a_y a_z b_y}{k\|\vec{a}\|} + \frac{k b_z}{\|\vec{a}\|} \end{bmatrix} \quad (10)$$

Rotate $\vec{b}'' \triangleq (\vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b})$ onto the xy plane

let $c \triangleq \sqrt{b_y''^2 + b_z''^2}$.

$$\vec{f}_{b''}^{xy}(\vec{v}) \triangleq \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} * v_x + \begin{bmatrix} 0 \\ \frac{b_y''}{c} \\ \frac{-b_z''}{c} \end{bmatrix} * v_y + \begin{bmatrix} 0 \\ \frac{b_z''}{c} \\ \frac{b_y''}{c} \end{bmatrix} * v_z \quad (11)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{b_y''}{c} & \frac{b_z''}{c} \\ 0 & \frac{-b_z''}{c} & \frac{b_y''}{c} \end{bmatrix} * \vec{v} \quad (12)$$

$$\vec{b}''' \triangleq (\vec{f}_{b''}^{xy} \circ \vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{b_y''}{c} & \frac{b_z''}{c} \\ 0 & \frac{-b_z''}{c} & \frac{b_y''}{c} \end{bmatrix} * (\vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b}) \quad (13)$$

Project $(\vec{f}_{b''}^{xy} \circ \vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b})$ onto the yz plane

Define $\vec{f}_{b'''}^y(\vec{v})$ to project any vector v onto the y axis.

$$\vec{f}_{b'''}^y(\vec{v}) \triangleq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} * v_x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} * v_y + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} * v_z \quad (14)$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} * \vec{v} \quad (15)$$

$$(\vec{f}_{b'''}^y \circ \vec{f}_{b''}^{xy} \circ \vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{b_y''}{c} & \frac{b_z''}{c} \\ 0 & \frac{-b_z''}{c} & \frac{b_y''}{c} \end{bmatrix} * (\vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b}) \quad (16)$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{b_y''}{c} & \frac{b_z''}{c} \\ 0 & 0 & 0 \end{bmatrix} * (\vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b}) \quad (17)$$

Rotate $(\vec{f}_{b'''}^y \circ \vec{f}_{b''}^{xy} \circ \vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b})$ **90 degrees on the yz plane.**

Define $\vec{f}_y^z(\vec{v})$ rotate any vector v around the yz plane.

$$\vec{f}_y^z(\vec{v}) \triangleq \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} * v_x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} * v_y + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} * v_z \quad (18)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} * \vec{v} \quad (19)$$

$$(\vec{f}_y^z \circ \vec{f}_{b'''}^y \circ \vec{f}_{b''}^{xy} \circ \vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} * \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{b_y''}{c} & \frac{b_z''}{c} \\ 0 & 0 & 0 \end{bmatrix} * (\vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b}) \quad (20)$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{b_y''}{c} & \frac{b_z''}{c} \end{bmatrix} * (\vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b}) \quad (21)$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{b_y''}{c} & \frac{b_z''}{c} \end{bmatrix} * \begin{bmatrix} b_x'' \\ b_y'' \\ b_z'' \end{bmatrix} \quad (22)$$

$$= \begin{bmatrix} 0 \\ 0 \\ \frac{b_y''^2 + b_z''^2}{\sqrt{b_y''^2 + b_z''^2}} \end{bmatrix} \quad (23)$$

$$= \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix} \quad (24)$$

Apply inverse of $\vec{f}_{b''}^{xy}$ to $(\vec{f}_y^z \circ \vec{f}_{b'''}^y \circ \vec{f}_{b''}^{xy} \circ \vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b})$

$$((\vec{f}_{b''}^{xy})^{-1} \circ \vec{f}_y^z \circ \vec{f}_{b''}^y \circ \vec{f}_{b''}^{xy} \circ \vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{b_y''}{c} & \frac{-b_z''}{c} \\ 0 & \frac{b_z''}{c} & \frac{b_y''}{c} \end{bmatrix} * \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix} \quad (25)$$

$$= \begin{bmatrix} 0 \\ -b_z'' \\ b_y'' \end{bmatrix} \quad (26)$$

$$= \begin{bmatrix} 0 \\ \frac{a_x a_z b_x}{k \|\vec{a}\|} + \frac{a_y a_z b_y}{k \|\vec{a}\|} + \frac{-k b_z}{\|\vec{a}\|} \\ \frac{-a_y b_x}{k} + \frac{a_x b_y}{k} \end{bmatrix} \quad (27)$$

$$= \frac{1}{k \|\vec{a}\|} * \begin{bmatrix} 0 \\ a_x a_z b_x + a_y a_z b_y + -k^2 b_z \\ -a_y b_x \|\vec{a}\| + a_x b_y \|\vec{a}\| \end{bmatrix} \quad (28)$$

Rotate the x axis back to \vec{a}

$$f \triangleq ((\vec{f}_a^{zx})^{-1} \circ (\vec{f}_{a'}^x)^{-1} \circ (\vec{f}_{b''}^{xy})^{-1} \circ \vec{f}_y^z \circ \vec{f}_{b'''}^y \circ \vec{f}_{b''}^{xy} \circ \vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b}) \quad (29)$$

$$= \begin{bmatrix} \frac{a_x}{\|\vec{a}\|} & \frac{-a_y}{k} & \frac{-a_x a_z}{k\|\vec{a}\|} \\ \frac{a_y}{\|\vec{a}\|} & \frac{a_x}{k} & \frac{-a_y a_z}{k\|\vec{a}\|} \\ \frac{a_z}{\|\vec{a}\|} & 0 & \frac{k}{\|\vec{a}\|} \end{bmatrix} * \begin{bmatrix} 0 \\ \frac{a_x a_z b_x}{k\|\vec{a}\|} + \frac{a_y a_z b_y}{k\|\vec{a}\|} + \frac{-k b_z}{\|\vec{a}\|} \\ \frac{-a_y b_x}{k} + \frac{a_x b_y}{k} \end{bmatrix} \quad (30)$$

$$= \frac{1}{k^2 \|\vec{a}\|^2} * \begin{bmatrix} a_x k & -a_y \|\vec{a}\| & -a_x a_z \\ a_y k & a_x \|\vec{a}\| & -a_y a_z \\ a_z k & 0 & k^2 \end{bmatrix} * \begin{bmatrix} 0 \\ a_x a_z b_x + a_y a_z b_y + -k^2 b_z \\ -a_y b_x \|\vec{a}\| + a_x b_y \|\vec{a}\| \end{bmatrix} \quad (31)$$

$$= \frac{1}{k^2 \|\vec{a}\|^2} * \begin{bmatrix} \|\vec{a}\| * (-a_x a_y a_z b_x + -a_y^2 a_z b_y + k^2 a_y b_z + a_x a_y a_z b_x + -a_x^2 a_z b_y) \\ \|\vec{a}\| * (a_x^2 a_z b_x + a_x a_y a_z b_y + -k^2 a_x b_z + a_y^2 a_z b_x + -a_x a_y a_z b_y) \\ k^2 \|\vec{a}\| * (-a_y b_x + a_x b_y) \end{bmatrix} \quad (32)$$

$$= \frac{1}{k^2 \|\vec{a}\|} * \begin{bmatrix} -a_y^2 a_z b_y + k^2 a_y b_z + -a_x^2 a_z b_y \\ a_x^2 a_z b_x + -k^2 a_x b_z + a_y^2 a_z b_x \\ k^2 \|\vec{a}\| * (-a_y b_x + a_x b_y) \end{bmatrix} \quad (33)$$

$$= \frac{1}{\|\vec{a}\|} * \begin{bmatrix} a_y b_z + \frac{-a_y^2 a_z b_y - a_x^2 a_z b_y}{k^2} \\ -a_x b_z + \frac{a_x^2 a_z b_x + a_y^2 a_z b_x}{k^2} \\ -a_y b_x + a_x b_y \end{bmatrix} \quad (34)$$

$$= \frac{1}{\|\vec{a}\|} * \begin{bmatrix} a_y b_z + \frac{-(a_x^2 + a_y^2) a_z b_y}{k^2} \\ -a_x b_z + \frac{(a_x^2 + a_y^2) a_z b_x}{k^2} \\ -a_y b_x + a_x b_y \end{bmatrix} \quad (35)$$

$$= \frac{1}{\|\vec{a}\|} * \begin{bmatrix} a_y b_z + \frac{-k^2 a_z b_y}{k^2} \\ -a_x b_z + \frac{k^2 a_z b_x}{k^2} \\ -a_y b_x + a_x b_y \end{bmatrix} \quad (36)$$

Scale $\vec{f}(\vec{b})$ by $\|\vec{a}\|$

$$\|\vec{a}\| * \vec{f}(\vec{b}) = \|\vec{a}\| * \frac{1}{\|\vec{a}\|} * \begin{bmatrix} a_y b_z - a_z b_y \\ -a_x b_z + a_z b_x \\ a_x b_y - a_y b_x \end{bmatrix} \quad (39)$$

$$= \begin{bmatrix} a_y b_z - a_z b_y \\ -a_x b_z + a_z b_x \\ a_x b_y - a_y b_x \end{bmatrix} \quad (40)$$

$$= a \times b \quad (41)$$

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License of proof

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