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Problem 1. Given vectors \vec{a} and \vec{b} in \mathbb{R}^3 , find a vector in \mathbb{R}^3 that is perpendicular both to \vec{a} and to \vec{b} .

Proof. Rotate \vec{a} and \vec{b} so that \vec{a} is on the x axis:

$$\text{Let } \vec{a} \triangleq \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \text{ and let } \vec{b} \triangleq \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

$$\text{let } k_1 \triangleq \sqrt{a_1^2 + a_2^2}.$$

Define $\vec{f}_a^{zx}(\vec{v})$ such that $\vec{f}_a^{zx}(\vec{a})$ is rotated onto the xz plane.

$$\vec{a}' \triangleq \vec{f}_a^{zx} \left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \right) \triangleq \begin{bmatrix} \frac{a_1}{k_1} \\ \frac{-a_2}{k_1} \\ 0 \end{bmatrix} * v_1 + \begin{bmatrix} \frac{a_2}{k_1} \\ \frac{a_1}{k_1} \\ 0 \end{bmatrix} * v_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} * v_3 \quad (1)$$

Define $\vec{f}_{a'}^x(\vec{v})$ such that $(\vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{a})$ is rotated onto the x axis.

$$\vec{f}_{a'}^x(\vec{v}) \triangleq \begin{bmatrix} \frac{k_1}{\|a\|} \\ 0 \\ \frac{-a_3}{\|a\|} \end{bmatrix} * v_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} * v_2 + \begin{bmatrix} \frac{a_3}{\|a\|} \\ 0 \\ \frac{k_1}{\|a\|} \end{bmatrix} * v_3 \quad (2)$$

$$\vec{a}'' \triangleq (\vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{v}) = \begin{bmatrix} \frac{k_1}{\|a\|} & 0 & \frac{a_3}{\|a\|} \\ 0 & 1 & 0 \\ \frac{-a_3}{\|a\|} & 0 & \frac{k_1}{\|a\|} \end{bmatrix} * \left(\begin{bmatrix} \frac{a_1}{k_1} & \frac{a_2}{k_1} & 0 \\ \frac{-a_2}{k_1} & \frac{a_1}{k_1} & 0 \\ 0 & 0 & 1 \end{bmatrix} * \vec{v} \right) \quad (3)$$

$$= \begin{bmatrix} \frac{a_1}{\|a\|} & \frac{a_2}{\|a\|} & \frac{a_3}{\|a\|} \\ \frac{-a_2}{k_1} & \frac{a_1}{k_1} & 0 \\ \frac{-a_1 a_3}{k_1 \|a\|} & \frac{-a_2 a_3}{k_1 \|a\|} & \frac{k_1}{\|a\|} \end{bmatrix} * \vec{v} \quad (4)$$

$$(\vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{a}) = \begin{bmatrix} \frac{a_1^2}{\|a\|} + \frac{a_2^2}{\|a\|} + \frac{a_3^2}{\|a\|} \\ \frac{-a_2a_1}{k_1} + \frac{a_1a_2}{k_1} \\ \frac{-a_1^2a_3}{k_1\|a\|} + \frac{-a_2^2a_3}{k_1\|a\|} + \frac{k_1a_3}{\|a\|} \end{bmatrix} \quad (5)$$

$$= \begin{bmatrix} \frac{a_1^2+a_2^2+a_3^2}{\|a\|} \\ 0 \\ \frac{1}{\|a\|} * \left(\frac{-a_1^2a_3}{k_1} + \frac{-a_2^2a_3}{k_1} + k_1a_3 \right) \end{bmatrix} \quad (6)$$

$$= \begin{bmatrix} \frac{\|a\|^2}{\|a\|} \\ 0 \\ \frac{1}{\|a\|} * \left(\frac{-a_1^2a_3}{k_1} + \frac{-a_2^2a_3}{k_1} + \frac{k_1^2a_3}{k_1} \right) \end{bmatrix} \quad (7)$$

$$= \begin{bmatrix} \|a\| \\ 0 \\ \frac{1}{\|a\|} * \left(\frac{-a_1^2a_3}{k_1} + \frac{-a_2^2a_3}{k_1} + \frac{((a_1^2+a_2^2))a_3}{k_1} \right) \end{bmatrix} \quad (8)$$

$$= \begin{bmatrix} \|a\| \\ 0 \\ 0 \end{bmatrix} \quad (9)$$

$$\vec{b}'' \triangleq (\vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b}) = \begin{bmatrix} \frac{a_1b_1}{\|a\|} + \frac{a_2b_2}{\|a\|} + \frac{a_3b_3}{\|a\|} \\ \frac{-a_2b_1}{k_1} + \frac{a_1b_2}{k_1} \\ \frac{-a_1a_3b_1}{k_1\|a\|} + \frac{-a_2a_3b_2}{k_1\|a\|} + \frac{k_1b_3}{\|a\|} \end{bmatrix} \quad (10)$$

Rotate $\vec{b}'' \triangleq (\vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b})$ onto the xy plane

let $k_2 \triangleq \sqrt{b_2''^2 + b_3''^2}$.

$$\vec{f}_{b''}^{xy}(\vec{v}) \triangleq \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} * v_1 + \begin{bmatrix} 0 \\ \frac{b_2''}{k_2} \\ \frac{-b_3''}{k_2} \end{bmatrix} * v_2 + \begin{bmatrix} 0 \\ \frac{b_3''}{k_2} \\ \frac{b_2''}{k_2} \end{bmatrix} * v_3 \quad (11)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{b_2''}{k_2} & \frac{b_3''}{k_2} \\ 0 & \frac{-b_3''}{k_2} & \frac{b_2''}{k_2} \end{bmatrix} * \vec{v} \quad (12)$$

$$\vec{b}''' \triangleq (\vec{f}_{b''}^{xy} \circ \vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{b_2''}{k_2} & \frac{b_3''}{k_2} \\ 0 & \frac{-b_3''}{k_2} & \frac{b_2''}{k_2} \end{bmatrix} * (\vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b}) \quad (13)$$

Project $(\vec{f}_{b''}^{xy} \circ \vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b})$ **onto the yz plane**

Define $\vec{f}_{b'''}^y(\vec{v})$ to project any vector v onto the y axis.

$$\vec{f}_{b'''}^y(\vec{v}) \triangleq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} * v_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} * v_2 + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} * v_3 \quad (14)$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} * \vec{v} \quad (15)$$

$$(\vec{f}_{b'''}^y \circ \vec{f}_{b''}^{xy} \circ \vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{b_2''}{k_2} & \frac{b_3''}{k_2} \\ 0 & \frac{-b_3''}{k_2} & \frac{b_2''}{k_2} \end{bmatrix} * (\vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b}) \quad (16)$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{b_2''}{k_2} & \frac{b_3''}{k_2} \\ 0 & 0 & 0 \end{bmatrix} * (\vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b}) \quad (17)$$

Rotate $(\vec{f}_{b'''}^y \circ \vec{f}_{b''}^{xy} \circ \vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b})$ **90 degrees on the yz plane.**

Define $\vec{f}_y^z(\vec{v})$ rotate any vector v around the yz plane.

$$\vec{f}_y^z(\vec{v}) \triangleq \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} * v_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} * v_2 + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} * v_3 \quad (18)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} * \vec{v} \quad (19)$$

$$(\vec{f}_y^z \circ \vec{f}_{b'''}^y \circ \vec{f}_{b''}^{xy} \circ \vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} * \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{b_2''}{k_2} & \frac{b_3''}{k_2} \\ 0 & 0 & 0 \end{bmatrix} * (\vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b}) \quad (20)$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{b_2''}{k_2} & \frac{b_3''}{k_2} \end{bmatrix} * (\vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b}) \quad (21)$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{b_2''}{k_2} & \frac{b_3''}{k_2} \end{bmatrix} * \begin{bmatrix} b_1'' \\ b_2'' \\ b_3'' \end{bmatrix} \quad (22)$$

$$= \begin{bmatrix} 0 \\ 0 \\ \frac{b_2''^2 + b_3''^2}{\sqrt{b_2''^2 + b_3''^2}} \end{bmatrix} \quad (23)$$

$$= \begin{bmatrix} 0 \\ 0 \\ k_2 \end{bmatrix} \quad (24)$$

Apply inverse of $\vec{f}_{b''}^{xy}$ to $(\vec{f}_y^z \circ \vec{f}_{b'''}^y \circ \vec{f}_{b''}^{xy} \circ \vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b})$

$$((\vec{f}_{b''}^{xy})^{-1} \circ \vec{f}_y^z \circ \vec{f}_{b'''}^y \circ \vec{f}_{b''}^{xy} \circ \vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{b_2''}{k_2} & \frac{-b_3''}{k_2} \\ 0 & \frac{b_3''}{k_2} & \frac{b_2''}{k_2} \end{bmatrix} * \begin{bmatrix} 0 \\ 0 \\ k_2 \end{bmatrix} \quad (25)$$

$$= \begin{bmatrix} 0 \\ -b_3'' \\ b_2'' \end{bmatrix} \quad (26)$$

$$= \begin{bmatrix} 0 \\ \frac{a_1 a_3 b_1}{k_1 \|a\|} + \frac{a_2 a_3 b_2}{k_1 \|a\|} + \frac{-k_1 b_3}{\|a\|} \\ \frac{-a_2 b_1}{k_1} + \frac{a_1 b_2}{k_1} \end{bmatrix} \quad (27)$$

$$= \frac{1}{k_1 \|a\|} * \begin{bmatrix} 0 \\ a_1 a_3 b_1 + a_2 a_3 b_2 + -k_1^2 b_3 \\ -a_2 b_1 \|a\| + a_1 b_2 \|a\| \end{bmatrix} \quad (28)$$

Rotate the x axis back to \vec{a}

$$f \triangleq ((\vec{f}_a^{zx})^{-1} \circ (\vec{f}_{a'}^x)^{-1} \circ (\vec{f}_{b''}^{xy})^{-1} \circ \vec{f}_y^z \circ \vec{f}_{b'''}^y \circ \vec{f}_{b''}^{xy} \circ \vec{f}_{a'}^x \circ \vec{f}_a^{zx})(\vec{b}) \quad (29)$$

$$= \begin{bmatrix} \frac{a_1}{\|a\|} & \frac{-a_2}{k_1} & \frac{-a_1 a_3}{k_1 \|a\|} \\ \frac{a_2}{\|a\|} & \frac{a_1}{k_1} & \frac{-a_2 a_3}{k_1 \|a\|} \\ \frac{a_3}{\|a\|} & 0 & \frac{k_1}{\|a\|} \end{bmatrix} * \begin{bmatrix} 0 \\ \frac{a_1 a_3 b_1}{k_1 \|a\|} + \frac{a_2 a_3 b_2}{k_1 \|a\|} + \frac{-k_1 b_3}{\|a\|} \\ \frac{-a_2 b_1}{k_1} + \frac{a_1 b_2}{k_1} \end{bmatrix} \quad (30)$$

$$= \frac{1}{k_1^2 \|a\|^2} * \begin{bmatrix} a_1 k_1 & -a_2 \|a\| & -a_1 a_3 \\ a_2 k_1 & a_1 \|a\| & -a_2 a_3 \\ a_3 k_1 & 0 & k_1^2 \end{bmatrix} * \begin{bmatrix} 0 \\ a_1 a_3 b_1 + a_2 a_3 b_2 + -k_1^2 b_3 \\ -a_2 b_1 \|a\| + a_1 b_2 \|a\| \end{bmatrix} \quad (31)$$

$$= \frac{1}{k_1^2 \|a\|^2} * \begin{bmatrix} \|a\| * (-a_1 a_2 a_3 b_1 + -a_2^2 a_3 b_2 + k_1^2 a_2 b_3 + a_1 a_2 a_3 b_1 + -a_1^2 a_3 b_2) \\ \|a\| * (a_1^2 a_3 b_1 + a_1 a_2 a_3 b_2 + -k_1^2 a_1 b_3 + a_2^2 a_3 b_1 + -a_1 a_2 a_3 b_2) \\ k_1^2 \|a\| * (-a_2 b_1 + a_1 b_2) \end{bmatrix} \quad (32)$$

$$= \frac{1}{k_1^2 \|a\|} * \begin{bmatrix} -a_2^2 a_3 b_2 + k_1^2 a_2 b_3 + -a_1^2 a_3 b_2 \\ a_1^2 a_3 b_1 + -k_1^2 a_1 b_3 + a_2^2 a_3 b_1 \\ k_1^2 \|a\| * (-a_2 b_1 + a_1 b_2) \end{bmatrix} \quad (33)$$

$$= \frac{1}{\|a\|} * \begin{bmatrix} a_2 b_3 + \frac{-a_2^2 a_3 b_2 - a_1^2 a_3 b_2}{k_1^2} \\ -a_1 b_3 + \frac{a_1^2 a_3 b_1 + a_2^2 a_3 b_1}{k_1^2} \\ -a_2 b_1 + a_1 b_2 \end{bmatrix} \quad (34)$$

$$= \frac{1}{\|a\|} * \begin{bmatrix} a_2 b_3 + \frac{-(a_1^2 + a_2^2) a_3 b_2}{k_1^2} \\ -a_1 b_3 + \frac{(a_1^2 + a_2^2) a_3 b_1}{k_1^2} \\ -a_2 b_1 + a_1 b_2 \end{bmatrix} \quad (35)$$

$$= \frac{1}{\|a\|} * \begin{bmatrix} a_2 b_3 + \frac{-k_1^2 a_3 b_2}{k_1^2} \\ -a_1 b_3 + \frac{k_1^2 a_3 b_1}{k_1^2} \\ -a_2 b_1 + a_1 b_2 \end{bmatrix} \quad (36)$$

$$= \frac{1}{\|a\|} * \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ -a_1 b_3 + a_3 b_1 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} \quad (37)$$

$$(38)$$

Scale $\vec{f}(\vec{b})$ by $\|a\|$

$$\|a\| * \vec{f}(\vec{b}) = \|a\| * \frac{1}{\|a\|} * \begin{bmatrix} a_2b_3 - a_3b_2 \\ -a_1b_3 + a_3b_1 \\ a_1b_2 - a_2b_1 \end{bmatrix} \quad (39)$$

$$= \begin{bmatrix} a_2b_3 - a_3b_2 \\ -a_1b_3 + a_3b_1 \\ a_1b_2 - a_2b_1 \end{bmatrix} \quad (40)$$

$$= a \times b \quad (41)$$

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