Develop differential equations for the cart-pole system

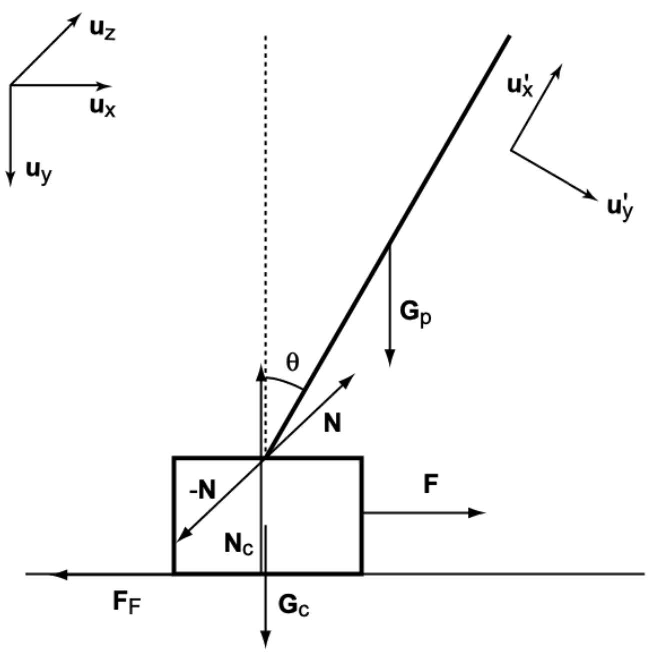
```
clear variables

results_dir = 'results';
if ~isfolder(results_dir)
    mkdir(results_dir)
end
```

Description of system

The diagram and derivations below are based on:

- R. V. Florian, 2007, Correct equations for the dynamics of the cart-pole system.



Note: Here, I will replace $\mathbf{u}'_x$ and $\mathbf{u}'_y$ with $\mathbf{u}_r$ and $\mathbf{u}_t$ to avoid confusion.

State variables

- Horizontal position of cart (from left) $x(t)$
- Horizontal velocity of cart $\dot{x}(t)$
- Angle of arm (clockwise from vertical up position) $\theta(t)$
- Angular velocity of cable $\dot{\theta}(t)$ (clockwise)

Other variables

- Upwards force on cart by track $N_c(t)$ (+ve upwards)
- Force exerted on arm by cart in x-direction $N_x(t)$ (+ve to right)

- Force exerted upwards on arm by cart (opposite to y-direction) $N_y(t)$ (+ve upwards)
- Frictional force exerted on cart by track $F_f(t)$ (+ve to left)
- Aerodynamic drag force on load opposite to x-direction $F_{d,x}(t)$ (+ve to left)
- Aerodynamic drag force on load opposite to y-direction $F_{d,y}(t)$ (+ve upwards)

Potential manipulatable input variables

- Horizontal driving force on cart $F(t)$

Potential disturbance input variables

- Horizontal wind velocity $v_{w,x}(t)$ - NOT YET IMPLEMENTED

Parameters

- Arm length L
- Cart mass m_c
- Pendulum mass m_p
- Acceleration due to gravity g
- Coefficient of friction for cart on track μ_c
- Radius of the object at the end of the pole, r (if it were a sphere)
- Aerodynamic drag coefficient of the object at the end of the pole c_d
- Mass density of air ρ_a

Derivation of equations

```
% Define symbolic variables
syms x(t) theta(t) N_c(t) F(t) F_f(t) F_d(t) Nx(t) L m_c m_p g muc r c_d rho
```

Co-ordinate systems:

$\mathbf{u}_x, \mathbf{u}_y$: unit vectors in the x direction (to the right), and y direction (down).

$\mathbf{u}_r = \mathbf{u}_x \sin\theta - \mathbf{u}_y \cos\theta$: unit vector in direction of pole (radially outwards)

$\mathbf{u}_t = \mathbf{u}_x \cos\theta + \mathbf{u}_y \sin\theta$: unit vector perpendicular to pole (clockwise tangential direction)

Force balance on cart (cartesian vectors):

$$\mathbf{F} + \mathbf{F}_f + \mathbf{G}_c - \mathbf{N} + \mathbf{N}_c = m_c \mathbf{a}_c \qquad (3)$$

where:

$\mathbf{F} = F\mathbf{u}_x$: external force applied to cart (to the right)

$\mathbf{F}_f = -F_f \mathbf{u}_x$: friction force on cart due to track (to the left when F_f is +ve)

$\mathbf{G}_c = m_c g \mathbf{u}_y$: downwards force due to weight of cart

$\mathbf{N} = N_x \mathbf{u}_x - N_y \mathbf{u}_y$: force on pole due to cart (upwards and to right when N_x and N_y are +ve)

$\mathbf{N}_c = -N_c \mathbf{u}_y$: Force on cart by track (upwards when N_c is +ve)

$\mathbf{a}_c = \ddot{x}\mathbf{u}_x$: acceleration of cart

Decompose (3) into x, y directions:

$$F - F_f - N_x = m_c \ddot{x} \qquad (4)$$

$$m_c g + N_y - N_c = 0 \qquad (5)$$

Coulomb friction model:

$$F_f = \mu_c |N_c| \mathrm{sign}(\dot{x}) = \mu_c N_c \mathrm{sign}(N_c \dot{x}) \qquad (6)$$

```
% Friction force of track on cart (to the left when F_f(t) is positive)
eqn1 = F_f(t) == muc * N_c * sign(N_c * diff(x(t)))
```

eqn1(t) =

$$F_f(t) = \mathrm{muc}\ \mathrm{sign}\left(N_c(t)\,\frac{\partial}{\partial t}\,x(t)\right) N_c(t)$$

Force balance on pole (cartesian vectors):

$$\mathbf{N} + \mathbf{F}_d + \mathbf{G}_p = m_p \mathbf{a}_p \qquad (7)$$

where:

$\mathbf{F}_d = -F_{d,x}\mathbf{u}_x - F_{d,y}\mathbf{u}_y$: Aerodynamic drag force on load (in opposite direction to velocity of load relative to air)

$\mathbf{G}_p = m_p g \mathbf{u}_y$: weight of load (downwards)

$\mathbf{a}_p$: acceleration of the load

Absolute magnitude of drag force on load:

$$F_d = \frac{1}{2}\rho|\mathbf{v}|^2 c_d A = \frac{1}{2}\rho c_d A\left((v_x + v_{w,x})^2 + v_y^2\right)$$

where

$\mathbf{v} = (v_x + v_{w,x})\mathbf{u}_x + v_y \mathbf{u}_y$: velocity of the load through the air

and

$$v_x = \dot{x} + L\dot{\theta}\cos\theta$$

$$v_y = L\dot{\theta}\sin\theta$$

```
% x and y components of velocity of load through the air
v_x = diff(x(t)) + L * diff(theta(t)) * cos(theta(t));
```

```
v_y = L * diff(theta(t)) * sin(theta(t));
```

$$F_{d,x} = \frac{1}{2}\rho c_d A v_x^2 = \frac{1}{2}\rho c_d A\left(\dot{x} + \dot{L}\sin\theta + L\dot{\theta}\cos\theta\right)^2$$

$$F_{d,y} = \frac{1}{2}\rho c_d A v_y^2 = \frac{1}{2}\rho c_d A\left(\dot{L}\cos\theta + L\dot{\theta}\sin\theta\right)^2$$

```
% Magnitudes of x and y components of drag force on load
F_dx = 0.5 * rho * c_d * pi * r^2  * v_x^2;  % TODO: Add wind as per above eqns
F_dy = 0.5 * rho * c_d * pi * r^2  * v_y^2;
% Absolute magnitude of aerodynamic drag force
eqn2 = F_d(t) == 0.5 * rho * c_d * pi * r^2 * (v_x^2 + v_y^2)
```

eqn2 =

$$F_d(t) = 1.5708\, c_d\, r^2\, \rho\, \left(\left(\sin(\theta(t))\, \frac{\partial}{\partial t}\, L(t) + \frac{\partial}{\partial t}\, x(t) + \cos(\theta(t))\, L(t)\, \frac{\partial}{\partial t}\, \theta(t)\right)^2 + \left(\cos(\theta(t))\, \frac{\partial}{\partial t}\, L(t) + \sin\right.\right.$$

The acceleration of the load is due to the composed effects of:

- acceleration of cart
- acceleration due to rotation of pole with angular velocity $\omega = \dot{\theta}\mathbf{u}_z$
- angular acceleration of the pole $\varepsilon = \ddot{\theta}\mathbf{u}_z$

$$\mathbf{a}_p = \mathbf{a}_c + \varepsilon \times \mathbf{r}_p + \omega \times (\omega \times \mathbf{r}_p) \qquad (8)$$

where:

$\mathbf{r}_p = L(\sin\theta\mathbf{u}_y - \cos\theta\mathbf{u}_x)$: position of centre of load relative to the pivot

Substitute (8) into (7) and decompose into x and y directions:

$$N_x + F_{d,x} = m_p\left(\ddot{x} + L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta\right) \qquad (11)$$

```
% Horizontal (x) component of force on pole by cart
N_x(t) = sign(v_x) * F_dx + m_p * ( diff(x(t), 2) + L * diff(theta(t), 2) * cos(theta(
```

N_x(t) =

$$m_p\, \left(\frac{\partial^2}{\partial t^2}\, x(t) - \sin(\theta(t))\, L(t)\, \left(\frac{\partial}{\partial t}\, \theta(t)\right)^2 + \cos(\theta(t))\, L(t)\, \frac{\partial^2}{\partial t^2}\, \theta(t)\right) + 1.5708\, c_d\, r^2\, \rho\, \mathrm{sign}(\sigma_1)\, {\sigma_1}^2$$

where

$$\sigma_1 = \sin(\theta(t))\, \frac{\partial}{\partial t}\, L(t) + \frac{\partial}{\partial t}\, x(t) + \cos(\theta(t))\, L(t)\, \frac{\partial}{\partial t}\, \theta(t)$$

$$m_p g - N_y + F_{d,y} = m_p\left(L\ddot{\theta}\sin\theta + L\dot{\theta}^2\cos\theta\right) \qquad (12)$$

```
% Vertical (y) component of force on pole by cart
N_y(t) = -sign(v_y) * F_dy - m_p * g - m_p * L * ( diff(x(t), 2) * sin(theta(t)) + dif
```

N_y(t) =

$$-g\, m_p - m_p\, L(t)\, \left(\cos(\theta(t))\, \frac{\partial^2}{\partial t^2}\, \theta(t) + \sin(\theta(t))\, \frac{\partial^2}{\partial t^2}\, x(t)\right) - 1.5708\, c_d\, r^2\, \rho\, \mathrm{sign}(\sigma_1)\, {\sigma_1}^2$$

where

$$\sigma_1 = \cos(\theta(t))\, \frac{\partial}{\partial t}\, L(t) + \sin(\theta(t))\, L(t)\, \frac{\partial}{\partial t}\, \theta(t)$$

Torque Balance of pole:

$\mathbf{M} = \mathbf{I}\varepsilon + \mathbf{r}_p \times \mathbf{a}_c$

where:

$\mathbf{M} = \mathbf{r}_p \times \mathbf{G}_p - \mathbf{r}_p \times \mathbf{F}_d$: sum of non-inertial torques acting on the pole relative to the pivot.

$\mathbf{I} = \frac{4}{3} m_p L^2$: moment of inertia of the pole relative to the pivot

$\mathbf{r}_p \times \mathbf{a}_c$: torque generated by the inertial force caused by the acceleration of the cart.

Hence:

$$(m_p \mathrm{g} - F_{d,y})\mathrm{L}\sin\theta - F_{d,x}\mathrm{L}\cos\theta = \frac{4}{3} m_p L^2 \ddot{\theta} + m_p \ddot{x} \mathrm{L}\cos\theta \qquad (14)$$

From (5), (6), and (12):

$$N_c = m_c g + N_y = m_c g + m_p g - m_p\left(L\ddot{\theta}\sin\theta + L\dot{\theta}^2\cos\theta\right) + F_{\mathrm{dy}}$$

$$= (m_c + m_p)g + m_p L\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right) + \frac{1}{2}\rho c_d A\left(\dot{L}\cos\theta + L\dot{\theta}\sin\theta\right)^2 \qquad (17)$$

```
% Downward force on track by cart
eqn3 = N_c == ...
    (m_c + m_p) * g ...
    - m_p * L * (diff(theta(t), 2) * sin(theta(t)) + diff(theta(t))^2 * cos(theta(t)))
    + F_dy
```

eqn3(t) =

$$N_c(t) = g\, (m_c + m_p) - m_p\, L(t)\, \left(\sin(\theta(t))\, \frac{\partial^2}{\partial t^2}\, \theta(t) + \cos(\theta(t))\, \left(\frac{\partial}{\partial t}\, \theta(t)\right)^2\right) + 1.5708\, c_d\, r^2\, \rho\, \left(\cos(\theta(t))\, \frac{\partial}{\partial t}\right.$$

$$\ddot{\theta} = \frac{g\sin\theta + \cos\theta\left\{\dfrac{-F - m_p L\dot{\theta}^2\left(\sin\theta - \mu_c\,\mathrm{sgn}\left(N_c\dot{x}\right)\cos\theta\right)}{m_c + m_p} + \mu_c g\,\mathrm{sgn}\left(N_c\dot{x}\right)\right\} + \dfrac{\pi^3\rho c_d L^2 r^2}{2m_p}\mathrm{sgn}\left(\dot{\theta}\right)\dot{\theta}^2}{L\left\{\dfrac{4}{3} - \dfrac{m_p\cos\theta}{m_c + m_p}\left(\cos\theta + \mu_c\,\mathrm{sgn}\left(N_c\dot{x}\right)\right)\right\}}$$

$$\ddot{x} = \frac{F + m_p L\left(\dot{\theta}^2\sin\theta - \ddot{\theta}\cos\theta\right) - \mu_c N_c \mathrm{sgn}\left(N_c\dot{x}\right)}{m_c + m_p}$$

```
% Horizontal acceleration of cart (+ve to right)
eqn4 = diff(x(t), 2) == ( ...
    F ...
    + m_p * L * (diff(theta(t))^2 * sin(theta(t)) - diff(theta(t), 2) * cos(theta(t)))
    - F_f(t) ...
    - F_dx ...
    - N_x(t) ...
) / (m_c + m_p)
```

eqn4(t) =

$$\sigma_3 = -\frac{F_f(t) - F(t) + m_p\,(\sigma_3 - \sin(\theta(t))\,L(t)\,\sigma_2 + \cos(\theta(t))\,L(t)\,\sigma_4) - m_p\,L(t)\,(\sin(\theta(t))\,\sigma_2 - \cos(\theta(t))\,\sigma}{m_c + m_p}$$

where

$$\sigma_1 = \sin(\theta(t))\,\frac{\partial}{\partial t}\,L(t) + \frac{\partial}{\partial t}\,x(t) + \cos(\theta(t))\,L(t)\,\frac{\partial}{\partial t}\,\theta(t)$$

$$\sigma_2 = \left(\frac{\partial}{\partial t}\,\theta(t)\right)^2$$

$$\sigma_3 = \frac{\partial^2}{\partial t^2}\,x(t)$$

$$\sigma_4 = \frac{\partial^2}{\partial t^2}\,\theta(t)$$

```
% Re-arrange in terms of D2xt(t)
eqn4 = isolate(eqn4, diff(x(t), 2))
```

eqn4 =

$$\frac{\partial^2}{\partial t^2}\,x(t) = \frac{F(t) - F_f(t) + m_p\,(\sin(\theta(t))\,L(t)\,\sigma_2 - \cos(\theta(t))\,L(t)\,\sigma_3) + m_p\,L(t)\,(\sin(\theta(t))\,\sigma_2 - \cos(\theta(t))\,\sigma_3}{(m_c + m_p)\left(\frac{m_p}{m_c + m_p} + 1\right)}$$

where

$$\sigma_1 = \sin(\theta(t))\,\frac{\partial}{\partial t}\,L(t) + \frac{\partial}{\partial t}\,x(t) + \cos(\theta(t))\,L(t)\,\frac{\partial}{\partial t}\,\theta(t)$$

$$\sigma_2 = \left(\frac{\partial}{\partial t}\,\theta(t)\right)^2$$

$$\sigma_3 = \frac{\partial^2}{\partial t^2}\,\theta(t)$$

```
return
% eqn2 = diff(theta(t), 2) == ...
%     (g * sin(theta(t)) + cos(theta(t)) * ( ...
%         (-F - mp * l * diff(theta(t))^2 * (sin(theta(t)) + muc * sign(Nc) * sign(dif
%      ) / (mc + mp) + muc * g * sign(Nc) * sign(diff(x(t)))) - mup * diff(theta(t)) /
%     / (l * (4/3 - mp * cos(theta(t)) / (mc + mp) * (cos(theta(t)) - muc * sign(Nc) *

% Cart-track friction force - in opposite direction to cart travel
%eqn2 = F_f == muc * N_c * sign(N_c * diff(x(t)))

% Angular acceleration
eqn5 = diff(theta(t), 2) == ( ...
        g * sin(theta(t)) ...
        + cos(theta(t)) * ( ...
            (-F - m_p * L * diff(theta(t))^2 * sin(theta(t)) + F_f + F_dr * sin(theta(
        ) / (m_c + m_p) ...
        - F_dt / m_p ...
    ) / (L * (4/3 - m_p * cos(theta(t))^2 / (m_c + m_p)))

eqns = [eqn1 eqn2 eqn3 eqn4 eqn5];
vars = [x(t); theta(t); N_c(t); F_f(t); F_d(t)];
origVars = length(vars);
```

Check Incidence of Variables

```
M = incidenceMatrix(eqns, vars)
```

Reduce Differential Order

```
[eqns, vars] = reduceDifferentialOrder(eqns, vars)
```

Check Differential Index of System

```
if ~isLowIndexDAE(eqns, vars)
    disp("Reducing Differential Index...")
    % Reduce Differential Index with reduceDAEIndex
    [DAEs, DAEvars] = reduceDAEIndex(eqns, vars)
    % Eliminate redundant equations and variables
    [DAEs, DAEvars] = reduceRedundancies(DAEs, DAEvars)
    % Check the differential index of the new system
    assert(isLowIndexDAE(DAEs, DAEvars))
else
    DAEs = eqns;
    DAEvars = vars;
end
```

Convert DAE system to MATLAB function

```
pDAEs = symvar(DAEs);
pDAEvars = symvar(DAEvars);
extraParams = setdiff(pDAEs, pDAEvars)
```

Create the function handle.

```
%f = daeFunction(DAEs, DAEvars, F, L, c_d, g, m_c, m_p, muc, r, rho);
```

To save the DAE equations as a function script use this option

```
filename = 'craneDAEFunction.m';
f = daeFunction(DAEs, DAEvars, F, L, c_d, g, m_c, m_p, muc, r, rho, 'File', filename);
```

Set parameter values

Define parameter values as struct

```
params = struct();
params.F = 10;
params.L = 5;
params.c_d = 0.47;  % drag coefficient of a sphere = 0.47
params.g = 9.8;
params.m_c = 5;
params.m_p = 2;
params.muc = 0.2;
params.r = 1;  % radius of load
params.rho = 1.293;  % density of air (kg/m3)
```

Checks of function calculations

```
% With no external force (F=0)
F_DAE = @(t, Y, YP) f(t, Y, YP, 0, params.L, params.c_d, params.g, params.m_c, params.r

% Equilibrium point 1: stationary at vertical down position
y0 = [0; 0; (params.m_c + params.m_p)*params.g; 0; 0; 0; 0];
yp0 = [0; 0; 0; 0; 0; 0; 0];
assert(all(F_DAE(0, y0, yp0) == 0))
% Check at any x position
y0 = [2; 0; (params.m_c + params.m_p)*params.g; 0; 0; 0; 0];
yp0 = [0; 0; 0; 0; 0; 0; 0];
assert(all(F_DAE(0, y0, yp0) == 0))
y0 = [-3; 0; (params.m_c + params.m_p)*params.g; 0; 0; 0; 0];
yp0 = [0; 0; 0; 0; 0; 0; 0];
assert(all(F_DAE(0, y0, yp0) == 0))

% Equilibrium point 2: stationary at vertical up position
y0 = [0; pi; (params.m_c + params.m_p)*params.g; 0; 0; 0; 0];
yp0 = [0; 0; 0; 0; 0; 0; 0];
assert(all(abs(F_DAE(0, y0, yp0)) < 1e-15))
% Check at any x position
y0 = [10; pi; (params.m_c + params.m_p)*params.g; 0; 0; 0; 0];
yp0 = [0; 0; 0; 0; 0; 0; 0];
assert(all(abs(F_DAE(0, y0, yp0)) < 1e-15))

% Accelerates towards stable equilibrium
y0est = [0; 0.99*pi; (params.m_c + params.m_p)*params.g; 0; 0; 0; 0];
yp0est = [0; 0; 0; 0; 0; 0; 0];
```

```
FIXED_Y0 = [1 1 0 0 0 1 1]';
opt = odeset('RelTol', 10.0^(-9), 'AbsTol', 10.0^(-9));
[y0, yp0] = decic(F_DAE, 0, y0est, FIXED_Y0, yp0est, [], opt);
disp(table(string(DAEvars), y0, yp0))
assert(all(yp0(1:5) == 0))
% yp0(6) = D2xt(t), yp0(6) = D2thetat(t)
assert(yp0(6) > 0 && yp0(7) > 0)

% Accelerates towards stable equilibrium
y0est = [0; 1.01*pi; (params.m_c + params.m_p)*params.g; 0; 0; 0; 0];
yp0est = [0; 0; 0; 0; 0; 0; 0];
FIXED_Y0 = [1 1 0 0 0 1 1]';
[y0, yp0] = decic(F_DAE, 0, y0est, FIXED_Y0, yp0est, [], opt);
disp(table(string(DAEvars), y0, yp0))
assert(all(yp0(1:5) == 0))
% yp0(6) = D2xt(t), yp0(6) = D2thetat(t)
assert(yp0(6) < 0 && yp0(7) < 0)

% Accelerates away from unstable equilibrium
y0est = [0; 0.01*pi; (params.m_c + params.m_p)*params.g; 0; 0; 0; 0];
yp0est = [0; 0; 0; 0; 0; 0; 0];
FIXED_Y0 = [1 1 0 0 0 1 1]';
[y0, yp0] = decic(F_DAE, 0, y0est, FIXED_Y0, yp0est, [], opt);
disp(table(string(DAEvars), y0, yp0))
assert(all(yp0(1:5) == 0))
% yp0(6) = D2xt(t), yp0(6) = D2thetat(t)
assert(yp0(6) < 0 && yp0(7) > 0)

% Accelerates away from unstable equilibrium
y0est = [0; -0.01*pi; (params.m_c + params.m_p)*params.g; 0; 0; 0; 0];
yp0est = [0; 0; 0; 0; 0; 0; 0];
FIXED_Y0 = [1 1 0 0 0 1 1]';
[y0, yp0] = decic(F_DAE, 0, y0est, FIXED_Y0, yp0est, [], opt);
disp(table(string(DAEvars), y0, yp0))
assert(all(yp0(1:5) == 0))
% yp0(6) = D2xt(t), yp0(6) = D2thetat(t)
assert(yp0(6) > 0 && yp0(7) < 0)

% With +ve external forcing
F_DAE = @(t, Y, YP) f(t, Y, YP, 10, params.L, params.c_d, params.g, params.m_c, params
% Accelerates away from unstable equilibrium
y0est = [0; 0; (params.m_c + params.m_p)*params.g; 0; 0; 0; 0];
yp0est = [0; 0; 0; 0; 0; 0; 0];
FIXED_Y0 = [1 1 0 0 0 1 1]';
[y0, yp0] = decic(F_DAE, 0, y0est, FIXED_Y0, yp0est, [], opt);
disp(table(string(DAEvars), y0, yp0))
assert(all(yp0(1:5) == 0))
% yp0(6) = D2xt(t), yp0(6) = D2thetat(t)
assert(yp0(6) > 0 && yp0(7) < 0)

% With -ve external forcing
F_DAE = @(t, Y, YP) f(t, Y, YP, -10, params.L, params.c_d, params.g, params.m_c, param
% Stable equilibrium
y0est = [0; pi; (params.m_c + params.m_p)*params.g; 0; 0; 0; 0];
```

```
yp0est = [0; 0; 0; 0; 0; 0; 0];
FIXED_Y0 = [1 1 0 0 0 1 1]';
[y0, yp0] = decic(F_DAE, 0, y0est, FIXED_Y0, yp0est, [], opt);
disp(table(string(DAEvars), y0, yp0))
assert(all(yp0(1:5) == 0))
% yp0(6) = D2xt(t), yp0(6) = D2thetat(t)
assert(yp0(6) < 0 && yp0(7) < 0)

% Check cart-track friction force and drag force
% With no cart-track friction or drag
%F_DAE = @(t, Y, YP) f(t, Y, YP, F, L, c_d, g, m_c, m_p, muc, r, rho);
F_DAE = @(t, Y, YP) f(t, Y, YP, 0, params.L, 0, params.g, params.m_c, params.m_p, 0, pa
% In vertical down position with horizontal velocity
y0est = [0; pi; (params.m_c + params.m_p)*params.g; 0; 0; 5; 0];
yp0est = [1; 0; 0; 0; 0; 0; 0];
FIXED_Y0 = [1 1 0 0 0 1 1]';
[y0, yp0] = decic(F_DAE, 0, y0est, FIXED_Y0, yp0est, [], opt);
disp(table(string(DAEvars), y0, yp0))
% yp0(6) = D2xt(t), yp0(6) = D2thetat(t)
assert(all(abs(yp0(6:7)) < 1e-15))
% With drag but no cart-track friction
F_DAE = @(t, Y, YP) f(t, Y, YP, 0, params.L, params.c_d, params.g, params.m_c, params.m
% In vertical down position with horizontal velocity
y0est = [0; pi; (params.m_c + params.m_p)*params.g; 0; 0; 5; 0];
yp0est = [5; 0; 0; 0; 0; 0; 0];
FIXED_Y0 = [1 1 0 0 0 1 1]';
[y0, yp0] = decic(F_DAE, 0, y0est, FIXED_Y0, yp0est, [], opt);
disp(table(string(DAEvars), y0, yp0))
% yp0(6) = D2xt(t), yp0(6) = D2thetat(t)
assert(abs(yp0(6)) < 1e-15 && yp0(7) < -1e-15)
% With drag but no cart-track friction
F_DAE = @(t, Y, YP) f(t, Y, YP, 0, params.L, 0, params.g, params.m_c, params.m_p, 0, pa
% In horizontal right position with horizontal velocity
y0est = [0; pi/2; (params.m_c + params.m_p)*params.g; 0; 0; 5; 0];
yp0est = [0; 0; 0; 0; 0; 0; 0];
FIXED_Y0 = [1 1 0 0 0 1 1]';
[y0, yp0] = decic(F_DAE, 0, y0est, FIXED_Y0, yp0est, [], opt);
% yp0(6) = D2xt(t), yp0(6) = D2thetat(t)
assert(abs(yp0(6)) < 1e-15 && abs(yp0(5)) < 1e-15)
% With drag force but no cart-track friction
F_DAE = @(t, Y, YP) f(t, Y, YP, 0, params.L, params.c_d, params.g, params.m_c, params.m
y0est = [0; pi; (params.m_c + params.m_p)*params.g; 0; 0; 5; 0];
yp0est = [0; 0; 0; 0; 0; 0; 0];
FIXED_Y0 = [1 1 0 0 0 1 1]';
[y0, yp0] = decic(F_DAE, 0, y0est, FIXED_Y0, yp0est, [], opt);
% yp0(6) = D2xt(t), yp0(6) = D2thetat(t)
assert(abs(yp0(6)) < 1e-15 && yp0(7) < -1e-13)
```

Find initial condition

Create function for ode15i

```
F_DAE = @(t, Y, YP) f(t, Y, YP, params.F, params.L, params.c_d, params.g, params.m_c, p
DAEvars
```

Note that `Dxt(t)`, `Dthetat(t)`, ... etc. are the first derivatives of `x(t)`, `theta(t)`.

Provide an estimate of the initial condition

```
% Variables
% 1 degrees = 0.0172
y0est = [-3; deg2rad(180 - 0); 0; 0; 0; 0; 0];
% Their derivatives
yp0est = [0; 0; 0; 0; 0; 0; 0];
```

Set tolerances and do numerical search.

```
opt = odeset('RelTol', 10.0^(-7), 'AbsTol', 10.0^(-7));
FIXED_Y0 = [1 1 0 0 0 1 1]';
FIXED_YP0 = [0 0 0 0 0 0 0]';
[y0, yp0] = decic(F_DAE, 0, y0est, FIXED_Y0, yp0est, FIXED_YP0, opt)
```

Draw initial position

```
figure(1); clf
draw_crane(y0([1 3 2 4]),params)
```

Solve DAEs Using ode15i

```
opt = odeset('RelTol', 10.0^(-7), 'AbsTol', 10.0^(-5));
[tSol, ySol] = ode15i(F_DAE, [0 2], y0, yp0, opt);
```

Plot solution

```
figure(2); clf
labels = arrayfun(@(i) char(DAEvars(i)), 1:origVars, 'UniformOutput', false);
y_scaling = [1 1 10 1 1];
labels{3} = strcat('0.1', labels{3});
plot(tSol, ySol(:, 1:origVars) ./ y_scaling, 'LineWidth', 2)
xlabel("Time ($t$)", 'Interpreter', 'latex')
legend(string2latex(labels), 'Location', 'best', 'Interpreter', 'latex')
grid on
```

Save results to file

```
labels = arrayfun(@(i) char(DAEvars(i)), 1:numel(DAEvars), 'UniformOutput', false);
sim_results = array2table([tSol ySol], 'VariableNames', [{'t'} labels]);
filename = "crane_benchmark_sim.csv";
writetable(sim_results, fullfile(results_dir, filename))
```

Solve using function script file

Solve the equations using the function script created by `daeFunction` above and check results are identical.

```
% Solve DAEs
F_DAE2 = @(t, Y, YP) craneDAEFunction(t, Y, YP, params.F, params.L, params.c_d, params
```

```
    params.m_c, params.m_p, params.muc, params.r, params.rho);
[tSol2, ySol2] = ode15i(F_DAE2, [0 2], y0, yp0, opt);
assert(isequal(tSol, tSol2))
assert(max(ySol - ySol2, [], [1 2]) < 1e-14)
```

Solve using the edited function in crane_DAEs.m

```
% Solve DAEs
F_DAE2 = @(t, Y, YP) crane_DAEs(t, Y, YP, params);
[tSol2, ySol2] = ode15i(F_DAE2, [0 2], y0, yp0, opt);
assert(isequal(tSol, tSol2))
assert(max(ySol - ySol2, [], [1 2]) < 1e-14)
```