Solve DAE for a gantry crane

See the pendulum example in documentation:

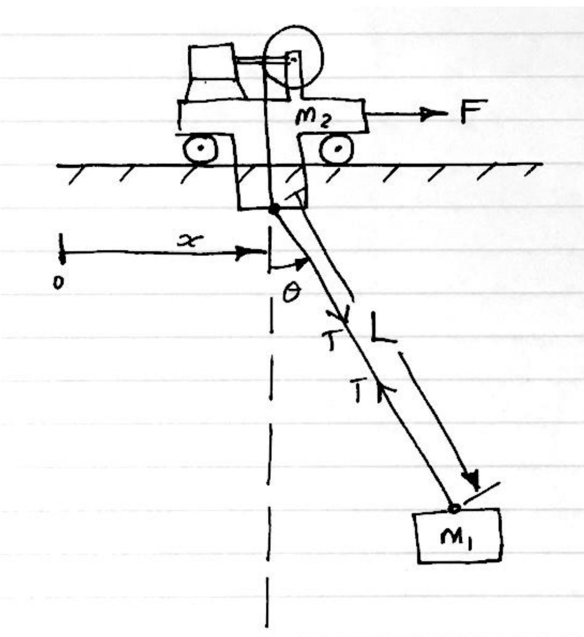
- https://www.mathworks.com/help/symbolic/solve-differential-algebraic-equations.html

```
clear variables

results_dir = 'results';
if ~isfolder(results_dir)
    mkdir(results_dir)
end
```

The equations below were developed from cart-pole equations presented by:

- R. V. Florian, 2007, Correct equations for the dynamics of the cart-pole system.



State variables

- Horizontal position of cart (from left) $x(t)$
- Horizontal velocity of cart $\dot{x}(t)$
- Angle of cable (anti-clockwise from vertical down position) $\theta(t)$
- Angular velocity of cable $\dot{\theta}(t)$
- Cable length $L(t)$

Other variables

- Downwards force on track from cart $N_c(t)$
- Frictional force exerted on cart by track $F_F(t)$

- Force exerted on cart by cable (and on load) $T(t)$

Potential input variables

- Horizontal driving force on cart $F(t)$
- Speed of cable leaving winch $\dot{L}(t)$

Parameters

- Cart mass m_c
- Load mass m_p
- Acceleration due to gravity g
- Coefficient of friction for cart and track μ_c
- Radius of the load, r
- Drag coefficient of load c_d
- Mass density of air ρ_a

System of DAEs

$$N_c = (m_c + m_p)g - m_p l\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right)$$

$$\ddot{\theta} = \frac{g\sin\theta + \cos\theta\left\{\dfrac{-F - m_p l\dot{\theta}^2\left(\sin\theta + \mu_c\,\mathrm{sgn}\left(N_c\dot{x}\right)\cos\theta\right)}{m_c + m_p} + \mu_c g\,\mathrm{sgn}\left(N_c\dot{x}\right)\right\} - \dfrac{\pi^3 c_d L^2 r^2}{2m_p}\dot{\theta}^2}{l\left\{\dfrac{4}{3} - \dfrac{m_p\cos\theta}{m_c + m_p}\left(\cos\theta - \mu_c\,\mathrm{sgn}\left(N_c\dot{x}\right)\right)\right\}}$$

$$\ddot{x} = \frac{F + m_p l\left(\dot{\theta}^2\sin\theta - \ddot{\theta}\cos\theta\right) - \mu_c N_c \mathrm{sgn}\left(N_c\dot{x}\right)}{m_c + m_p}$$

First, define an autonomous system

i.e. `F(t) = 0`, `L(t) = l` (constant)

```
% Define symbolic variables
syms x(t) theta(t) Nc(t) F mc mp L g muc r cd rho

eqn1 = 1000 * Nc == ...
    (mc + mp) * g - mp * L * (diff(theta(t), 2) * sin(theta(t)) + diff(theta(t))^2 * co
```

eqn1(t) =

$$1000\,\mathrm{Nc}(t) = g\,(\mathrm{mc} + \mathrm{mp}) - L\,\mathrm{mp}\left(\sin(\theta(t))\,\frac{\partial^2}{\partial t^2}\,\theta(t) + \cos(\theta(t))\,\left(\frac{\partial}{\partial t}\,\theta(t)\right)^2\right)$$

```
eqn2 = diff(theta(t), 2) == ...
    (g * sin(theta(t)) + cos(theta(t)) * ( ...
        (-F - mp * L * diff(theta(t))^2 * (sin(theta(t)) + muc * sign(Nc) * sign(diff(x
     ) / (mc + mp) + muc * g * sign(Nc) * sign(diff(x(t)))) - pi^3 * rho * cd * (L * r
```

```
    / (L * (4/3 - mp * cos(theta(t)) / (mc + mp) * (cos(theta(t)) - muc * sign(Nc) * s
```

```
eqn2(t) =
```

$$\frac{\partial^2}{\partial t^2}\,\theta(t) = \frac{\cos(\theta(t))\,\left(\frac{F + L\,\mathrm{mp}\,\left(\sin(\theta(t)) + \mathrm{muc}\,\mathrm{sign}(\mathrm{Nc}(t))\,\sigma_1\cos(\theta(t))\right)\sigma_2}{\mathrm{mc}+\mathrm{mp}} - g\,\mathrm{muc}\,\mathrm{sign}(\mathrm{Nc}(t))\,\sigma_1\right) -}{L\,\left(\frac{\mathrm{mp}\cos(\theta(t))\,\left(\cos(\theta(t)) - \mathrm{muc}\,\mathrm{sign}(\mathrm{Nc}(t))\,\sigma_1\right)}{\mathrm{mc}+\mathrm{mp}} - 1.3333\right.}$$

where

$$\sigma_1 = \mathrm{sign}\left(\frac{\partial}{\partial t}\,x(t)\right)$$

$$\sigma_2 = \left(\frac{\partial}{\partial t}\,\theta(t)\right)^2$$

```
eqn3 = diff(x(t), 2) == ...
    (F + mp * L * (diff(theta(t))^2 * sin(theta(t)) - diff(theta(t), 2) * cos(theta(t)
     - muc * 1000 * Nc * sign(Nc) * sign(diff(x(t)))) ...
    / (mc + mp)
```

```
eqn3(t) =
```

$$\frac{\partial^2}{\partial t^2}\,x(t) = \frac{F + L\,\mathrm{mp}\,\left(\sin(\theta(t))\,\left(\frac{\partial}{\partial t}\,\theta(t)\right)^2 - \cos(\theta(t))\,\frac{\partial^2}{\partial t^2}\,\theta(t)\right) - 1000\,\mathrm{muc}\,\mathrm{sign}(\mathrm{Nc}(t))\,\mathrm{sign}\left(\frac{\partial}{\partial t}\,x(t)\right)}{\mathrm{mc}+\mathrm{mp}}$$

```
eqns = [eqn1 eqn2 eqn3];
vars = [x(t); theta(t); Nc(t)];
origVars = length(vars)
```

```
origVars = 3
```

Check Incidence of Variables

```
M = incidenceMatrix(eqns, vars)
```

```
M = 3×3
     0     1     1
     1     1     1
     1     1     1
```

Reduce Differential Order

```
[eqns, vars] = reduceDifferentialOrder(eqns, vars)
```

```
eqns =
```

$$\left(\begin{array}{c}\dfrac{\partial}{\partial t}\,\mathrm{Dthetat}(t) - \dfrac{\cos(\theta(t))\left(\dfrac{L\,\mathrm{mp}\,(\sin(\theta(t)) + \mathrm{muc}\,\mathrm{sign}(\mathrm{Dxt}(t))\,\mathrm{sign}(\mathrm{Nc}(t))\cos(\theta(t)))\,\mathrm{Dthetat}(t)^2 + F}{\mathrm{mc}+\mathrm{mp}}\right.}{L\left(\dfrac{\mathrm{mp}\cos(\theta(t))\,(\cos(\theta(t)) - \mathrm{muc\ sig}}{\mathrm{mc}+\mathrm{mp}}\right.} \quad 1000\,\mathrm{Nc}(t) - g\,(\mathrm{mc}+\mathrm{mp}) + L\,\mathrm{mp}\left(\sin(\theta(t))\dfrac{\partial}{\partial t}\,\mathrm{Dth}\right. \\ \dfrac{L\,\mathrm{mp}\left(\cos(\theta(t))\dfrac{\partial}{\partial t}\,\mathrm{Dthetat}(t) - \sin(\theta(t))\,\mathrm{Dthetat}(t)^2\right) - F + 1000\,\mathrm{r}}{\mathrm{mc}+\mathrm{mp}} \\ \mathrm{Dxt}(t) - \dfrac{\partial}{\partial t}\,x(t) \\ \mathrm{Dthetat}(t) - \dfrac{\partial}{\partial t}\,\theta(t)\end{array}\right.$$

vars =

$$\begin{pmatrix} x(t) \\ \theta(t) \\ \mathrm{Nc}(t) \\ \mathrm{Dxt}(t) \\ \mathrm{Dthetat}(t) \end{pmatrix}$$

Check Differential Index of System

```
if ~isLowIndexDAE(eqns, vars)
    disp("Reducing Differential Index...")
    % Reduce Differential Index with reduceDAEIndex
    [DAEs, DAEvars] = reduceDAEIndex(eqns, vars)
    % Eliminate redundant equations and variables
    [DAEs, DAEvars] = reduceRedundancies(DAEs, DAEvars)
    % Check the differential index of the new system
    assert(isLowIndexDAE(DAEs, DAEvars))
else
    DAEs = eqns;
    DAEvars = vars;
end
```

Convert DAE system to MATLAB function

```
pDAEs = symvar(DAEs);
pDAEvars = symvar(DAEvars);
extraParams = setdiff(pDAEs, pDAEvars)
```

extraParams = $\begin{pmatrix} F & L & \mathrm{cd} & g & \mathrm{mc} & \mathrm{mp} & \mathrm{muc} & r & \rho \end{pmatrix}$

Create the function handle.

```
%f = daeFunction(DAEs, DAEvars, F, L, cd, g, mc, mp, muc, r, rho);
```

To save the DAE equations as a function script use this option

```
filename = 'craneDAEFunction.m';
```

```
f = daeFunction(DAEs, DAEvars, F, L, cd, g, mc, mp, muc, r, rho, 'File', filename);
```

Set parameter values

```
F = 0;
L = 3;
cd = 0.47;  % drag coefficient of a sphere = 0.47
g = 10;
mc = 5;
mp = 2;
muc = 0.2;
r = 0.25;  % radius of load
rho = 1.293;  % density of air = 1.293

% Parameter values as struct
params = struct();
params.F = F;
params.L = L;
params.cd = cd;
params.g = g;
params.l = L;
params.mc = mc;
params.mp = mp;
params.muc = muc;
params.r = r;
params.rho = rho;
```

Create function for ode15i

```
F_DAE = @(t, Y, YP) f(t, Y, YP, F, L, cd, g, mc, mp, muc, r, rho);
```

Find initial condition

```
DAEvars
```

DAEvars =

$$\begin{pmatrix} x(t) \\ \theta(t) \\ \mathrm{Nc}(t) \\ \mathrm{Dxt}(t) \\ \mathrm{Dthetat}(t) \end{pmatrix}$$

Note that `Dxt(t)`, `Dthetat(t)`, ... etc. are the first derivatives of `x(t)` ... etc.

Provide an estimate of the initial condition

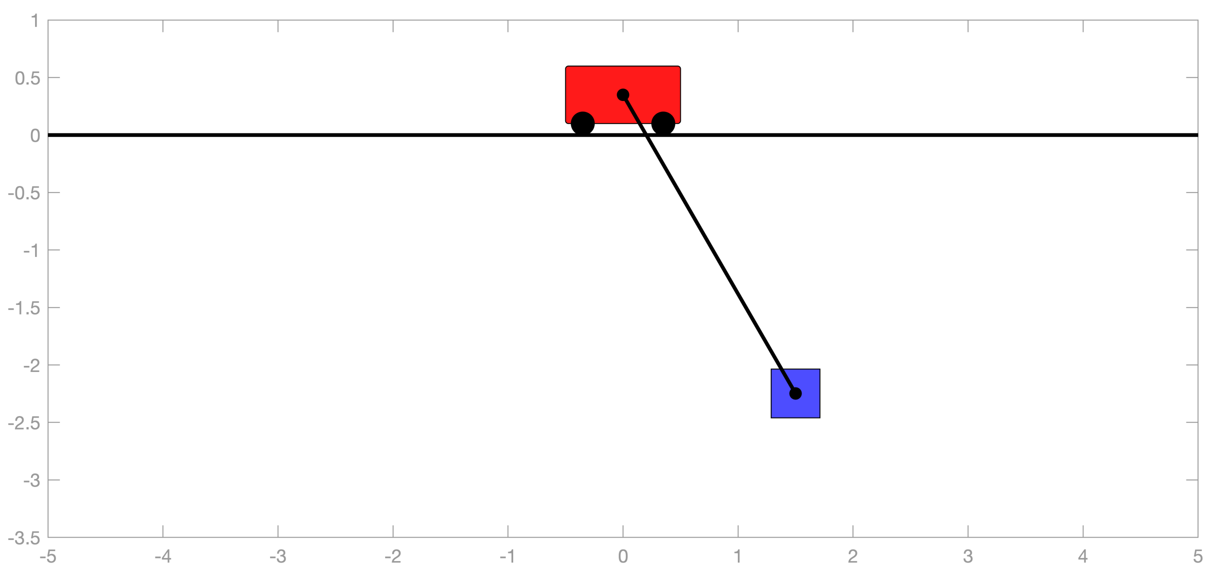
```
% Variables
% 1 degrees = 0.0172
y0est = [0; deg2rad(30); 0; 0; 0];
% Their derivatives
yp0est = [0; 0; 0; 0; 0];
```

Set tolerances and do numerical search.

```
opt = odeset('RelTol', 10.0^(-7), 'AbsTol', 10.0^(-7));
FIXED_Y0 = [1 1 0 1 1]';
FIXED_YP0 = [0 0 0 0 0]';
[y0, yp0] = decic(F_DAE, 0, y0est, FIXED_Y0, yp0est, FIXED_YP0, opt)
```

```
y0 = 5×1
         0
    0.5236
    0.0655
         0
         0
yp0 = 5×1
         0
         0
         0
   -1.1056
    1.4894
```

```
figure(1); clf
draw_crane(y0([1 3 2 4]),params)
```



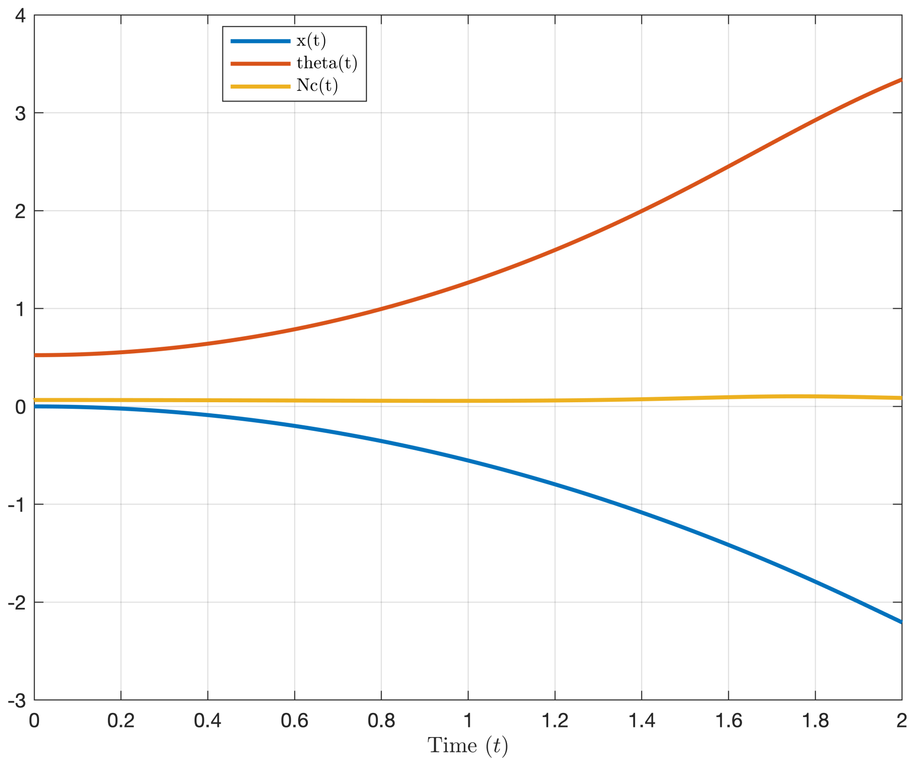
Solve DAEs Using ode15i

```
[tSol, ySol] = ode15i(F_DAE, [0 2], y0, yp0, opt);
```

Plot solution

```
figure(2); clf
```

```
plot(tSol, ySol(:, 1:origVars), 'LineWidth', 2)
xlabel("Time ($t$)", 'Interpreter', 'latex')
labels = arrayfun(@(i) char(DAEvars(i)), 1:origVars, 'UniformOutput', false);
legend(labels, 'Location', 'Best', 'Interpreter', 'latex')
grid on
```



Save results to file

```
labels = arrayfun(@(i) char(DAEvars(i)), 1:numel(DAEvars), 'UniformOutput', false);
sim_results = array2table([tSol ySol], 'VariableNames', [{'t'} labels]);
filename = "crane_benchmark_sim.csv";
writetable(sim_results, fullfile(results_dir, filename))
```

Solve using function script file

Solve the equations using the function script created by `daeFunction` above and check results are identical.

```
% Solve DAEs
F_DAE2 = @(t, Y, YP) craneDAEFunction(t, Y, YP, F, L, cd, g, mc, mp, muc, r, rho);
[tSol, ySol] = ode15i(F_DAE2, [0 1], y0, yp0, opt);
```

Solve using the edited function in crane_DAEs.m

```
% Solve DAEs
F_DAE2 = @(t, Y, YP) crane_DAEs(t, Y, YP, params);
[tSol2, ySol2] = ode15i(F_DAE2, [0 1], y0, yp0, opt);
assert(isequal(tSol, tSol2))
assert(max(ySol - ySol2, [], [1 2]) < 1e-14)
```