Advanced Analysis of Algorithms - Homework III

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1 Problems

1. ```
int countInv(int A[], int start, int end) {
  if(start == end || start + 1 == end) {
    return 0;
  }
  int a, b;
  a = countInv(A, start, (start+end)/2);
  b = countInv(A, start+(start+end)/2, end);
  int i = start, j;
  int count = 0;
  while(i < (start+end)/2) {
    j = start+(start+end)/2;
    while(j < end) {
      if(A[i] > A[j]) {
        count++;
      }
      j++;
    }
    i++;
  }
  return a+b+count;
}
```

2. a) The optimal substructure property does not hold when there are negative weight cycles.

   b) ```
int floyd(int n, const float W[][], float D[][]) {
  int i, j, k;
  D = W; //do a deep copy, not shallow
  for(k=0; k<n; k++) {
    for(i=0; i<n; i++) {
      for(j=0; j<n; j++) {
        D[i][j] = min(D[i][j], D[i][k]+D[k][j]);
      }
    }
  }
  for(i=0; i<n; i++) {
    if(D[i][i] < 0) {
```

```
        return FALSE; //if path from i to i is negative, there has to be a negative cycl
      }
    }
    return TRUE;
  }
```

3. a)

$$X = \begin{pmatrix} 9 & 3 \\ 2 & -1 \end{pmatrix} Y = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

For Strassen's matrix multiplication algorithm, we have to calculate matrices m_1 through m_7.

$$\begin{aligned}
\mathrm{m}_1 &= (a_{11} + a_{22})(b_{11} + b_{22}) \\
\mathrm{m}_2 &= (a_{21} + a_{22})b_{11} \\
\mathrm{m}_3 &= a_{11}(b_{12} - b_{22}) \\
\mathrm{m}_4 &= a_{22}(b_{21} - b_{11} \\
\mathrm{m}_5 &= (a_{11} + a_{12})b_{22} \\
\mathrm{m}_6 &= (a_{21} - a_{11})(b_{11} + b_{12}) \\
\mathrm{m}_7 &= (a_{12} - a22)(b_{21} + b_{22})
\end{aligned}$$

$X \times Y = C$ where

$$C = \begin{pmatrix} \mathrm{m}_1 + m_4 - m_5 + m_7 & \mathrm{m}_3 + m_5 \\ \mathrm{m}_2 + m_4 & \mathrm{m}_1 + m_3 - m_2 + m_6 \end{pmatrix}$$

So, in this case,

$$\begin{aligned}
\mathrm{m}_1 &= (9 + -1)(1 + -1) = 0 \\
\mathrm{m}_2 &= (2 + -1)(1) = 1 \\
\mathrm{m}_3 &= 9(2 - -1) = 27 \\
\mathrm{m}_4 &= -1(2 - 1) = -1 \\
\mathrm{m}_5 &= (9 + 3)(-1) = -12 \\
\mathrm{m}_6 &= (2 - 9)(1 + 2) = -21 \\
\mathrm{m}_7 &= (3 - -1)(2 + -1) = 4
\end{aligned}$$

and

$$C = \begin{pmatrix} 0 + -1 - (-12) + 4 & 27 + (-12) \\ 1 + -1 & 0 + 27 - 1 + (-21) \end{pmatrix} = \begin{pmatrix} 15 & 15 \\ 0 & 5 \end{pmatrix}$$

b)

4.

5. a) For each binary tree of n nodes, there is exactly one ordering of the distinct keys which yields a binary search tree. So this problem reduces to show that the number of binary trees of n nodes is $\frac{1}{n+1}2nn$ Let T_n be the sequence which gives the number of binary trees of n nodes. $T_n = \sum_{k=0}^{n-1} T_k \times T_{n-k-1}$ and $T_0 = 1$. This is because once we choose the root node, we either put 0 nodes in the left subtree and $n-1$ in the right, 1 in the left and $n-2$ in the right $\ldots$. So $T_n = \{1, 1, 2, 5, 14, 42, \ldots\} \forall n 0$. Let $T(z) = \sum_{N=0}^{\infty} T_N z^N$. So $T(z) = T_0 + T_1 z + T_2 z^2 + T_3 z^3 + \ldots$. Squaring both sides we obtain $T(z)^2 = (T_0 T_0) + (T_0 T_1 + T_1 T_0)z + (T_0 T_2 + T_1 T_1 + T_2 T_0)z^2 + \ldots$ and by the definition of T_n, we get $T(z)^2 = T_1 + T_2 z + T_3 z^2 + \ldots$. Next by algebra: $zT(z)^2 = T_1 z + T_2 z^2 + \ldots = -T_0 + T_0 + T_1 z + T_2 z^2 + \ldots = -T_0 + T(z)$. So $zT(z)^2 - T(z) + 1 = 0$ and by the quadratic formula, $T(z) = \frac{1 \pm \sqrt{1-4z}}{2z}$. Since $T(z) = T_0 = 1$, we can simplify this to $T(z) = \frac{1-\sqrt{1-4z}}{2z}$ which goes to 1 as $z \to 0$. Now expand $(1 - 4z)^{\frac{1}{2}} = 1 - \frac{1}{2}4z + \frac{(\frac{1}{2})(-\frac{1}{2})}{2!}(4z)^2 - \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!}(4z)^3 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{4!}(4z)^4 - \ldots = 1 - 2z - \frac{1}{2!}4z^2 - \frac{3\times1}{3!}8z^3 - \frac{5\times3\times1}{4!}16z^4 - \ldots$. Plugging this into $T(z)$, we get $T(z) = 1 + \frac{1}{2!}2z + \frac{3\times1}{3!}4z^2 + \frac{5\times3\times1}{4!}8z^3 + \ldots = 1 + \frac{1}{2}(\frac{2!}{1!1!})z + \frac{1}{3}(\frac{4!}{2!2!})z^2 + \frac{1}{4}(\frac{6!}{3!3!}z^3 + \ldots = \sum_{k=0} \infty \frac{1}{k+1} 2kkz^k$. Therefore, $T_n = \frac{1}{n+1}2nn$ as desired.

   b)