Cutting cuts

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What we want to prove

Lemma

(Basic Elimination Lemma) If $\vdash^{\alpha}_{\rho+1} \Delta$ *then* $\vdash^{\omega^{\alpha}}_{\rho} \Delta$

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(Generalized Elimination Lemma) If $\vdash^{\alpha}_{\beta+\omega^{\rho}}$ *then* $\vdash^{\phi_{\rho}(\alpha)}_{\beta} \Delta$

Veblen functions (defined inductively)

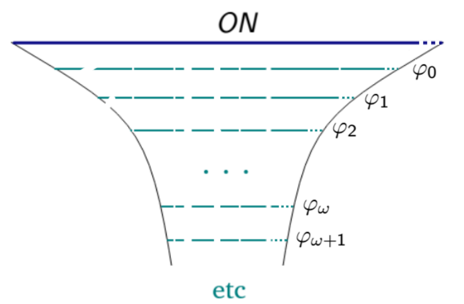

Figure 1: The range (image) of the first Veblen functions. Not on scale

Zero ordinal: $\varphi_0(\alpha) := \omega^\alpha$

Successor ordinals ρ: $\varphi_{\rho+1}(\alpha) := Enum_{FIX(\varphi(\rho))}(\alpha)$

Limit ordinals λ: $\varphi_\lambda(\alpha) := Enum_{FIX(\bigcap_{\rho<\lambda} \varphi(\rho))}(\alpha)$

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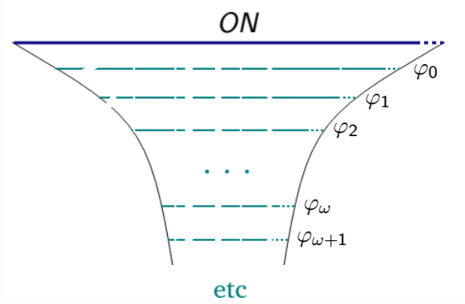

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Remark: if $\rho_1 < \rho_2$ then φ_{ρ_2} enumerates a subset of fixed points of φ_{ρ_1}

Proof (cont)

Now assume that the last inference was

$$\vdash^{\alpha_0}_{\beta+\omega^\rho} \Delta, F \text{and} \vdash^{\alpha_0}_{\beta+\omega^\rho} \Delta, \neg F \Rightarrow \vdash^{\alpha}_{\beta+\omega^\rho} \Delta$$

for $\alpha_0 < \alpha$, and formula F such that $rank(F) \in [\beta, \beta + \omega^\rho)$.

Let γ be an ordinal such that $rank(F) = \beta + \gamma$. We decompose γ into its Cantor Normal Form and get,

$$rank(F) = \beta + \gamma = \beta + \omega^{\sigma_1} + ... + \omega^{\sigma_n} < \beta + \omega^\rho$$

such that $\rho > \sigma_1 \geq \sigma_2 \geq ... \geq \sigma_n$, and thus,

$$rank(F) < \beta + \omega^{\sigma_1}(n+1).$$

We now do a side induction on α. When $\alpha = 0$, the claim is trivial. Otherwise, we use the induction hypothesis to get,

$$\vdash^{\varphi_\rho(\alpha_0)}_{\beta} \Delta, F \text{and} \vdash^{\varphi_\rho(\alpha_0)}_{\beta} \Delta, \neg F.$$

Thus, we get $\vdash^{\varphi_\rho(\alpha_0)+1}_{\beta+\omega^{\sigma_1}(n+1)} \Delta$ by a cut.

Since $\sigma_1 < \rho$, we apply the induction hypothesis to get

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By induction on n.

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$n \Rightarrow n+1 : \varphi_{\sigma_1}^{n+1}(\varphi_\rho(\alpha_0)+1) = \varphi_{\sigma_1}(\varphi_{\sigma_1}^n(\varphi_\rho(\alpha_0)+1))$

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By I.H., this is less than $\varphi_{\sigma_1}(\varphi_\rho(\alpha))$ and by definition this is equal to $\varphi_\rho(\alpha)$.

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By induction on n.

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By I.H., this is less than $\varphi_{\sigma_1}(\varphi_\rho(\alpha))$ and by definition this is equal to $\varphi_\rho(\alpha)$.
And we are done!