The Forgotten Quaternions

Ryan J. Wisnesky
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1 Introduction

A quick glance through the Mathematics section of the 2004 Stanford University Bulletin reveals five courses devoted to complex analysis, but no courses even mentioning quaternions. How is it that complex analysis, a subject that has suffered through hundreds of years of skepticism and distrust, came to be so widely accepted today, while quaternionic analysis, a modern subject immediately accepted as a straight-forward extension to complex analysis, has fallen by the way-side? I contend that the modern lack of interest in quaternions exists because *quaternions solved no problem that could not be solved more easily by another method.* I will (briefly) argue that complex numbers were too useful to be ignored, and thus developed into their modern conception, and offer it as an example of a subject accepted and developed because of its utility. The bulk of the paper will then be devoted to showing that quaternions did not achieve a level of utility sufficient to warrant its development into a mainstream subject. I will argue that quaternions did not achieve this level of utility because they were eclipsed by a related field of vector analysis that eventually became modern Vector Calculus.

2 Complex Numbers

Many modern courses on complex analysis begin with the instructor offering some variation of "complex numbers emerged when mathematicians needed a way to specify the roots of irreducible quadratics such as $x^2+1=0$" as the reason why complex numbers were 'invented'. Sometimes, the instructor even goes so far as to say that i was then *defined* to be a root of that equation. Despite the popularity of such an introduction, complex numbers developed along a much more controversial path than that introduction would indicate.

2.1 A Brief History of Complex Numbers

Although precursors of complex numbers can be traced all the way back to the ancient Greek Diophantus in his arithmetic of pairs [7, p383], the modern conception of complex numbers begins in sixteenth century Italy, where the mathematicians del Ferro and Tartaglia independently discovered a solution to the cubic equation $y^3 = py + q$ [7, p92]. The mathematician Cardano was given the solution from Tartaglia, and we now have Cardano's formula for a solution to the reduced cubic:

$$y = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}}$$

Because of the difference under the square-root, it is possible that the square-root of a negative number will be extracted during the computation of the root. But because this formula is valid for any coefficients p and q, and because cubics must have at least one real root, this formula will still yield a real root despite the appearance of complex numbers. In fact, in 1896 Bombelli showed that any algebraic formula for the general cubic must involve the square roots of quantities that are negative for certain values of the coefficients [7, p258].

The fact that the formula would yield correct answers using complex numbers, which were at the time considered 'impossible', mathematicians were forced to cope with the philosophical implications of 'impossible' numbers at a time when they were still uneasy with, and avoiding, negative numbers. In fact, at that time, the two equations

$$y^3 + px + q = 0$$

and

$$y^3 + px = q$$

were regarded as two different equations, requiring two different solutions, precisely because mathematicians wanted to avoid introducing negative quantities into their equations. Despite this first taste of complex numbers, the solution to the quadratic $x^2 + 1 = 0$ was still considered 'impossible', despite the fact that i is a root, because mathematicians had not yet found a *need* to specify its roots. Thus, at the time Cardano's formula was discovered — circa 1540 — mathematicians regarded complex numbers strictly as aids for real computation. The newly discovered utility of complex numbers for calculations stimulated some research in the area, and by 1572 the mathematician Bombelli had worked out the formal algebra of complex numbers to the point that he could reduce expressions of the form $\sqrt{a + b\sqrt{-1}}$ to the form $c + d\sqrt{-1}$ [7, p258]. However, the application of complex numbers to fields other than the solution of cubics was still centuries away.

The algebra of complex numbers did not advance much until 1673, when the English mathematician Wallis, who among other things gave one of the first infinite products for π, almost discovered the modern interpretation of complex numbers as points in the plane. (Wallis's construction was plagued by his belief that the negative of a complex number must lay to the left of it in the complex plane) [7, p261]. It appears that the development of algebra around this time brought a new need for complex numbers. As the eighteenth century progressed, mathematicians began to use complex numbers to achieve interesting results about real numbers, although some problems remained, and the new methods lacked rigor. For instance, in 1740, Euler discovered the identity $e^{ix} = \cos x + i \sin x$, but in 1770, Euler gave an erroneous proof that $\sqrt{-2} \times \sqrt{-3} = \sqrt{6}$ [7, p261]. Given the deep relationship between algebra and complex numbers, and the sudden explosion of complex methods for algebra around this time, it seems unlikely that either algebra or complex analysis could have advanced independently of each other.

By the end of the eighteenth century, the modern geometric interpretation of complex numbers as points in the plane was discovered by Gauss, d'Alembert, and Argand. Once this discovery was made, the study of complex numbers as an independent subject began in earnest, and the philosophical questions associated with complex numbers began to fade. Questions about what complex numbers were could be answered by an appeal to geometric intuition. After the discovery of non-euclidean geometries in the early to mid nineteenth centuries, the euclidean plane was rejected by some mathematicians as a firm basis for complex numbers. These mathematicians, including Gauss, who had earlier been a proponent of the plane as a foundation for complex numbers, rejected the old interpretation of complex numbers as points in the plane in favor of an interpretation of complex numbers as ordered pairs of real numbers [2, p26]. Today, this difference seems trivial, but at the time, the difference between a point and an ordered pair of numbers carried significant philosophical implications.

2.2 The Utility of Complex Analysis

The first truly modern use of complex analysis — as opposed to the simple use of complex numbers — occurred in the late eighteenth century when properties of complex conjugates were used in proofs of the Fundamental Theorem of Algebra by d'Alembert, Gauss, and others. In connection with this proof, mathematicians began to investigate questions of continuity of functions in the complex plane that would prove key in the development of continuity proofs in real analysis and the modern set-theoretic definitions of real numbers [7, p267].

By the early-nineteenth century, Gauss and others had defined the meaning of complex integration and differentiation. The equations

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 0$$
$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 0$$

relating the velocity components P and Q in a two-dimensional steady non-rotational fluid flow discovered by d' Alembert in 1752 turned out to be precisely the same equations defining an analytic function of a complex variable. When the concept of analycity for complex functions was developed by Cauchy in the early and mid nineteenth century, the result was a mathematical bombshell [7, p296]. For the first time, a physical phenomenon was more easily describable in terms of a single complex function than as seemingly unrelated pairs of real functions. In modern terms, this was equivalent to thinking of fluid flow as given by $f(z) = u(x, y) + iv(x, y)$ rather than two separate functions $u(x, y)$ and $v(x, y)$. However, because results arrived at via complex analysis of $f(z)$ were, in principle, reducible to proofs about the real functions $u(x, y)$ and $v(x, y)$, mathematicians and physicists were able to continue treating complex numbers as useful tools and beautiful mathematics while at the same time being able to dodge philosophical questions about complex numbers.

Since this first result from mathematical physics, imaginary numbers have come to play key roles in physics, ranging from electrical engineering, where complex quantities called 'phasers' are used to describe voltages and currents in a circuit, to Special Relativity, where time is often multiplied by an imaginary quantity in order to preserve the special distance metric of Einstein's space-time. In electrical engineering, the use of complex numbers is so fundamental that the imaginary quantity i is given its own special symbol j to avoid confusion with current, which is often denoted by I. Use of complex numbers in Electrical Engineering is now so common that this author's Introductory Electronics course reader did not even attempt to justify the definition of current as a complex quantity.

On the strictly mathematical side, the advent of complex function theory has allowed investigations of subjects ranging from elliptic curves to conformal mappings, and the Calculus of Residues pioneered by Cauchy has allowed the determination of a myriad of definite real integrals. Complex analysis is used routinely to arrive at results in traditional number theory, and the recent (unchecked) proof offered by Arenstorf of the Twin Prime Conjecture relies heavily on complex analysis.

2.3 The Acceptance of Complex Methods

When considering the value of the above results, it is important to note that there are two distinct classes of problems solved by complex analysis: problems that depend in some way on what complex numbers inherently 'are', and problems where complex analysis is used as a 'short-cut' to obtain results that could, in theory, be obtained without the use of complex numbers. As an example of the first, consider the Fundamental Theorem of Algebra, whose domain of validity necessarily extends over the complex plane. As an example of the second, complex analysis allows us to easily compute certain integrals that can be also be computed using real analysis only [1, p154].

I believe it is this extremely wide range of applicability that has lead complex analysis to be called the single most beautiful subject of mathematics. Moskowitz, in the introduction to his textbook on single-variable complex analysis, sums up this view nicely: "Complex analysis is a beautiful subject - perhaps the single most beautiful, and striking, in mathematics. It presents completely unforeseen results that are of a dramatic, even magical, nature" [5].

It is not my intention to argue vigorously for the above view that complex numbers were simple too useful and too beautiful to be discarded. Rather, I believe that most mathematicians, having surveyed the vast expanses of mathematics for themselves, and having seen subjects both useful and not, and subjects both beautiful and not, will recognize the truth of the view at once. Readers still unconvinced are encouraged to look at [6].

With this background in mind, let us now proceed to contrast the historical development and utility of complex numbers with that of quaternions.

3 Quaternions

3.1 Basic Definitions

Having read a number of introductions to the subject of quaternions, I do not believe any of them compare with the own words of William Hamilton, the discoverer of quaternions. In [4], Hamilton defines quaternions follows:

> Let an expression of the form
>
> $$Q = w + ix + jy + kz$$
>
> be called a *quaternion*, when w, x, y, z, which we shall call the four *constituents* of the quaternion Q, denote any real quantities, positive or negative or null, but i, j, k are symbols of three imaginary quantities, which we shall call *imaginary units*, and shall suppose to be unconnected by any linear relation with each other; in such a manner that if there be another expression of the same form
>
> $$Q' = w' + ix' + jy' + kz'$$
>
> the supposition of an equality between these two quaternions,
>
> $$Q = Q'$$
>
> shall be understood to involve four separate equations between their respective constituents, name, the four following,
>
> $$w = w' \quad x = x' \quad y = y' \quad z = z'$$

Hamilton then goes on to define addition and subtraction of quaternions as the piecewise sum and difference of the constituents:

$$Q \pm Q' = w \pm w' + i(x \pm x') + j(y \pm y') + k(z \pm z')$$

He also defines multiplication of quaternions by treating i, j, and k as ordinary numbers and using standard algebraic multiplication:

$$\begin{aligned}
QQ' = {} & ww' + iwx' + jwy' + kwz' \\
& + ixw' + i^2xx' + ijxy' + ikxz' \\
& + jyw' + jiyx' + j^2yy' + jkyz' \\
& + kzw' + kizx' + kjzy' + k^2zz'
\end{aligned}$$

and he defines

$$i^2 = j^2 = k^2 = -1$$

$$ij = k, \quad jk = i, \quad ki = j$$

$$ji = -k, \quad kj = -i, \quad ik = -j$$

From these basic definitions, Hamilton develops over 80 pages of mathematics in this paper [4] alone.

3.2 History of Quaternion Development

Hamilton, a veritable giant among English scientists [2, p19], began to develop quaternions in 1830, around the time that two important trends were emerging: the modern view of complex numbers as an extension to the system of real numbers and the view of *an* algebra as a means of manipulating symbols as a generalization of regular algebra [2, p23]. In some sense a product of these trends, Hamilton sought an extension of the complex numbers, which he viewed at the time *only* as pairs of real numbers [2, p25], into triplets of real numbers. In essence, Hamilton was attempting to extend the "Theory of Couplets" (his words) to the "Theory of Triplets". Hamilton was motivated in his quest by his somewhat unusual philosophical beliefs about time [2, p27], which we will not discuss here. After nearly ten years of unsuccessfully searching for a three-dimensional extension of complex numbers, Hamilton had little to show for it, although he did explicitly distill the properties he wanted his new set of numbers to have [2, p28]:

1. The associative property for addition and multiplication. He desired $N + (N' + N'') = (N + N') + N''$ and $(NN')N'' = N(N'N'')$.

2. The commutative property for addition and multiplication, namely, $N + N' = N' + N$ and $NN' = N'N$.

3. The distributive property of multiplication over addition, that $N(N' + N'') = NN' + NN''$.

4. Unambiguous division. Or, put another way, unique inverses. This guarantees that for any $N \neq 0$ and N', there is only one X such that $NX = N'$.

5. The Law of the Moduli. That is, if

$$(a_1 + b_1 i + c_1 j)(a_2 + b_2 i + c_2 j) = a_3 + b_3 i + c_3 j$$

then

$$(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2) = a_3^2 + b_3^2 + c_3^2$$

6. That the numbers have an interpretation in three dimensional euclidean space.

All of these properties hold for complex numbers (restricted to two dimensions, of course). Unfortunately for Hamilton, no such system of triplets exists. The mathematician C. S. Peirce proved in 1881 that real algebra, complex algebra, and quaternionic algebra are the only associative algebras in which division is unambiguous [2, p28]. Hamilton was unaware of the number-theoretic results of Legendre's *Théorie des Nombres*, a consequence of which is that only 1,2,4, and 8 tuples obey the law of the moduli. A simple observation might have saved Hamilton a decade of work. Ironically, less than a year after Hamilton discovered quaternions, his colleague John Graves wrote to him informing him of Legendre's work.

In 1843, while still searching for a system of triplets, Hamilton discovered quaternions. He was forced to give up the commutativity of multiplication (although he gained anti-commutativity), but was confident enough in his new discovery that he "felt it might be worthy my while to expend the labour of at least ten (or it might be fifteen) years to come" [2, p30]. Note that Hamilton still retained property 6: Hamilton viewed a quaternion as being made up of a scalar part (w) and a three-dimensional vector part ($ix+jy+kz$). From our modern point of view, it is tempting to consider quaternions as vectors in four-space, but *the properties of quaternions and four-vectors are different, although related.* This somewhat unusual interpretation of a quaternion as a sum of scalar and vector parts was not overlooked by physicists, as we shall see later.

Hamilton went on to spend the last twenty-two years of his life working on quaternions almost exclusively.

3.3 Useful Properties of Quaternions

Being the sum of a scalar and a vector, Hamilton introduced notation for the scalar part of a quaternion Q as SQ and the vector part of the quaternion as VQ. Thus, $Q = SQ + VQ$. Interestingly, in the case of a quaternion consisting only of a vector part $ix + jy + kz$, Hamilton defined close analogs of modern dot and cross products. Let $\alpha = ix + jy + kz$ and $\beta = ix' + jy' + kz'$. Then we have

$$S\alpha\beta = xx' + yy' + zz'$$

(the analog of the dot product), and

$$V\alpha\beta = i(yz' - zy') + j(zx' - xz') + k(xy' - yx')$$

(the analog of the cross product). Hamilton also defined

$$\triangleleft = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

and

$$-\triangleleft^2 = \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 + \left(\frac{\partial}{\partial z}\right)^2$$

These definitions are similar enough to modern vector notation that Hamilton was, in fact, able to apply essentially modern vector methods to physical problems in three-space. James Clerk Maxwell made extensive use of quaternions in his *Treatise* about electromagnetism. Lord Kelvin also (grudgingly) used quaternions in his investigations of heat [2, p120]. The above definitions led most use of quaternions to center around the vector part while neglecting the scalar part. Moreover, some of the more interesting algebraic properties of quaternions, like division (which modern vectors lack), were not as often employed.

The most useful properties of quaternions, from a mathematician's point of view, are undoubtedly their algebraic properties. The ability to multiply and divide quaternions and the law of the moduli lets the mathematician treat quaternions as numbers in the same way as complex numbers, and enables interesting investigations into algebraic systems in general. However, most physicists were concerned primarily with the vector part of quaternions, and the algebraic properties that mathematicians so enjoyed became a hinderance, and thus the modern system of vectors developed.

4 Modern Vectors

The relationship between quaternions and modern vectors is a complex one. As we have seen, the notion of a quaternion in some sense encompasses the notion of a vector in euclidean three space. But in another way, a vector in four space is a close analog to a quaternion. Before we can analyze the relationship between the two systems, let us first give a few definitions.

4.1 Definitions

It is standard to define a vector $\mathbf{x}$ in n dimensional Euclidean space as the ordered n-tuple $(x_1, x_2, ..., x_n)$. Addition of vectors is defined term-wise just as for quaternions. For three space, we may write $\mathbf{v} = \mathbf{i}v_1 + \mathbf{j}v_2 + \mathbf{k}v_3$, where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors along the three coordinate axes. Define the dot product $\cdot$ of two vectors as

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + ... + u_nv_n$$

. The cross product $\times$ is defined only in three space as

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

.

The similarities (and differences) between modern vector definitions and quaternions are quite striking and will be discussed in a following section. And so it should be, for vectors and quaternions posses a shared history.

4.2 History of Modern Vector Development

Hamilton was not the only person working on algebras of multiple quantities. However, Hamilton aimed primarily to extend the complex numbers rather than solve particular problems, and many others, including, Möbius, Comte de Saint-Venant, and Cauchy, developed other vectorial systems to solve a variety of problems, including the need to solve vector-like equations. Note, however, that none of these systems would be considered a part of modern linear algebra, but instead were aids to simplify notation, and these early systems did not have a large impact in the subsequent development of modern vectors [2, p47]. Despite the complex relationships between the two subjects, it is safe to say that the discovery of modern vectors was motivated primarily by physics, and can be viewed as a combination of Hamilton's ideas and another vectorial system developed by Grassman.

Hermann Grassman, a German physicist, published a work on the theory of ocean tides entitled *Theorie der Ebbe und Flut* in 1840. To develop his theory of tides, Grassman also developed a *purely geometrical* vector system, and developed analogs of the modern dot and cross products. Realizing that the value of such a system extended far beyond his own theory of tides, in 1844 he published his *Ausdehnungslehre*, which contained an extensive system of vectors that contains much of modern vector analysis [2, p65]. However, Grassman's work was extraordinarily difficult to read, and appeared ahead of its time. Grassman discussed the modern concept of a *basis* for arbitrary dimensions, and developed inner and outer vector products. However, to develop the modern vector analysis from Grassman's work, one would have to

> 1) Read and understand the book (no small task), (2) delete major mathematical portions of the book (such as point analysis), (3) limit the presentation to three-dimensional space [since at the time vectors in higher spaces were not yet taken seriously] , (4) redefine some of the fundamental ideas (such as the outer product), (5) change the structure and emphasis of the work, (6) detach the presentation from the philosophical ideas contained in it, and (7) attach to it ideas already in the literature of the times, but unknown to Grassman, such as the development of the geometrical representation of complex numbers and the theorems of Green and Gauss. [2, p77]

Decades passed before Grassman's work was fully understood. Grassman had considerable trouble using his work to acquire a university position (he spent much of his career teaching only elementary mathematics at Stettin), and was disappointed that he was unable to pass on his ideas to a new generation of physicists and mathematicians as Hamilton was doing [2, p92]. However, Grassman understood that his theories were ahead of their time, and with amazing foresight and not misplaced confidence wrote the following in the forward of the second edition of his *Ausdehnungslehre* in 1861 that I cannot but help giving here:

> For I remain completely confident that the labor which I have expended on the science presented here and which has demanded a significant part of my life

> as well as the most strenuous application of my powers will not be lost. It is true that I am aware that the form which I have given the science is imperfect and must be imperfect. But I know and feel obliged to state (though I run the risk of seeming arrogant) that even if this work should again remain unused for another seventeen years or even longer, without entering into the actual development of science still that time will come when it will be brought forth from the dust of oblivion, and when ideas now dormant will bring forth fruit. I know that if I also fail to gather around me in a position (which I have up to now desired in vain) a circle of scholars, whom I could fructify with these ideas, and whom I could stimulate to develop and enrich further these ideas, nevertheless there will come a time when these ideas, perhaps in a new form, will arise anew and will enter into living communication with contemporary developments. For truth is eternal and divine, and no phase in the development of truth, however small may be the region encompassed, can pass on without leaving a trace; truth remains, even though the garment in which poor mortals clothe it may fall to dust. [2, p254]

Grassman's work was not in vain. The American mathematician and physicist Josiah Gibbs and English electrician Oliver Heaviside (who lacked any university training) independently and simultaneously discovered the modern system of vectors late in the nineteenth century, and their work may be considered a sort of combination of quaternion and Grassmanian ideas, in that the notation was primarily borrowed from quaternions but the geometric interpretation was borrowed from Grassman's system. Peter Tait, the successor to Hamilton in the development of quaternions, went so far as to call Gibbs's system of vectors "a sort of hermaphrodite monster, compounded of the notions of Hamilton and Grassman" [2, p150].

Both Heaviside and Gibbs became interested in quaternions from Maxwell's *Treatise* [2, p163]. Both developed their systems of vector analysis from quaternions by a process of "elimination and simplification" [2, p163]. Only Gibbs, however, *knowingly* incorporated elements of Grassman's system. Heaviside summed up the reason for his development of a new vector system of as follows:

> Against the above stated great advantages of Quaternions [their algebraic properties] has to be set the fact that the operations met with are much more difficult than the corresponding ones in the ordinary [scalar] system, so that the saving of labour is, in a great sense, imaginary. There is much more thinking to be done, for the mind has to do what in scalar algebra is done almost mechanically [i.e. take vector parts, scalar parts, etc]. At the same time, when working with vectors by the scalar system, there is great advantage to be found in continually bearing in mind the fundamental ideas of the vector system. Make a compromise; look behind the easily-managed but complex scalar equations, and see the single vector one behind them, expressing the real things. [2, p164].

Heaviside was expressing a fundamental difficulty in using quaternions — which are only part vector — in expressing purely vectoral relations. In essence, their usefulness for physical applications was restrained by the very algebraic properties that made them an interesting extension to complex numbers. The difficulty, however, was a purely pragmatic one, which Heaviside was expressing when he wrote that "there is much more thinking to be done [to set up quaternion equations]". In principle, most everything done with the new system of vectors could be done with quaternions, but the operations required to make the quaternions behave like vectors added difficulty to using them and provided little benefit to the physicist. And when Heaviside and Gibbs removed the offending parts of the quaternions that Hamilton cherished so deeply using a purely pragmatic justification, the stage was thus set for a showdown between the two different systems for the hearts and minds of physicists and mathematicians alike.

5 Competing Systems

Before we can investigate the great debate about the merits of quaternions and modern vectors, let us first examine the commonalities and make a comparison between the two systems.

5.1 Algebraic Comparison

Given that Hamilton was seeking an algebra of extended complex numbers, one might expect that quaternions would be completely superior to modern vectors in terms of algebraic properties. However, that is not entirely the case; both systems have their advantages.

Although quaternions do not form a field, because they are not commutative, the quaternions do obey the law of the moduli, are uniquely divisible, and have a definite product (although modern vectors have the dot and cross product, the standard definition of the direct vector product yields a tensor rather than a vector). In this sense quaternions are superior to vectors of any dimension, which do not posses these properties. However, quaternions are restricted to being four-tuples of real numbers, whereas vectors can be extended to any number of dimensions. Matrix multiplication can be used to transform vectors from one dimension to another; no such operation exists for quaternions.

Quaternions do generalize in a straightforward manner to *octonions*, or eight-tuples of real numbers, but loose associativity in the process. The differences between complex numbers, quaternions, and octonions have direct applications in projective geometry [7, p397], Clifford Algebras, and spinors, and are related to modern development of Cayley Algebra. Modern vectors, however, are related in a deeply intimate way to linear algebra, giving them applications in differential equations, differential geometry, and many other fields. In short, both systems posses interesting algebraic qualities, although in different fields, but at the time most of these related fields did not exist in modern form and their relative merits were hotly debated.

Quaternionic analysis has been developed to the point where it includes direct extensions of conformal mappings, power series, and Cauchy's theorem for complex analysis. However, these results are little known. The interested reader is encouraged to look at [8].

5.2 Geometric Comparison

Hamilton himself understood that quaternions could be described in terms of a four dimensional space [2, p31]. However, he seemed to prefer to treat quaternions as made up of two distinct parts — the scalar and vector parts — than as a four dimensional quantity. In fact, it is awkward to treat a quaternion as a modern four-dimensional vector, because if we naïvely let Hamilton's i, j, and k become the basis vectors of three-space, then the scalar part of the quaternion is not multiplied by a basis vector. The situation is somewhat analogous to treating a complex number as a vector in the plane: possible, but awkward despite the similarities of the two systems. However, it is *possible* to give quaternions an interpretation in terms of four dimensional space, as the direct analog of the interpretation of complex numbers as points in the plane.

Although Hamilton desired for his quaternions to have a geometric interpretation, he did not define quaternions in terms of euclidean space. Modern vectors, on the other hand, were *designed* by Gibbs and Heaviside with exactly this in mind. Thus, in general, it is easier to use modern vectors to describe physical situations in euclidean three-space, and at the time this application of vectors and quaternions was of paramount importance for physics. Most damaging to quaternions is that the geometric intuition of a quaternion with a non-zero scalar part in terms of euclidean three-space is far from obvious.

In modern terms, we can think of a quaternion as the outer tensor product of a scalar and a three vector, which gives it some useful properties in regards to rotations that vectors lack. However, the usefulness of this property only relates to advanced mathematics that was not to be developed for many decades after the introduction of quaternions.

5.3 Physical Comparison

James Clerk Maxwell used quaternions extensively in his *Treatise*, but the fundamental advantages of modern vectors for physical applications was never quite in doubt. For some applications, like describing stress and strain, the superiority of vectors is apparent. In a solid, stress and strain requires a tensor of three vectors to even be represented. A corresponding system for quaternions is much more limited, especially because of the close connection between matrices and vectors that has developed in modern linear algebra.

However, in some circumstances, quaternions do lead to more compact forms of physical laws. For instance, in Electromagnetism, it is possible to write one quaternion equation where the scalar part of the equations states that no magnetic monopoles exist and the vector part gives Faraday's law [9]. There has been a modern development of quaternions for physics, but this development exists primarily to show that quaternions *can* be used in physics, and most modern physics is not done this way. The higher-geometric properties resulting from a quaternion being the outer tensor product of a scalar and a three vector do have applications to advanced physics, i.e. in General Relativity and Quantum Mechanics. In most cases, the modern theory of tensors has been used as a successor to the relevant pieces of vector analysis, linear algebra, and quaternions. The interested reader is encouraged to look at [9].

Despite the usefulness of quaternions for describing physical problems, modern vectors were on better philosophical ground. Most mathematicians and physicists viewed vectors as corresponding directly to physical quantities like force or direction. Quaternions, however, had no direct physical analog and suffered criticisms similar to the ones aimed at complex numbers centuries earlier. The mathematician John Graves summed up the situation nicely in a letter to Hamilton [2, p34]:

> There is still something in the system which gravels me. I have not yet any clear views as to the extent to which we are at liberty arbitrarily to create imaginaries, and endow them with supernatural properties. You are certainly justified by the event [the discovery]. You have got an instrument that facilitates the working of trigonometrical theorems and suggests new ones, and it seems hard to ask more; but I am glad that you have glimpses on physical analogies.

Grave's worry is an echo of the ancient debate between creation and discovery in mathematics. Although the philosophical debate over quaternions never approached a level that would endanger the acceptance of quaternions [2, p34], from a modern perspective it is somewhat amusing to see a mathematician express relief that a physical justification for a mathematical concept was found.

6 The Great Debate

By the end of the nineteenth century, the development of two similar and competing systems that were both used in the same fields of physics led directly to a confrontation over which was to be the preferred system for discourse by physicists. Use by physicists would, in turn, lead mathematicians to further develop the winning theory. Thus, proponents of both systems, aware of the immense consequences of such a contest, threw themselves into a great debate that was to last years, span eight journals, and involve a dozen of the leading minds of the time [2, p182]. The contest was described by Gibbs in 1888 as nothing less than a "struggle for existence" between the two systems [2, p182].

6.1 Roots of the Debate

The debate between proponents of both systems was much more than a debate over notation. In a sense, the debate was about whether a beautiful mathematical theory, applied to physics, was superior in utility to a system created by physicists specifically to solve physical problems.

Peter Tait, Hamilton's chosen heir to quaternion development, began the debate in 1889 by addressing the Physical Society of the University of Edinburgh. He asked "whether experiment or mathematics is more important to the progress of physics" [2, p183]. Tait argued that because "everything penetrable must, some day, give up its secrets" the question of which system to use was related to the ways in which the systems corresponded to physical reality. He thought that a method should be chosen based not on compactness or elegance, but instead on "expressiveness" [2, p183]. He argued that quaternions were "transcendently expressive" and that vectors were "uniquely adapted to Euclidean Space" and thus not as expressive. In essence, he argued that quaternions were natural, and vectors unnatural [2, p184]. With this opening salvo, the debate began in earnest the next year.

6.2 Height of the Battle: 1890-1894

The year 1890 was dominated by the proponents of quaternions, especially Tait. The arguments centered primarily around the "naturalness" of quaternions, and many arguments used Hamilton's reputation as leverage. For instance, Tait offered a quote of Hamilton's as support for his arguments: "*Could* anything be simpler or more satisfactory? Don't you *feel*, as well as think, that were are on a *right track*, and shall be *thanked* hereafter? Never mind when" [2, p185]. It is unfortunate that, partially as a result of this, Hamilton's stature was subsequently diminished [2, p185]. Tait also attempted to establish Hamilton's priority in publishing quaternions before Grassman published his system, and argued that the use of any Cartesian methods (modern vector methods) would cause mathematical and scientific progress to be slowed.

The proponents of modern vectors responded in 1891, when Gibbs replied to Tait's arguments is a considerably reasoned voice:

> It seems to be assumed that a departure from quaternionic usage in the treatment of vectors is an enormity. If this assumption is true, it is an important truth; if not, it would be unfortunate if it should remain unchallenged, especially when supported by so high an authority [Hamilton and Tait]. The criticism relates particularly to notations, but I believe there is a deeper question of notions underlying that of notations. Indeed, if my offence had been solely in the matter of notation, it would have been less accurate to describe my production as a monstrosity, than to characterize its dress as uncouth. [Here Gibbs is refering to Tait's earlier comparison of modern vectors to a "hermaphrodite monster"] [2, p185]

Gibbs argued that the scalar and cross products were more fundamental than the quaternion product because they represent the most important relations in physics. He also gave an account of transformations that could be used to more easily represent rotations in vector notation — which had been a weakness of the vector system — and argued that vectors could be extended to spaces higher than three. Gibbs, at that time, was a proponent of development of both systems, and ended his response by saying that "these considerations are sufficient, I think, to show that the position of the quaternionist is not the only one from which the subject of vector analysis may be viewed, and that a method that may be monstrous from one point of view, may be normal and inevitable from another." [2, p186].

Tait and Gibbs wrangled with each other for a while longer, discussing the relative merits of the "indeterminate vector product" against the more natural product of quaternions, and also arguing over the priorities of Hamilton and Grassman. Tait appeared to gain a slight advantage by pointing out that quaternion products are associative, whereas the cross product is not:

$$\mathbf{i} \times (\mathbf{j} \times \mathbf{j}) = \mathbf{0} \neq (\mathbf{i} \times \mathbf{j}) \times \mathbf{j} = -\mathbf{i}$$

Tait also asked of Gibbs, "What have students of physics, as such, to do with spaces of more than three dimensions?" [2, p186]. Today, of course, we know that the answer is a great deal.

Another article by Gibbs contained the following statement:

> To what extent are the geometrical methods which are usually called quaternionic peculiar to Hamilton, and to what extent are they common to Grassman? This is a question which anyone can easily decide for himself. It is only necessary to run one's eye over the equations used by quaternionic writers in the discussion of geometrical or physical subjects, and see how far they necessarily involve the idea of the quaternion, and how far they would be intelligible to one understanding the functions $S\alpha\beta$ and $V\alpha\beta$, but having no conception of the quaternion $\alpha\beta$, or at least could be made so by trifling changes of notation, as by writing S or V in places they would not affect the value of the expressions. For such a test the examples and illustrations in treatises on quaternions would be manifestly inappropriate, so far as they are chosen to illustrate quaternionic principles, since the object may influence the form of presentation. But we may use any discussion of geometrical or physical subjects, where the writer is free to choose the form most suitable to the subject. [2, p188]

Gibbs was acutely aware that quaternionic methods contained the most important pieces of his vector methods. He was able to turn the tables on the proponents of quaternions by suggesting that the scalar and cross products are more useful and fundamental than the quaternion product, and he used evidence from Tait's own texts to show the loose coupling of the quaternionic product to the scalar and cross products. His carefully reasoned arguments appealed to both sides while alienating neither. After a very brief response by Tait, the battle grew quiet. [2, p189].

The lull was not to last, however. Sensitive to the lack of development of quaternions in the recent past, the mathematician Alexander McAulay wrote in 1892 a piece directly challenging physicists using modern vector methods. McAulay posed the hypothetical question, "can any cause be assigned for this extraordinary case of arrested development [of quaternions]?" [2, p189]. He speculated that most physicists would reply that quaternions were not being developed because they were not leading to any physical discoveries. McAulay's stated goal was to "shake" the belief that quaternions were useless to physicists. He accused physicists of only studying quaternions once they were committed to the system of modern vectors, and even then not studying them in sufficient detail to understand their power and utility. He also accused James Clerk Maxwell, who had earlier used quaternions extensively in his *Treatise*, of discrediting quaternions among younger physicists when he switched to modern vector methods.

That same year the mathematician Alexander MacFarlane developed another system of vectors that unified the concepts of modern vectors and quaternions. He had sought "an algebra which will apply directly to physical quantities, will include and unify the several branches of analysis, and when specialized become ordinary algebra" [2, p190]. But by that time, however, the proponents of both sides were unwilling to compromise, and MacFarlane's system was largely ignored despite its potential [2, p191]. Later that year, the first reviews of textbooks about modern vector methods began to trickle into journals, signaling a milestone in the acceptance of the modern vector methods. Also that year a number of inflammatory articles appeared, including one in which McAulay accused the proponents of modern vectors of "spoon-feeding the physical public", whereas the proponents of quaternions were providing "strong meat" [2, p193]. Despite the polemics, by this time the battle had turned against the proponents of quaternions.

By 1893 the proponents of quaternions became aware that they were losing the battle. McAulay gave the following depiction of the state of quaternion development in Cambridge, the center of British mathematics at the time:

> There is no lack in Cambridge of the cultivation of Quaternions *as an algebra*, but this cultivation is not Hamiltonian... Hamilton looked upon Quaternions as a *geometrical* method, and it is in this respect that he has yet failed to find worthy followers resident in Cambridge. [2, p195]

As the battle became more one-sided, so did the rhetoric; McAulay pleaded with Cambridge students to "steep" themselves in the "delirious pleasures" of quaternions and promised that they would then become more happy than millionaires [2, p195]. Hamilton's quaternions were also called "the greatest mathematical work of the century" and the modern vector system was compared to "an arid waste" [2, p196]. Proponents of modern vectors were described as ingrates by Tait:

> Intuitively recognising its power, he [the quaternionist] snatches up the magnificent weapon which Hamilton tenders to all, and at once dashes off to the jungle on the quest of big game. Others, more cautious or perhaps more captious, meanwhile sit pondering gravely on the fancied imperfections of the arm; and endeavor to convince a bewildered public (if they cannot convince themselves) that, like the Highlander's musket, it requires to be treated to a brand-new stock, lock, and barrel, *of their own devising*, before it can be safely regarded as fit for service. [2, p197].

The proponents of modern vectors were also aware that they were winning the debate. Gibbs suggested a new law of mathematical evolution:

> Whatever is special, accidental, and individual, will die, as it should; but that which is universal and essential should remain as an organic part of the whole intellectual acquisition. If that which is essential dies with the accidental, it must be because the accidental has been given the prominence which belongs to the essential. [2, p198]

Gibbs's argument that the modern system of vectors was an underlying piece of quaternions that was more connected to nature seemed to persuade many minds. Heaviside's glib writing style and humorous nature contrasted markedly with the styles of the quaternion proponents: "A difficulty in the way [of McAulay] is that he has got used to quaternions. I know what it is, as I was in the quaternionic slough myself once" [2, p200]. Heaviside's antics also contrasted quite well with Gibbs's reasoned arguments, and he could not resist poking-fun at his opponents:

> The quaternionic calm and peace have been disturbed. There is confusion in the quaternionic citadel; alarms and excursions, and hurling of stones and pouring of boiling water upon the invading host. [2, p200]

6.3 The Aftermath

By 1894 the debate had largely been settled in favor of modern vectors, although this was not realized by all parties involved, and debate continued at a reduced level for many years. Arthur Cayley was one of the first mathematicians that espoused a truly modern view of quaternions:

> It must always be remembered that Cartesian methods are mere particular cases of quaternions where most of the distinctive features have disappeared; and that when, in the treatment of any particular question, scalars have to be adopted, the quaternion solution becomes identical with the Cartesian one. Nothing, therefore, is ever lost, though much is generally gained, by employing quaternions in the place of ordinary methods. In fact, even when quaternions degrade to scalars, they give the solution of the most general statement of the problem they are applied to, quite independently of any limitations as to choice of particular coordinate axes...
>
> My own view is that quaternions are merely a particular method, or say a theory, in coordinates. I have the highest admiration for the notion of a quaternion; but... as I consider the full moon far more beautiful than any moonlit view, so I regard the notion of a quaternion as far more beautiful than any of its applications. As another illustration... I compare a quaternion formula to a pocket-map — a capital thing to put in one's pocket, but which for use must be

> unfolded: the formula, to be understood, must be translated into coordinates. [2, p212]

Cayley illustrated his point with a number of examples, the most telling of which involves a quaternion equation that, although concise, is representative of no intuition until translated into a cartesian/vector equivlanent. Consider two lines OA and OB; to find the line OC perpendicular to the plane containing OA and OB, the quaternion solution would be

$$m\gamma = V\alpha\beta$$

where

$$\gamma = OC, \quad \alpha = OA, \quad \beta = OB$$

and m is a scalar. Although this equation is shorter than its vectoral equivalent, Cayley contended that is was essentially unintelligible until actually translated into the vector equivalent [2, p212].

Over the course of the debate, both Gibbs and Heaviside attained great reputations as physicists. The quaternions, having begun the debate widely known, soon faded into the history.

7 Conclusion

The forgotten quaternions, one of the first mathematical systems explicitly constructed to extend a previously discovered system, was the culmination of decades of work by the great scientist and mathematician William Hamilton in the mid nineteenth century. The quaternions, four-tuples of real numbers that extended the complex number system, were made up of a scalar part and a vector part, and possessed many properties, both algebraic and geometric, that made them useful for investigations into mathematics and physics alike. By the last quarter of the nineteenth century, the quaternions were well known and used widely within a few emerging domains of scientific inquiry.

Despite their apparent popularity, the quaternions fell victim to their own success. The modern theory of vectors developed from the quaternions and incorporated their most useful properties for scientific applications. But because the modern theory of vectors was more than just an evolutionary step for quaternions — the theory also contained elements from other early systems of vectors — a great struggle ensued between the proponents of both systems to determine which system would dominate future mathematical and scientific research. The struggle peaked between 1890 and 1894 and involved some of the most fiery rhetoric ever seen in mathematical debate. At the end of the debate, modern vectors emerged victorious, and quaternions faded into history.

Although the halcyon years of quaternions were brief, the quaternions motivated developments in algebra, analysis, geometry, and physics. Indeed, without quaternions, there might be no modern system of vectors at all. And despite their relegation to the dust-bin of history, their place among the great discoveries of mathematics is well deserved.

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