

A big $J\Omega$ mp in the Ordinals

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Building things from above

- Cantor Normal Form, Veblen Hierarchy

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 - impredicative :(

The Collapsing Function

$$B(\alpha) := \left\{ \begin{array}{l} \text{closure of } \{0, \Omega\} \text{ under:} \\ + \\ \xi \mapsto \omega^\xi \\ \xi, \eta \mapsto \phi_\xi(\eta) \\ \xi \mapsto \psi_\Omega(\xi) \upharpoonright_{\xi < \alpha} \end{array} \right.$$

$$\psi_\Omega(\alpha) := \min\{\rho < \Omega \mid \rho \notin B(\alpha)\}$$

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Fact

$$\alpha \notin B_0(\alpha) \Rightarrow B_0(\alpha) = B_0(\alpha + 1) \text{ and } \psi_{\Omega,0}(\alpha) = \psi_{\Omega,0}(\alpha + 1)$$

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Corollary $B_0(\Omega + 1) = B_0(\Omega)$. In fact, it actually halts from now on. And $\psi_{\Omega,0}$ too.

What about $B(\Omega + 1)$?

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So Ω is useful after all.

Lets construct the enumeration system!

Definition: 9.13 The set $OT(\Omega)$ and $G\alpha$ for $\alpha \in OT(\Omega)$ are inductively defined by the following clauses:

(R1) $0, \Omega \in OT(\Omega)$ and $G0 = G\Omega := 0$

(R2) If $\alpha =_{NF} \alpha_1 + \dots + \alpha_n, n > 1$ and $\alpha_1, \dots, \alpha_n \in OT(\Omega)$ then $\alpha \in OT(\Omega)$ and $G\alpha = \max(G\alpha_1, \dots, G\alpha_n) + 1$

(R3) If $\alpha =_{NF} \phi_\beta(\delta), \beta, \delta < \Omega$ and $\beta, \delta \in OT(\Omega)$ then $\alpha \in OT(\Omega)$ and $G\alpha = \max(G\beta, G\delta) + 1$

(R4) If $\alpha =_{NF} \omega^\beta, \beta > \Omega$ and $\beta \in OT(\Omega)$ then $\alpha \in OT(\Omega)$ and $G\alpha = (G\beta)$

(R5) If $\alpha =_{NF} \psi_\Omega(\Omega), \beta \in OT(\Omega)$ and $\beta \in B(\beta)$ then $\alpha \in OT(\Omega)$ and $G\alpha = (G\beta) + 1$

Why stay inside of B?

From before:

Fact

$$\alpha \notin B(\alpha) \Rightarrow B(\alpha) = B(\alpha + 1) \text{ and } \psi_{\Omega}(\alpha) = \psi_{\Omega}(\alpha + 1)$$

The Bachmann Howard ordinal

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Proposition: $\psi_\Omega(\varepsilon_{\Omega+1}) = B(\varepsilon_{\Omega+1}) \cap \Omega \subset OT(\Omega) \subset B(\varepsilon_{\Omega+1}) \cap \varepsilon_{\Omega+1}$

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How far can we get?

$$\varepsilon_0, \Gamma_0, \dots, \Gamma_{\psi(\Omega)} \in \text{OT}(\Omega)$$

The system KP

Extensionality: $a = b \rightarrow (F(x) \leftrightarrow F(b))$

Foundation: $\exists x G(x) \leftarrow \exists x (G(x) \wedge \forall y (y \in x \rightarrow \neg G(y)))$

Pair: $\exists x (x = \{a, b\})$

Union: $\exists x (x = \bigcup a)$

Infinity: $\exists x (x \neq \emptyset \wedge \forall y \exists z (y \in x \wedge z \in x \rightarrow y \in z))$

Δ_0 Separation: $\exists x (x = \{y \in a : F(y)\})$ for all Δ_0 – formulas F
in which x does not occur free

Δ_0 Collection: $\forall x \exists y (x \in a \rightarrow G(x, y)) \rightarrow \exists z \forall x \exists y (x \in a \wedge y \in z \rightarrow G(x, y))$ for all Δ_0 -formulas G

The Constructable Hierarchy

$$L_0 := \emptyset$$

$$L_{\alpha+1} := \{X \subseteq L_\alpha \mid X \text{ definable over } \langle L_\alpha, \in \rangle\}$$

$$L_\alpha := \bigcup_{\beta < \alpha} L_\beta \text{ for limit } \alpha$$

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$$\|KP\| = \psi_\Omega(\varepsilon_{\Omega+1})$$