A big JΩmp in the Ordinals

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Building things from above

- Cantor Normal Form, Veblen Hierarchy

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  - impredicative :(

The Collapsing Function

$$B(\alpha) := \begin{cases} \text{closure of } \{0, \Omega\} \text{ under:} \\ + \\ \xi \mapsto \omega^{\xi} \\ \xi, \eta \mapsto \phi_{\xi}(\eta) \\ \xi \mapsto \psi_{\Omega}(\xi) \restriction_{\xi < \alpha} \end{cases}$$

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A nice little lemma

Fact

$\alpha \notin B_0(\alpha) \Rightarrow B_0(\alpha) = B_0(\alpha + 1)$ and $\psi_{\Omega,0}(\alpha) = \psi_{\Omega,0}(\alpha + 1)$

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Corollary $B_0(\Omega + 1) = B_0(\Omega)$. In fact, it actually halts from now on. And $\psi_{\Omega,0}$ too.

What about B(Ω + 1)?

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So Ω is useful after all.

Lets construct the enumeration system!

Definition: 9.13 The set OT(Ω) and Gα for $\alpha \in$ OT(Ω) are inductively defined by the following clauses:

(R1) 0,$\Omega \in$ OT(Ω) and G0 $=$ G$\Omega := 0$

(R2) If $\alpha =_{NF} \alpha_1 + \cdots + \alpha_n, n > 1$ and $\alpha_1, \cdots, \alpha_n \in OT(\Omega)$
then $\alpha \in OT(\Omega)$ and $G\alpha = max(G\alpha_1, \cdots, G\alpha_n) + 1$

(R3) If $\alpha =_{NF} \phi_\beta(\delta), \beta, \delta < \Omega$ and $\beta, \delta \in OT(\Omega)$
then $\alpha \in OT(\Omega)$ and $G\alpha = max(G\beta, G\delta) + 1$

(R4) If $\alpha =_{NF} \omega^\beta, \beta > \Omega$ and $\beta \in OT(\Omega)$ then $\alpha \in OT(\Omega)$ and $G\alpha = (G\beta)$

(R5) If $\alpha =_{NF} \psi_\Omega(\Omega), \beta \in OT(\Omega)$ and $\beta \in B(\beta)$
then $\alpha \in OT(\Omega)$ and $G\alpha = (G\beta) + 1$

Why stay inside of B?

From before:

Fact

$\alpha \notin B(\alpha) \Rightarrow B(\alpha) = B(\alpha + 1)$ and $\psi_\Omega(\alpha) = \psi_\Omega(\alpha + 1)$

The Bachmann Howard ordinal

$$B(\alpha) = \begin{cases} \text{closure of } \{0, \Omega\} \text{ under:} \\ + \\ \xi \mapsto \omega^{\xi} \\ \xi, \eta \mapsto \phi_{\xi}(\eta) \\ \xi \mapsto \psi_{\Omega}(\xi) \restriction_{\xi < \alpha} \end{cases}$$

Proposition: $\psi_{\Omega}(\varepsilon_{\Omega+1}) = B(\varepsilon_{\Omega+1}) \cap \Omega \subset OT(\Omega) \subset B(\varepsilon_{\Omega+1}) \cap \varepsilon_{\Omega+1}$

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How far can we get?

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How far can we get?

$\varepsilon_0, \Gamma_0, \cdots, \Gamma_{\psi(\Omega)} \in \mathrm{OT}(\Omega)$

The system KP

Extensionality: $a = b \rightarrow (F(x) \leftrightarrow F(x))$

Foundation: $\exists\ x\ G(x) \leftarrow \exists\ x\ (G(x) \wedge \forall y(y \in x \rightarrow \neg G(y)))$

Pair: $\exists x\ (x = \{a, b\})$

Union: $\exists x\ (x = \bigcup a)$

Infinity: $\exists\ x\ (x \neq \emptyset \wedge \forall\ y \exists\ z(y \in x \wedge z \in x \rightarrow y \in z))$

Δ_0 Separation: $\exists\ x\ (x = \{y \in a : F(y)\})$ for all Δ_0 – formulas F
in which x does not occur free

Δ_0 Collection: $\forall\ x\ \exists\ y(x \in a \rightarrow\ G(x, y)) \rightarrow \exists\ z \forall\ x \exists\ y(x \in a \wedge y \in z \rightarrow G(x, y))$ for all Δ_0-formulas G

The Constructable Hierarchy

$$
\begin{aligned}
L_0 :=& \emptyset \\
L_{\alpha+1} :=& \{X \subseteq L_\alpha \| \ X \text{ definable over } \langle L_\alpha, \in \rangle\} \\
L_\alpha :=& \bigcup_{\beta<\alpha} L_\beta \text{ for limit } \alpha \\
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$\|KP\| = \psi_\Omega(\varepsilon_{\Omega+1})$