Cutting cuts

Cut down to 10 minutes.

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January 9, 2019

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What we want to prove

Lemma

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(Generalized Elimination Lemma) If $\vdash^{\alpha}_{\beta+\omega^{\rho}} \Delta$ *then* $\vdash^{\varphi_{\rho}(\alpha)}_{\beta} \Delta$.

Veblen functions (defined inductively)

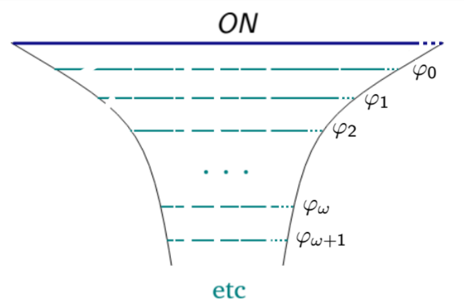

Figure 1: The range (image) of the first Veblen functions. Not to scale.

- Zero ordinal: $\varphi_0(\alpha) := \omega^\alpha$
- Successor ordinals $\rho + 1$: $\varphi_{\rho+1}(\alpha) := Enum(FIX(\varphi_\rho), \alpha)$
- Limit ordinals λ: $\varphi_\lambda(\alpha) := Enum\big(\cap_{\rho<\lambda} FIX(\varphi(\rho)), \alpha\big)$

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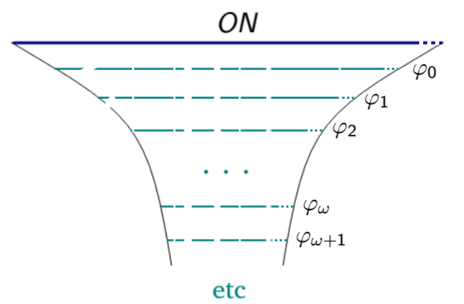

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Remark: If $\rho_1 < \rho_2$ then φ_{ρ_2} enumerates a subset of fixed points of φ_{ρ_1}

Proof (1/5)

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For $\rho = 0$, we have $\vdash^{\alpha}_{\beta+\omega^{0}} \Delta \equiv \vdash^{\alpha}_{\beta+1} \Delta$. By the Basic Elimination Lemma, we get $\vdash^{\omega^{\alpha}}_{\beta} \Delta \equiv \vdash^{\varphi_{0}(\alpha)}_{\beta} \Delta$.

Proof (2/5)

Now assume $\rho > 0$. If the last inference was not a cut, we have,

$$\left|\frac{\alpha_\iota}{\beta+\omega^\rho}\right. \Delta_\iota \text{ for } \iota \in I \Rightarrow \left|\frac{\alpha}{\beta+\omega^\rho}\right. \Delta$$

with $\alpha_\iota < \alpha$ for all $\iota \in I$.

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By the induction hypothesis, we get,

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By the induction hypothesis, we get,

$$\vdash^{\varphi_\rho(\alpha_\iota)}_{\beta} \Delta_\iota \text{ for } \iota \in I.$$

Since for every $\iota \in I$, $\alpha_\iota < \alpha$, we have $\varphi_\rho(\alpha_\iota) < \varphi_\rho(\alpha)$ (since the Veblen functions are strictly increasing). Thus, we get,

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using the same inference.

Similarly, if the last inference was a cut of rank $< \beta$, a similar argument proves the statement (exercise!).

Proof (3/5)

Now assume that the last inference was a cut of the form

$$\vdash^{\alpha_0}_{\beta+\omega^\rho} \Delta, F \text{ and } \vdash^{\alpha_0}_{\beta+\omega^\rho} \Delta, \neg F \Rightarrow \vdash^{\alpha}_{\beta+\omega^\rho} \Delta$$

for $\alpha_0 < \alpha$, and formula F such that $rank(F) \in [\beta, \beta + \omega^\rho)$.

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for $\alpha_0 < \alpha$, and formula F such that $rank(F) \in [\beta, \beta + \omega^\rho)$.

Let γ be an ordinal such that $rank(F) = \beta + \gamma$. We decompose γ into its Cantor Normal Form and get,

$$rank(F) = \beta + \gamma = \beta + \omega^{\sigma_1} + ... + \omega^{\sigma_n} < \beta + \omega^\rho$$

such that $\rho > \sigma_1 \geq \sigma_2 \geq ... \geq \sigma_n$, and thus,

$$rank(F) < \beta + \omega^{\sigma_1}(n + 1).$$

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to get

$$\vdash^{\varphi_\rho(\alpha_0)}_{\beta} \Delta, F \text{ and } \vdash^{\varphi_\rho(\alpha_0)}_{\beta} \Delta, \neg F. \Rightarrow \vdash^{\varphi_\rho(\alpha_0)+1}_{\beta+\omega^{\sigma_1}(n+1)} \Delta$$

by a cut.

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Since $\sigma_1 < \rho$, we apply the main induction hypothesis multiple times to get,

$$\vdash^{\varphi_{\sigma_1}(\varphi_{\sigma_1}(\ldots(\varphi_\rho(\alpha_0)+1)))}_{\beta} \Delta = \vdash^{\varphi^{n+1}_{\sigma_1}(\varphi_\rho(\alpha_0)+1)))}_{\beta} \Delta.$$

Proof: What's left? (5/5)

We have: $\left|\!\frac{\varphi^{n+1}_{\sigma 1}(\varphi_\rho(\alpha_0)+1)))}{\beta}\right.\!\Delta.$

And want to get: $\left|\!\frac{\varphi_\rho(\alpha)}{\beta}\right.\!\Delta.$

Proof: What's left? (5/5)

We have: $\vdash^{\varphi_{\sigma_1}^{n+1}(\varphi_\rho(\alpha_0)+1)))}_{\beta} \Delta$.

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Claim: $\varphi_{\sigma_1}^{n}(\varphi_\rho(\alpha_0)+1) < \varphi_\rho(\alpha)$

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Claim: $\varphi_{\sigma_1}^{n}(\varphi_\rho(\alpha_0) + 1) < \varphi_\rho(\alpha)$

By induction on n.

When $n = 0, \varphi_{\sigma_1}^{0}(\varphi_\rho(\alpha_0) + 1) = \varphi_\rho(\alpha_0) + 1$

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When $n = 0, \varphi_{\sigma_1}^0(\varphi_\rho(\alpha_0) + 1) = \varphi_\rho(\alpha_0) + 1 < \varphi_\rho(\alpha)$.

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$n \Rightarrow n+1 : \varphi^{n+1}_{\sigma_1}(\varphi_\rho(\alpha_0)+1) = \varphi_{\sigma_1}(\varphi^n_{\sigma_1}(\varphi_\rho(\alpha_0)+1))$

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When $n = 0, \varphi^0_{\sigma_1}(\varphi_\rho(\alpha_0)+1) = \varphi_\rho(\alpha_0)+1 < \varphi_\rho(\alpha)$.

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So by a weakening lemma, we get:

$$\vdash^{\varphi_\rho(\alpha)}_{\beta} \Delta.$$

And we are done!

Let's cut it here.

Thanks for listening! Any questions?