recap

Logic and Proof

COURSE SYLLABUS ON MY COURSES

Course Syllabus

> *This course syllabus will continue to be updated throughout the semester, please check back often.*

Week	Lecture Notes and Practice Exercises	Office Hours	Quizzes
Wk00	Lec00 - Course Information		Quiz Lec00 Aug12-13 (*trial*)
Wk01	Lec01 - Propositional Logic	W Aug16, 11am	Quiz Lec01 Aug19-20
Wk02	Lec02 - Proof by Mathematical Induction	M Aug21, 11am	Quiz Lec02 Aug26-27
Wk03	Lec03 - Sets and Inclusion-Exclusion Principle		Quiz Lec03 Sep2-3
Wk04	Lec04 - Counting and Combinatorics		Quiz Lec04 Sep9-10
Wk05	Lec05 - The Pigeonhole Principles		Quiz Lec05 Sep16-17
Wk06	Lec06 - Divisibility and Modular Arithmetic		Quiz Lec06 Sep23-24
Wk07	Lec07 - Integer Representations		Quiz Lec07 Sep30-Oct01
Wk08	Lec08 - Number Base Arithmetic		
	MIDTERM Examination		

CLASS WEEKLY TIMETABLE

MON	TUE	WED	THU	FRI	SAT	SUN
S1 8-11		S2 8-11	S3 9-12	S4 9-12		
OH TBA (TO BE ANNOUNCED)		OH TBA (TO BE ANNOUNCED)			QUIZ *(a week after)*	
11-11:30am		11-11:30am			0005--	--2355

Quiz: **1 hour**, **1 submission**, by **ONE** student, no make-up

EXAM

10+1(xtra credit) points *Submit a quiz* = ATTENDANCE

Closed book, default calculator, scratch paper

In a lab, similar to MyCourses, no internet, no phone/iPad

OFFICE HOUR

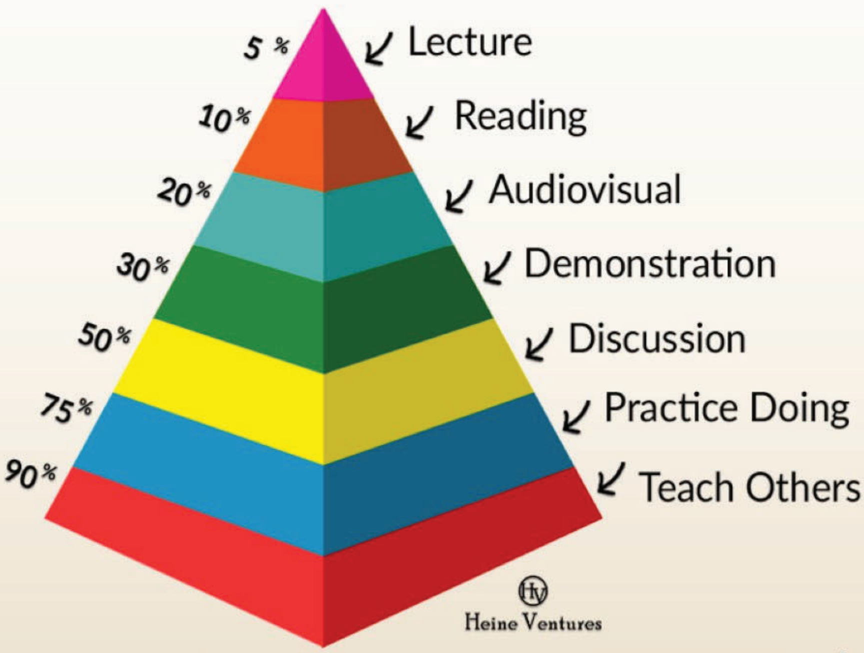
Solutions for HW & quiz
Q & A from lectures
Catching up, discussion
with friends & instructors

@IT311, schedule will be adjusted according to your attendance

Review in-class materials
Work together on practice problems

If you cannot attend a scheduled session, email me to setup another time.

THE LEARNING PYRAMID



Reflection! Know what you learn. Rate your understanding.

- Summary, a checklist of key topics
- Keep catching up with the course
- Do HW & master these before a quiz
- Help others or get help if you need it

Confident	Got it	Okay	Fuzzy	Not a clue

Propositional and Logical Reasoning

p, q, r,... **propositional variables**, representing statements

¬, ∧, ∨, ⊕ **logical connectives** combined to **compound statements**

		not negation	and conjunction	or disjunction	xor exclusive disjunction	if-then implication	if-and-only-if
p	q	$\neg p$	$p \land q$	$p \lor q$	$p \oplus q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	F	T	T	F	T	T
T	F	F	F	T	T	F	F
F	T	T	F	T	T	T	F
F	F	T	F	F	F	T	T

Implication	Inverse	Converse	Contraposition
$p \to q$	$\neg p \to \neg q$	$q \to p$	$\neg q \to \neg p$

Tautology	Contradiction or Absurdity	Contingency
always true	always false	some true some false

Logically Equivalence	Valid Arguments
$p \equiv q$ iff $p \leftrightarrow q$ is a tautology	premises $\to$ conclusion is a tautology
always have the same truth values	*when all premises true, conclusion must also be true*

Property	Addition	Multiplication
Commutative Property	$a+b=b+a$	$a \cdot b = b \cdot a$
Associative Property	$a+(b+c)=(a+b)+c$	$a \cdot (b \cdot c) = (a \cdot b) \cdot c$
Distributive Property	$a \cdot (b+c) = a \cdot b + a \cdot c$	
Identity Property	$a+0=a$	$a \cdot 1 = a$
Inverse Property	$a+(-a)=0$	$a \cdot \frac{1}{a} = 1$

$$(p_1 \land p_2 \land \cdots \land p_n) \to q$$

Premises Conclusion

premise 1
premise 2
⋮
premise n

∴ conclusion

Equivalence vs Validity

$P \equiv Q$ iff $P \leftrightarrow Q$ is a tautology

$(p_1 \land p_2 \land \cdots \land p_n) \to q$ is a tautology

p	q	p→q	¬p∨q	(p→q) ↔ (¬p∨q)
T	T	T	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

Construct a truth table

$\neg(p \to q)$
$\equiv \neg(\neg p \lor q)$ *Cond. disjunction*
$\equiv \neg(\neg p) \land \neg q$ *De Morgan's law*
$\equiv p \land \neg q$ *Double negation*

By deduction – applying equivalence properties

Proof of equivalence /validity

- Construct a truth table
- Use a short truth table
- By deduction (applying equivalence properties and/or rules of inference)

MODUS PONENS	MODUS TOLLENS	RESOLUTION	CONJUNCTION
p p→q ∴q	¬q p→q ∴¬p	p∨q ¬p∨r ∴q∨r	p q ∴p∧q

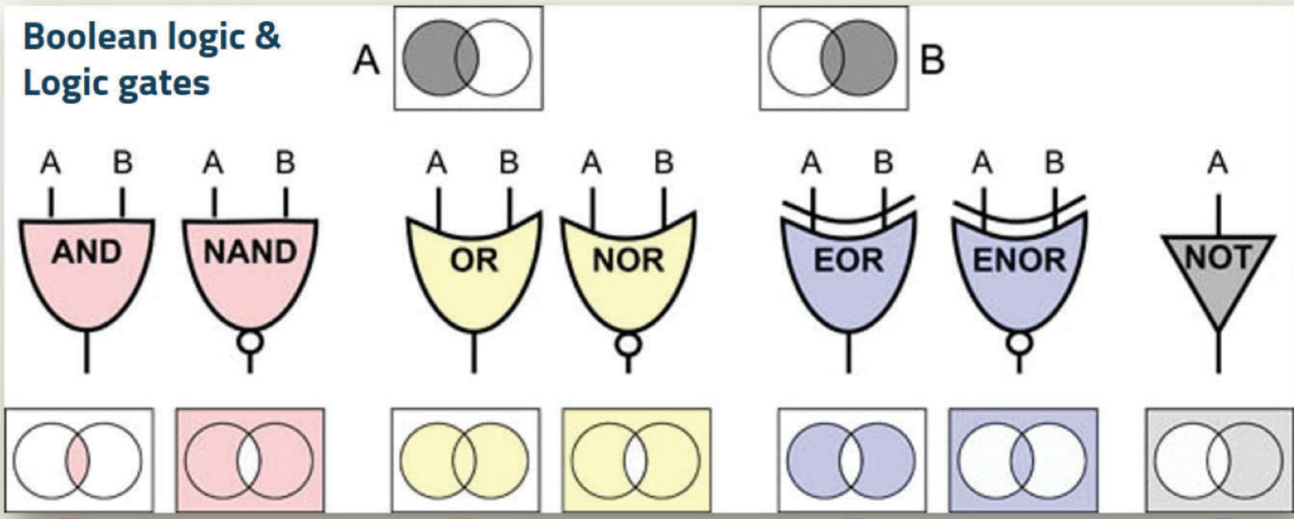
... and a few other rules

Give counterexample – assign truth values so P evaluates to true and Q false or vice versa (they disagree)

Proof non-equivalence / invalidity

Give counterexample – assign truth values that make an argument false (all premises true, conclusion false)

Where else do we find logic?



$X=AB$ $X=A+B$ $X=A\oplus B$ $X=\overline{A}$

https://www.practicallyscience.com/understanding-boolean-logic-gates/

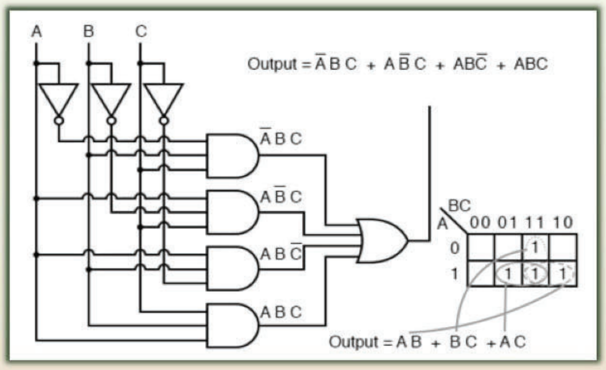
Introduction to Digital Systems

Determine whether the following Boolean equation is true or false.
Write the correct forms of the false ones (making them true).

- A + A' = 1
- A + 0 = A
- A + AB = A
- (A+B)' = A' + B

Show using Boolean algebra that the given statements are equivalent, i.e. simplify the Boolean expressions

- AB + AB' + A'B = A + B
- ABC' + BC'D' + BC + C'D = B + C'D



MATHEMATICAL INDUCTION

Logic and Proof

Learning objectives! Know what you will learn today

Self-Reflection! Rate levels of your understanding

❍ **Checklist of key topics. Keep catching up with the course.**

- ☐ **Intuition** – intuitive understanding of the principles of mathematical induction
- ☐ **Template** of a proof by math induction – base case, induction step, conclusion
- ☐ **Basis step** – clearly and explicitly show that the base case $P(n_0)$ is true
  n_0 is not necessarily equal to 0. There may also be more than one value for the base case.
- ☐ **Induction step** – show if P(k) is true [induction hypothesis], P(k+1) must be true
  Be sure to use IH when proving that P(k+1) is true. Do not get the variables k and n mixed-up.

Confident	Got it	Okay	Fuzzy	Not a clue

A mathematical proof is a sequence of logical statements, one implying another, that provides an explanation why one set of assumptions can lead to a conclusion that some statement must be true.

Proofs are in many ways like programs – they have their own vocabulary, terminology, and structure, and you will need to train yourself to think differently in order to understand and synthesize them

Induction gives us a way to show that some property is true for all $n \in \mathbb{N}$ not by directly showing that it must be true, but instead by showing that we could incrementally build up the result one piece at a time.

Today's Agenda

01 Intuition
Understand logic behind principles of math induction

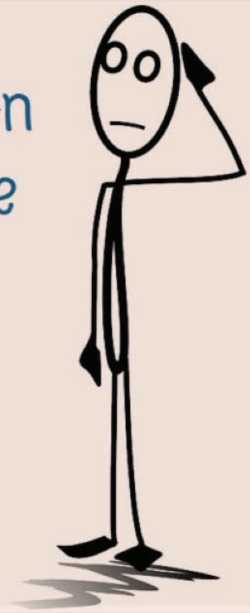
02 Template
Formalizing and writing a proof by induction

03 Practice
Learning by doing. Give it a try and master it

Induction: show something is true for some $n \in \mathbb{N}$ not by directly showing that it must be true, but by showing that we could incrementally build up the result one piece at a time.

Can you climb this (infinite) ladder?

Think in an inductive way!



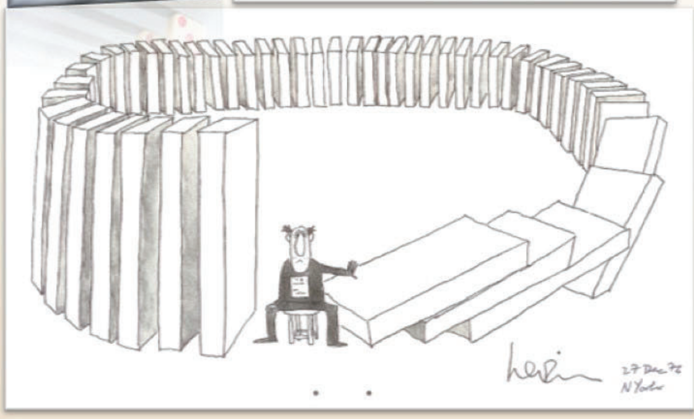
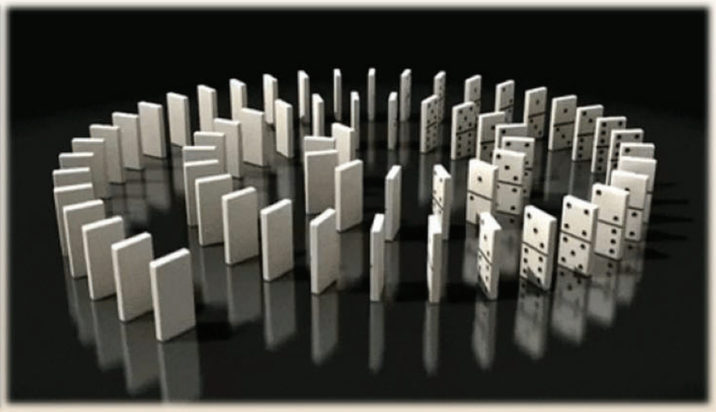
With these assumptions, are you able to get to any step on the ladder?

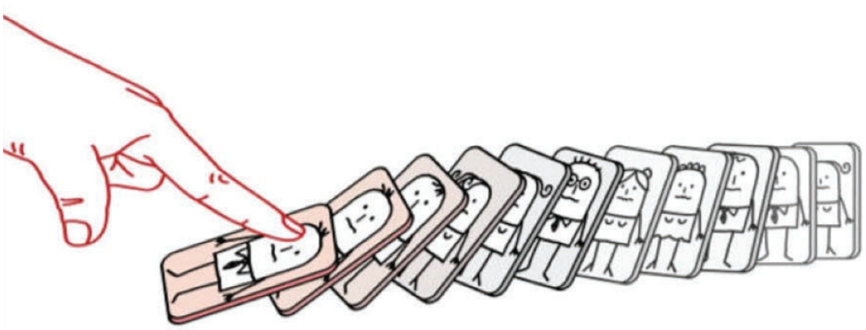
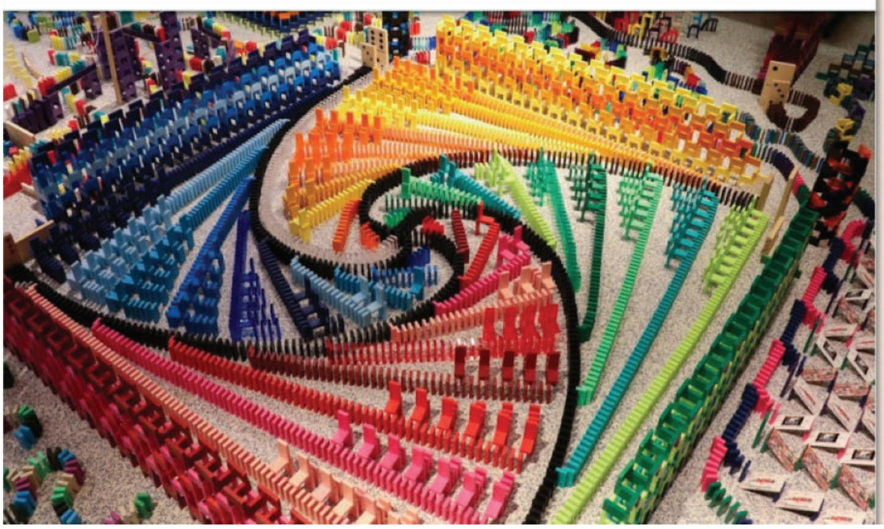
1. You know how to get onto the first step of the ladder.
2. If you find yourself at any step on the ladder, you know how to get onto the next step of the ladder.

Example: climb to the 10^{th} step

Domino Effect! Making all of them fall.

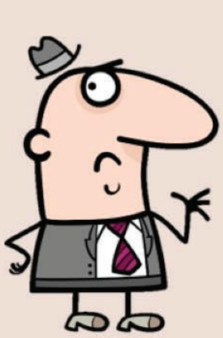
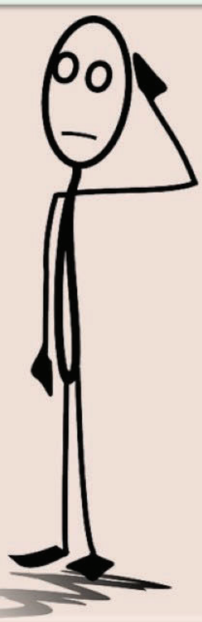

Have you ever stack dominos so you can knock them all down? How many have you every tried?

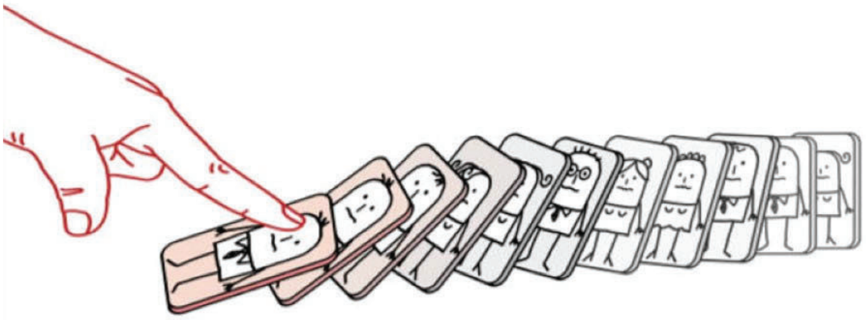



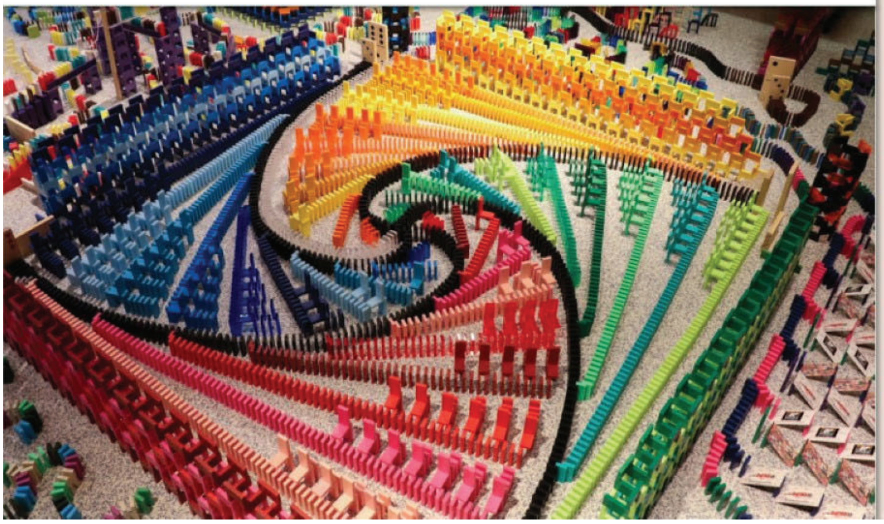



Can you knock down all the dominoes?

Think in an inductive way!



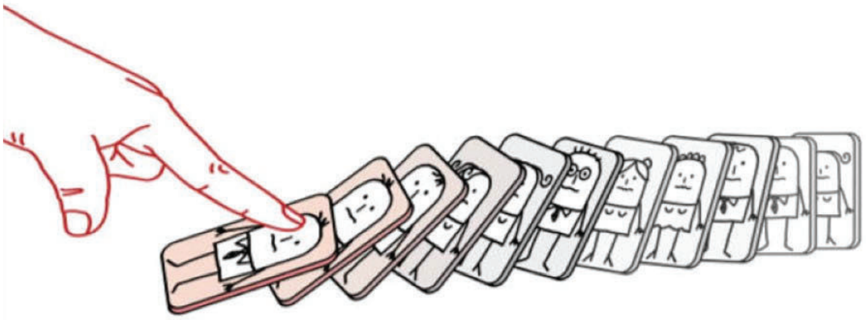


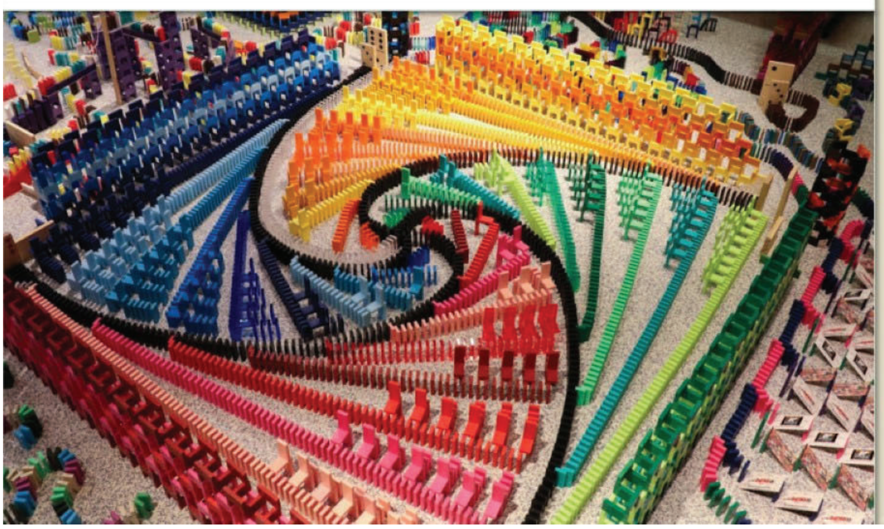
Can you knock down all the dominoes?

Assume the following conditions:

1. The first domino is knocked down.
   How? This is straightforward. 🙂
2. All dominoes are lined up in such a way that when one falls, it knocks the next one down.
   Can you explain how to do this? 🤔

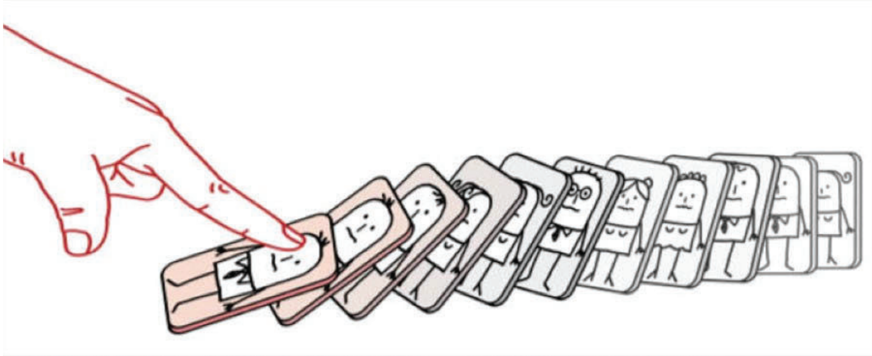
Following these assumptions, can you knock down any n^{th} dominoes?

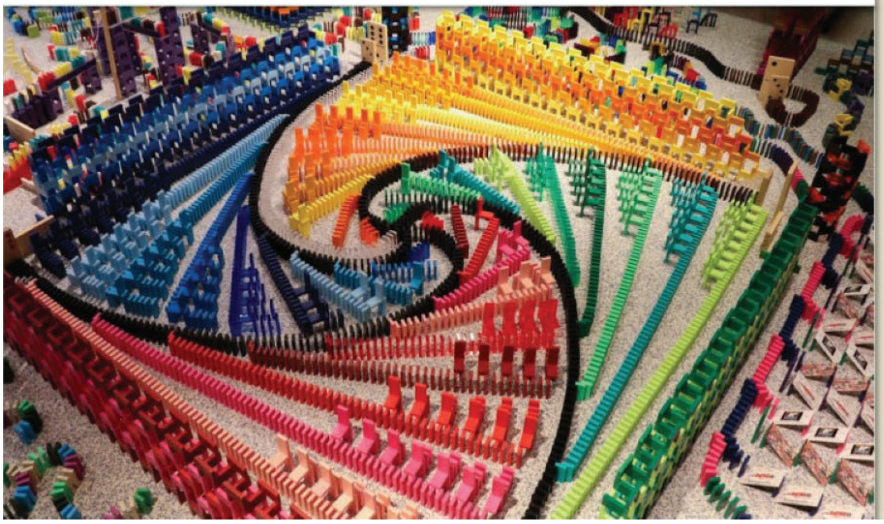




Can you knock down all the dominoes?

Ex: knock down the 10^{th} domino

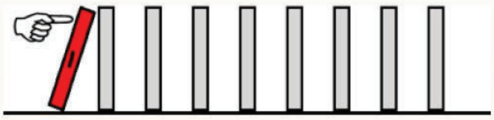




Can you knock down all the dominoes?

Important Notice: in #2, did we actually make any domino fall?

We want to show that all dominoes can be knocked down. Let P(n) be the statement the n^{th} domino is knocked down. Prove that P(n) is true for all n $\in \mathbb{Z}^+$.



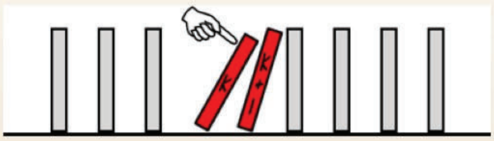
If the first domino falls ...

Show that P(1) is true

AND

... and every domino that falls knocks the next one down

Show that if P(k) is true, then P(k+1) is also true



We are not showing directly that anything is true. We are not showing P(k) is true, neither are we showing P(k+1) is true.

THEN

then all dominoes fall

Conclude that P(n) is true for all n $\in \mathbb{N}$

CAN YOU PROVE IT?

Prove by mathematical induction

Prove: $P(n)\ \forall n \geq n_0$

The TEMPLATE!

Base case: show that $P(n_0)$ is true

Induction step: $\forall k \geq n_0$, if $P(k)$ is true, then $P(k+1)$ must also be true

Assume $P(k)$ is true for some $k \geq n_0$ (*this is* **IH - induction hypothesis**)

Show that $P(k+1)$ must also be true *Be sure to use IH in your arguments*

Conclusion: $P(n)$ is true for all $n \geq n_0$

CAN YOU PROVE IT?

Suppose there are 2 types of coins: a 5 and a 7 Baht coins. Prove that, with these coins only, you can pay any amount greater than 23 Baht.

Prove: $P(n)\ \forall n \geq n_0$

Base case: show that $P(n_0)$ is true

Induction step: $\forall k \geq n_0$, if $P(k)$ is true, then $P(k+1)$ must also be true

Assume $P(k)$ is true for some $k \geq n_0$ (*this is* **IH - induction hypothesis**)

Show that $P(k+1)$ must also be true

Conclusion: $P(n)$ is true for all $n \geq n_0$

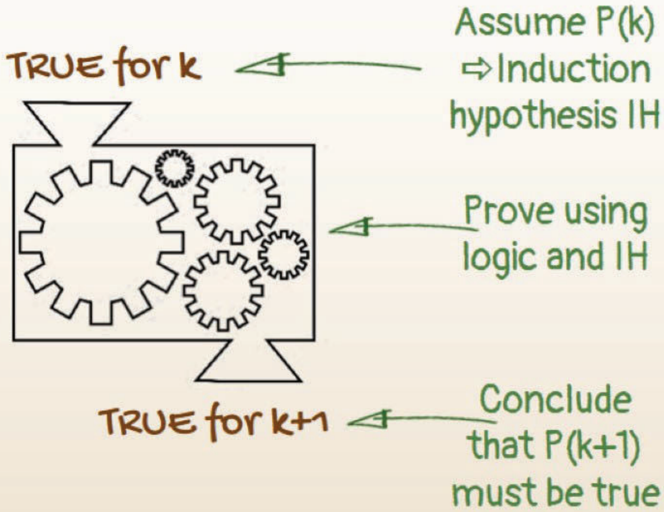
Induction step: $\forall k \geq 24$, if P(k) is true, then P(k+1) must also be true

Assume P(k) is true for some $k \geq 24$ (*this is* **IH – induction hypothesis**)

Show that P(k+1) must also be true

The main part of the proof!

Technically, we are not actually showing anything is true. All we are doing is building a machine, which shows that something is true for k+1 if you assert into it a true statement for k.

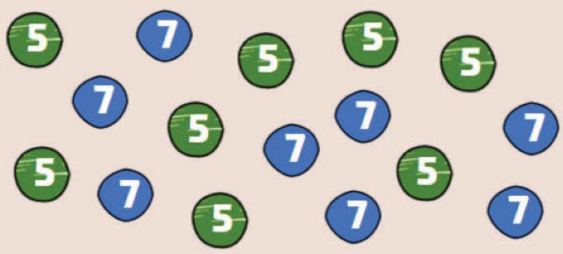


Induction step: $\forall k \geq 24$, if P(k) is true, then P(k+1) must also be true

Assume P(k) is true for some $k \geq 24$ (*this is* **IH – induction hypothesis**)

Show that P(k+1) must also be true

- Assuming P(k) means we can make k Baht using some combination 5B and 7B coins
- **Attention!** We do not know exactly what combination of coins it is but we know that such combination totals to k Baht



- Given P(k) true, to show that P(k+1) must also be true, we need to change the coin combination so that its total increases by 1. We can do …

PRACTICE PROBLEMS Challenge!

Find the largest amount which cannot be paid exactly using only 3 and 5 Baht coins.

Proving by Mathematical Induction

Induction is used to prove various results about natural numbers, discrete structures, and games, to reason about correctness of algorithms and formal models of computation.

We will write a few induction proofs together, then you will practice writing a few more.

Prove some statement P(n) about natural numbers $n \geq n_0$

If it starts true ...

... and it stays true

then it is true for all

Base case

$P(n_0)$ is true, i.e. P is true for n_0

AND

Induction step

For any $k \in \mathbb{N}$, $k \geq n_0$ if P is true for k, then P is true for $k+1$

THEN

Conclusion

P is true for every $n \in \mathbb{N}$, $n \geq n_0$

The TEMPLATE! A formal proof by mathematical induction

- **Prove** a statement P(n) is true $\forall n \geq n_0$ *clearly state the predicate P(n)*
- **Base case**: show that the first proposition $P(n_0)$ is true

  Demonstrate clearly, this is usually simple and straightforward

- **Induction step**: $\forall k \geq n_0$, if P(k) is true, then P(k+1) must also be true

  State the **induction hypothesis**: **assume** P(k) is true for some $k \geq n_0$

  Show that P(k+1) has to also be true *use IH in your arguments*

- **Conclusion**: by the principle of math induction, P(n) is true $\forall n \geq n_0$

P(n): $1 + 2 + 3 + \ldots + n = (½)(n)(n+1)$ for all integers $n \geq 1$

BASE CASE we show that P(1) is true

Work out the (simple) equation.
Show clearly that LHS = RHS.

Ch2.4, Ex.1

P(n): $1 + 2 + 3 + \ldots + n = (½)(n)(n+1)$ for all integers $n \geq 1$

INDUCTION STEP $\forall k \geq n_0$, if P(k) is true, then P(k+1) must also be true

IH: ASSUME

SHOW

Ch2.4, Ex.1

P(n): $1 + 2 + 3 + \ldots + n = (½)(n)(n+1)$ for all integers $n \geq 1$

INDUCTION STEP $\forall k \geq n_0$, if P(k) is true, then P(k+1) must also be true

IH: ASSUME for some $k \geq 1$, P(k): $1 + 2 + 3 + \ldots + k = (½)(k)(k + 1)$

SHOW P(k+1): $1 + 2 + 3 + \ldots + k + (k+1) = (½)(k+1)((k + 1)+1)$

Know how to write P(k+1)

What we want to show is P(k+1).
Write it by substituting k+1 for n.
Include the term k on the LHS.

What you need to do!

Start with the LHS of P(k+1)
Slowly turn it into the RHS
Remember to use IH, P(k)

P(n): $1 + 2 + 3 + \ldots + n = (½)(n)(n+1)$ for all integers $n \geq 1$

INDUCTION STEP $\forall k \geq n_0$, if P(k) is true, then P(k+1) must also be true

IH: ASSUME for some $k \geq 1$, P(k): $1 + 2 + 3 + \ldots + k = (½)(k)(k + 1)$

SHOW P(k+1): $1 + 2 + 3 + \ldots + k + (k+1) = (½)(k+1)((k + 1)+1)$

Writing a Proof by Mathematical Induction

Induction variable: n versus k

- Use n in the statement that you attempt to prove.
- Use k and/or other letters for variables appearing in the inductive step.
- These variables, n and k, play different roles in the proof. They cannot be mixed and cannot be used interchangeably — never use one variable to represent more than one thing at a time.

The role of the induction hypothesis (IH)

- IH is P(n) when n = k. It is what you assume at the beginning of the induction step.
- Use IH in the induction step of the proof — if not, you are doing something wrong.
- Where IH is used is the most crucial step in any induction argument. Clearly point it out within your chain of equations, e.g. "Applying/By the IH, ...", or "... (by IH)".

https://faculty.math.illinois.edu/~hildebr/213/induction.pdf

Prove by induction P(n): $1 + 2^n < 3^n$ for $n \geq 2$

BASE CASE we show that P(2) is true

n_0 is not always equal to one and in some cases, you may be asked to determine the value of n_0 that makes the statement true

Ch2.4,Q.10

Prove by induction P(n): $1 + 2^n < 3^n$ for $n \geq 2$

INDUCTION STEP $\forall k \geq n_0$, if P(k) is true, then P(k+1) must also be true

IH: ASSUME

SHOW

Ch2.4,Q.10

P(n): $3 \mid (n^3 - n)$ for every positive integer n

pipe

$a \mid b$

read as

"a **divides** b"

b is divisible by a

$a \mid b$

can be written as

$b = ar$

an integer

$\frac{b}{a}$ is an integer

Examples:

$2 \mid 10$

$3 \mid 15$

$4 \nmid 22$

$5 \nmid 27$

$3 \mid (n^3 - n)$ means __?

https://www.chilimath.com/lessons/basic-math-proofs/if-a-divides-b-and-b-divides-c-then-a-divides-c/

P(n): 3 | (n^3 – n) for every positive integer n

BASE CASE we show that P(1) is true

Ch2.4,Q.16

P(n): 3 | (n^3 – n) for every positive integer n

INDUCTION STEP $\forall k \geq n_0$, if P(k) is true, then P(k+1) must also be true

IH: ASSUME

SHOW

Ch2.4,Q.16

PRACTICE PROBLEMS

Prove the following statements by mathematical induction:

- $2 + 4 + 6 + \ldots + 2n = n(n + 1)$ for all integers $n \geq 1$
- $3^n < (n+1)!$ for all integers $n > 3$
- $6^n - 1$ is divisible by 5 for any positive integer n
- $6 \mid n^3 + 5n$ for every positive integer n
- $1 + x + \ldots + x^n = (1 - x^{n+1}) / (1 - x)$ for any real number x and n is a positive integer
- $(1 + \frac{1}{1})(1 + \frac{1}{2})(1 + \frac{1}{3}) \ldots (1 + \frac{1}{n}) = n + 1$ for all integers $n \geq 1$

ATTENTION PLEASE

A quiz on induction proof will be a drag-and-drop question. So you can work on the entire proof effectively, using a mobile phone is **NOT** recommended for this quiz.

DON'T FORGET!

A WEEKLY QUIZ

DON'T FORGET!

Reading
KBR, Rosen, Levin

Textbook exercises

HW - Practice Problems