16. Evaluating Definite Integrals by Substitution

An integral of the form

$$\int_a^b f(g(x))g'(x)dx$$

can be evaluated by making the *u*-substitution using two different methods.

Method 1 Leave off the limits of integration, evaluate the indefinite integral, and then put the limits back.

Let $u = g(x)$, then

$du = g'(x)dx$

$$\int_a^b f(g(x))g'(x)dx = \left[\int f(g(x))g'(x)dx\right]_{x=a}^{b}$$

$$= \left[\int f(u)du\right]_{x=a}^{b}$$

Method 2 Change the limits of integration.

Let $u = g(x)$, then

$du = g'(x)dx$

Also, if $x = a$, then $u = g(a)$,

and if $x = b$, then $u = g(b)$. Thus

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Note that the new definite integral is expressed entirely in terms of *u*.

Example 1 Evaluate $\int_{1/2}^{1} (2x-1)^{10}dx$

Solution Using method 1, let $u = 2x-1$, then $du/dx = 2$ and $dx = du/2$. Thus,

$$\int (2x-1)^{10}dx = \frac{1}{2}\int u^{10}du = \frac{u^{11}}{22} + C = \frac{(2x-1)^{11}}{22} + C$$

Then,

$$\int_{1/2}^{1} (2x-1)^{10}dx = \left.\frac{(2x-1)^{11}}{22}\right]_{x=1/2}^{1} = \frac{1}{22}$$

Using method 2, let $u = 2x - 1$, then $du/dx = 2$ and $dx = du/2$.
If $x = ½$, then $u = 0$. If $x = 1$, then $u = 1$.
Both upper and lower limits of integration with respect to x are changed to those with respect to u. Thus,

$$\begin{aligned}\int_{1/2}^{1} (2x-1)^{10} dx &= \int_0^1 u^{10} \frac{du}{2} \\ &= \left.\frac{u^{11}}{22}\right]_{u=0}^{1} \\ &= \frac{1}{22}\end{aligned}$$

Exercises: Evaluate the integrals.

1. $\int_0^1 (3x-1)^4 \, dx$

2. $\int_{-5}^{0} x\sqrt{4-x}\,dx$

3. $\int_0^{\pi/6} 2\cos 3x \, dx$

4. $\int_0^{\pi} \sin^2 x \cos x \, dx$