Permutation

Pigeonhole Principle

Drawing from http://web.stanford.edu/class/archive/cs/cs103/cs103.1184/lectures/11/Small11.pdf

Learning objectives! Know what you will learn today

Self-Reflection! Rate levels of your understanding

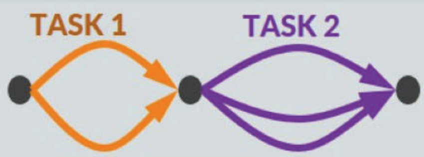
❍ Checklist of key topics. Keep catching up with the course.

- ☐ Circular permutation (rotation invariance) with flip/reflection invariance
- ☐ Permutation when there are non-distinct objects with various constraints
- ☐ Logical thinking and problem solving practice in combinatorics
- ☐ Understanding of the pigeonhole principle and its generalization
- ☐ Realization of real-world problems involving the Pigeonhole principle

Confident	Got it	Okay	Fuzzy	Not a clue

Apply the sum rule for disjoint cases

SUM $n_1 + n_2 + \ldots + n_k$ ways



Tasks that require sequentially steps

PRODUCT $n_1 \times n_2 \times \ldots \times n_k$ ways

When overcounting occurs, use **subtraction** or **division** rules to remove them
Use the **complement** rule when unwanted cases are fewer & less complicated

	REPETITION allowed	NO REPEAT
PERMUTATIONS Arrangement, Order matters	n^r	${}_n\mathrm{P}_r = \dfrac{n!}{(n-r)!}$
COMBINATIONS Grouping/Selection, No ordering	${}_{n+r-1}\mathrm{C}_r = \dfrac{(n+r-1)!}{r!(n-1)!}$	${}_n\mathrm{C}_r = \dfrac{n!}{(n-r)!r!}$

- Select a group/collection
  - Apply division rule to make order not matter
  - All things →
  - r of n →

COMBINATIONS

- Putting n persons in a row, order matter
  - All things →
  - r of n things →

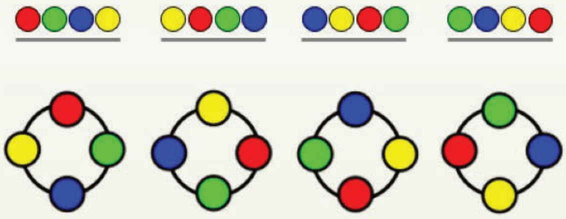
PERMUTATIONS

- Arranging in a circle
  - Division rule make it rotation invariance
  - All things →
  - r of n →

CIRCULAR PERM.

How many ways to sit 4 people at a 4-seated round table?

We know how to seat 4 people in a row.



Seating in a circle can be rotated! Each set of ______ arrangements are the same.

COUNT

All = sample space = count what we know how

OVERCOUNT

Overcount = some arrangements are the same

DIVISION RULE

ANSWER
= All / overcount factor

Circular Permutation – Rotation Invariant

- **What if there are more people than the number of seats?**
- How many different ways to sit 6 people at a 4-seated round table?
  - COUNT

    Start with seating 4 of 6 peo in a row.

    That is, a 4-permutation of 6 people.
  - REDUCE by what have OVERCOUNTED

    Divide by the rotation invariant factor.

WORKED EXAMPLES Approach 1 – apply the complement rule

- In how many ways can we arrange 20 family members at a round table if two brothers refuse to sit next to each other?

WORKED EXAMPLES Approach 2 – seat others then the brothers

- In how many ways can we arrange 20 family members at a round table if two brothers refuse to sit next to each other?

WORKED EXAMPLES

Approach 2 – seat others then the brothers

- In how many ways can we arrange 15 out of 20 family members at a round table if two brothers refuse to sit next to each other?

PRACTICE PROBLEMS

- In how many ways the letters of the word CLIPBOARDS can be written around a circle?

- In how many ways 6-letter words formed from the letters of the word CLIPBOARDS can be written around a circle?

PRACTICE PROBLEMS

- In how many ways the letters of the word CLIPBOARDS can be written around a circle if the vowels are to be together in any arrangement?

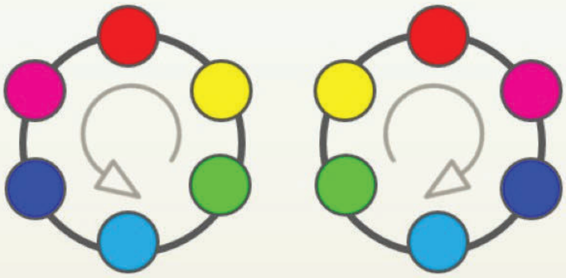
- ... what if the vowels are to be separated in any arrangement?

PRACTICE PROBLEMS

- From the letters of the word CLIPBOARDS, in how many ways can we write 5 consonants and 3 vowels around a circle if the vowels are to be together?

- ... what if the vowels are to be separated?

Circular permutation with reflection/flip invariant



Are these two the same?

They are the same IF we say flipping them over does not make a difference!

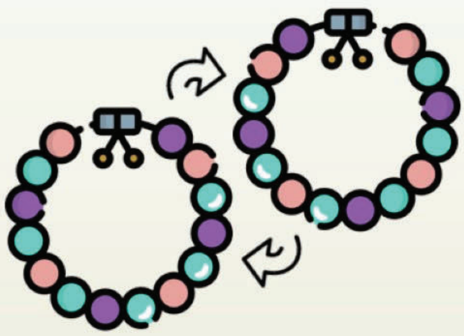
⇨ 2 circular arrangements counting as 1 unique one

⇨ Division rule → divided by ____

- Other terms that indicate circular arrangements with flip invariant
  - No difference between the clockwise and counterclockwise arrangements
  - Having the same neighbors in any two arrangements

Circular permutation with reflection/flip invariant

- How many 6-bead bracelets made from 8 beads of different colors?
  - COUNT Following circular permutation, ...
  - DIVISION divide by the flip invariant factor, ...



PRACTICE PROBLEMS

- A couple wants to plant 7 different flowers around a circular walkway. How many different ways can they plant their flowers so no two arrangements look the same clockwise and counterclockwise?

PRACTICE PROBLEMS (cont.)

- How would your calculation change if the couple has in their home collection, not 7 but 9 flowers to choose from? Note: their circular walkway is still the same, having enough space for only 7 plants.

Permutation of non-distinct objects

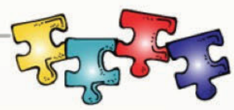
How many different permutations are there of all the letters in a word **BANANA** ?

- COUNT: permute all 6 letters ⇨ _____ ways

  rearrange all 6 letters in a row

- DIVISION: divide by the factor that we OVERCOUNTED

Ch3.1,Ex.9



WORKED EXAMPLES

How many arrangements of the letters of the word ENGINEERING ...

- ❑ are possible (no constraint)?
- ❑ the vowels must be together?
- ❑ have at least two adjacent N's?

PRACTICE PROBLEMS

How many arrangements of the letters of the word MISSISSIPPI …

- ❑ are possible (no constraint)?

- ❑ exist such that the two P's are separated?

- ❑ have at least two adjacent S's?

Circular permutation

	n different things	n things, taken r at a time
Rotation invariance	$\frac{n!}{n} = (n-1)!$	$\frac{_nP_r}{r}$
Flip/symmetric invariance	$\frac{(n-1)!}{2}$	$\frac{_nP_r}{2r}$

The number of permutations, counting all possible arrangements of all of its n elements $_nP_n = n!$

Permutation of all elements in a collection

If some of the n elements are the same (not distinguishable)

The number of distinguishable permutations that can be formed from a collection of n objects where the first object appears k_1 times, the second object k_2 times, and so on, is

$$\frac{n!}{k_1!\,k_2!\cdots k_t!} \text{ where } k_1 + k_2 + \ldots + k_t = n$$

PRACTICE PROBLEMS

- There are 4 identical pens and 7 identical books. In how many ways can a person select at least one object from this set?

Hint: the Cartesian product set and how we find positive divisors of an integer

PRACTICE PROBLEMS

- How many different 4-letter words can be formed (the words need not be meaningful) using the letters of the word MEDITERRANEAN such that the first letter is E and the last letter is R?

Hint: case analysis – the middle two letters can either be the same or different

PRACTICE PROBLEMS

- An insurance company wants to assign 15 new clients equally to 3 of its salespersons. In how many different ways can this be done?

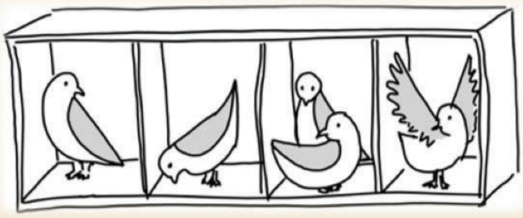
Hint: from basic (combination) task analysis to (permutation) of non-distinct objects

PRACTICE PROBLEMS

- How many ways to put five different toys into three toyboxes? The toyboxes are colored red, green, and blue. No boxes should be empty.

PIGEONHOLE PRINCIPLE

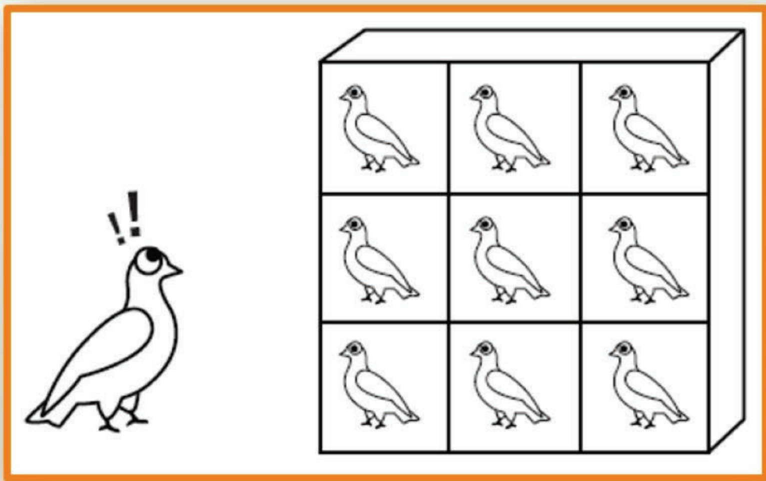
Counting and Combinatorics



The Pigeonhole Principle

If there are more pigeons than pigeonholes, then there must be at least one pigeonhole with at least two pigeons in it

- If $k \in \mathbb{Z}^+$ and $k + 1$ or more objects are placed into k boxes, then there must be some box that contains two or more of the objects



https://www.expii.com/t/pigeonhole-principle-3587

WORKED EXAMPLES

If eight people are chosen, at least two of them will have been born on the same day of the week.

Ch3.3,Ex.1

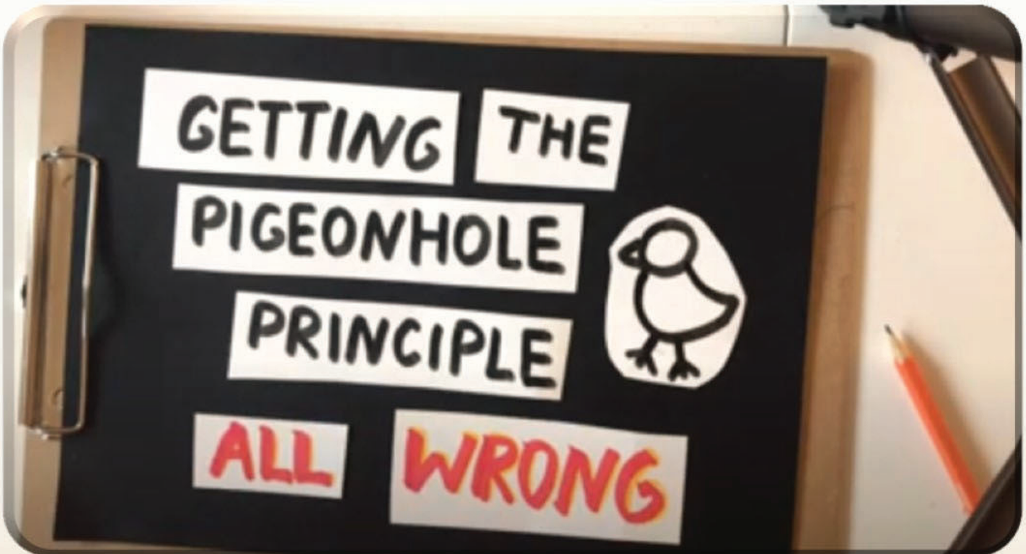
WORKED EXAMPLES

How many numbers from 1 to 8 do we have to choose so that at least two of them sum to 9?

PRACTICE PROBLEMS

- At the minimum, how many integers must be chosen from the set {1, 2, . . . , 20} so that one of them will be a multiple of another?

Ch3.3,Ex.3



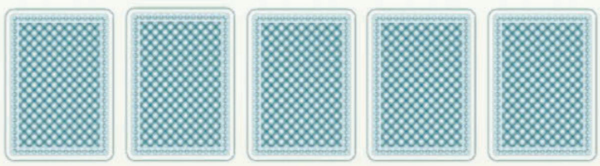
A cartoon-like video on a common misconception of the Pigeonhole
https://www.youtube.com/watch?v=_MWp3lBDiwE

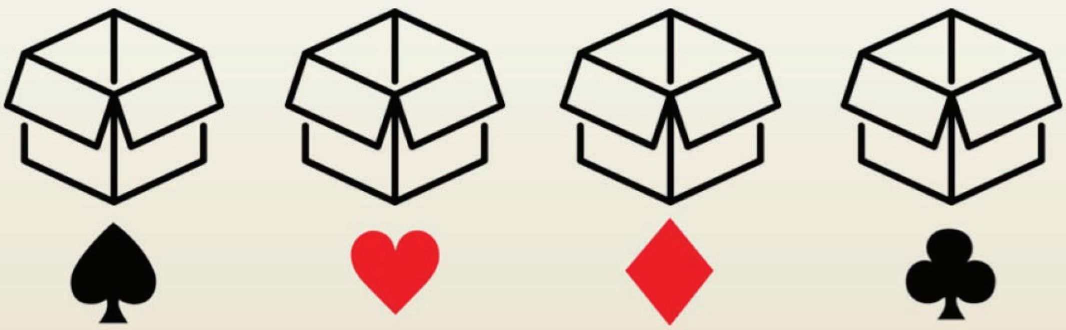
At first glance, **the pigeonhole principle** (also called *the Dirichlet's drawer principle*) may appear too simple and too obvious to be useful.

The opposite is however true; it is a remarkably powerful tool in combinatorial mathematics. Its power comes from cleverly defining the **boxes** and the **objects**.

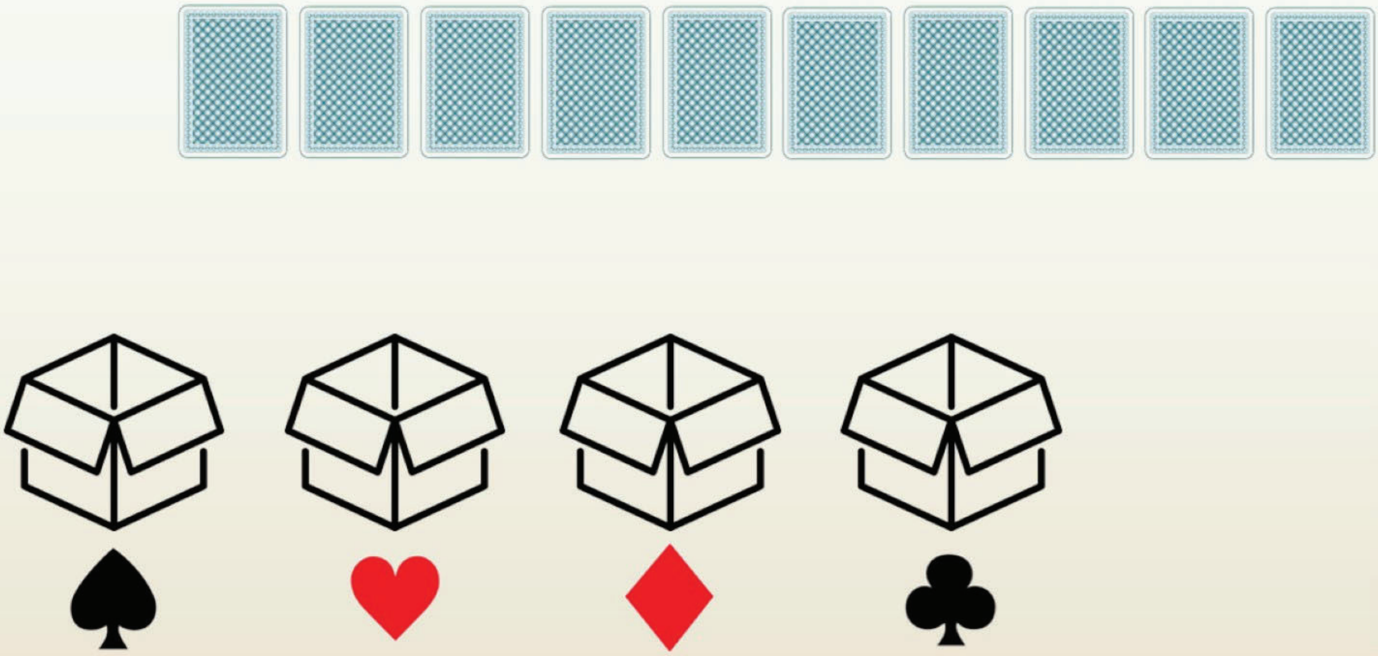
Draw 5 cards, how many must be of the same suit?

No matter how you put the five cards into the suit boxes below, there will be a box with k or more cards in it. What is k?





Draw 10 cards, how many must be of the same suit?



Draw 10 cards, how many must be of the same suit?

- Draw 10 cards from a standard deck, at least 3 must be of the same suit

  Where does this 3 come from?

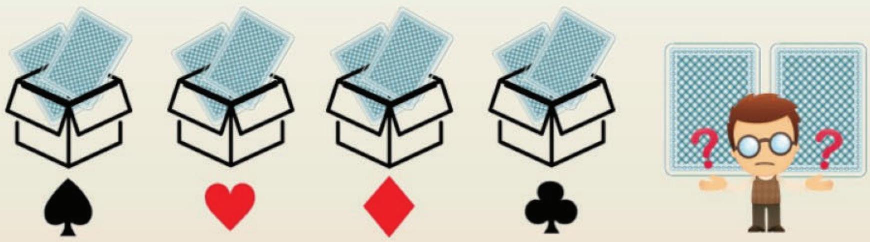
- Distribute the cards evenly into the four boxes. The average number of cards per box is 10/4 = 2.5 cards. Since you cannot have half the card, there must be a box with $\lceil 10/4 \rceil = 3$ cards.

$$k = \left\lceil \frac{n}{m} \right\rceil = \left\lceil \frac{\text{n objects}}{\text{m containers}} \right\rceil$$

$\lceil x \rceil$ is a ceiling function

$\lceil 2.1 \rceil = \lceil 2.5 \rceil = \lceil 2.9 \rceil = 3$

$\lceil 3.001 \rceil = \lceil 3.999 \rceil = 4$



mitcourseware

Solve these logically **Hint: consider the worst case scenario**

Marbles are drawn blindfolded: there are 6 red, 6 green, and 6 blue.

- ❑Take 8 marbles, how many must we have of the same color?

- ❑Find the minimum number of marbles we must take to get 5 of the same color

Solve these logically **Hint: consider the worst case scenario**

Marbles are drawn blindfolded: there are 6 red, 6 green, and 6 blue.

- ❑Find the minimum number of marbles we must take to get at least 3 blue

- ❑Find the min number of marbles we must take to get at least one of every color

PRACTICE PROBLEMS

You have 10 black socks and 10 white socks and you are picking socks randomly

- If you pick 10 socks, how many matching pairs must you have?
- At the minimum, how many will you need to pick to find a matching pair?
- At the minimum, how many will you need to pick to guarantee that you get a pair of white socks?

PRACTICE PROBLEMS

4 suits (♠♥♣♦), each has 13 cards (A, 2, 3, …, 10, J, Q, K)

- How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?
- How many must be selected to guarantee that at least three hearts are selected?

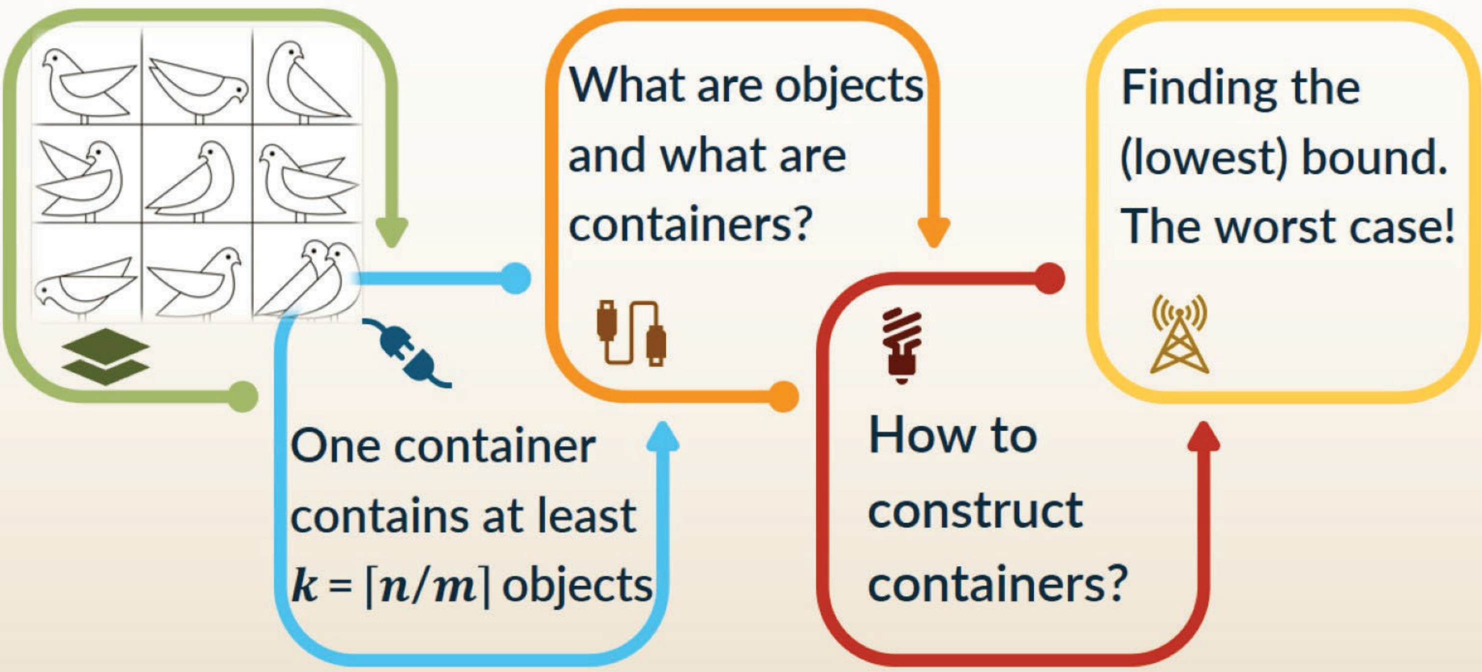
The Generalized Pigeonhole Principle

If n objects are placed into m boxes, then there is **at least** one box containing at least $k = \lceil n/m \rceil$ objects.

There are m kind of objects. Drawing blindfolded, if we want to guarantee getting **at least** k of the same kind, we need to take at least $n = (k-1)(m) + 1$.

The two formulas are mathematically the same. The top one is a straightforward generalization of the pegionhole principle, while the other reverses engineering it.

A Simple but elegant principle! Many real world applications!



PRACTICE PROBLEMS

REAL WORLD PROBLEM SOLVING

- There are 38 different time periods during which classes at a university can be scheduled. If there are 677 different classes, what is the minimum number of different rooms that will be needed?

PRACTICE PROBLEMS

REAL WORLD PROBLEM SOLVING

- Use pigeonhole principle to reason that at least 18 cables are required to connect 8 computers to 3 printers in order to guarantee that any set of 3 computers have access to printers at the same time.

PRACTICE PROBLEMS

REAL WORLD PROBLEM SOLVING

- How would 18 cables be connected between 8 computers and 3 printers in order to guarantee that any set of 3 computers have access to printers at the same time.

PRACTICE PROBLEMS

REAL WORLD PROBLEM SOLVING

- The computer classroom has 12 PCs and 5 printers. What is the minimum number of connections that must be made to guarantee that any set of 5 or fewer PCs can access printers at the same time?

Ch3.3,Q.25

PRACTICE PROBLEMS

REAL WORLD PROBLEM SOLVING

- In a computer science department, a student club can be formed with either 10 members from first year or 8 members from second year or 6 from third year or 4 from final year. What is the minimum number of students we have to choose randomly from department to ensure that a student club is formed?

What's next?

DON'T FORGET!

A WEEKLY QUIZ

DON'T FORGET!

Reading
KBR, Rosen, Levin

Textbook exercises

HW - Practice Problems