ITCS111 Linear Algebra and Calculus

Fall 2023 Exercise Problems

Thitivatr PatanasakPinyo

1 Exercise Problems

1. Let $\vec{a}$ and $\vec{b}$ be 2 vectors in $\mathbb{R}^3$. $|\vec{b}| = \sqrt{673}$ and $\vec{a} \bullet \vec{b} = 2019$. Compute all numbers $\lambda \in \mathbb{R}$ such that there can exist $\vec{c} \in \mathbb{R}^3$ that makes $\vec{a} - \lambda\vec{b} = \vec{c} \times \vec{b}$ satisfies true.

2. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

Let S be a set of real numbers t that:

$$t^3 A\text{adj}(A - \frac{1}{3}I) = \text{adj}(tA - I)$$

Compute $\sum_{t \in S} t$

3. Let a, b, c be positive integers where $a < b < c$ and

$$\det\left(\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}\right) = 16$$

Compute $ac - b^2$

4. Let $\vec{u} = \langle 6, 5 \rangle$ and $\vec{a} = \langle -3, 4 \rangle$. (1) Find the vector component of $\vec{u}$ parallel to $\vec{a}$ and (2) find the vector component of $\vec{u}$ orthogonal to $\vec{a}$.

5. Prove a formula of orthogonal projection:

$$\vec{w} = \text{proj}_{\vec{b}}\vec{u} = \left(\frac{\vec{u} \bullet \vec{b}}{|\vec{b}|^2}\right)\vec{b}$$

6. Find an equation for the plane parallel to the line:

$$
\begin{aligned}
x &= 1 - 2t \\
y &= 2 + 3t \\
z &= 1 + 2t
\end{aligned}
$$

and contains the line of intersection of $x + 2y - 7z = 2$ and $3x + 11y - 17z = 2$

2 Solution to Exercise Problems

1. Let $\vec{a}$ and $\vec{b}$ be 2 vectors in $\mathbb{R}^3$. $|\vec{b}| = \sqrt{673}$ and $\vec{a} \bullet \vec{b} = 2019$. Compute all numbers $\lambda \in \mathbb{R}$ such that there can exist $\vec{c} \in \mathbb{R}^3$ that makes $\vec{a} - \lambda\vec{b} = \vec{c} \times \vec{b}$ satisfies true.

   - Solution

$$\begin{aligned}
\vec{a} - \lambda\vec{b} &= \vec{c} \times \vec{b} \\
(\vec{a} - \lambda\vec{b}) \bullet \vec{b} &= (\vec{c} \times \vec{b}) \bullet \vec{b} \\
\vec{a} \bullet \vec{b} - \lambda|\vec{b}|^2 &= 0 \\
\vec{a} \bullet \vec{b} &= \lambda|\vec{b}|^2 \\
2019 &= 673\lambda \\
\lambda &= 3 \qquad \blacksquare
\end{aligned}$$

2. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

Let S be a set of real numbers t that:

$$t^3 A\text{adj}(A - \frac{1}{t}I) = \text{adj}(tA - I)$$

Compute $\sum_{t \in S} t$

- Solution

$$\begin{aligned}
\text{let } B &= tA - I \\
\frac{1}{t}B &= A - \frac{1}{t}I \\
t^3 A\text{adj}(A - \frac{1}{t}I) &= \text{adj}(tA - I) \\
t^3 A\text{adj}(\frac{1}{t}B) &= \text{adj}(B) \\
t^3 A \left| \frac{1}{t}B \right| (\frac{1}{t}B)^{-1} &= \text{adj}(B) \\
t^3 A(\frac{1}{t^3}|B|)(\frac{1}{t}B)^{-1} &= \text{adj}(B) \\
A|B|(\frac{1}{t}B)^{-1} &= \text{adj}(B) \\
A|B|(tB^{-1}) &= \text{adj}(B) \\
tA\text{adj}(B) &= \text{adj}(B) \\
tA &= I \\
t \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
t &= 0, 1, \frac{1}{4}, \frac{1}{6} \qquad * * * \\
t &= \cancel{0}, 1, \frac{1}{4}, \frac{1}{6} \\
\therefore S &= \{1, \frac{1}{4}, \frac{1}{6}\} \\
\sum_{t \in S} t &= 1 + \frac{1}{4} + \frac{1}{6} \\
&= \frac{17}{12} \qquad \blacksquare
\end{aligned}$$

Note that the line $***$ is just a bypass to get t since our purpose is just to find t that makes the given equation satisfy true. Line $* * *$ is not completely rigorous. If you would like to see the completely rigorous work, contact me after the final exam week.

3. Let a, b, c be positive integers where $a < b < c$ and

$$\det\left(\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}\right) = 16$$

Compute $ac - b^2$

- Solution

By row reduction technique

(a) Apply $R_2 \leftarrow R_2 + (-a)R_1$. The result is:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ a^2 & b^2 & c^2 \end{bmatrix}$$

The det is still 16.

(b) Apply $\frac{1}{b-a}R_2$. The result is

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{c-a}{b-a} \\ a^2 & b^2 & c^2 \end{bmatrix}$$

The det is $\frac{16}{b-a}$.

(c) Apply $R_3 \leftarrow R_3 + (-a^2)R_1$. The result is:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{c-a}{b-a} \\ 0 & b^2-a^2 & c^2-a^2 \end{bmatrix}$$

The det is still $\frac{16}{b-a}$.

(d) Apply $R_3 \leftarrow R_3 + (a^2 - b^2)R_2$. The result is:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{c-a}{b-a} \\ 0 & 0 & c^2 - a^2 + \frac{(a^2-b^2)(c-a)}{b-a} \end{bmatrix}$$

The det is still $\frac{16}{b-a}$.

After we get a triangular matrix, we compute a det.

$$
\begin{aligned}
\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{c-a}{b-a} \\ 0 & 0 & c^2-a^2+\frac{(a^2-b^2)(c-a)}{b-a} \end{vmatrix} &= \frac{16}{b-a} \\
c^2-a^2+\frac{(a^2-b^2)(c-a)}{b-a} &= \frac{16}{b-a} \\
(c-a)(c+a)+\frac{(a-b)(a+b)(c-a)}{b-a} &= \frac{16}{b-a} \\
(c-a)(c+a)+(a+b)(a-c) &= \frac{16}{b-a} \\
(c-a)(c+a-b-a)(b-a) &= 16 \\
(c-b)(c-a)(b-a) &= 16 \\
(c-b)(c-a)(b-a) &= (2)(4)(2) && \because (c-b)+(b-a)=(c-a) \\
c-a &= 4 \\
c &= 4+a \\
b-a &= 2 \\
b &= 2+a \\
ac-b^2 &= a(4+a)-(2+a)^2 \\
&= 4a+a^2-(a^2+4a+4) \\
&= -4 && \blacksquare
\end{aligned}
$$

4. Let $\vec{u} = \langle 6, 5\rangle$ and $\vec{a} = \langle -3, 4\rangle$. (1) Find the vector component of $\vec{u}$ parallel to $\vec{a}$ and (2) find the vector component of $\vec{u}$ orthogonal to $\vec{a}$.

- (1)

$$\begin{aligned}
\text{vector component parallel} &= \text{proj}_{\vec{a}}\vec{u} \\
&= \left(\frac{\vec{u} \bullet \vec{a}}{|\vec{a}|^2}\right)\vec{a} \\
&= \left(\frac{\langle 6, 5\rangle \bullet \langle -3, 4\rangle}{(\sqrt{(-3)^2 + 4^2})^2}\right)\langle -3, 4\rangle \\
&= \left(\frac{2}{25}\right)\langle -3, 4\rangle \\
&= \langle -\frac{6}{25}, \frac{8}{25}\rangle \qquad \blacksquare
\end{aligned}$$

- (2)

$$\begin{aligned}
\text{vector component orthogonal} &= \vec{u} - \text{proj}_{\vec{a}}\vec{u} \\
&= \langle 6, 5\rangle - \langle -\frac{6}{25}, \frac{8}{25}\rangle \qquad \blacksquare
\end{aligned}$$

5. Prove a formula of orthogonal projection:

$$\vec{w} = \text{proj}_{\vec{b}}\vec{u} = \left(\frac{\vec{u} \bullet \vec{b}}{|\vec{b}|^2}\right)\vec{b}$$

- Solution

Let an angle between $\vec{u}$ and $\vec{b}$ be θ. Since $\vec{w} \parallel \vec{b}$, an angle between $\vec{u}$ and $\vec{w}$ is also θ.

$$\begin{aligned}
\text{unit vector} &= \text{unit vector} && \triangleright \text{ assume same direction} \\
\frac{\vec{w}}{|\vec{w}|} &= \frac{\vec{b}}{|\vec{b}|} && \because \vec{w} \parallel \vec{b} \\
\vec{w} &= \left(\frac{|\vec{w}|}{|\vec{b}|}\right)\vec{b} \\
\vec{w} &= \left(\frac{|\vec{b}|}{|\vec{b}|}\right)\left(\frac{|\vec{w}|}{|\vec{b}|}\right)\vec{b} \\
\vec{w} &= \left(\frac{|\vec{w}||\vec{b}|}{|\vec{b}|^2}\right)\vec{b} \\
\vec{w} &= \left(\frac{|\vec{u}|\cos(\theta)|\vec{b}|}{|\vec{b}|^2}\right)\vec{b} && \because \text{ trigonometric ratio} \\
\vec{w} &= \left(\frac{|\vec{u}||\vec{b}|\cos(\theta)}{|\vec{b}|^2}\right)\vec{b} \\
\vec{w} &= \left(\frac{\vec{u} \bullet \vec{b}}{|\vec{b}|^2}\right)\vec{b} && \blacksquare
\end{aligned}$$

6. Find an equation for the plane parallel to the line:

$$\begin{aligned} x &= 1 - 2t \\ y &= 2 + 3t \\ z &= 1 + 2t \end{aligned}$$

and contains the line of intersection of $x + 2y - 7z = 2$ and $3x + 11y - 17z = 2$

- Solution

$$\begin{aligned}
\langle 1, 2, -7\rangle \times \langle 3, 11, -17\rangle &= \left|\begin{array}{ccc|cc} i & j & k & i & j \\ 1 & 2 & -7 & 1 & 2 \\ 3 & 11 & -17 & 3 & 11 \end{array}\right| \\
&= -34i - 21j + 11k - 6k + 77i + 17j \\
&= 43i - 4j + 5k \\
\text{try } z = 0 : x + 2y &= 2 \\
3x + 11y &= 2 \\
3(2 - 2y) + 11y &= 2 \\
6 - 6y + 11y &= 2 \\
6 + 5y &= 2 \\
y &= -\frac{4}{5} \\
x &= 2 - 2(-\frac{4}{5}) \\
&= \frac{18}{5} \\
\langle -2, 3, 2\rangle \times \langle 43, -4, 5\rangle &= \langle 23, 96, -121\rangle \\
ax + by + cz + d &= 0 \\
23x + 96y - 121z + d &= 0 \\
(23)(\frac{18}{5}) + (96)(-\frac{4}{5}) + d &= 0 \\
d &= -6 \\
\therefore 23x + 96y - 121z - 6 &= 0
\end{aligned}$$

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