L'Hopital Rule

ITCS111 Linear Algebra and Calculus for Computing

1 L'Hopital Rule

- Suppose that $f(a) = g(a) = 0$, that $f()$ and $g()$ are differentiable on an open interval I containing a and that $g'(x) \neq 0$ on I if $x \neq a$, then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

2 Proof

- We first establish the limit equation for the case $x \to a^+$. The method needs almost no change to apply to $x \to a^-$, and the combination of these two cases establishes the result.
- Suppose that x lies to the right of a. Then $g'(x) \neq 0$, and we can apply **Cauchy's Mean Value Theorem** to the closed interval from a to x. This step produces a number c between a and x such that:

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(a)}{g(x) - g(a)}$$

- But $f(a) = g(a) = 0$, so:

$$\frac{f'(c)}{g'(c)} = \frac{f(x)}{g(x)}$$

- As x approaches a, c approaches a because it always lies between a and x. Therefore,

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{c \to a^+} \frac{f'(c)}{g'(c)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)},$$

which establishes L'Hopital's Rule for the case where x approaches a from above. The proof of the case where x approaches a from below is intentionally left for students' exercise. ■

3 Example

1. Find $\lim_{x \to 0} \frac{3x - \sin(x)}{x}$

$$\begin{aligned} \lim_{x \to 0} \frac{3x - \sin(x)}{x} &= \lim_{x \to 0} \frac{\frac{d}{dx}(3x - \sin(x))}{\frac{d}{dx}x} \\ &= \lim_{x \to 0} \frac{3 - \cos(x)}{1} \\ &= 3 - \cos(0) \\ &= 3 - 1 \\ &= 2 \qquad \blacksquare \end{aligned}$$

2. Find $\lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x}$

$$\begin{aligned} \lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x} &= \lim_{x \to 0} \frac{\frac{d}{dx}(\sqrt{1+x} - 1)}{\frac{d}{dx}x} \\ &= \lim_{x \to 0} \frac{d}{dx}((1+x)^{\frac{1}{2}} - 1) \\ &= \lim_{x \to 0} \frac{1}{2}(1+x)^{-\frac{1}{2}} - 0 \\ &= \lim_{x \to 0} \frac{1}{2\sqrt{1+x}} \\ &= \frac{1}{2\sqrt{1+0}} \\ &= \frac{1}{2} \qquad \blacksquare \end{aligned}$$

3. Find $\lim_{x \to 0} \frac{x - \sin(x)}{x^3}$

$$\begin{aligned} \lim_{x \to 0} \frac{x - \sin(x)}{x^3} &= \lim_{x \to 0} \frac{1 - \cos(x)}{3x^2} \\ &= \lim_{x \to 0} \frac{\sin(x)}{6x} && \because \text{Apply L'Hopital again} \\ &= \lim_{x \to 0} \frac{\cos(x)}{6} && \because \text{Apply L'Hopital again} \\ &= \frac{\cos(0)}{6} \\ &= \frac{1}{6} \qquad \blacksquare \end{aligned}$$