Chapter 1 Exercise Set

Introduction to Systems of Linear Equations

1. In each part, determine whether the equation is linear in x_1, x_2, and x_3:
   (a) $3x_1 - \sqrt{3}x_2 + x_3 = 0$
   (b) $-2x_1 - 4x_2 + x_2x_3 = 5$
   (c) $3x_1 = 5x_2 - 7x_3$
   (d) $x_1^{-1} - 3x_2 + 4x_3 = 17$
   (e) $4x_1 + x_2^{2/7} - 3x_3 = -1$
   (f) $\sqrt{17}x_1 - \pi x_2 + 1.4x_3 = 14^{1/5}$

2. In each part, determine whether the equations form a linear system.
   (a) $\begin{aligned} -2x + 4y + z &= 2 \\ 3x - \frac{2}{y} &= 0 \end{aligned}$
   (b) $\begin{aligned} x &= 4 \\ 2x &= 8 \end{aligned}$
   (c) $\begin{aligned} 4x - y + 2z &= -1 \\ -x + (\ln 2)y - 3z &= 0 \end{aligned}$
   (d) $\begin{aligned} 3z + x &= -4 \\ y + 5z &= 1 \\ 6x + 2z &= 3 \\ -x - y - z &= 4 \end{aligned}$

3. In each part, determine whether the equations form a linear system.
   (a) $\begin{aligned} x_1 - x_2 + x_3 &= \cos(\pi) \\ 3x_1 - x_2 \quad x_3 &= 2 \end{aligned}$
   (b) $\begin{aligned} 5y + w &= 1 \\ 2x + 5y - 4z + w &= 1 \end{aligned}$
   (c) $\begin{aligned} 7x_1 - x_2 + 2x_3 &= 0 \\ 2x_1 + x_2 - x_3x_4 &= 3 \\ -x_1 + 5x_2 - x_4 &= -1 \end{aligned}$
   (d) $x_1 + x_2 = x_3 + x_4$

4. For each system in Exercise 2 that is linear, determine whether it is consistent.

5. For each system in Exercise 3 that is linear, determine whether it is consistent.

6. Write a system of linear equations consisting of three equations in three unknowns with
   (a) no solutions.
   (b) exactly one solution.
   (c) infinitely many solutions.

7. In each part, determine whether the given vector is a solution of the linear system

$$\begin{aligned} 3x_1 + 2x_2 - 2x_3 &= 1 \\ 2x_1 - x_2 + x_3 &= 2 \\ x_1 + 3x_2 - 3x_3 &= -1 \end{aligned}$$

   (a) $(5, -4, 0)$ (b) $(\frac{5}{7}, \frac{-4}{7}, 0)$ (c) $(3, -2, 2)$
   (d) $(\frac{5}{7}, \frac{3}{7}, 1)$ (e) $(-3, 0, -5)$

8. In each part, determine whether the given vector is a solution of the linear system

$$\begin{aligned} x_1 + 2x_2 - 2x_3 &= 3 \\ 3x_1 - x_2 + x_3 &= 1 \\ -x_1 + 5x_2 - 5x_3 &= 5 \end{aligned}$$

   (a) $(\frac{5}{7}, \frac{8}{7}, 1)$ (b) $(\frac{5}{7}, \frac{8}{7}, 0)$ (c) $(5, 8, 1)$
   (d) $(\frac{5}{7}, \frac{10}{7}, \frac{2}{7})$ (e) $(\frac{5}{7}, \frac{22}{7}, 2)$

9. In each part, find the solution set of the linear equation by using parameters as necessary.
   (a) $2x + 4y = 3$
   (b) $3x_1 - 5x_2 + x_3 + 4x_4 = 9$

10. In each part, find the solution set of the linear equation by using parameters as necessary.
    (a) $3x_1 - 5x_2 + 4x_3 = 7$
    (b) $3v - 8w + 2x - y + 4z = 0$

11. In each part, find a system of linear equations corresponding to the given augmented matrix.
    (a) $\begin{bmatrix} 2 & 5 & 6 \\ 0 & 1 & 2 \\ -1 & 0 & 0 \end{bmatrix}$
    (b) $\begin{bmatrix} 3 & 0 & -2 & 5 \\ 7 & 1 & 4 & -3 \\ 0 & -2 & 1 & 7 \end{bmatrix}$
    (c) $\begin{bmatrix} 1 & 5 & 7 & -1 & 3 \\ 2 & 2 & 1 & 1 & 0 \end{bmatrix}$
    (d) $\begin{bmatrix} 1 & 0 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$

12. In each part, find a system of linear equations corresponding to the given augmented matrix.
    (a) $\begin{bmatrix} 2 & -2 \\ -3 & 4 \\ 3 & -2 \\ 4 & 0 \end{bmatrix}$
    (b) $\begin{bmatrix} 0 & 4 & 2 & -1 & -2 \\ -6 & 0 & -3 & 5 & -2 \end{bmatrix}$
    (c) $\begin{bmatrix} 1 & 3 & 5 & 7 \\ -7 & -5 & -3 & -1 \\ -5 & 6 & -1 & -1 \\ 8 & 0 & 0 & -2 \end{bmatrix}$
    (d) $\begin{bmatrix} -3 & 0 & 1 & -4 & -2 \\ 4 & 0 & -4 & -1 & 2 \\ 1 & -3 & 0 & -4 & 0 \\ 0 & 0 & 0 & 2 & 3 \end{bmatrix}$

13. In each part, find the augmented matrix for the given system of linear equations.

(a) $\begin{aligned} -2x_1 &= 6 \\ 3x_1 &= 8 \\ 9x_1 &= -3 \end{aligned}$ (b) $\begin{aligned} 3x_1 \qquad - x_3 + 6x_4 &= 0 \\ 2x_2 - x_3 - 5x_4 &= -2 \end{aligned}$

(c) $\begin{aligned} 2x_2 \qquad - 3x_4 + x_5 &= 0 \\ -3x_1 - x_2 + x_3 \qquad\qquad &= -1 \\ 6x_1 + 2x_2 - x_3 + 2x_4 - 3x_5 &= 6 \end{aligned}$

(d) $\begin{aligned} x_1 - x_3 &= 4 \\ x_2 + x_4 &= 9 \end{aligned}$

14. In each part, find the augmented matrix for the given system of linear equations.

(a) $\begin{aligned} 3x_1 - 2x_2 &= -1 \\ 4x_1 + 5x_2 &= 3 \\ 7x_1 + 3x_2 &= 2 \end{aligned}$ (b) $\begin{aligned} 2x_1 \qquad + 2x_3 &= 1 \\ 3x_1 - x_2 + 4x_3 &= 7 \\ 6x_1 + x_2 - x_3 &= 0 \end{aligned}$

(c) $\begin{aligned} x_1 + 2x_2 \qquad - x_4 + x_5 &= 1 \\ 3x_2 + x_3 \qquad - x_5 &= 2 \\ x_3 + 7x_4 \qquad &= 1 \end{aligned}$

(d) $\begin{aligned} x_1 \qquad\qquad &= 1 \\ x_2 \qquad &= 2 \\ x_3 &= 3 \end{aligned}$

Gaussian Elimination

15. In each part, determine whether the matrix is in row echelon form, reduced row echelon form, both, or neither.

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 6 & 3 & 4 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} 0 & 1 & 3 & 4 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ (g) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

16. In each part, determine whether the matrix is in row echelon form, reduced row echelon form, both, or neither.

(a) $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(f) $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ (g) $\begin{bmatrix} 1 & 2 & 4 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$

17. In each part, suppose that the augmented matrix for a system of linear equations has been reduced by row operations to the given row echelon form. Solve the system.

(a) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 6 & 3 & -4 \\ 0 & 1 & 3 & 7 & 2 \\ 0 & 0 & 1 & 1 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & -3 & 2 & 0 & 6 & 1 \\ 0 & 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

18. In each part, suppose that the augmented matrix for a system of linear equations has been reduced by row operations to the given row echelon form. Solve the system.

(a) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 & 3 & 2 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -4 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & 0 & 2 & -2 \\ 0 & 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

► In Exercises 19–22, solve the linear system by Gauss–Jordan elimination. ◄

19. $\begin{aligned} x_1 + 2x_2 - 3x_3 &= 6 \\ 2x_1 - x_2 + 4x_3 &= 1 \\ x_1 - x_2 + x_3 &= 3 \end{aligned}$

20. $\begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 4 \\ -2x_1 + 5x_2 + 2x_3 &= 1 \\ 8x_1 + x_2 + 4x_3 &= 11 \end{aligned}$

21. $\begin{aligned} 3x - y + z + 7w &= 13 \\ -2x + y - z - 3w &= -9 \\ -2x + y \qquad - 7w &= -8 \end{aligned}$

22. $\begin{aligned} -2y + 3z &= 3 \\ 3x + 6y - 3z &= -2 \\ 6x + 6y + 3z &= 4 \end{aligned}$

► In Exercises 23–26, solve the linear system by Gaussian elimination. ◄

23. Exercise 19

24. Exercise 20

25. Exercise 21

26. Exercise 22

► In Exercises 27–30, determine whether the homogeneous system has nontrivial solutions by inspection (without pencil and paper). ◄

27. $$\begin{aligned} 3x_1 + 2x_2 - x_3 + 6x_4 &= 0 \\ 2x_1 \qquad - 5x_3 - x_4 &= 0 \\ -6x_1 - 2x_2 + 3x_3 - 3x_4 &= 0 \end{aligned}$$

28. $$\begin{aligned} 4x_1 - 3x_2 - x_3 &= 0 \\ 3x_2 - 5x_3 &= 0 \\ 3x_3 &= 0 \end{aligned}$$

29. $$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= 0 \end{aligned}$$

30. $$\begin{aligned} 3x_1 - 2x_2 &= 0 \\ 6x_1 - 4x_2 &= 0 \end{aligned}$$

In Exercises 31–37, solve the given linear system by any method.

31. $$\begin{aligned} 2x + y + 4z &= 0 \\ 3x + y + 6z &= 0 \\ 4x + y + 9z &= 0 \end{aligned}$$

32. $$\begin{aligned} 3x + y - z &= 0 \\ -x + 2y - 2z &= 0 \\ x + y - z &= 0 \end{aligned}$$

33. $$\begin{aligned} x_1 - x_2 + 7x_3 + x_4 &= 0 \\ x_1 + 2x_2 - 6x_3 - x_4 &= 0 \end{aligned}$$

34. $$\begin{aligned} v - 2w + 2x &= 0 \\ 2u - v + 4w - 3x &= 0 \\ 4u - v + 6w - 4x &= 0 \\ -2u + 2v - 6w + 5x &= 0 \end{aligned}$$

35. $$\begin{aligned} 3w + 3x \qquad + 5z &= 0 \\ -x + y - 3z &= 0 \\ 2w - x + 3y - z &= 0 \\ -3w + x - 4y + 5z &= 0 \end{aligned}$$

36. $$\begin{aligned} x_1 + 3x_2 \qquad - x_4 &= 0 \\ -x_1 + 4x_2 + 2x_3 \qquad &= 0 \\ -x_2 - x_3 - x_4 &= 0 \\ 2x_1 - 4x_2 + x_3 + x_4 &= 0 \\ x_1 - 2x_2 - x_3 + x_4 &= 0 \end{aligned}$$

37. $$\begin{aligned} 4I_1 + 3I_2 - 2I_3 - I_4 &= 0 \\ -I_2 + 6I_3 - 4I_4 &= 0 \\ -2I_1 - I_2 \qquad + I_4 &= 0 \\ -I_1 + I_2 + I_3 - I_4 &= 0 \end{aligned}$$

In Exercises 38–41, determine the values of a for which the system has no solutions, exactly one solution, or infinitely many solutions.

38. $$\begin{aligned} x + 2y + z &= 2 \\ 2x - 2y + 3z &= 1 \\ x + 2y - az &= a \end{aligned}$$

39. $$\begin{aligned} x + 2y + \qquad z &= 2 \\ 2x - 2y + \qquad 3z &= 1 \\ x + 2y - (a^2 - 3)z &= a \end{aligned}$$

40. $$\begin{aligned} x + 2y - \qquad 3z &= 4 \\ 3x - y + \qquad 5z &= 2 \\ 4x + y + (a^2 - 2)z &= a + 4 \end{aligned}$$

41. $$\begin{aligned} x + y + \qquad 7z &= -7 \\ 2x + 3y + \qquad 17z &= 11 \\ x + 2y + (a^2 + 1)z &= 6a \end{aligned}$$

In Exercises 42–43, solve the following systems, where a, b, and c are constants.

42. $$\begin{aligned} 2x - y &= 0 \\ 3x + 2y &= 0 \end{aligned}$$

43. $$\begin{aligned} x_1 + x_2 + 2x_3 &= a \\ 2x_1 \qquad + x_3 &= b \\ x_2 + 3x_3 &= c \end{aligned}$$

44. Find two different row echelon forms of

$$\begin{bmatrix} 1 & 4 \\ 3 & 11 \end{bmatrix}$$

This exercise shows that a matrix can have multiple row echelon forms.

45. Let

$$\begin{bmatrix} a & 0 & b & 2 \\ a & a & 4 & 4 \\ 0 & a & 2 & b \end{bmatrix}$$

be the augmented matrix for a linear system. Find for what values of a and b the system has

(a) a unique solution.

(b) a one-parameter solution.

(c) a two-parameter solution. (d) no solution.

46. For which value(s) of a does the following system have zero solutions? One solution? Infinitely many solutions?

$$\begin{aligned} x_1 + x_2 + x_3 &= 4 \\ x_3 &= 2 \\ (a^2 - 4)x_3 &= a - 2 \end{aligned}$$

47. Solve the following system of nonlinear equations for the unknown angles α, β, and γ, where $0 \le \alpha \le \frac{\pi}{2}$, $0 \le \beta < 2\pi$, and $0 \le \gamma \le \pi$.

$$\begin{aligned} 2\sin\alpha + \cos\beta - \tan\gamma &= 1 \\ -4\sin\alpha + \cos\beta + \tan\gamma &= 0 \\ -2\sin\alpha + 3\cos\beta + 2\tan\gamma &= 4 \end{aligned}$$

48. Solve the following system of nonlinear equations for x, y, and z.

$$\begin{aligned} 2x^2 + y^2 - 3z^2 &= -8 \\ x^2 - y^2 + 2z^2 &= 7 \\ x^2 + 2y^2 - z^2 &= 1 \end{aligned}$$

49. Find positive integers that satisfy

$$\begin{aligned} x + y + z &= 9 \\ x + 5y + 10z &= 44 \end{aligned}$$

50. Find values of a, b, and c such that the graph of the polynomial $p(x) = ax^2 + bx + c$ passes through the points $(1, 2)$, $(-1, 6)$, and $(2, 3)$.

51. Use Gauss–Jordan elimination to solve for x' and y' in terms of x and y.

$$\begin{aligned} x &= \tfrac{3}{5}x' - \tfrac{4}{5}y' \\ y &= \tfrac{4}{5}x' + \tfrac{3}{5}y' \end{aligned}$$

52. Use Gauss–Jordan elimination to solve for x' and y' in terms of x and y.

$$\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta \end{aligned}$$

53. (a) If A is a 4×6 matrix, what is the maximum possible number of leading 1's in its reduced row echelon form?

(b) If B is a 4×7 matrix whose last column has all zeros, what is the maximum possible number of parameters in the general solution of the linear system with augmented matrix B?

(c) If C is a 6×3 matrix, what is the minimum possible number of rows of zeros in any row echelon form of C?

54. (*Calculus required*) Find values of a, b, and c such that the graph of $p(x) = ax^2 + bx + c$ passes through the point $(-1, 0)$ and has a horizontal tangent at $(2, -9)$.

55. (a) Find a system of two linear equations in the variables x, y, and z whose solutions are given parametrically by $x = 3 + t$, $y = t$, and $z = 7 - 2t$.

(b) Find another parametric solution to the same system in which the parameter is r, and $x = r$.

56. Let A be a 3×3 matrix. Express the following sequence of row operations on A in a simpler form:

Add the first row to the third row
Subtract the third row from the first row
Add the first row to the third row
Multiply the first row by -1.

Matrices and Matrix Operations

57. Suppose that A, B, C, D, and E are matrices with the following sizes:

A	B	C	D	E
(5×6)	(5×6)	(6×3)	(5×3)	(6×5)

In each part, determine whether the given matrix expression is defined. For those that are defined, give the size of the resulting matrix.

(a) BA (b) $AC + D$ (c) $B + EA$
(d) $B + AB$ (e) $E(B + A)$ (f) $(EA)C$
(g) $A^T E$ (h) $D^T(A + E^T)$

58. Suppose that A, B, C, D, and E are matrices with the following sizes:

A	B	C	D	E
(4×1)	(4×5)	(5×3)	(3×5)	(1×4)

In each part, determine whether the given matrix expression is defined. For those that are defined, give the size of the resulting matrix.

(a) AE (b) $A^T B$ (c) $B^T(E^T + A)$
(d) $3B + D$ (e) $B(C + D^T)$ (f) $(EB)^T + CD$
(g) $C(DB^T)$ (h) $EA + DC$

59. Consider the matrices

$$A = \begin{bmatrix} 2 & 0 \\ -4 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -7 & 2 \\ 5 & 3 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & 9 \\ -3 & 0 \\ 2 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} -2 & 1 & 8 \\ 3 & 0 & 2 \\ 4 & -6 & 3 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 3 & 0 \\ -5 & 1 & 1 \\ 7 & 6 & 2 \end{bmatrix}$$

In each part, compute the given expression (where possible).

(a) $D + E$ (b) $D - E$ (c) $5A$
(d) $-9D$ (e) $2B - C$ (f) $7E - 3D$
(g) $2(D + 5E)$ (h) $B - B$ (i) $\operatorname{tr}(D)$
(j) $\operatorname{tr}(D - E)$ (k) $2\operatorname{tr}(4B)$ (l) $\operatorname{tr}(A)$

60. Using the matrices in Exercise 59, in each part compute the given expression (where possible).

(a) AB (b) BA (c) $(3E)D$
(d) $(AB)C$ (e) $A(BC)$ (f) CC^T
(g) $(DC)^T$ (h) $(C^T B)A^T$ (i) $\operatorname{tr}(DD^T)$
(j) $\operatorname{tr}(4E^T - D)$ (k) $\operatorname{tr}(A^T C^T + 2E^T)$ (l) $\operatorname{tr}((E^T C)B)$

▶ In Exercises 61–64 the given matrix represents an augmented matrix for a linear system. Write the corresponding set of linear equations for the system, and use Gaussian elimination to solve the linear system. Introduce free parameters as necessary. ◀

61. $\begin{bmatrix} 3 & -1 & 0 & 4 & 1 \\ 2 & 0 & 3 & 3 & -1 \end{bmatrix}$

62. $\begin{bmatrix} 1 & 4 & -1 \\ -2 & -8 & 2 \\ 3 & 12 & -3 \\ 0 & 0 & 0 \end{bmatrix}$

63. $\begin{bmatrix} 2 & -4 & 1 & 6 \\ -4 & 0 & 3 & -1 \\ 0 & 1 & -1 & 3 \end{bmatrix}$

64. $\begin{bmatrix} 3 & 1 & -2 \\ -9 & -3 & 6 \\ 6 & 2 & 1 \end{bmatrix}$

65. Let

$$A = \begin{bmatrix} 3 & 2 & -5 \\ 4 & 5 & 6 \\ 2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -6 & 2 & 3 \\ 0 & -1 & 2 \\ 5 & 5 & 4 \end{bmatrix}$$

Use the row method or column method (as appropriate) to find

(a) the first row of AB (b) the third row of AB
(c) the second column of AB (d) the first column of BA.
(e) the third row of AA (f) the third column of AA

66. Referring to the matrices in Exercise 65, use the row method or column method (as appropriate) to find

(a) the first column of AB. (b) the third column of BB.

(c) the second row of BB. (d) the first column of AA.

(e) the third column of AB. (f) the first row of BA.

67. Referring to the matrices in Exercise 65 and Example 9 of Section 1.3,

(a) express each column vector of AA as a linear combination of the column vectors of A.

(b) express each column vector of BB as a linear combination of the column vectors of B.

68. Referring to the matrices in Exercise 65 and Example 9 of Section 1.3,

(a) express each column vector of AB as a linear combination of the column vectors of A.

(b) express each column vector of BA as a linear combination of the column vectors of B.

69. In each part, find matrices A, $\mathbf{x}$, and $\mathbf{b}$ that express the given system of linear equations as a single matrix equation $A\mathbf{x} = \mathbf{b}$, and write out this matrix equation.

(a) $\begin{aligned} 5x + y + z &= 2 \\ 2x \quad\quad + 3z &= 1 \\ x + 2y \quad\quad &= 0 \end{aligned}$ (b) $\begin{aligned} x_1 + x_2 - x_3 - 7x_4 &= 6 \\ -x_2 + 4x_3 + x_4 &= 1 \\ 4x_1 + 2x_2 + x_3 + 8x_4 &= 0 \end{aligned}$

70. In each part, find matrices A, $\mathbf{x}$, and $\mathbf{b}$ that express the given system of linear equations as a single matrix equation $A\mathbf{x} = \mathbf{b}$, and write out this matrix equation.

(a) $\begin{aligned} 2x_1 - x_2 + 3x_3 &= 4 \\ x_1 + 3x_2 \quad\quad &= -2 \\ 2x_2 - x_3 &= 1 \\ -x_1 \quad\quad + 2x_3 &= 0 \end{aligned}$

(b) $\begin{aligned} 4x_1 + 4x_2 + 4x_3 &= 4 \\ -2x_2 - 3x_2 - x_3 &= 0 \\ 4x_2 - 2x_3 &= -2 \end{aligned}$

71. In each part, express the matrix equation as a system of linear equations.

(a) $\begin{bmatrix} 5 & 6 & -7 \\ -1 & -2 & 3 \\ 0 & 4 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 5 & -3 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -9 \end{bmatrix}$

72. In each part, express the matrix equation as a system of linear equations.

(a) $\begin{bmatrix} 4 & 0 & -1 \\ 3 & 2 & 4 \\ -2 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & -1 & 1 & 3 \\ 4 & -1 & 0 & 2 \\ -2 & 1 & 3 & -2 \\ 2 & -5 & -1 & -6 \end{bmatrix}\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

▶ In Exercises 73–74, find all values of k, if any, that satisfy the equation. ◀

73. $\begin{bmatrix} k & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & -3 \end{bmatrix}\begin{bmatrix} k \\ 1 \\ 1 \end{bmatrix} = 0$

74. $\begin{bmatrix} 3 & 3 & k \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 4 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \\ k \end{bmatrix}$

▶ In Exercises 75–76, solve the matrix equation for a, b, c, and d.

75. $\begin{bmatrix} 3 & a \\ 1 & a+b \end{bmatrix} = \begin{bmatrix} b & c-2d \\ c+2d & 0 \end{bmatrix}$

76. $\begin{bmatrix} a-b & b+a \\ 4d+c & 2d-2c \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 7 & 6 \end{bmatrix}$

77. Let A be any $m \times n$ matrix and let 0 be the $m \times n$ matrix each of whose entries is zero. Show that if $kA = 0$, either $k = 0$ or $A = 0$.

78. Show that if a square matrix A satisfies

$$A^3 + 4A^2 - 2A + 7I = 0$$

then so does A^T.

79. Prove: If A is an $m \times n$ matrix and B is the $n \times 1$ matrix each of whose entries is $1/n$, then

$$AB = \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \vdots \\ \bar{r}_m \end{bmatrix}$$

where $\bar{r}_i$ is the average of the entries in the ith row of A.

80. (a) Show that if B is any matrix with a column of zeros and A is any matrix for which AB is defined, then AB also has a column of zeros.

(b) Find a similar result involving a row of zeros.

81. Find the 4×4 matrix $A = [a_{ij}]$ whose entries satisfy the stated condition.

(a) $a_{ij} = i - j$ (b) $a_{ij} = (-1)^i ij$

(c) $a_{ij} = \begin{cases} 0 & |i-j| \geq 1 \\ -1 & |i-j| < 1 \end{cases}$

82. Consider the function $y = f(x)$ defined for 2×1 matrices x by $y = Ax$, where

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Plot $f(x)$ together with x in each case below. How would you describe the action of f?

(a) $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (b) $x = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

(c) $x = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ (d) $x = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$

83. Let I be the $n \times n$ matrix whose entry in row i and column j is

$$\begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

Show that $AI = IA = A$ for every $n \times n$ matrix A.

84. How many 3×3 matrices A can you find such that

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} xy \\ y \\ z \end{bmatrix}$$

for all choices of x, y, and z? (Note that A may also depend on x, y, and z).

85. If A and B are $n \times n$ matrices, then

(a) $\operatorname{tr}(cA) = c \operatorname{tr}(A)$ where c is a real number,

(b) $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

86. Show that there are no 2×2 matrices A and B with $AB - BA$ equal to the 2×2 identity matrix I. [*Hint:* use the previous exercise.]

Inverses; Algebraic Properties of Matrices

87. Let

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 5 \\ -2 & 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 8 & -3 & -5 \\ 0 & 1 & 2 \\ 4 & -7 & 6 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & -2 & 3 \\ 1 & 7 & 4 \\ 3 & 5 & 9 \end{bmatrix}, \quad a = 4, \quad b = -7$$

Show that

(a) $A + (B + C) = (A + B) + C$

(b) $(AB)C = A(BC)$ (c) $(a + b)C = aC + bC$

(d) $a(B - C) = aB - aC$

88. Using the matrices and scalars in Exercise 87, verify that

(a) $a(BC) = (aB)C = B(aC)$

(b) $A(B - C) = AB - AC$ (c) $(B + C)A = BA + CA$

(d) $a(bC) = (ab)C$

89. Using the matrices and scalars in Exercise 87, verify that

(a) $(B^T)^T = B$ (b) $(A + C)^T = A^T + C^T$

(c) $(bA)^T = bA^T$ (d) $(CA)^T = A^T C^T$

▶ In Exercises 90–93, use Theorem 1.4.5 to compute the inverses of the following matrices. ◀

90. $A = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ 91. $B = \begin{bmatrix} 6 & 3 \\ -5 & -2 \end{bmatrix}$

92. $C = \begin{bmatrix} 4 & 9 \\ 1 & 3 \end{bmatrix}$ 93. $D = \begin{bmatrix} 0 & -3 \\ 7 & 2 \end{bmatrix}$

94. Find the inverse of

$$\begin{bmatrix} \frac{1}{2}(e^x - e^{-x}) & \frac{1}{2}(e^x + e^{-x}) \\ \frac{1}{2}(e^x + e^{-x}) & \frac{1}{2}(e^x - e^{-x}) \end{bmatrix}$$

95. Use the matrix C in Exercise 92 to verify that $(A^T)^{-1} = (A^{-1})^T$.

96. Use the matrices A and B in Exercises 90 and 91 to verify that $(AB)^{-1} = B^{-1}A^{-1}$.

97. Use the matrices A, B, and C in Exercises 90–92 to verify that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

▶ In Exercises 98–101, use the given information to find A. ◀

98. $A^{-1} = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$ 99. $(5A)^{-1} = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$

100. $(3A^T)^{-1} = \begin{bmatrix} -5 & 1 \\ -9 & 2 \end{bmatrix}$ 101. $(I + 2A)^{-1} = \begin{bmatrix} -1 & 3 \\ 4 & 5 \end{bmatrix}$

102. How should the coefficients a, b, and c be chosen so that the system

$$\begin{aligned} ax + by - 3z &= -3 \\ -2x - by + cz &= -1 \\ ax + 3y - cz &= -3 \end{aligned}$$

has the solution $x = 1$, $y = -1$, and $z = 2$?

103. Let A be the matrix

$$\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

In each part, compute the given quantity.

(a) A^3 (b) A^{-3} (c) $A^2 - 2A + I$

(d) $p(A)$, where $p(x) = x - 2$

(e) $p(A)$, where $p(x) = 2x^2 - x + 1$

(f) $p(A)$, where $p(x) = x^3 - 2x + 4$

104. Repeat Exercise 103 for the matrix

$$A = \begin{bmatrix} 3 & 0 & -1 \\ 0 & -2 & 0 \\ 5 & 0 & 2 \end{bmatrix}$$

105. Repeat Exercise 103 for the matrix

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & -3 & -1 \end{bmatrix}$$

106. Let $p_1(x) = x^2 - 9$, $p_2(x) = x + 3$, and $p_3(x) = x - 3$. Show that $p_1(A) = p_2(A)p_3(A)$ for the matrix A in Exercise 105.

107. Show that if $p(x) = x^2 - (a + d)x + (ad - bc)$ and

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then $p(A) = 0$.

108. Show that if $p(x) = x^3 - (a + b + c)x^2 + (ab + ae + be - cd)x - a(be - cd)$ and

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & d & e \end{bmatrix}$$

then $p(A) = 0$.

109. Consider the matrix

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

where $a_{11}a_{22} \cdots a_{nn} \neq 0$. Show that A is invertible and find its inverse.

110. Show that if a square matrix A satisfies the equation $A^2 + 5A - 2I = 0$, then $A^{-1} = \frac{1}{2}(A + 5I)$.

111. (a) Show that a matrix with a row of zeros cannot have an inverse.

(b) Show that a matrix with a column of zeros cannot have an inverse.

112. Assuming that all matrices are $n \times n$ and invertible, solve for D.

$$ABC^TDBA^TC = AB^T$$

113. Assuming that all matrices are $n \times n$ and invertible, solve for D.

$$C^TB^{-1}A^2BAC^{-1}DA^{-2}B^TC^{-2} = C^T$$

114. If A is a square matrix and n is a positive integer, is it true that $(A^n)^T = (A^T)^n$? Justify your answer.

115. Simplify:

$$D^{-1}CBA(BA)^{-1}C^{-1}(C^{-1}D)^{-1}$$

▶ In Exercises 116–117, determine whether A is invertible, and if so, find the inverse. [*Hint:* Solve $AX = I$ for X by equating corresponding entries on the two sides.] ◀

116. $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

117. $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

▶ In Exercises 118–121, use the method of Example 8 of Section 1.4 to find the unique solution of the given linear system. ◀

118. $\begin{aligned} 3x_1 + 2x_2 &= 1 \\ 4x_1 - 5x_2 &= 2 \end{aligned}$

119. $\begin{aligned} x_1 + 3x_2 &= 0 \\ 2x_1 - 5x_2 &= 3 \end{aligned}$

120. $\begin{aligned} 7x_1 + 2x_2 &= 3 \\ 3x_1 + x_2 &= 0 \end{aligned}$

121. $\begin{aligned} 3x_1 - 2x_2 &= 6 \\ -x_1 + 4x_2 &= 1 \end{aligned}$

122. Prove: If B is invertible, then $AB^{-1} = B^{-1}A$ if and only if $AB = BA$.

123. Prove: If A is invertible, then $A + B$ and $I + BA^{-1}$ are both invertible or both not invertible.

124. Find a matrix K such that $AKB = C$ given that

$$A = \begin{bmatrix} 1 & 4 \\ -2 & 3 \\ 1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix},$$

$$C = \begin{bmatrix} 8 & 6 & -6 \\ 6 & -1 & 1 \\ -4 & 0 & 0 \end{bmatrix},$$

125. (a) Show that if A, B, and $A + B$ are invertible matrices with the same size, then

$$A(A^{-1} + B^{-1})B(A + B)^{-1} = I$$

(b) What does the result in part (a) tell you about the matrix $A^{-1} + B^{-1}$?

126. A square matrix A is said to be ***idempotent*** if $A^2 = A$.

(a) Show that if A is idempotent, then so is $I - A$.

(b) Show that if A is idempotent, then $2A - I$ is invertible and is its own inverse.

127. Show that if A is a square matrix such that $A^k = 0$ for some positive integer k, then the matrix A is invertible and

$$(I - A)^{-1} = I + A + A^2 + \cdots + A^{k-1}$$

Elementary Matrices and a Method for Finding A^{-1}

128. Decide whether each matrix below is an elementary matrix.

(a) $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix}$

129. Decide whether each matrix below is an elementary matrix.

(a) $\begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

130. Find a row operation and the corresponding elementary matrix that will restore the given elementary matrix to the identity matrix.

(a) $\begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 9 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

131. Find a row operation and the corresponding elementary matrix that will restore the given elementary matrix to the identity matrix.

(a) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

132. In each part, an elementary matrix E and a matrix A are given. Write down the row operation corresponding to E and show that the product EA results from applying the row operation to A.

(a) $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $A = \begin{bmatrix} 1 & -3 & 4 & 0 \\ -2 & 5 & 1 & -1 \end{bmatrix}$

(b) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$,

$A = \begin{bmatrix} -2 & 1 & 0 & 3 & 3 \\ 1 & -3 & 0 & 2 & 6 \\ 3 & 0 & -1 & 2 & 2 \end{bmatrix}$

(c) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 4 & 2 \\ 5 & 1 \\ -1 & 3 \end{bmatrix}$

133. In each part, an elementary matrix E and a matrix A are given. Write down the row operation corresponding to E and show that the product EA results from applying the row operation to A.

(a) $E = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 2 & -5 & 1 \\ -3 & 6 & 6 & 6 \end{bmatrix}$

(b) $E = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $A = \begin{bmatrix} -2 & -1 & 0 & 3 & 3 \\ -1 & 2 & 0 & 5 & 3 \\ 2 & 0 & 1 & 3 & 1 \end{bmatrix}$

(c) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $A = \begin{bmatrix} -1 & 5 \\ -2 & 4 \\ -3 & 7 \end{bmatrix}$

In Exercises 134–135, use the following matrices.

$$A = \begin{bmatrix} 2 & 6 & -8 \\ 0 & 5 & 3 \\ 4 & 7 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 6 & -8 \\ 0 & 5 & 3 \\ 0 & -5 & 25 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 3 & -4 \\ 0 & 5 & 3 \\ 4 & 7 & 9 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 6 & -8 \\ 0 & 0 & 28 \\ 0 & -5 & 25 \end{bmatrix}$$

$$F = \begin{bmatrix} 2 & 6 & -8 \\ 6 & 23 & -21 \\ 0 & -5 & 25 \end{bmatrix}$$

134. Find an elementary matrix E that satisfies the equation.

(a) $EA = B$ (b) $EB = A$

(c) $EA = C$ (d) $EC = A$

135. Find an elementary matrix E that satisfies the equation.

(a) $EB = D$ (b) $ED = B$

(c) $EB = F$ (d) $EF = B$

In Exercises 136–150, use the inversion algorithm to find the inverse of the given matrix, if the inverse exists.

136. $\begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}$ **137.** $\begin{bmatrix} -3 & 6 \\ 4 & 5 \end{bmatrix}$ **138.** $\begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}$

139. $\begin{bmatrix} 6 & -4 \\ -3 & 2 \end{bmatrix}$ **140.** $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 6 & 4 \\ 0 & -2 & 2 \end{bmatrix}$

141. $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$ **142.** $\begin{bmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{bmatrix}$

143. $\begin{bmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{bmatrix}$

144. $\begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

145. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3\sqrt{2} & \sqrt{2} \\ 0 & \sqrt{2} & -4\sqrt{2} \end{bmatrix}$

146. $\begin{bmatrix} 1 & 4 & 4 \\ 1 & 2 & 4 \\ 1 & 3 & 2 \end{bmatrix}$

147. $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 0 \\ 1 & 4 & 7 \end{bmatrix}$

148. $\begin{bmatrix} -8 & 17 & 2 & \frac{1}{3} \\ 4 & 0 & \frac{2}{5} & -9 \\ 0 & 0 & 0 & 0 \\ -1 & 13 & 4 & 2 \end{bmatrix}$

149. $\begin{bmatrix} -1 & 0 & 1 & 0 \\ 2 & 3 & -2 & 6 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}$

150. $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 3 & 0 \\ 2 & 1 & 4 & -3 \end{bmatrix}$

151. Find the inverse of each of the following 3×3 matrices, where k_1, k_2, k_3, k_4, and k are all nonzero.

(a) $\begin{bmatrix} 0 & 0 & k_1 \\ 0 & k_2 & 0 \\ k_3 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} k & 1 & 0 \\ 0 & k & 1 \\ 0 & 0 & k \end{bmatrix}$

▶ In Exercises 152–153, find all values of c, if any, for which the given matrix is invertible. ◀

152. $\begin{bmatrix} c & -c & c \\ 1 & c & 1 \\ 0 & 0 & c \end{bmatrix}$

153. $\begin{bmatrix} c & 2 & 0 \\ 1 & c & 2 \\ 0 & 1 & c \end{bmatrix}$

▶ In Exercises 154–156, write the given matrix as a product of elementary matrices. ◀

154. $\begin{bmatrix} -2 & 3 \\ 1 & 0 \end{bmatrix}$

155. $\begin{bmatrix} 1 & 0 \\ 3 & -4 \end{bmatrix}$

156. $\begin{bmatrix} 1 & 0 & 3 \\ 0 & -2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

▶ In Exercises 157–159, write the *inverse* of the given matrix as a product of elementary matrices. ◀

157. The matrix in Exercise 154.

158. The matrix in Exercise 155.

159. The matrix in Exercise 156.

▶ In Exercises 160–161, show that the given matrices A and B are row equivalent, and find a sequence of elementary row operations that produces B from A. ◀

160. $A = \begin{bmatrix} 7 & 1 & -2 \\ -1 & 3 & 4 \\ 5 & 6 & 8 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & -11 & -18 \\ 5 & 6 & 8 \\ -1 & 3 & 4 \end{bmatrix}$

161. $A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 1 & 0 \\ 3 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & 9 & 4 \\ -5 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$

162. Show that if

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{bmatrix}$$

is an elementary matrix, then at least one entry in the third row must be zero.

163. Show that

$$\begin{bmatrix} 0 & 0 & 0 & a & 0 \\ 0 & 0 & b & 0 & c \\ 0 & d & 0 & e & 0 \\ f & 0 & g & 0 & 0 \\ 0 & h & 0 & 0 & 0 \end{bmatrix}$$

is not invertible for any values of the entries.

164. In each part, solve the matrix equation for X.

(a) $X \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ -3 & 1 & 5 \end{bmatrix}$

(b) $X \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & -1 & 0 \\ 6 & -3 & 7 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} X - X \begin{bmatrix} 1 & 4 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 5 & 4 \end{bmatrix}$

More on Linear Systems and Invertible Matrices

▶ In Exercises 165–168, solve the system by inverting the coefficient matrix and using Theorem 1.6.2. ◀

165. $\begin{aligned} 3x_1 + 5x_2 &= -2 \\ x_1 + 2x_2 &= 3 \end{aligned}$

166. $\begin{aligned} 4x_1 - 3x_2 &= 7 \\ -6x_1 + 5x_2 &= -2 \end{aligned}$

167. $\begin{aligned} x_1 \quad\quad - x_3 &= 6 \\ x_1 + x_2 + x_3 &= -3 \\ -x_1 + x_2 \quad\quad &= 12 \end{aligned}$

168. $\begin{aligned} x_1 + x_2 &= b_1 \\ 5x_1 + 6x_2 &= b_2 \end{aligned}$

▶ In Exercises 169–171, solve the linear systems together by reducing the appropriate augmented matrix. ◀

169. $\begin{aligned} x_1 + 4x_2 + x_3 &= b_1 \\ -x_1 - 3x_2 - 2x_3 &= b_2 \\ 2x_1 + 6x_2 + 6x_3 &= b_3 \end{aligned}$

(i) $b_1 = 1, b_2 = 1, b_3 = 0$
(ii) $b_1 = -1, b_2 = 5, b_3 = 6$

170. $\begin{aligned} 6x_1 + 5x_2 &= b_1 \\ 5x_1 + 4x_2 &= b_2 \end{aligned}$

(i) $b_1 = 0, \ b_2 = 1$ (ii) $b_1 = -4, \ b_2 = 6$
(iii) $b_1 = -1, \ b_2 = 3$ (iv) $b_1 = -5, \ b_2 = 1$

171. $$\begin{aligned} x_1 + 3x_2 + 5x_3 &= b_1 \\ -x_1 - 2x_2 \quad &= b_2 \\ 2x_1 + 5x_2 + 4x_3 &= b_3 \end{aligned}$$

(i) $b_1 = 1,\ b_2 = 0,\ b_3 = -1$
(ii) $b_1 = 0,\ b_2 = 1,\ b_3 = 1$
(iii) $b_1 = -1,\ b_2 = -1,\ b_3 = 0$

► In Exercises 172–175, determine conditions on the b_i's, if any, in order to guarantee that the linear system is consistent. ◄

172. $$\begin{aligned} x_1 - 3x_2 &= b_1 \\ 4x_1 - 12x_2 &= b_2 \end{aligned}$$

173. $$\begin{aligned} 2x_1 - 5x_2 &= b_1 \\ 3x_1 + 6x_2 &= b_2 \end{aligned}$$

174. $$\begin{aligned} x_1 - 2x_2 - 2x_3 &= b_1 \\ -4x_1 + 5x_2 + 4x_3 &= b_2 \\ -4x_1 + 7x_2 + 8x_3 &= b_3 \end{aligned}$$

175. $$\begin{aligned} x_1 + 3x_2 - x_3 + 2x_4 &= b_1 \\ -2x_1 + x_2 + 5x_3 + x_4 &= b_2 \\ 3x_1 - 2x_2 - 2x_3 + x_4 &= b_3 \\ 5x_1 - 7x_2 - 3x_3 \quad &= b_4 \end{aligned}$$

176. Consider the matrices

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & -2 \\ 3 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(a) Show that the equation $A\mathbf{x} = \mathbf{x}$ can be rewritten as $(A - I)\mathbf{x} = \mathbf{0}$ and use this result to solve $A\mathbf{x} = \mathbf{x}$ for $\mathbf{x}$.

(b) Solve $A\mathbf{x} = 4\mathbf{x}$.

► In Exercises 177–178, solve the given matrix equation for X. ◄

177. $$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 7 & 6 \\ 1 & 0 & 8 \end{bmatrix} X = \begin{bmatrix} 1 & 4 & -2 & 0 & 3 \\ 0 & -1 & 5 & 2 & 7 \\ -3 & 6 & 8 & 9 & 0 \end{bmatrix}$$

178. $$\begin{bmatrix} -2 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 1 & 0 \end{bmatrix} X = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 9 & 8 & 7 & 6 \\ 2 & 1 & 0 & -1 \end{bmatrix}$$

179. Let $A\mathbf{x} = \mathbf{0}$ be a homogeneous system of n linear equations in n unknowns that has only the trivial solution. Show that if k is any positive integer, then the system $A^k\mathbf{x} = \mathbf{0}$ also has only the trivial solution.

180. Let $A\mathbf{x} = \mathbf{0}$ be a homogeneous system of n linear equations in n unknowns, and let Q be an invertible $n \times n$ matrix. Show that $A\mathbf{x} = \mathbf{0}$ has just the trivial solution if and only if $(QA)\mathbf{x} = \mathbf{0}$ has just the trivial solution.

181. Let $A\mathbf{x} = \mathbf{b}$ be any consistent system of linear equations, and let $\mathbf{x}_1$ be a fixed solution. Show that every solution to the system can be written in the form $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_0$, where $\mathbf{x}_0$ is a solution to $A\mathbf{x} = \mathbf{0}$. Show also that every matrix of this form is a solution.

182. Use part (*a*) of Theorem 1.6.3 to prove part (*b*).

Diagonal, Triangular, and Symmetric Matrices

► In Exercises 183–186, determine whether the given matrix is invertible. ◄

183. $$\begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$$

184. $$\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

185. $$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

186. $$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

► In Exercises 187–190, determine the product by inspection. ◄

187. $$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -2 & 0 \\ 4 & -2 \end{bmatrix}$$

188. $$\begin{bmatrix} -3 & 2 & 8 \\ 4 & 1 & 6 \end{bmatrix} \begin{bmatrix} 7 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

189. $$\begin{bmatrix} -5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 & 1 & -1 \\ -1 & 2 & 3 & 0 & -2 \\ -2 & 0 & 4 & -3 & 1 \end{bmatrix}$$

190. $$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 4 & 1 & -3 \\ -1 & 2 & 0 \\ 5 & 1 & -2 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

► In Exercises 191–194, find A^2, A^{-2}, and A^{-k} (where k is any integer) by inspection. ◄

191. $$A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

192. $$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

193. $$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$$

194. $$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

► In Exercises 195–201, decide whether the given matrix is symmetric. ◄

195. $$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

196. $$\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$$

197. $$\begin{bmatrix} 0 & -7 \\ -7 & 7 \end{bmatrix}$$

198. $$\begin{bmatrix} 3 & 4 \\ 4 & 0 \end{bmatrix}$$

199. $$\begin{bmatrix} 2 & 3a & 0 \\ 3a & 1 & b \\ 0 & b & 0 \end{bmatrix}$$

200. $$\begin{bmatrix} 2 & -1 & 3 \\ -1 & 5 & 1 \\ 3 & 1 & 7 \end{bmatrix}$$

201. $$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

In Exercises 202–204, decide by inspection whether the given matrix is invertible.

202. $\begin{bmatrix} 2 & 6 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

203. $\begin{bmatrix} 9 & 0 & 0 & 1 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 4 & -5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

204. $\begin{bmatrix} -2 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 7 & 9 & 0 & 0 \\ 0 & 6 & -2 & 4 \end{bmatrix}$

In Exercises 205–206, find all values of the unknown constant(s) in order for A to be symmetric.

205. $A = \begin{bmatrix} -3 & a^2 \\ 4 & 0 \end{bmatrix}$

206. $A = \begin{bmatrix} 7 & a+b-c & a-b \\ 4 & 6 & 2a-b-c \\ 1 & 3 & 4 \end{bmatrix}$

207. Find all values of x in order for A to be invertible.

$$A = \begin{bmatrix} 2-x & 5 & x^2 \\ 0 & x+3 & x-1 \\ 0 & 0 & x \end{bmatrix}$$

208. Find a diagonal matrix A that satisfies

$$A^{-2} = \begin{bmatrix} 16 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

209. Verify Theorem 1.7.1(*b*) for the product AB, where

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ -3 & 4 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 & 0 \\ 6 & 2 & 0 \\ 1 & 1 & 5 \end{bmatrix}$$

210. Verify Theorem 1.7.4 for the given matrix A.

(a) $A = \begin{bmatrix} -4 & 2 \\ 2 & 3 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 4 \\ -2 & 4 & 0 \end{bmatrix}$

211. Find all 2×2 diagonal matrices A that satisfy the equation $A^2 - 3A + 2I = 0$.

212. Let A be a square matrix.

(a) Show that $(I - A)^{-1} = I + A + A^2 + A^3$ if $A^4 = 0$.

(b) Show that

$$(I - A)^{-1} = I + A + A^2 + \cdots + A^n$$

if $A^{n+1} = 0$.

213. Let J_n be the $n \times n$ matrix each of whose entries is 1. Show that if $n > 1$, then

$$(I - J_n)^{-1} = I - \frac{1}{n-1} J_n$$

214. A square matrix A is called *skew-symmetric* if $A^T = -A$. Prove:

(a) If A is an invertible skew-symmetric matrix, then A^{-1} is skew-symmetric.

(b) If A and B are skew-symmetric matrices, then so are $A^T, A+B, A-B$, and kA for any scalar k.

(c) Every square matrix A can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix. [*Hint:* Note the identity $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$.]

In Exercises 215–216, fill in the missing entries (marked with $\times$) to produce a skew-symmetric matrix.

215. $A = \begin{bmatrix} \times & \times & -3 \\ 1 & \times & \times \\ \times & 0 & \times \end{bmatrix}$

216. $A = \begin{bmatrix} \times & 3 & \times \\ \times & \times & 0 \\ -2 & \times & \times \end{bmatrix}$

217. Find all values of a, b, c, and d for which A is skew-symmetric.

$$A = \begin{bmatrix} 0 & 2a-3b+c & 3a-5b+5c \\ -2 & 0 & 5a-8b+6c \\ -3 & -5 & d \end{bmatrix}$$

218. We showed in the text that the product of symmetric matrices is symmetric if and only if the matrices commute. Is the product of commuting skew-symmetric matrices skew-symmetric? Explain. [*Note:* See Exercise 214 for the definition of *skew-symmetric*.]

219. If the $n \times n$ matrix A can be expressed as $A = LU$, where L is a lower triangular matrix and U is an upper triangular matrix, then the linear system $A\mathbf{x} = \mathbf{b}$ can be expressed as $LU\mathbf{x} = \mathbf{b}$ and can be solved in two steps:

Step 1. Let $U\mathbf{x} = \mathbf{y}$, so that $LU\mathbf{x} = \mathbf{b}$ can be expressed as $L\mathbf{y} = \mathbf{b}$. Solve this system.

Step 2. Solve the system $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$.

In each part, use this two-step method to solve the given system.

(a) $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 3 & 0 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ -3 & -2 & 3 \end{bmatrix} \begin{bmatrix} 3 & -5 & 2 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ 2 \end{bmatrix}$

220. Find a lower triangular matrix that satisfies

$$A^3 = \begin{bmatrix} 8 & 0 \\ 9 & -1 \end{bmatrix}$$