PROPOSITIONS AND LOGICAL OPERATIONS

Propositional Logic and Proofs

Learning objectives! Know what you will learn today

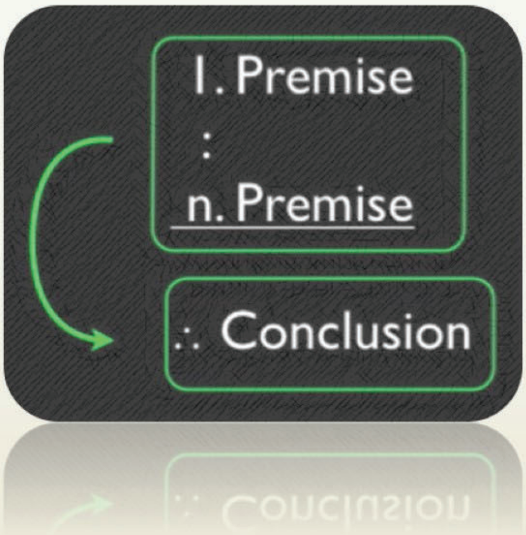
Self-Reflection! Rate levels of your understanding

❍ Checklist of key topics. Keep catching up with the course.

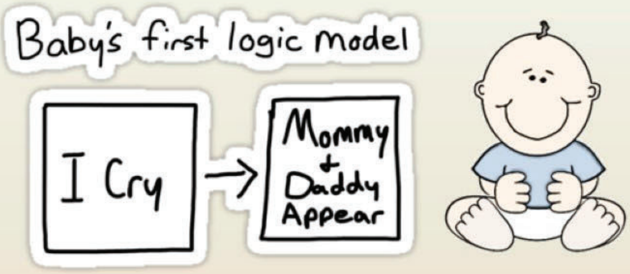
- ☐ Compound propositions, logical operators, and conditional statements
- ☐ Tautologies, contradictions, and contingencies
- ☐ Logical equivalences and validity of arguments
- ☐ Proof by truth tables & by series of logical equivalences/rules of inference
- ☐ Disproof by counterexample, finding an example when it is not satisfied

Confident	Got it	Okay	Fuzzy	Not a clue

Logic in brief



- **Applying logic :** assumes you know some things for sure and allows you to use that knowledge to arrive at further conclusions.
- **Logic :** is a system used for distinguishing between correct and incorrect arguments.



https://freshspectrum.com/babys-first-logic-model/

Logic and algorithms are essential to CS and ICT

- **Good news:** humans already have an innate, intuitive understanding of both logic and algorithms.
- **Not so good news:** logic and algorithms are both mathematical concepts in nature. Consequently, each has its own set of rules, procedures, and definitions, which are very precise and systematic.
- **What this means:** you cannot rely solely on your intuition when dealing with these topics, otherwise you will make mistakes.
- **What to do:** since computers essentially automate your reasoning, you must learn precise concepts of logic and to perform logic correctly before writing a computer solution.

The basic building blocks of logic – propositions

- A **proposition** is a declarative sentence (that is, a sentence that declares a fact) that is **either true or false, but not both**.

Is it a proposition?
5 is a prime number
12 + 7 = 8
Do you speak French?
Happy Birthday!
Please pass the salt.
3 – x = 5

Compound statements and logical operators

Math

x, y, z, … conventional (common) variables representing real numbers
+, −, ×, ÷, … mathematical operators that are used to combined variables

Logic

p, q, r, s, … **propositional variables**, representing propositions
$\neg, \wedge, \vee, \oplus, \rightarrow, \leftrightarrow$, … **logical operators** are used to form **compound propositions**

- The negation operator, $\neg p$: *it is not the case that p*, constructs a new proposition from a single existing proposition. Other logical operators, i.e., $\wedge$, $\vee$, $\oplus$, $\rightarrow$, $\leftrightarrow$, …, are **connectives**, combing two or more existing propositions.
- **Atomic propositions** are ones that cannot be expressed in a simple proposition

Unary operation

- The **negation** (i.e. not) logical operator
- ¬p is a proposition if p is a proposition
- ¬p means *it is not the case that p*

not

negation

p	¬p
T	F
F	T

A truth table: giving the truth value of a compound proposition for every combination of its component parts

Logical connectives

and

conjunction

p	q	p ∧ q
T	T	
T	F	
F	T	
F	F	

or

disjunction

p	q	p ∨ q
T	T	
T	F	
F	T	
F	F	

xor

exclusive disjunction

p	q	p ⊕ q
T	T	
T	F	
F	T	
F	F	

A truth table for $(p \wedge q) \vee (\neg p)$

p	q	$p \wedge q$	$\neg p$	$(p \wedge q) \vee (\neg p)$
T	T			
T	F			
F	T			
F	F			

Know this …

If a compound statement contains n propositional statements, then how many rows will there be in the truth table?

WORKED EXAMPLES

Find the truth value of each proposition if p and r are true and q is false

- $p \vee q \vee r$
- $(\neg p \vee q) \wedge r$

Ch2.1, Q.8

Conditional statement (implication)

- "**if p then q**" is denoted $p \to q$
- p is called the **premise** or hypothesis or antecedent
- q is called the **conclusion** or consequent

if-then

p	q	$p \to q$
T	T	
T	F	
F	T	
F	F	

Biconditional (double-implication)

- If p and q are statements, the compound statement **p if and only if q** is called an **equivalence** or **biconditional**, denoted as $p \leftrightarrow q$
- $p \leftrightarrow q$ asserts that p and q has the same truth value
- $p \leftrightarrow q$ or **p iff q** is equivalent to $(p \to q) \wedge (q \to p)$
- p is a **necessary and sufficient condition** for q

if-and-only-if

p	q	$p \leftrightarrow q$
T	T	
T	F	
F	T	
F	F	

Precedence of logical operators

- $\neg p \land q$ means _______________
- $p \lor q \land r$ means_____________
- $p \rightarrow q \lor r$ means____________
- To make it clear, **use parenthesis!**

operators	precedence
$\neg$	1
$\land$	2
$\lor$	3
$\rightarrow$	4
$\leftrightarrow$	5

Analogously, we do ^ (exponent or power) before ×, ÷ before +, − Math

PRACTICE PROBLEMS

Find the truth value of each statement if p, q are true and r, s, t are false

- $(q \rightarrow (r \rightarrow s)) \land ((p \rightarrow s) \rightarrow (\neg t))$
- $(r \land s \land t) \rightarrow (p \lor q)$
- $(r \land s \land t) \leftrightarrow (p \lor q)$
- $(\neg t) \rightarrow (r \lor (\neg s \rightarrow (q \oplus p)))$

Ch2.2, Q.22c,d

Statements related to the original conditional statement

p	q	implication $p \to q$	inverse $\neg p \to \neg q$	converse $q \to p$	contraposition $\neg q \to \neg p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

same truth values

same truth values

Tautology, contradiction, and contingency

Tautology	Contradiction or Absurdity	Contingency
Always true	Always false	Some true some false
$p \vee \neg p$	$p \wedge \neg p$	$p \to \neg p$

p	$p \vee \neg p$
T	T
F	T

p	$p \wedge \neg p$
T	F
F	F

p	$p \to \neg p$
T	F
F	T

WORKED EXAMPLES

Construct truth tables to determine whether the given statement is a tautology, a contingency, or an absurdity.

- $p \wedge q \wedge (q \rightarrow \neg p)$

p	q	$p \wedge q$	$q \rightarrow \neg p$	$p \wedge q \wedge (q \rightarrow \neg p)$
T	T			
T	F			
F	T			
F	F			

PRACTICE PROBLEMS 2 variables

Construct truth tables to determine whether the given statement is a tautology, a contingency, or an absurdity.

- $(p \wedge \neg q) \leftrightarrow (p \rightarrow q)$
- $(q \wedge p) \vee (q \wedge \neg p)$

Ch2.2, Q.12

PRACTICE PROBLEMS 2 variables

Construct truth tables to determine whether the given statement is a tautology, a contingency, or an absurdity.

- $(p \wedge q) \rightarrow p$
- $p \rightarrow (q \wedge p)$

Ch2.2, Q.12

PRACTICE PROBLEMS 3 variables

Is the given proposition is a tautology, a contingency, or a contradiction. If it is a contingency, give one truth assignment for which the proposition is true and one for which it is false.

- $[\neg(\neg p \vee \neg(q \wedge \neg r))] \vee \neg\neg(q \vee \neg p)$

PRACTICE PROBLEMS 3 variables

Is the given proposition is a tautology, a contingency, or a contradiction. If it is a contingency, give one truth assignment for which the proposition is true and one for which it is false.

- $\neg([\neg(p \wedge \neg r) \wedge \neg(q \wedge \neg s)] \wedge \neg[\neg(p \vee q) \vee (r \vee s)])$

PROPOSITIONAL EQUIVALENCES

Propositional Logic and Proofs

Two compound propositions are **logically equivalent** or simply equivalent if **$P \leftrightarrow Q$ is a tautology**, denoted by **$P \equiv Q$**

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

two compound propositions always have the same truth values, regardless of the truth values of its propositional variables

Logically equivalence

Example: verify a logical equivalence property for $(p \rightarrow q) \equiv (\neg p \lor q)$

How? -- join the two statements using $\leftrightarrow$ (iff), and construct a truth table

The two statements are equivalent if all cases evaluate to true (a tautology)

p	q	$p \rightarrow q$	$\neg p \lor q$	$(p \rightarrow q) \leftrightarrow (\neg p \lor q)$
T	T			
T	F			
F	T			
F	F			

Ch2.2, Ex.7

PRACTICE PROBLEMS

Are two statements logically equivalent: $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$?

Use a truth table

Methods of proof for logical equivalence

- **Use a truth table** to show that two propositions are equivalent – join them by iff and that all cases of $p \leftrightarrow q$ evaluate to true (a tautology)
- **Too many row**, when there are more than a few variables, a truth table is impractical, e.g. when n=10, a table has $2^n = 2^{10} = 1024$ rows.

- **Use common logical equivalences** as a building block to prove/show/verify or to construct new logical equivalences.

This is just like how we solve equations in algebra using algebraic properties

Statements	Reasons
1. $6x - 3 = 4x + 1$	1. Given
2. $6x - 4x - 3 = 4x - 4x + 1$	2. Subtraction prop
3. $2x - 3 = 1$	3. Substitution
4. $2x - 3 + 3 = 1 + 3$	4. Addition prop.
5. $2x = 4$	5. Substitution
6. $2x/2 = 4/2$	6. Division prop.
7. $x = 2$	7. Substitution

Logically equivalence

- The equivalence we have just verified using a truth table,

$$p \rightarrow q \equiv \neg p \lor q,$$

  is known as the **conditional disjunction equivalence**.

- Another commonly used equivalence are the **De Morgan's Laws**

$$\neg(p \land q) \equiv \neg p \lor \neg q$$

$$\neg(p \lor q) \equiv \neg p \land \neg q$$

Try verifying these using truth tables

IDENTITY

Identity does not change the value of the operand

$a + 0 = a$
$a \times 1 = a$

INVERSE

Inverse changes the number to the identity

$a + (-a) = 0$
$a \times (1/a) = 1$

COMMUTATIVE

Changing the order of the operands

$a + b = b + a$
$a \times b = b \times a$

ASSOCIATIVE

Changing the group of the operands

$a + (b + c) = (a + b) + c$
$a \times (b \times c) = (a \times b) \times c$

DISTRIBUTIVE

This involves two operators, e.g. combining addition & multiplication

$a \times (b + c) = (a \times b) + (a \times c)$

Common logically equivalent statements

Identity	$p \wedge T \equiv p$	$p \vee F \equiv p$
Domination	$p \wedge F \equiv F$	$p \vee T \equiv T$
Idempotent	$p \wedge p \equiv p$	$p \vee p \equiv p$
Negation	$p \wedge \neg p \equiv F$	$p \vee \neg p \equiv T$
Double negation	$\neg(\neg p) \equiv p$	

IDENTITY
does not change the value of the operand
a + 0 = a
a × 1 = a

INVERSE
changes the number to the identity
a + (-a) = 0
a × (1/a) = 1

Commutation	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
Association	$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$	$p \vee (q \vee r) \equiv (p \vee q) \vee r$
Distribution	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
De Morgan's	$\neg(p \wedge q) \equiv \neg p \vee \neg q$	$\neg(p \vee q) \equiv \neg p \wedge \neg q$
Absorption	$p \wedge (p \vee q) \equiv p$	$p \vee (p \wedge q) \equiv p$
Exportation	$p \rightarrow (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$	

COMMUTATIVE
Changing the order of the operands
a + b = b + a
a × b = b × a

ASSOCIATIVE
Changing the group of the operands
a + (b + c) = (a + b) + c
a × (b × c) = (a × b) × c

DISTRIBUTIVE
This involves two operators, e.g. combining addition & multiplication
a × (b + c) = (a × b) + (a × c)

WORKED EXAMPLES

Use series of logical equivalences to show that $\neg(p \rightarrow q) \equiv p \wedge \neg q$

R.Ch1.3,Ex.6

WORKED EXAMPLES

Use series of logical equivalences to show that $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$

PRACTICE PROBLEMS

Use series of logical equivalences to show that $(p \land q) \rightarrow r \equiv p \rightarrow (q \rightarrow r)$

PRACTICE PROBLEMS

Part 1: use series of logical equivalences to show $p \leftrightarrow q \equiv (\neg p \land \neg q) \lor (p \land q)$

PRACTICE PROBLEMS

Part 2: use series of logical equivalences to show $\neg(p \leftrightarrow q) \equiv \neg q \leftrightarrow p$

Hint: for part 2, there is no need to start from scratch.
Instead, make use of the equivalence already verified in part 1.

PRACTICE PROBLEMS

Use series of logical equivalences to show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology

Hint: Showing that a statement is a tautology is to show that it is logically equivalence to the truth value "true" (it is always true)

Are they logically equivalent?	❍ Is the bicondition (if-and-only-if) a tautology? ❍ Can you conclude … ? Prove or disprove.
Proof of logically equivalence	Do you need to check every row in a truth table? ❍ Construct a truth table (iff is all true) ❍ Applying series of logical equivalences
Proof of NON equivalence	Give a counterexample – truth values that break the iff tautology (when two statements DISAGREE)

NOT logically equivalence

Example: show that the negation of $p \leftrightarrow q$ is not equivalent to $(\neg p \rightarrow q)$

How? -- join the two statements using $\leftrightarrow$ (iff), and construct a truth table

The two statements are NOT equivalent if there is a row evaluated to false

p	q	$\neg(p \leftrightarrow q)$	$\neg p \rightarrow q$	$\neg(p \leftrightarrow q) \leftrightarrow (\neg p \rightarrow q)$
T	T	F	T	F
T	F	T	T	T
F	T	T	T	T
F	F	F	F	T

Ch2.2, Ex.7

PRACTICE PROBLEMS

Show that $(p \to q) \to r$ and $p \to (q \to r)$ are not equivalent?

Give a counterexample

RULE OF INFERENCE

Valid Arguments in Propositional Logic

Propositional Logic and Proofs

An argument is a sequence of statements that ends with a conclusion

An argument is valid when the implication is a tautology

q logically follows from p_i's

$$(p_1 \land p_2 \land \cdots \land p_n) \rightarrow q$$

Premises   Conclusion

premise 1
premise 2
⋮
premise n

∴ conclusion

Is the given argument valid? "implication is a tautology"

$p \land q$
$\neg q \rightarrow r$

∴ r

p	q	r	$p \land q$	$\neg q \rightarrow r$	$(p \land q) \land (\neg q \rightarrow r)$	r	→
T	T	T					
T	T	F					
T	F	T					
T	F	F					
F	T	T					
F	T	F					
F	F	T					
F	F	F					

Answer with the least amount of work (fill in as few cells as you can)!!!

What does it mean? "implication is a tautology"

PREMISES $(p_1 \land p_2 \land \cdots \land p_n)$	$\rightarrow$	CONCLUSION q	THE IMPLICATION
T		T	T
T		F	F
F		T	T
F		F	T

No need to check when premises are false and when the conclusion is true

Proof of validity: show that when all premises p_i's are true, the conclusion q must also be true.

Proof of invalidity: find a case when all premises p_i's are true, but the conclusion q is false.

Equivalence vs Validity

P≡Q iff P↔Q is a tautology

$(p_1 \land p_2 \land \cdots \land p_n) \rightarrow q$ is a tautology

p q	p→q	¬p∨q	(p→q) ↔ (¬p∨q)
T T	T	T	T
T F	F	F	T
F T	T	T	T
F F	T	T	T

Construct a truth table

$\neg(p \rightarrow q)$
$\equiv \neg(\neg p \lor q)$ *Cond. disjunction*
$\equiv \neg(\neg p) \land \neg q$ *De Morgan's law*
$\equiv p \land \neg q$ *Double negation*

By deduction – applying equivalence properties

Proof of equivalence /validity

- Construct a truth table (implication is all true)
- Use a short truth table
- By deduction (apply equivalence properties and/or rules of inference)

Give counterexample – assign truth values so P evaluates to true and Q false or vice versa (they disagree)

Proof non-equivalence / invalidity

Give counterexample – assign truth values that make an argument false (all premises true, conclusion false)

Short truth table method for validity

- In a full truth table, an argument is invalid when we find one or more rows where all premises are true and the conclusion is false
- In a **short truth table** method, we attempt to find this **"invalid" row** without constructing the entire table
- **HOW**: assume such a row exists where *all premises true and conclusion false*
  - Make every premise true (assign it a T) and the conclusion false (assign it an F)
  - Work logically, assigning one variable after another, see if you can build this row:
    - If so, we **find a truth assignment** that make the argument false. It is **INVALID**.
    - If not (we **arrive at a contradiction**), then we conclude that it is not possible to make the argument false. That is, the argument is always true. It is **VALID**.

Is the given argument valid?

$p \wedge q$

$\neg q \rightarrow r$

$\therefore r$

p q r	p ∧ q	¬ q → r	r

Is the given argument valid?

$$p \lor (q \lor r)$$
$$\neg r$$
$$\therefore p \lor q$$

p	q	r	p	∨	(q	∨	r)	¬	r	p	∨	q

Try verifying this using a truth table (fill in as few cells as you can).

PRACTICE PROBLEMS

$$p \lor (q \lor r)$$
$$\neg r$$
$$\therefore p \lor q$$

Is the argument valid or invalid? Prove it.

Answer with the least amount of work (fill in as few cells as you can)!!!

Use a full truth table

PRACTICE PROBLEMS

$p \rightarrow q$

$r \rightarrow F$

$\therefore (p \wedge r) \rightarrow (q \vee F)$

Is the argument valid or invalid? Prove it.

PRACTICE PROBLEMS

$p \rightarrow (q \vee \neg r)$

$q \rightarrow (p \wedge r)$

$\therefore p \rightarrow r$

Is the argument valid or invalid? Prove it.

PRACTICE PROBLEMS

Is the given argument valid? Prove it.

$$\begin{array}{l} p \leftrightarrow q \\ p \vee \neg q \\ \hline \therefore p \wedge q \end{array}$$

Sometimes you may have to guess. Make sure you try *all* possible cases.

PRACTICE PROBLEMS

- Use the short truth table to show that $\neg q \rightarrow (r \rightarrow \neg(p \wedge q))$ is a tautology.
  Hint: assume it is not a tautology, i.e. the statement is false, then see if you can find a truth assignment without arriving at any contradictions.

Methods of proof for validity of arguments

- **Use a truth table** to show that an argument form is valid – whenever the premises are true, the conclusion must also be true (→ is a tautology) 
- **Use a short truth table**, a kind of indirect proof, **a proof by contradiction** – assume the argument is false then solve it to find possible truth assignments.
- Next, we establish validity of simple arguments, called **rules of inference**. Then, use them as building blocks to prove or to build other arguments.
- Validity of rules of inference depends only on **the form of the statements** involved and not on the truth values of the variables they contain.

Again, this is just like how we solve algebraic equations.

Rules of inference – known valid argument forms

SIMPLIFICATION

$$\begin{array}{l} p \wedge q \\ \hline \therefore p \end{array}$$

MODUS PONENS

$$\begin{array}{l} p \\ p \to q \\ \hline \therefore q \end{array}$$

MODUS TOLLENS

$$\begin{array}{l} \neg q \\ p \to q \\ \hline \therefore \neg p \end{array}$$

p	q	$p \wedge q \to p$
T	T	T
T	F	T
F	T	T
F	F	T

p	q	$p \wedge (p \to q) \to q$
T	T	T
T	F	T
F	T	T
F	F	T

p	q	$\neg q \wedge (p \to q) \to \neg p$
T	T	T
T	F	T
F	T	T
F	F	T

Prove by deduction that the argument is VALID

$\neg p \rightarrow (r \wedge s)$
$p \rightarrow q$
$\neg q$
$\therefore r$

Step	Reason
1	
2	
3	
4	
5	
6	

Since the conclusion **r** is logically followed from the premises $\neg p \rightarrow (r \wedge s)$, $p \rightarrow q$, $\neg q$, therefore the argument is valid.

Rules of Inference – known valid argument forms

MODUS PONENS

p
$p \rightarrow q$
$\therefore q$

MODUS TOLLENS

$\neg q$
$p \rightarrow q$
$\therefore \neg p$

HYPOTHETICAL SYLLOGISM

$p \rightarrow q$
$q \rightarrow r$
$\therefore p \rightarrow r$

DISJUNCTIVE SYLLOGISM

$p \vee q$
$\neg p$
$\therefore q$

RESOLUTION

$p \vee q$
$\neg p \vee r$
$\therefore q \vee r$

CONJUNCTION

p
q
$\therefore p \wedge q$

SIMPLIFICATION

$p \wedge q$
$\therefore p$

ADDITION

p
$\therefore p \vee q$

Prove by deduction that the argument is VALID

$(p \vee q) \to s$
$q \wedge r$
$\neg p$

$\therefore r \wedge s$

Step	Reason
1	premise
2	simplification (1)
3	simplification (1)
4	premise
5	premise
6	disjunctive syllogism (4,5)
7	modus ponens (2,6)
8	conjunction

Since the conclusion $r \wedge s$ is logically followed from the premises $(p \vee q) \to s$, $q \wedge r$, $\neg p$, therefore the argument is valid.

Prove by deduction that the argument is VALID

- p := You send me an email
- q := I finish my program
- r := I go to sleep early
- s := I wake up refreshed

Premises: $p \to q$, $\neg p \to r$, $r \to s$

Conclusion: $q \vee s$

From the proof, it is valid

	Step	Reason
1	$p \to q$	
2	$\neg q \to \neg p$	
3	$\neg p \to r$	
4	$\neg q \to r$	
5	$r \to s$	
6	$\neg q \to s$	
7	$q \vee s$	

Reasons can be common logical equivalences or rules of inference

PRACTICE PROBLEMS

Prove by deduction that the argument is VALID

$p \leftrightarrow q$

$\neg q$

$\therefore \neg p \vee r$

	Step	Reason
1	$p \leftrightarrow q$	premise
2	$(p \rightarrow q) \wedge (q \rightarrow p)$	bicondition equivalence (1)
3		
4		
5		
6		

Since the conclusion $\neg p \vee r$ is logically followed from the premises $p \leftrightarrow q$, $\neg q$, therefore the argument is valid.

PRACTICE PROBLEMS

Prove by deduction that the argument is VALID

$\neg(p \wedge \neg q)$

$r \rightarrow p$

$\therefore q \vee \neg r$

	Step	Reason
1		
2		
3		
4		
5		
6		

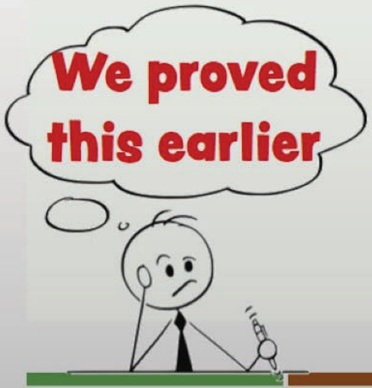
Since the conclusion $q \vee \neg r$ is logically followed from the premises $\neg(p \wedge \neg q)$, $r \rightarrow p$, therefore the argument is valid.

PRACTICE PROBLEMS

Prove by deduction that the argument is VALID

$$\begin{array}{l} p \vee (q \vee r) \\ \neg r \\ \hline \therefore p \vee q \end{array}$$

Step	Reason
1	
2	
3	
4	



Since the conclusion **p ∨ q** is logically followed from the premises **p ∨ (q ∨ r)**, **¬r**, therefore the argument is valid.

PRACTICE PROBLEMS

Show that the following argument is invalid.

Premises: $(q \vee r) \rightarrow p$, $\neg q$, p

Conclusion: r

Recall: to prove invalidity, find a truth assignment (give each variable a truth value) that makes an argument false, i.e. all premises true and the conclusion false

PRACTICE PROBLEMS

Determine if the following statements are valid or invalid. Detail a proof of validity or give a truth-assignment that makes the argument false (invalid).

- $(p \rightarrow q) \wedge (q \rightarrow r), (\neg q) \wedge r \therefore p$
- $\neg(p \rightarrow q), p \therefore \neg q$
- $(p \vee q), \neg(q \wedge \neg\neg r) \therefore \neg(r \vee p)$
- $(p \vee q), (\neg p \vee r), \neg(q \wedge s) \therefore \neg(\neg r \wedge s)$
- $(q \wedge r) \leftrightarrow (p \vee r), (q \rightarrow \neg p) \therefore (r \rightarrow \neg p)$

Ch2.3, Q.6,12

Quiz00 Same as others, submitting it is your attendance. Score & solutions posted after the quiz closes (*since quiz00 is a trail, it does not effect your grade*). Different from other quiz, you may do it multiple times (*normally, only* ONE *attempt is allowed*).

QUIZ00 THIS WEEKEND

Reading
KBR, Rosen, Levin

Textbook exercises

HW - Practice Problems