ITCS 111
Chapter 2: *Chain Rule and Implicit Differentiation*

Some of the material in these slides is from *Calculus* 10/E by Howard Anton, Irl Bivens, and Stephen Davis

2.6 Derivatives of Composite Functions – *Chain Rule*

2.6.1 THEOREM (*The Chain Rule*) *If g is differentiable at x and f is differentiable at $g(x)$, then the composition $f \circ g$ is differentiable at x. Moreover, if*

$$y = f(g(x)) \quad \text{and} \quad u = g(x)$$

then $y = f(u)$ and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \tag{1}$$

$$\frac{d}{dx}[f(g(x))] = (f \circ g)'(x) = f'(g(x))g'(x)$$

The derivative of $f(g(x))$ is the derivative of the outside function evaluated at the inside function times the derivative of the inside function.

Derivatives of Composite Functions *– Chain Rule*

Example: Find *dy/dx* by the chain rule given that

$$y = \frac{u-1}{u+1} \qquad \text{and} \qquad u = x^2.$$

$$\frac{dy}{du} = \frac{(u+1)(1)-(u-1)(1)}{(u+1)^2} = \frac{2}{(u+1)^2}$$

$$\frac{du}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{dy}{dx} = \left[\frac{2}{(u+1)^2}\right]2x = \frac{4x}{(x^2+1)^2}$$

Derivatives of Composite Functions – *Chain Rule*

Example: Find $\frac{dy}{dx}$ of $y = (x^2\text{-}1)^{100}$

Should we expand $(x^2\text{-}1)^{100}$?

By Chain Rule:

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

$$let \quad u = x^2 - 1$$

$$\frac{d\,(x^2-1)^{100}}{dx} = \frac{du^{100}}{dx} = 100u^{100-1}\frac{du}{dx} = 100(x^2-1)^{99}(2x) = 200x(x^2-1)^{99}$$

Derivatives of Composite Functions – *Chain Rule*

Exercises: Find the derivatives, $\dfrac{dy}{dx}$.

1) Let $y = \sin u$ and $u = 2x + \pi$

2) Let $y = u^{10}$ and $u = 3x^4 + x$

Derivatives of Composite Functions – *Chain Rule*

Exercises: Differentiate y with respect to x.

$$y = (x^4 + x^3 - 1)^{-3}$$

$$y = \tan(4x - 1)$$

$$y = \cos^5 x$$

$$y = \frac{1}{\sin(x^2+1)}$$

Exercises#10: The Chain Rule

2.7 Implicit Differentiation

Implicit differentiation is a method for differentiating functions for which it is **inconvenient** or **impossible** to express them in the form $y = f(x)$.

Implicit Differentiation

Example 2 (p 163): Use implicit differentiation to find $\frac{dy}{dx}$ if $5y^2 + \sin y = x^2$

Implicit Differentiation

Example 3 (p 163): Use implicit differentiation to find $\frac{d^2y}{dx^2}$ if $4x^2 - 2y^2 = 9$

Implicit Differentiation

Exercises: Use implicit differentiation to find the derivatives, $\frac{dy}{dx}$.

1) $x^2 + y^4 = 1$
2) $x + x^2y + 3x^3y^4 = 0$
3) $\sin x + \cos(x + y) = 0$
4) $xy = 1$

Implicit Differentiation

Exercise#11: Implicit Differentiation

Basic differentiation formulas

DIFFERENTIATION FORMULA	DIFFERENTIATION FORMULA
1. $\frac{d}{dx}[x] = 1$	8. $\frac{d}{dx}[-\csc x] = \csc x \cot x$
2. $\frac{d}{dx}\left[\frac{x^{r+1}}{r+1}\right] = x^r \quad (r \neq -1)$	9. $\frac{d}{dx}[e^x] = e^x$
3. $\frac{d}{dx}[\sin x] = \cos x$	10. $\frac{d}{dx}\left[\frac{b^x}{\ln b}\right] = b^x \quad (0 < b, b \neq 1)$
4. $\frac{d}{dx}[-\cos x] = \sin x$	11. $\frac{d}{dx}[\ln \lvert x \rvert] = \frac{1}{x}$
5. $\frac{d}{dx}[\tan x] = \sec^2 x$	12. $\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1+x^2}$
6. $\frac{d}{dx}[-\cot x] = \csc^2 x$	13. $\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}}$
7. $\frac{d}{dx}[\sec x] = \sec x \tan x$	14. $\frac{d}{dx}[\sec^{-1} \lvert x \rvert] = \frac{1}{x\sqrt{x^2-1}}$