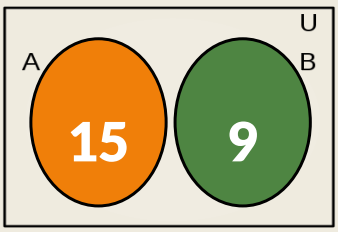
COUNTING WITH SETS

Principle of Inclusion-Exclusion

Counting and Combinatorics

The cardinality of $A \cup B$ when A and B are disjoint: $|A \cup B| = |A| + |B|$

$A \cap B = \varnothing$



Find the cardinality of the union of A and B.

Answer =>

The same question asked in different ways

- In a box, there are 15 orange and 9 green balls. How many balls are there in total?
- How many ways can we pick a ball from a box of 15 orange and 9 green balls (each ball is unique)?

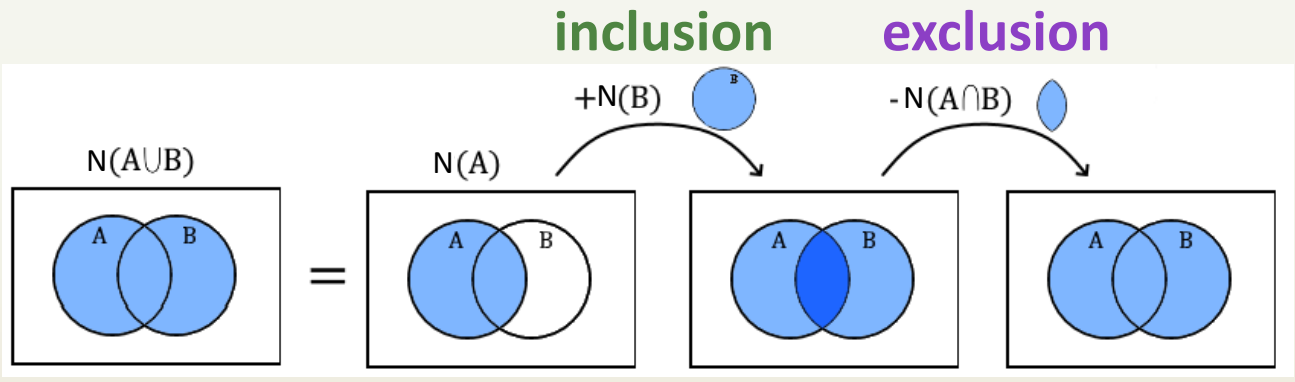
PROBLEM SOLVING

Find the cardinality of the set $A \cup B$

PRACTICE PROBLEMS

- A student can choose a project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. How many possible projects are there to choose from?

PIE: finding $|A \cup B|$ when the sets are not disjoint …

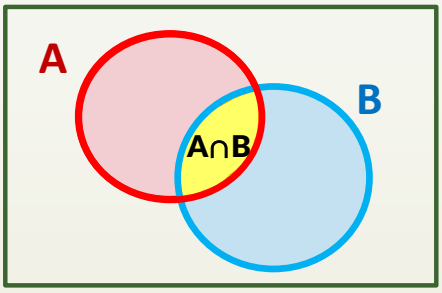


Principle of inclusion and exclusion

If A and B are finite sets, then $|A \cup B| = |A| + |B| - |A \cap B|$

https://www.usu.edu/math/schneit/StatsStuff/Probability/probability4.html

PIE: finding $|A \cup B|$ when the sets are not disjoint …



Principle of inclusion and exclusion

If A and B are finite sets, then

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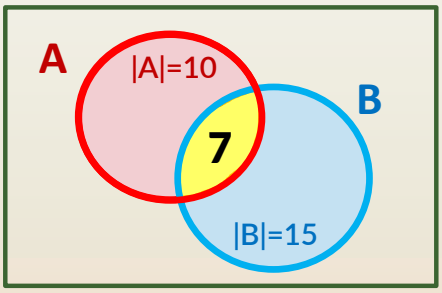


Given the sets as shown, find $|A \cup B|$

WORKED EXAMPLES

In a class, there are

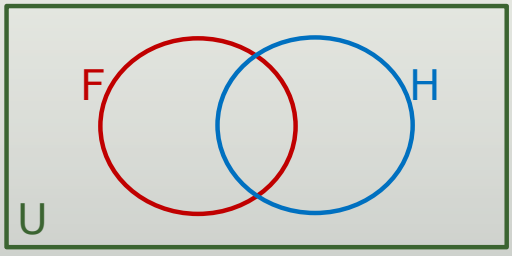
8 students who play football and hockey

7 students who do not play football or hockey

13 students who play hockey

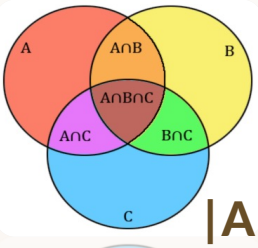
19 students who play football.

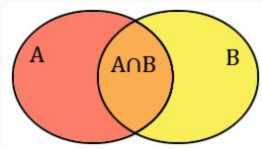
How many students are there in the class?



PRACTICE PROBLEMS

A computer company wants to hire 25 programmers to handle systems programming jobs and 40 programmers for applications programming. Of those hired, 10 will be expected to perform jobs of both types. How many programmers must be hired?





Principle of inclusion-exclusion (**PIE**)

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

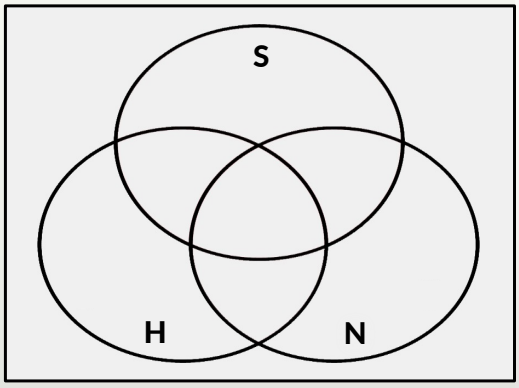
When there're more than two sets

- Draw a Venn diagram that represents a given scenario
- Determine what each (**disjoint**) region corresponds to
- Fill-in the values, usually from inside out
- Use variables when there are unknowns

PROBLEM SOLVING PRACTICE

WORKED EXAMPLES

- A company is hiring. Every candidate has at least one skill. Out of 75 candidates, 48 are software engineers, 35 hardware engineers, 42 network engineers, and 18 have skills in all three areas.
- If the company hires everyone with exactly two skills, how many candidates are hired?



PRACTICE PROBLEMS

A survey was taken on methods of traveling to work. Each respondent was asked to check BUS, TRAIN, or CAR. More than one answer allowed. The results were: BUS 30, TRAIN 35, CAR 100, BUS & TRAIN 15, BUS & CAR 15, TRAIN & CAR 20, and all three methods 5 people. How many people completed a survey?

PRACTICE PROBLEMS

- In a town, 85% speak Tamil, 40% English, 20% Hindi. Also, 32% speak Tamil & English, 13% Tamil & Hindi, and 10% English & Hindi, find the percentage of people who can speak all three languages.

THE SUM AND PRODUCT RULES

How many ?

Counting and Combinatorics

THE SUM RULE

If there are tasks $T_1, T_2, \ldots, T_k$ that can be done in $n_1, n_2, \ldots, n_k$ ways,

No tasks are done at the same time

then, we can do it in $n_1 + n_2 + \ldots + n_k$ ways

Additive principle

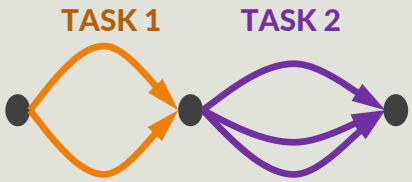
THE PRODUCT RULE

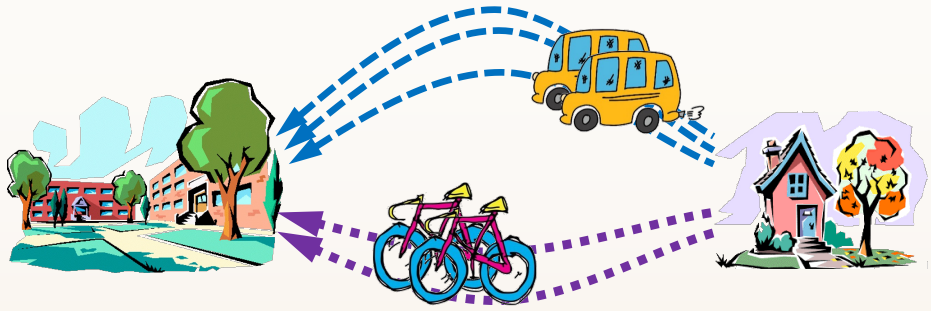
If there are tasks $T_1, T_2, \ldots, T_k$ that can be done in $n_1, n_2, \ldots, n_k$ ways,

Tasks are performed in sequence

then, we can do it in $n_1 \times n_2 \times \ldots \times n_k$ ways

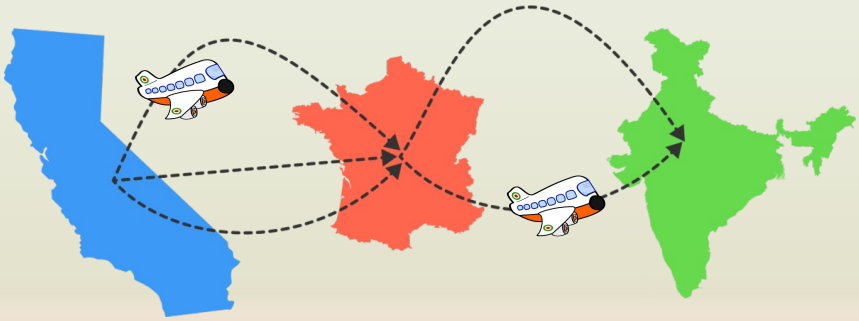
Multiplicative principle





There are 3 bus services from home to school and 2 bike routes. How many ways for a student to go to school? $3 + 2 = 5$

Sum VS. Product Rules



There are 3 flights from USA to EU, and 2 from EU to Asia. How many ways can one fly from US to Asia? $3 \times 2 = 6$

WORKED EXAMPLES

- There are 12 boys in the class and 13 girls. How many possible representatives can the class select if it needs …
  - only one representative, either a boy or a girl.
  - a pair of representatives, one boy and one girl.

WORKED EXAMPLES

- Using only lowercase English alphabets, a label can be either a single letter or a letter followed by a digit. Find a number of possible labels?

Many problems require a combination of both sum & product rules

PRACTICE PROBLEMS

There are 3 routes between Atlanta and Athens, 4 routes between Athens and Augusta, and 2 routes between Atlanta and Augusta.

- How many ways are there to travel from Atlanta to Augusta?
- How many ways are there to travel from Athens to Atlanta?
- How many different ways can the round trip between Augusta and Athens be made if the trip does not go through Atlanta?
- How many different ways can the round trip between Augusta and Athens be made if the trip does not go through Atlanta and each route is used only once?

PRACTICE PROBLEMS

At a gym, there are 7 exercise classes offered every Monday, 14 on Wednesday, and 12 on Friday. Max will join on either Mon & Wed, or Mon & Fri, or Mon & Wed & Fri. How many different combinations of exercise classes can Max pick?

PERMUTATIONS COMBINATIONS

Counting and Combinatorics

ways to select (arrange/order) or group elements from a set to form subsets with or without replacement (repetition)

How many lock combinations are there?

Applying the product rule with four subtasks, one for each digit.

The lock has four-dial with 10 digits (0-9) on each

Repetition of digits on any of the 4 dials is **allowed**

The **order** in which the code is entered **matters**, switching them will not open the lock!

Given 7 letters, how many ways can they be rearranged?

A B C D E F G

Order matter
No repeat

One by one, in any order, pick a letter that is left and put it in place

Task 1: 1st position, 7 ways
Task 2: 2nd position,
Task 3: 3rd position,
⋮ ⋮
Task 7: 7th position,

Given n=7 letters of English alphabet, choose r=3

A B C D E F G

Order matter
No repeat

Choose in order, each letter can be chosen only once (no repeat)

WORKED EXAMPLES

How many 4-digit pin number are there if the first digit must be a prime?

How many 4-digit pin number can be made using digits 1-7, repetition of digits is not allowed, and the digit 4 is always there in the resulting 4-digit pin?

PRACTICE PROBLEMS

A homeowner has two artwork collections: 6 photographs and 4 paintings. How many ways can he decorate his living room if he displays on ___

- only one side of the wall with either all of his 6 photos or all 4 painting?

- two sides of the wall, putting all 6 photographs on one side and all 4 painting on the other?

PRACTICE PROBLEMS

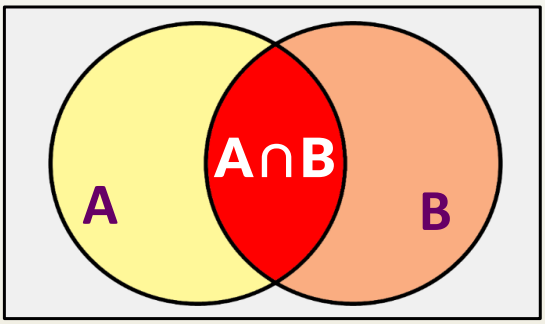
- There are 5 American, 4 British, and 3 Japanese, to be seated in row. How many ways are there to ... seat them so that all person of the same nationality sit together?
- ... seat 2 from each country so that representatives of the same nationality sit together?
- ... seat 2 representatives from each country?

AVOID OVERCOUNTING

Subtraction Rule
Division Rule
Complement Rule

Counting and Combinatorics

The Subtraction Rule



Principle of inclusion and exclusion

If A and B are finite sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

inclusion — exclusion

We "**overcount**" the middle (the intersection region), so we need to "**subtract**" it out.

WORKED EXAMPLES

- How many 7-bit strings are there?

NOTES

A bit string is a sequence of bits, i.e. a sequence of binary digits 0 and 1. We can use it to represent sets in computing, e.g. when U is a set of positive integers less than 6, the odd ones are 10101.

WORKED EXAMPLES

How many 7-bit strings have **exactly one zero**?

The Division Rule

How many 7-bit strings have **exactly two zeros**?

The Complement Rule

How many 7-bit strings have **at least one zero**?

PRACTICE PROBLEMS

- How many 7-bit strings have exactly three zeros?

- How many 7-bit strings do not begin with 101?

WORKED EXAMPLES

Use the complement rule to find the number of 8-bit strings that start with a 0 bit or do not end with the two bits of 00.

PRACTICE PROBLEMS

Use the complement rule to find the number of 8-bit strings that start with a 0 bit and do not end with the two bits of 00. Hint: use De Morgan's and PIE