# Systems of Linear Equations and Matrices

## Introduction to Systems of Linear Equations

# Introduction to Systems of Equations

- Recall that a system of two linear equations in two variables may be written in the general form

$$\begin{aligned} ax + by &= h \\ cx + dy &= k \end{aligned}$$

where $a$, $b$, $c$, $d$, $h$, and $k$ are real numbers and neither $a$ and $b$ nor $c$ and $d$ are both zero.

- Recall that the graph of each equation in the system is a straight line in the plane, so that geometrically, the solution to the system is the point(s) of intersection of the two straight lines $L_1$ and $L_2$, represented by the first and second equations of the system.

# Introduction to Systems of Equations

- We define a linear equation in the n variables $x_1$, $x_2$, …, $x_n$ to be one that can expressed in the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

where *a1*, *a2*, …, *an*, and *b* are real numbers and the *a's* are not all zero.

- In the special case where b = 0. It is called a ***homogeneous linear equation***.

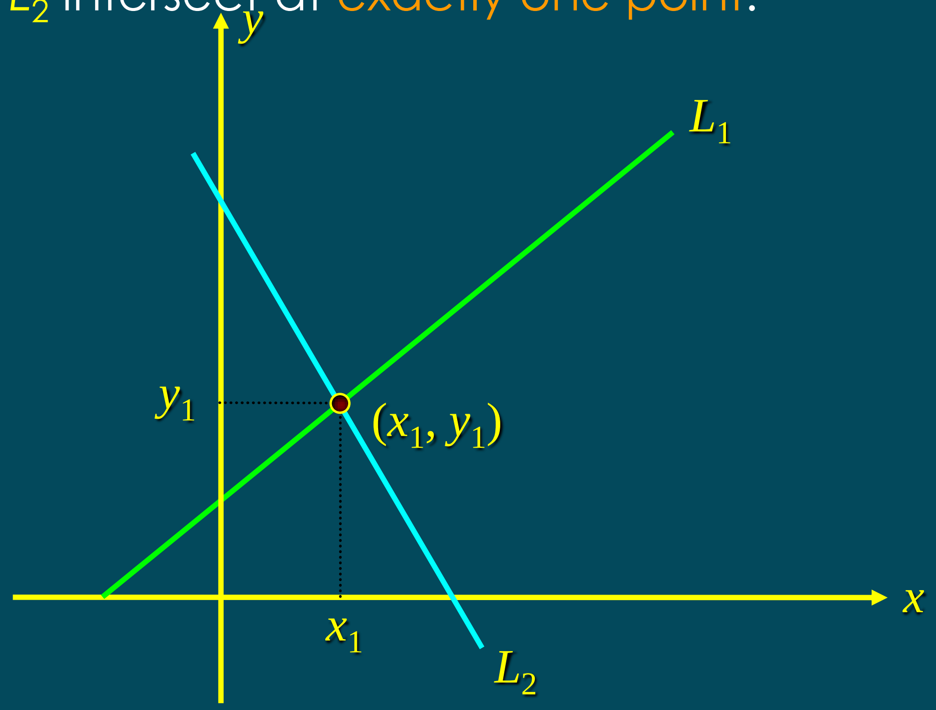
$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0$$

# Introduction to Systems of Equations

- A finite set of linear equations is called a **system of linear equations** (or a **linear system**).
- The variables $x_1, x_2, \ldots, x_n$ are called unknowns.
- A general linear system of $m$ equations in the $n$ unknown
- $$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$
- A solution of a linear system in $n$ unknown can be written as $(s_1, s_2, \ldots, s_n)$ which is called an *ordered n-tuple*, n=2 called an **ordered pair**, n=3 called an **ordered triple**.

# Introduction to Systems of Equations : Linear Systems with Two unknowns

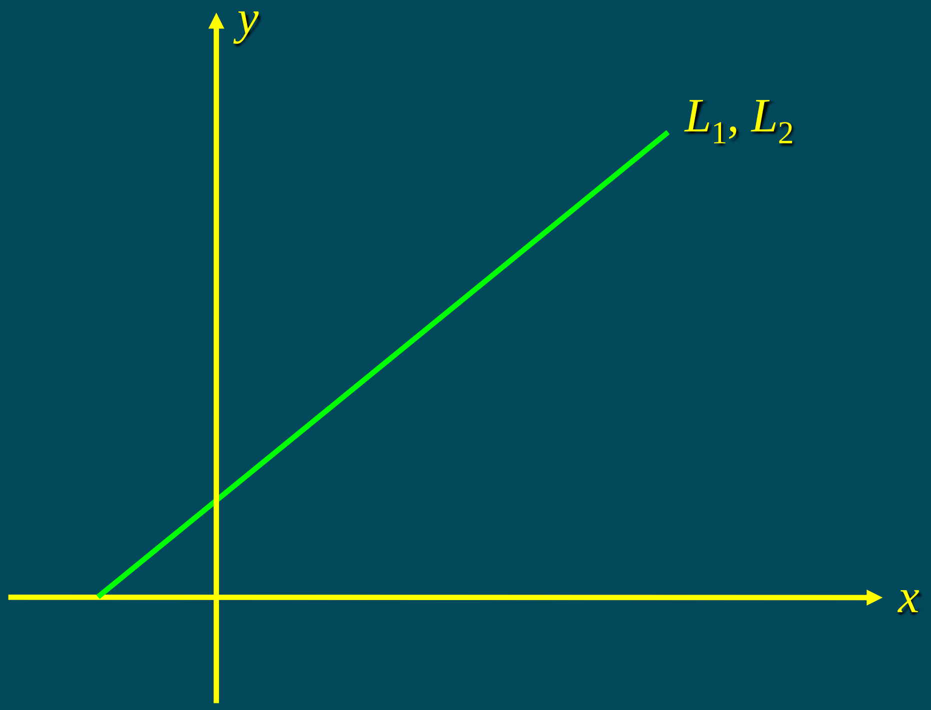
- Given the two straight lines $L_1$ and $L_2$, one and only one of the following may occur:
    1. $L_1$ and $L_2$ intersect at exactly one point.




**Unique solution**
$(x_1, y_1)$

# Introduction to Systems of Equations : Linear Systems with Two unknowns

- Given the two straight lines $L_1$ and $L_2$, one and only one of the following may occur:
  2. $L_1$ and $L_2$ are coincident (coincident lines).

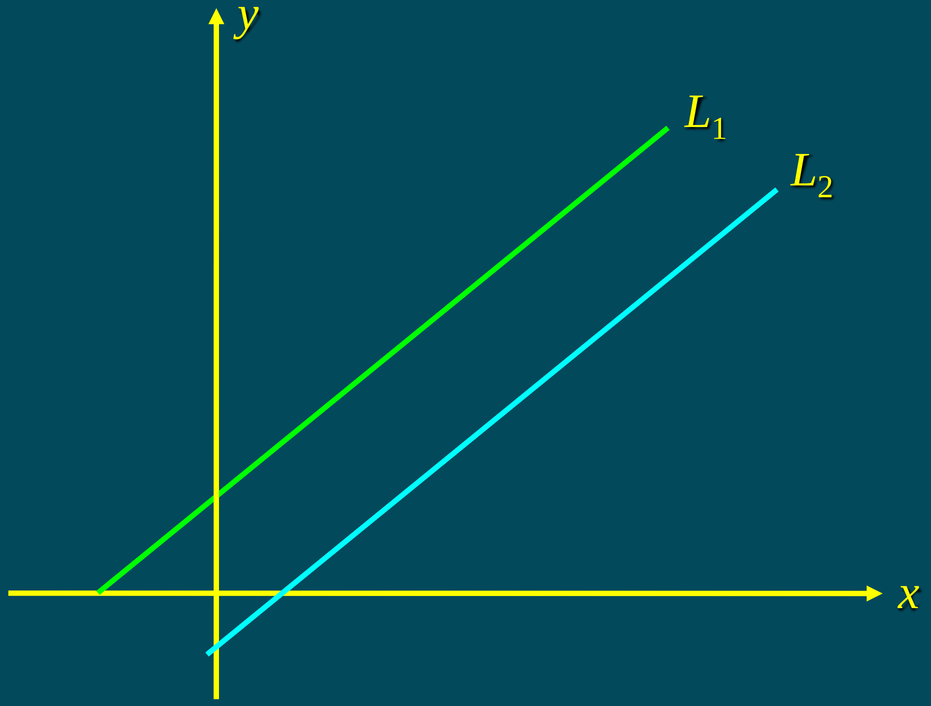



**Infinitely many solutions**

# Introduction to Systems of Equations : Linear Systems with Two unknowns

- Given the two straight lines $L_1$ and $L_2$, one and only one of the following may occur:

   3. $L_1$ and $L_2$ are parallel.




**No solution**

# *Example 1:*

## A System of Equations With Exactly One Solution

- Consider the system

$$\begin{aligned} 2x - y &= 1 \\ 3x + 2y &= 12 \end{aligned}$$

- Solving the first equation for y in terms of x, we obtain

$$y = 2x - 1$$

- Substituting this expression for y into the second equation yields

$$\begin{aligned} 3x + 2(2x - 1) &= 12 \\ 3x + 4x - 2 &= 12 \\ 7x &= 14 \\ x &= 2 \end{aligned}$$

# *Example 1:*

A System of Equations With Exactly One Solution

- Finally, substituting this value of x into the expression for *y* obtained earlier gives

$$\begin{aligned} y &= 2x - 1 \\ &= 2(2) - 1 \\ &= 3 \end{aligned}$$

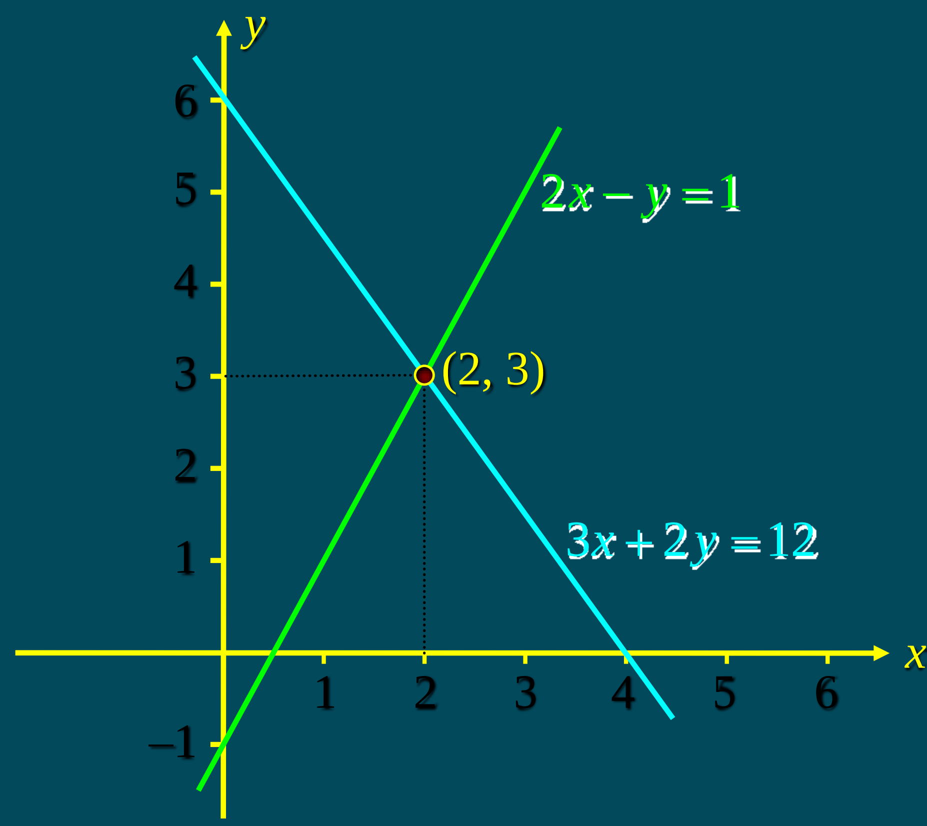
- Therefore, the unique solution of the system is given by

  $x = 2$ and $y = 3$.

# Example 1:

## A System of Equations With Exactly One Solution

- Geometrically, the two lines represented by the two equations that make up the system intersect at the point (2, 3):



# *Example 2:*

## A System of Equations With Infinitely Many Solutions

- Consider the system

$$2x - y = 1$$
$$6x - 3y = 3$$

- Solving the first equation for $y$ in terms of $x$, we obtain

$$y = 2x - 1$$

- Substituting this expression for $y$ into the second equation yields

$$6x - 3(2x - 1) = 3$$
$$6x - 6x + 3 = 3$$
$$0 = 0$$

  which is a true statement.

- This result follows from the fact that the second equation is equivalent to the first.

# *Example 2:*

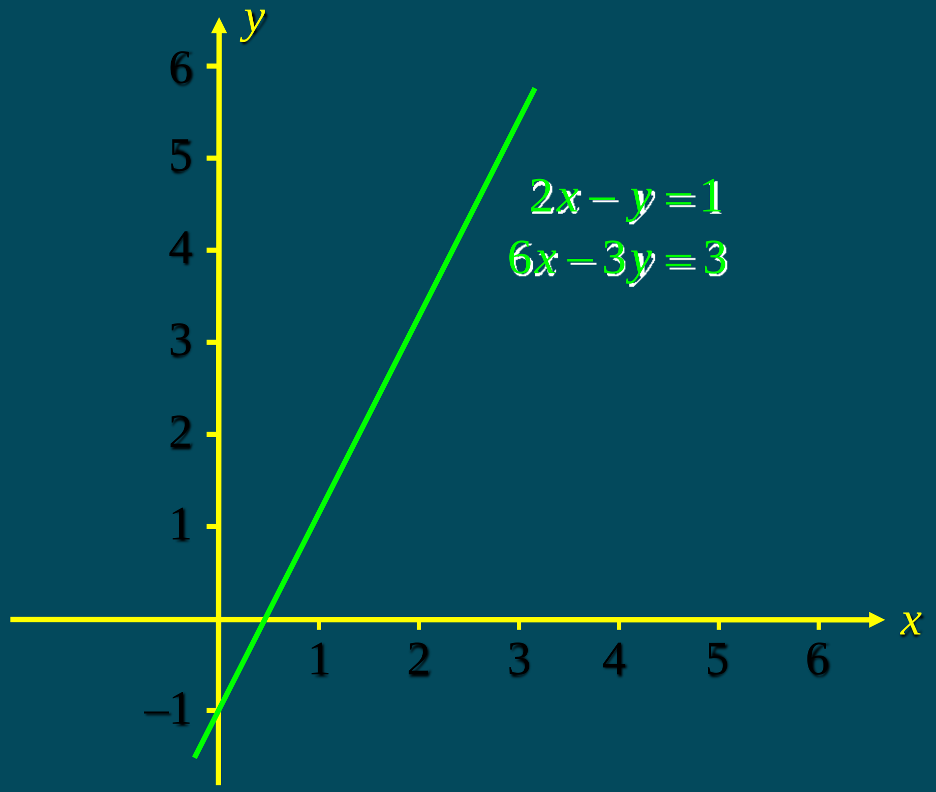
## A System of Equations With Infinitely Many Solutions

- Thus, any order pair of numbers $(x, y)$ satisfying the equation $y = 2x - 1$ constitutes a solution to the system.
- By assigning the value $t$ to $x$, where $t$ is any real number, we find that $y = 2t - 1$ and so the ordered pair $(t, 2t - 1)$ is a solution to the system.
- The variable $t$ is called a parameter.
- *For example:*
  - Setting $t = 0$, gives the point $(0, -1)$ as a solution of the system.
  - Setting $t = 1$, gives the point $(1, 1)$ as another solution of the system.

# Example 2:

## A System of Equations With Infinitely Many Solutions

- Since $t$ represents any real number, there are infinitely many solutions of the system.
- Geometrically, the two equations in the system represent the same line, and all solutions of the system are points lying on the line:



# *Example 3:*

## A System of Equations That Has No Solution

- Consider the system

$$\begin{aligned} 2x - \phantom{3}y &= 1 \\ 6x - 3y &= 12 \end{aligned}$$

- Solving the first equation for $y$ in terms of $x$, we obtain

$$y = 2x - 1$$

- Substituting this expression for $y$ into the second equation yields

$$\begin{aligned} 6x - 3(2x - 1) &= 12 \\ 6x - 6x + 3 &= 12 \\ 0 &= 9 \end{aligned}$$

which is clearly impossible.

- Thus, there is no solution to the system of equations.

# *Example 3:*

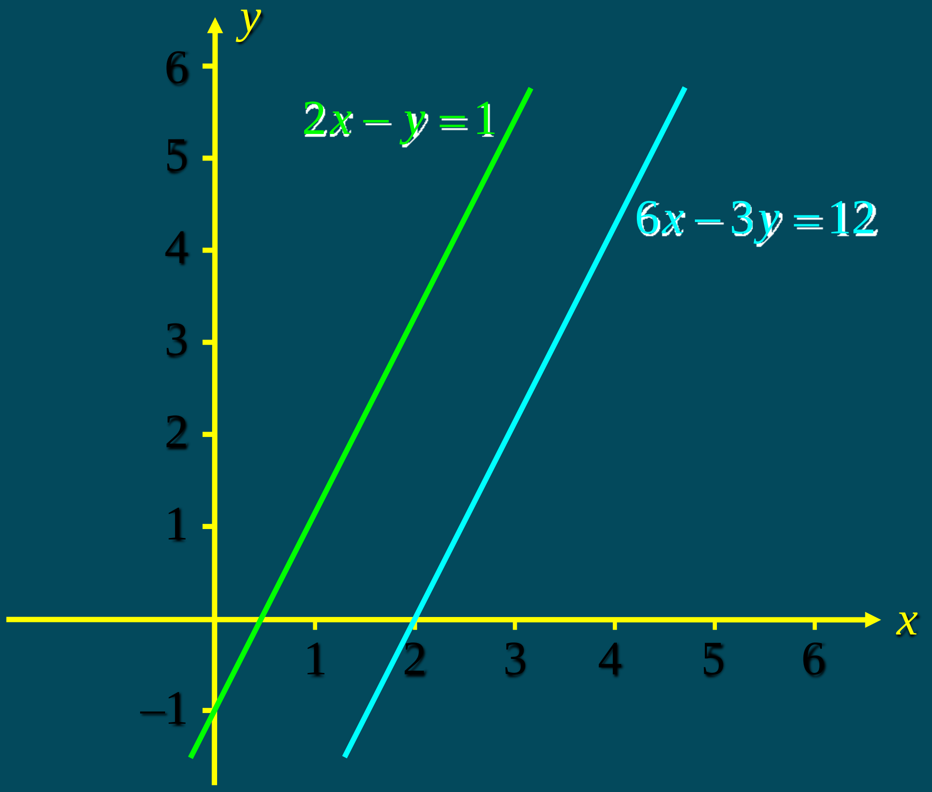
## A System of Equations That Has No Solution

- To interpret the situation geometrically, cast both equations in the slope-intercept form, obtaining

$$y = 2x - 1 \quad \text{and} \quad 3y = 6x - 12$$

which shows that the lines are parallel.

- Graphically:



# Introduction to Systems of Equations : Linear Systems with **Two** unknowns

There are three possibilities: no solutions, one solution or infinitely many solution.

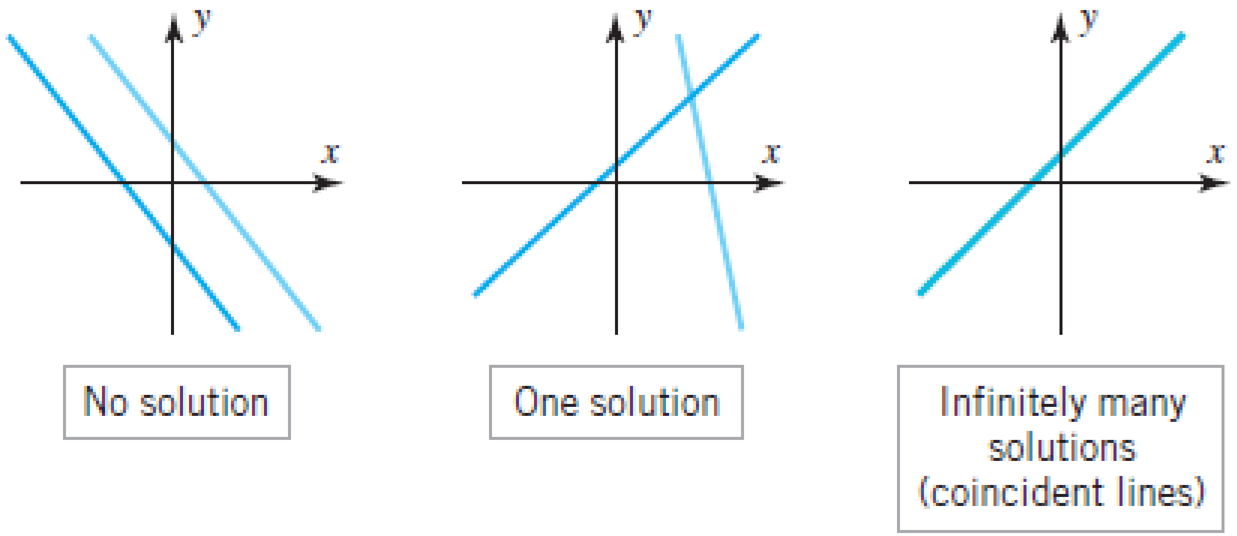



Figure 1.1.1

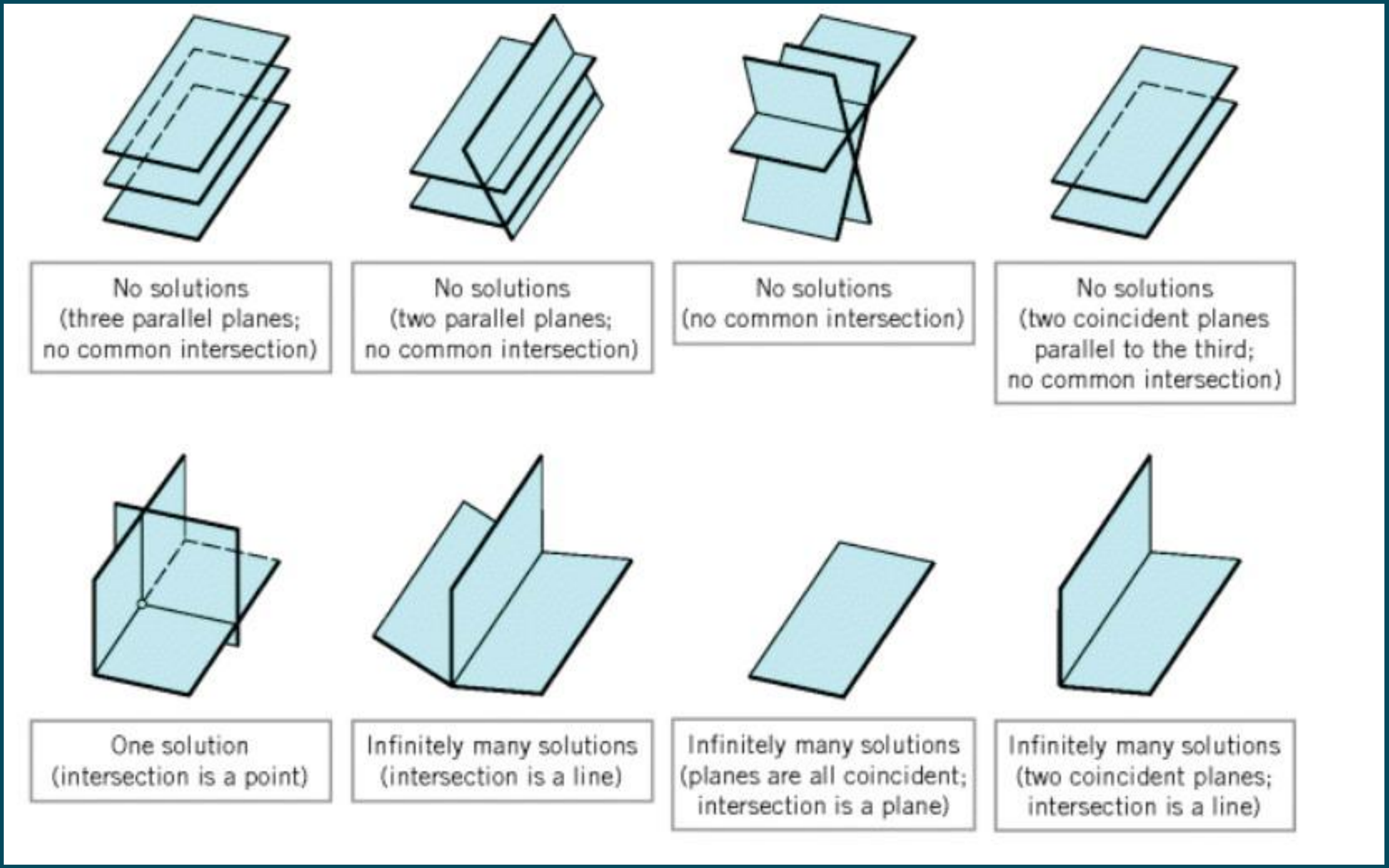
# Introduction to Systems of Equations : Linear Systems with **Three** unknowns

- A linear system of three equations in three unknowns in which the graphs of the equations are planes.

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

- There are three possibilities: no solutions, one solution or infinitely many solution
- Graphically (see next slide)
- In general, we say that a linear system is **consistent** if it has at least one solution (*one solution* or *infinitely many solutions*), and **inconsisten**t if it has no solutions.

# Introduction to Systems of Equations : Linear Systems with **Three** unknowns



# *Example*

- Solve the following system of equations:

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + \ \ y + 2z &= \ \ 2 \end{aligned}$$

<u>Solution</u>

# Example

- Solve the following system of equations:

$$
\begin{aligned}
2x + 4y + 6z &= 22 \\
3x + 8y + 5z &= 27 \\
-x + \phantom{4}y + 2z &= \phantom{2}2
\end{aligned}
$$

<u>Solution</u>

- First, we transform this system into an equivalent system in which the coefficient of x in the first equation is 1:

Toggle slides back and forth to compare before and changes

$$
\begin{aligned}
2x + 4y + 6z &= 22 \quad \longleftarrow \text{Multiply the equation by } 1/2 \\
3x + 8y + 5z &= 27 \\
-x + \phantom{4}y + 2z &= \phantom{2}2
\end{aligned}
$$

# *Example*

- Solve the following system of equations:

$$
\begin{aligned}
2x + 4y + 6z &= 22 \\
3x + 8y + 5z &= 27 \\
-x + \phantom{4}y + 2z &= \phantom{2}2
\end{aligned}
$$

<u>Solution</u>

- First, we transform this system into an equivalent system in which the coefficient of x in the first equation is 1:

**Toggle slides back and forth to compare before and changes**

$$
\begin{aligned}
x + 2y + 3z &= 11 \\
3x + 8y + 5z &= 27 \\
-x + \phantom{2}y + 2z &= \phantom{1}2
\end{aligned}
$$

**Multiply the first equation by** 1/2

# Example

- Solve the following system of equations:

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + \;\; y + 2z &= \;\; 2 \end{aligned}$$

<u>Solution</u>

- Next, we eliminate the variable x from all equations except the first:

Toggle slides back and forth to compare before and changes

$$\begin{aligned} x + 2y + 3z &= 11 \\ 3x + 8y + 5z &= 27 \\ -x + \;\; y + 2z &= \;\; 2 \end{aligned}$$

Replace by the sum of
– 3 X the first equation
+ the second equation

# Example

- Solve the following system of equations:

$$
\begin{aligned}
2x + 4y + 6z &= 22\\
3x + 8y + 5z &= 27\\
-x + \phantom{4}y + 2z &= \phantom{2}2
\end{aligned}
$$

<u>Solution</u>

- Next, we eliminate the variable x from all equations except the first:

Toggle slides back and forth to compare before and changes

$$
\begin{aligned}
x + 2y + 3z &= 11\\
2y - 4z &= -6\\
-x + \phantom{2}y + 2z &= \phantom{-}2
\end{aligned}
$$

Replace by the sum of
– 3 X the first equation
+ the second equation

# Example

- Solve the following system of equations:

$$
\begin{aligned}
2x+4y+6z&=22\\
3x+8y+5z&=27\\
-x+\ \ y+2z&=\ 2
\end{aligned}
$$

<u>Solution</u>

- Next, we eliminate the variable x from all equations except the first:

**Toggle slides back and forth to compare before and changes**

$$
\begin{aligned}
x+2y+3z&=11\\
2y-4z&=-6\\
-x+\ \ y+2z&=\ 2
\end{aligned}
$$

**Replace by the sum of the first equation + the third equation**

# Example

- Solve the following system of equations:

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + \ \ y + 2z &= \ \ 2 \end{aligned}$$

<u>Solution</u>

- Next, we eliminate the variable x from all equations except the first:

Toggle slides back and forth to compare before and changes

$$\begin{aligned} x + 2y + 3z &= 11 \\ 2y - 4z &= -6 \\ 3y + 5z &= 13 \end{aligned}$$

Replace by the sum of the first equation + the third equation

# Example

- Solve the following system of equations:

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + \ \ y + 2z &= \ \ 2 \end{aligned}$$

Solution

- Then we transform so that the coefficient of $y$ in the second equation is 1:

Toggle slides back and forth to compare before and changes

$$\begin{aligned} x + 2y + 3z &= 11 \\ 2y - 4z &= -6 \\ 3y + 5z &= 13 \end{aligned}$$

Multiply the second equation by 1/2

# Example

- Solve the following system of equations:

$$
\begin{aligned}
2x+4y+6z&=22\\
3x+8y+5z&=27\\
-x+\ \ y+2z&=\ \ 2
\end{aligned}
$$

Solution

- Then we transform so that the coefficient of $y$ in the second equation is 1:

Toggle slides back and forth to compare before and changes

$$
\begin{aligned}
x+2y+3z&=11\\
y-2z&=-3\\
3y+5z&=13
\end{aligned}
$$

**Multiply the second equation by** 1/2

# Example

- Solve the following system of equations:

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + \quad y + 2z &= \ \ 2 \end{aligned}$$

Solution

- We now eliminate $y$ from all equations except the second:

Toggle slides back and forth to compare before and changes

$$\begin{aligned} x + 2y + 3z &= 11 \\ y - 2z &= -3 \\ 3y + 5z &= 13 \end{aligned}$$

**Replace by the sum of the first equation +** (–2) × **the second equation**

# Example

- Solve the following system of equations:

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + \phantom{4}y + 2z &= \phantom{2}2 \end{aligned}$$

<u>Solution</u>

- We now eliminate *y* from all equations except the second:

Toggle slides back and forth to compare before and changes

$$\begin{aligned} x \qquad\quad + 7z &= 17 \\ y - 2z &= -3 \\ 3y + 5z &= 13 \end{aligned}$$

**Replace by the sum of the first equation + (–2) × the second equation**

# Example

- Solve the following system of equations:

$$2x + 4y + 6z = 22$$
$$3x + 8y + 5z = 27$$
$$-x + y + 2z = 2$$

<u>Solution</u>

- We now eliminate $y$ from all equations except the second:

**Toggle slides back and forth to compare before and changes**

$$x \qquad + 7z = 17$$
$$y - 2z = -3$$
$$3y + 5z = 13$$

**Replace by the sum of the third equation + (–3) × the second equation**

# Example

- Solve the following system of equations:

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + \quad y + 2z &= \ \ 2 \end{aligned}$$

Solution

- We now eliminate $y$ from all equations except the second:

Toggle slides back and forth to compare before and changes

$$\begin{aligned} x \qquad\quad + 7z &= 17 \\ y - 2z &= -3 \\ 11z &= 22 \end{aligned}$$

**Replace by the sum of the third equation + (–3) × the second equation**

# Example

- Solve the following system of equations:

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + y + 2z &= 2 \end{aligned}$$

Solution

- Now we transform so that the coefficient of z in the third equation is 1:

Toggle slides back and forth to compare before and changes

$$\begin{aligned} x \qquad + 7z &= 17 \\ y - 2z &= -3 \\ 11z &= 22 \end{aligned}$$

Multiply the third equation by 1/11

# Example

- Solve the following system of equations:

$$
\begin{aligned}
2x + 4y + 6z &= 22\\
3x + 8y + 5z &= 27\\
-x + \phantom{4}y + 2z &= \phantom{2}2
\end{aligned}
$$

Solution

- Now we transform so that the coefficient of $z$ in the third equation is 1:

Toggle slides back and forth to compare before and changes

$$
\begin{aligned}
x \qquad\quad + 7z &= 17\\
y - 2z &= -3\\
z &= 2
\end{aligned}
$$

**Multiply the third equation by** 1/11

# Example

- Solve the following system of equations:

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + \quad y + 2z &= \ \ 2 \end{aligned}$$

<u>Solution</u>

- We now eliminate z from all equations except the third:

Toggle slides back and forth to compare before and changes

$$\begin{aligned} x \qquad\quad + 7z &= 17 \\ y - 2z &= -3 \\ z &= 2 \end{aligned}$$

Replace by the sum of the first equation + (–7) × the third equation

# Example

- Solve the following system of equations:

$$
\begin{aligned}
2x + 4y + 6z &= 22 \\
3x + 8y + 5z &= 27 \\
-x + \ \ y + 2z &= \ \ 2
\end{aligned}
$$

<u>Solution</u>

- We now eliminate z from all equations except the third:

$$
\begin{aligned}
x \qquad\qquad &= 3 \\
y - 2z &= -3 \\
z &= 2
\end{aligned}
$$

**Replace by the sum of the first equation + (–7) × the third equation**

**Toggle slides back and forth to compare before and changes**

# Example

- Solve the following system of equations:

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + \quad y + 2z &= \quad 2 \end{aligned}$$

<u>Solution</u>

- We now eliminate z from all equations except the third:

**Toggle slides back and forth to compare before and changes**

$$\begin{aligned} x \qquad\qquad &= 3 \\ y - 2z &= -3 \\ z &= 2 \end{aligned}$$

**Replace by the sum of the second equation + 2 X the third equation**

# Example

- Solve the following system of equations:

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + \ \ y + 2z &= \ \ 2 \end{aligned}$$

<u>Solution</u>

- We now eliminate z from all equations except the third:

Toggle slides back and forth to compare before and changes

$$\begin{aligned} x \qquad\qquad &= 3 \\ y \qquad &= 1 \\ z &= 2 \end{aligned}$$

**Replace by the sum of the second equation +** 2 X **the third equation**

# Example

- Solve the following system of equations:

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + \quad y + 2z &= \ \ 2 \end{aligned}$$

Solution

- Thus, the solution to the system is $x = 3$, $y = 1$, and $z = 2$.

$$\begin{aligned} x \qquad\qquad &= 3 \\ y \qquad &= 1 \\ z &= 2 \end{aligned}$$

# Augmented Matrices

- Matrices are rectangular arrays of numbers that can aid us by eliminating the need to write the variables at each step of the reduction.
- For example, the system

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + \phantom{4}y + 2z &= \phantom{2}2 \end{aligned}$$

may be represented by the augmented matrix

**Coefficient Matrix**

$$\left[\begin{array}{ccc|c} 2 & 4 & 6 & 22 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{array}\right]$$

# Elementary Row Operations On A Matrix

1. Interchange any two rows ($R_i$ and $R_j$).

   **Notation**: $R_i \leftrightarrow R_j$

2. Replace any row ($R_i$) by a nonzero constant ($k$) multiple of itself.

   **Notation**: $kR_i$

3. Replace any row by the sum of that row and a constant multiple of any other row.

   **Notation**: $kR_i + R_j \rightarrow R_j$

   (Add $k$ times row $R_i$ to row $R_j$ (but row $R_i$ remains the same)

# Example 1:

Use elementary row operations and augmented matrix to solve the following linear system:

$$\begin{aligned} 3x - y &= 1 \\ x + 2y &= 5 \end{aligned}$$

# Example 2:

Use elementary row operations and augmented matrix to solve the following linear system:

$$
\begin{aligned}
x + y + 2z &= 9 \\
2x + 4y - 3z &= 1 \\
3x + 6y - 5z &= 0
\end{aligned}
$$

# Diagonal, Triangular and Symmetric Matrices

A general $n \times n$ diagonal matrix $D$ can be written as

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

A general $4 \times 4$ upper triangular matrix

A general $4 \times 4$ lower triangular matrix

**DEFINITION 1** A square matrix $A$ is said to be *symmetric* if $A = A^T$.

# Trace of a Matrix

**DEFINITION 8** If $A$ is a square matrix, then the ***trace of*** $A$, denoted by $\operatorname{tr}(A)$, is defined to be the sum of the entries on the main diagonal of $A$. The trace of $A$ is undefined if $A$ is not a square matrix.

## ▶ EXAMPLE 11 Trace of a Matrix

The following are examples of matrices and their traces.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{bmatrix}$$

$$\operatorname{tr}(A) = a_{11} + a_{22} + a_{33} \qquad \operatorname{tr}(B) = -1 + 5 + 7 + 0 = 11$$ ◀