29. Determinants

Let **A be a square matrix, then the *determinant of A*, denoted by** $|\mathbf{A}|$, is a real number.

Definitions

1) If $\mathbf{A} = [a_{11}]$ is a square matrix of order 1, then $|\mathrm{A}| = |a_{11}| = a_{11}$.
2) If $\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then $|\mathrm{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

Example 1: Evaluate each determinant.

1.1) $\begin{vmatrix} \mathbf{7} & \mathbf{1} \\ \mathbf{2} & \mathbf{3} \end{vmatrix}$.

1.2) $\begin{vmatrix} \mathbf{-2} & \mathbf{4} \\ \mathbf{0} & \mathbf{5} \end{vmatrix}$

Minor and Cofactor of the entry a_{ij}

Given the matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

The **minor** of a_{ij} is the determinant obtained by deleting the entries in row i and column j. For examples,

the minor of $a_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$

and the minor of $a_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$

The **cofactor** of a_{ij}, denoted by c_{ij}, is the product of $(-1)^{i+j}$ and the minor of a_{ij}

For examples, the cofactor of a_{21} is $c_{21} = (-1)^{2+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$

and the cofactor of a_{23} is $c_{23} = (-1)^{2+3}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$

Example 2: For each determinant, find the minor and cofactor of a_{21}.

a) $\begin{vmatrix} 2 & -1 & 3 \\ 3 & 0 & -5 \\ 2 & 1 & 1 \end{vmatrix}$. b) $\begin{vmatrix} 0 & 1 & 0 & 3 \\ 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 1 & 2 & 3 & 0 \end{vmatrix}$

Determinant of a Square Matrix

To find the determinant of any square matrix **A** of order $n > 2$, select *any* row (or column) of **A**, and multiply each entry in the row (or column) by its cofactor. The sum of these products is defined to be the determinant of **A** and is called a **determinant of order *n***.

Example 3: Evaluate the determinant of $\begin{vmatrix} 2 & -1 & 3 \\ 3 & 0 & -5 \\ 2 & 1 & 1 \end{vmatrix}$.

Applying the rule above to the first row, we obtain

$$\begin{aligned}
\begin{vmatrix} 2 & -1 & 3 \\ 3 & 0 & -5 \\ 2 & 1 & 1 \end{vmatrix} &= 2(-1)^{1+1}\begin{vmatrix} 0 & -5 \\ 1 & 1 \end{vmatrix} + (-1)(-1)^{1+2}\begin{vmatrix} 3 & -5 \\ 2 & 1 \end{vmatrix} + (3)(-1)^{1+3}\begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix} \\
&= 2(1)5 + (-1)(-1)13 + (3)(1)3 \\
&= 10 + 13 + 9 \\
&= 32
\end{aligned}$$

Example 4: In Example 3, one can expand along the second column.

$$\begin{aligned}
\begin{vmatrix} 2 & -1 & 3 \\ 3 & 0 & -5 \\ 2 & 1 & 1 \end{vmatrix} &= (-1)(-1)^{1+2}\begin{vmatrix} 3 & -5 \\ 2 & 1 \end{vmatrix} + 0 + (1)(-1)^{3+2}\begin{vmatrix} 2 & 3 \\ 3 & -5 \end{vmatrix} \\
&= 13 + 0 + 19 \\
&= 32
\end{aligned}$$

We can also evaluate a determinant of order 3 by copying the first and second columns of the determinant to its right, thus giving

$$\begin{array}{|ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

$$- \quad - \quad - \qquad + \quad + \quad +$$

Then take the sum of the three products of the entries on the arrows extending to the right, and subtract from this the sum of the three products of the entries on the arrows extending to the left. The result is

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31}$$

Note: The formula above works only for a determinant of order 3.

Example 5: Find $\begin{vmatrix} 0 & 1 & 0 & 3 \\ 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 1 & 2 & 3 & 0 \end{vmatrix}$.

Properties of Determinant

1. If each of the entries in a row (or column) of **A** is 0, then $|\mathbf{A}| = 0$.

   For example, $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{vmatrix} = 0$.

2. If two rows (or columns) of **A** are identical, then $|\mathbf{A}| = 0$.

   For example, $\begin{vmatrix} 3 & 1 & 4 \\ 2 & 7 & 7 \\ 2 & 7 & 7 \end{vmatrix} = 0$.

3. If **A** is upper (or lower) triangular, then $|\mathbf{A}|$ is equal to the product of the main diagonal entries.

   For example,. $\begin{vmatrix} 3 & 1 & 4 & 0 \\ 0 & 7 & 2 & -1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 2 \end{vmatrix} = 3(7)(1)2 = 42$

4. If **B** is the matrix obtained by adding a multiple of one row (or column) of **A** to another row (or column), then $|\mathbf{B}| = |\mathbf{A}|$.

   For example, if $|\mathbf{A}| = \begin{vmatrix} \mathbf{-2} & \mathbf{4} \\ \mathbf{0} & \mathbf{5} \end{vmatrix} = -10$, and let **B** be the matrix obtained from **A** by adding 2 times row 1 to row 2, then $\mathbf{B} = \begin{bmatrix} -2 & 4 \\ -4 & 13 \end{bmatrix}$. Thus,

   $$|\mathbf{B}| = \begin{vmatrix} -2 & 4 \\ -4 & 13 \end{vmatrix} = -26-(-16) = -10$$

5. If **B** is the matrix obtained by interchanging two rows (or columns) of **A**, then $|\mathbf{B}| = -|\mathbf{A}|$, or equivalently, $|\mathbf{A}| = -|\mathbf{B}|$.
6. If **B** is the matrix obtained by multiplying each entry of a row(or column) of **A** by the same number k, then $|\mathbf{B}| = k|\mathbf{A}|$.
7. If k is a constant and **A** has order n, then $|k\mathbf{A}| = k^n|\mathbf{A}|$.
8. The determinant of the product of two matrices of order n is the product of their determinants. That is, $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$.