Systems of Linear Equations and Matrices

Reduced Row Echelon Form
&
Elimination Methods

Row Echelon and Reduced Row Echelon Form

To be of the **reduced row echelon** form, a matrix must have the following properties:

1. If a row does not consists entirely of zeros, then the first nonzero number in the row is a 1. We call this a leading 1.
2. If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix .
3. In any two successive rows that do not consists entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.
4. Each column that contains a leading 1 has zeros everywhere else in that column

A matrix that has the first three properties is said to be row echelon form.

Example 1: Determine the matrices are in **Row echelon** form or **Reduced row echelon** form.

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 1: Determine the matrices are in **Row echelon** form or **Reduced row echelon** form.

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Reduced row echelon

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{vmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

Row echelon

Example 2: Determine the matrices are in **Row echelon** form or **Reduced row echelon** form.

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

The *'s represent any real numbers..

Example 2: Determine the matrices are in **Row echelon** form or **Reduced row echelon** form.

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Row echelon

The *'s represent any real numbers..

Example 3: Determine the matrices are in **Row echelon** form or **Reduced row echelon** form.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

The *'s represent any real numbers..

Example 3: Determine the matrices are in **Row echelon** form or **Reduced row echelon** form.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Reduced row echelon

The *'s represent any real numbers..

Elimination Methods

Once the **augmented matrix** of a *system of linear equations* is in **reduced row echelon** form, it is easy to solve the system.

- Two Elimination procedure used to reduce any matrix to row echelon
  - Gaussian Elimination
  - Gauss – Jordan Elimination

Step-by-Step Elimination Procedure

- **Step 1**: Locate the leftmost column that does not consist entirely zeros.
- **Step 2**: Interchange the top row with another row, if necessary, to bring a nonzero entry to the top of the column found in Step 1
- **Step 3**: If the entry that is now at the top of the column found in Step 1 is a, multiply the first row by 1/a in order to introduce a leading 1.
- **Step 4**: Add suitable multiple of the top row to the rows below so that all entries below the leading 1 become zeros.
- **Step 5**: Now cover the top row in the matrix and begin again with Step 1 applied to the sub-matrix that remains. Continue in this way until the entire is in row echelon form.

Step-by-Step Elimination Procedure

- **Step 6**: Beginning with the last nonzero row and working upward, add suitable multiples of each row to the rows above to introduce zeros above the leading 1's. The last matrix is in reduced row echelon form.

- The procedure (algorithm) for reducing a matrix to a row echelon form is called **Gaussian elimination**. (Step 1 to 5)
- The procedure (algorithm) for reducing a matrix to reduced row echelon form is called **Gaussian - Jordan elimination**. (Step 1 to 6)

Step-by-Step Elimination Procedure

- **Gaussian - Jordan elimination** consists of two parts
  - **Forward phase** in which zeros are introduces below the leading 1's.
  - **Backward phase** in which zeros are introduced above the leading 1's.

- **Gaussian elimination** consists of only one part, the <u>forward phase</u>.

Summary of the Gauss-Jordan Method

- The Gauss-Jordan elimination method is a technique for solving systems of linear equations of any size.
- The operations of the Gauss-Jordan method are

1. Interchange any two equations.
2. Replace an equation by a nonzero constant multiple of itself.
3. Replace an equation by the sum of that equation and a constant multiple of any other equation.

Summary of Gaussian & Gauss-Jordan Methods

Gaussian

↓

Row Echelon Form

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

Gauss-Jordan

↓

Reduced Row Echelon Form

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Example

- Solve the following system of equations:

$$\begin{aligned} 2x+4y+6z&=22\\ 3x+8y+5z&=27\\ -x+\ \ y+2z&=\ \ 2 \end{aligned}$$

<u>Solution</u>

Augmented Matrices

- Matrices are rectangular arrays of numbers that can aid us by eliminating the need to write the variables at each step of the reduction.
- For example, the system

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + y + 2z &= 2 \end{aligned}$$

may be represented by the augmented matrix

$$\begin{array}{c} \textbf{Coefficient Matrix} \\ \left[\begin{array}{ccc|c} 2 & 4 & 6 & 22 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{array}\right] \end{array}$$

Matrices and Gaussian, Gauss-Jordan

- Every step in the Gaussian, Gauss-Jordan elimination methods can be expressed with matrices, rather than systems of equations, thus simplifying the whole process:
- Steps expressed as systems of equations:

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + y + 2z &= 2 \end{aligned}$$

- Steps expressed as augmented matrices:

$$\left[\begin{array}{ccc|c} 2 & 4 & 6 & 22 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{array}\right]$$

Toggle slides back and forth to compare before and changes

Gaussian Elimination

- Every step in the Gauss-Jordan elimination method can be expressed with matrices, rather than systems of equations, thus simplifying the whole process:
- Steps expressed as systems of equations:

$$\begin{aligned} x + 2y + 3z &= 11 \\ 3x + 8y + 5z &= 27 \\ -x + y + 2z &= 2 \end{aligned}$$

- Steps expressed as augmented matrices:

Toggle slides back and forth to compare before and changes

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{array}\right]$$

Gaussian Elimination

- Every step in the Gauss-Jordan elimination method can be expressed with matrices, rather than systems of equations, thus simplifying the whole process:
- Steps expressed as systems of equations:

$$
\begin{aligned}
x + 2y + 3z &= 11 \\
2y - 4z &= -6 \\
-x + y + 2z &= 2
\end{aligned}
$$

- Steps expressed as augmented matrices:

$$
\left[\begin{array}{ccc|c}
1 & 2 & 3 & 11 \\
0 & 2 & -4 & -6 \\
-1 & 1 & 2 & 2
\end{array}\right]
$$

Toggle slides back and forth to compare before and changes

Gaussian Elimination

- Every step in the Gauss-Jordan elimination method can be expressed with matrices, rather than systems of equations, thus simplifying the whole process:
- Steps expressed as systems of equations:

$$\begin{aligned} x + 2y + 3z &= 11 \\ 2y - 4z &= -6 \\ 3y + 5z &= 13 \end{aligned}$$

- Steps expressed as augmented matrices:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ 0 & 3 & 5 & 13 \end{array}\right]$$

Toggle slides back and forth to compare before and changes

Gaussian Elimination

- Every step in the Gauss-Jordan elimination method can be expressed with matrices, rather than systems of equations, thus simplifying the whole process:
- Steps expressed as systems of equations:

$$\begin{aligned} x + 2y + 3z &= 11 \\ y - 2z &= -3 \\ 3y + 5z &= 13 \end{aligned}$$

- Steps expressed as augmented matrices:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 0 & 1 & -2 & -3 \\ 0 & 3 & 5 & 13 \end{array}\right] \qquad \ldots?$$

Toggle slides back and forth to compare before and changes

Gaussian → Gauss-Jordan

- Every step in the Gauss-Jordan elimination method can be expressed with matrices, rather than systems of equations, thus simplifying the whole process.
- Steps expressed as systems of equations:

$$
\begin{aligned}
x + 2y + 3z &= 11 \\
y - 2z &= -3 \\
3y + 5z &= 13
\end{aligned}
$$

- Steps expressed as augmented matrices:

Toggle slides back and forth to compare before and changes

$$
\left[\begin{array}{ccc|c}
1 & 2 & 3 & 11 \\
0 & 1 & -2 & -3 \\
0 & 3 & 5 & 13
\end{array}\right]
$$

...?

Gauss-Jordan Elimination

- Every step in the Gauss-Jordan elimination method can be expressed with matrices, rather than systems of equations, thus simplifying the whole process:
- Steps expressed as systems of equations:

$$\begin{aligned} x \qquad\qquad &= 3 \\ y \qquad &= 1 \\ z &= 2 \end{aligned}$$

- Steps expressed as augmented matrices:

Toggle slides back and forth to compare before and changes

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Reduced Row Echelon Form