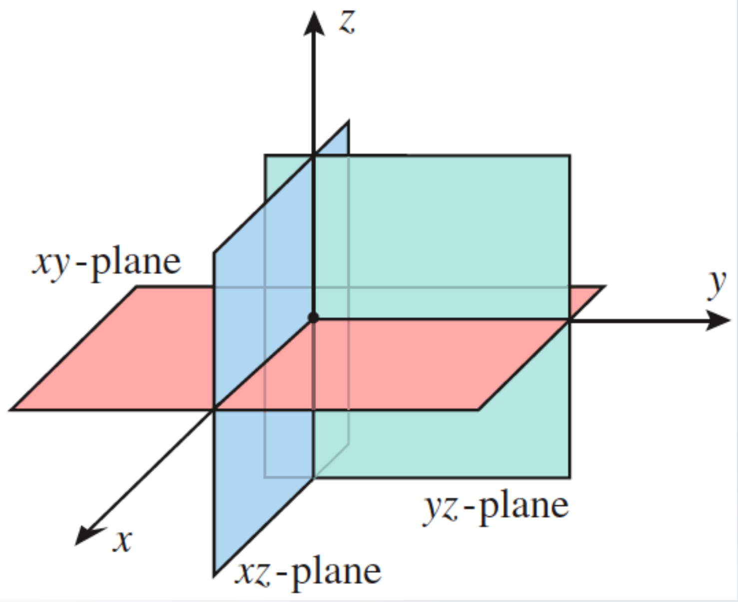
ITCS 175 - Advance Mathematics I for Computer Science

# Three-Dimensional Space; Vectors and Lines

# 1. Rectangular Coordinates in 3-Space




- Coordinate Axes

Three mutually perpendicular coordinate lines *x*-axis, *y*-axis, *z*-axis (intersecting at the origin)

- Coordinate Planes

Three planes determined by coordinate axes *xy*-plane, *xz*-plane, *yz*-plane.

# Point in a Plane (2-space)

In a ***two-dimensional space*** (or 2-space) to locate a point in a plane we represent any point in the plane by an ordered pair ($a$,$b$) where $a$ and $b$ are real numbers.

($a$ is the ***x*-coordinate** and $b$ is the ***y*-coordinate**)

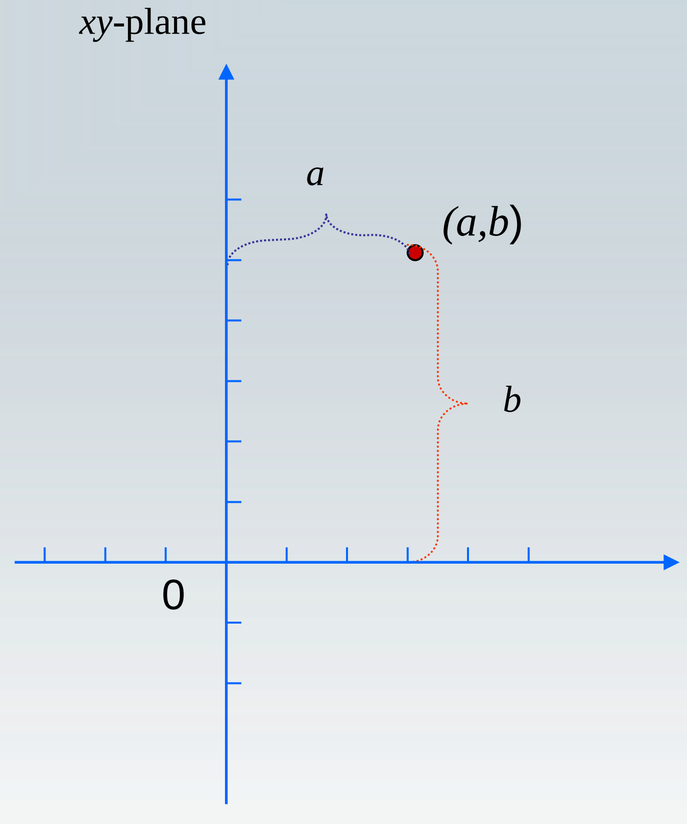



Figure 1: The point in the plane

# Example 1

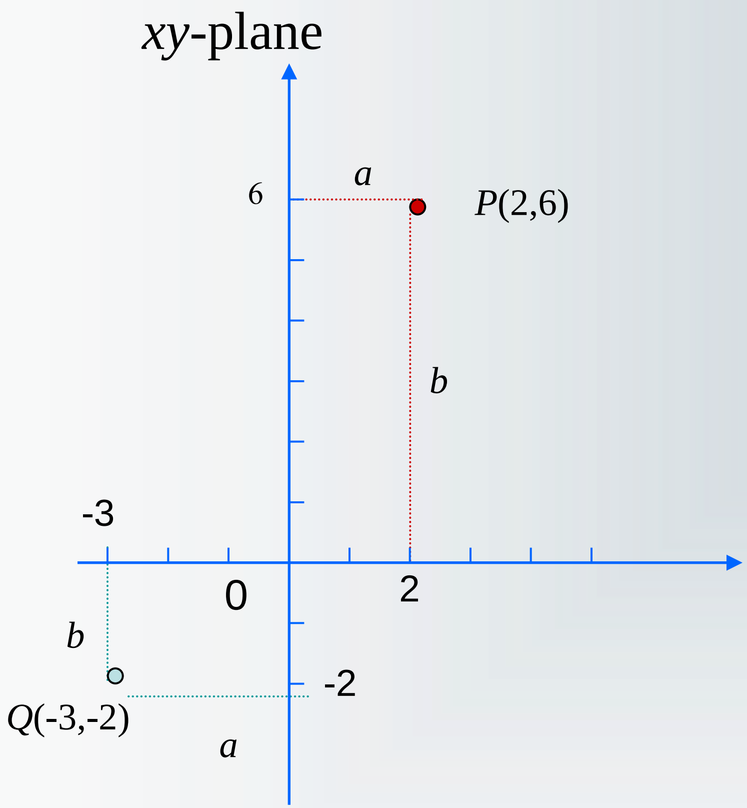



Figure 2: Sample of the points

In Figure 2 :

The point $P$ is represented by (2, 6) that is $a = 2$ and $b = 6$.

The point $Q$ is represented by (-3, -2) that is $a = -3$ and $b = -2$.

# Point in 3-Space

In a ***three-dimensional space*** (or 3-space), we represent each point *P* by an ordered triple

$P(a, b, c)$.

*a* is the ***x*-coordinate,**
*b* is the ***y*-coordinate,**
and *c* is the ***z*-coordinate**

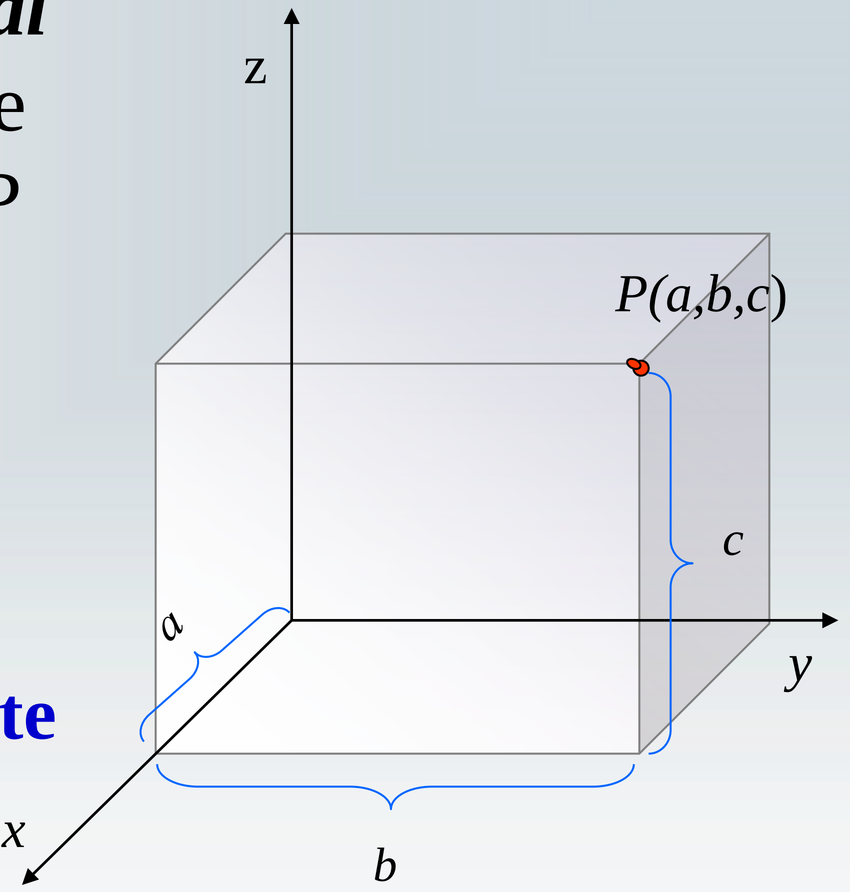
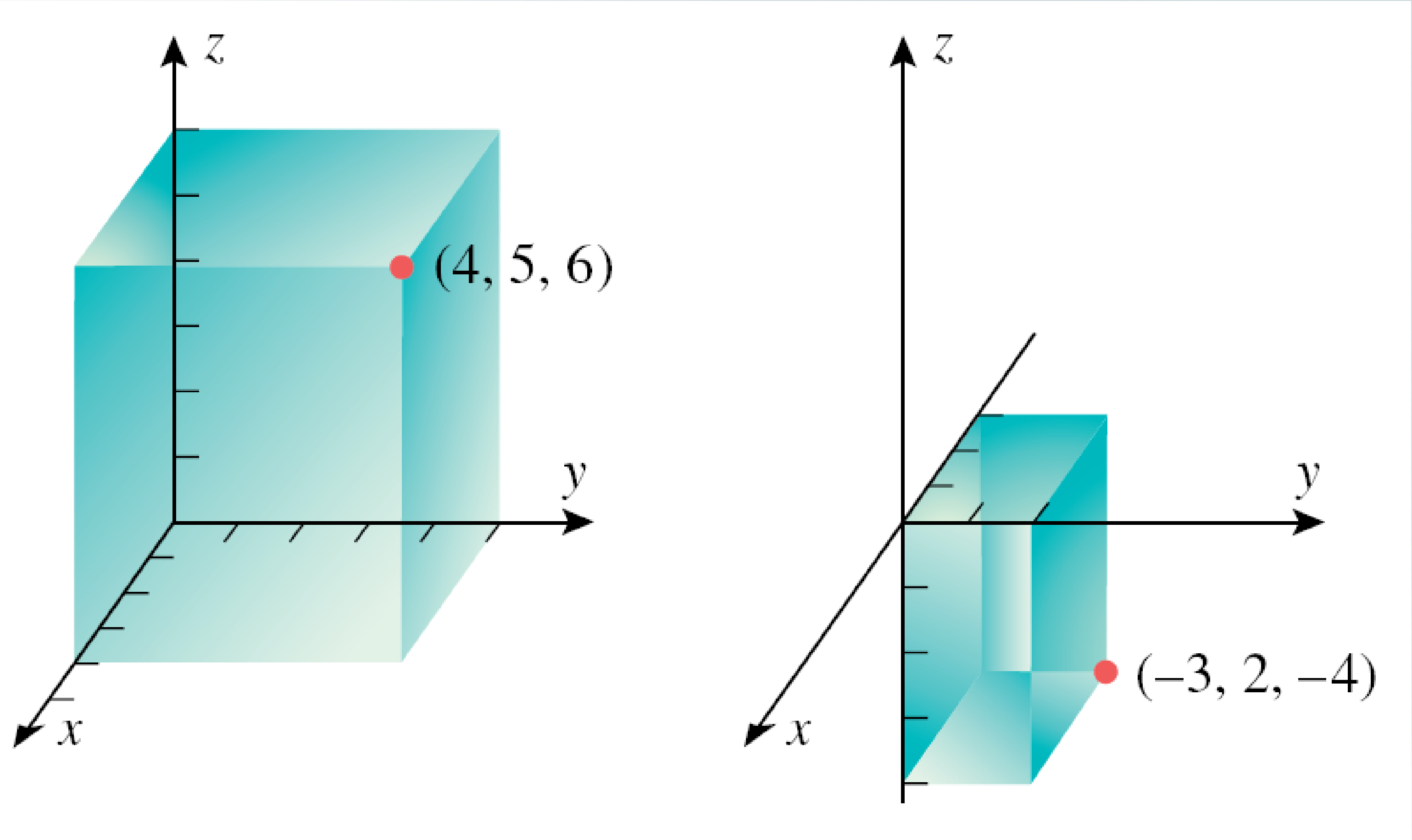


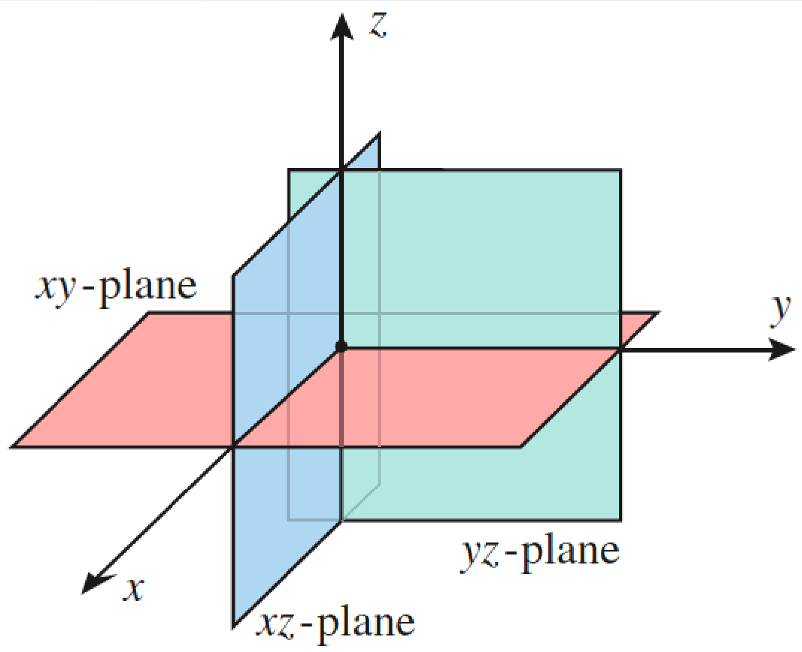

Figure 3 :The point in 3- space

# Point in 3-Space



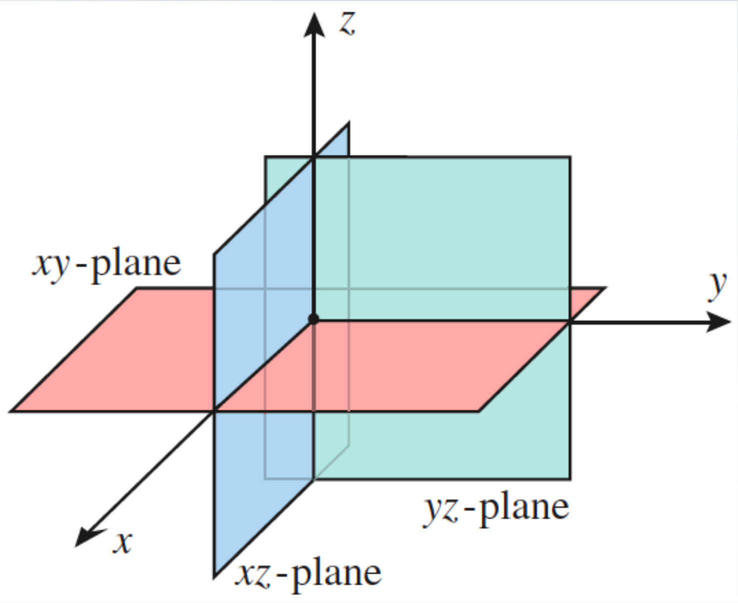
# Some Facts about Three-Dimensional Rectangular coordinate System

| Region | Description |
|---|---|
| $xy$-plane | Consist of all points of the form $(x, y, 0)$ |
| $xz$-plane | Consist of all points of the form $(x, 0, z)$ |
| $yz$-plane | Consist of all points of the form $(0, y, z)$ |



# Some Facts about Three-Dimensional Rectangular coordinate System

| **Axis** | **Description** |
|---|---|
| $x$-axis | Consist of all points of the form $(x, 0, 0)$ |
| $y$-axis | Consist of all points of the form $(0, y, 0)$ |
| $z$-axis | Consist of all points of the form $(0, 0, z)$ |



# Distance Formula in Two-Dimension

- Recall that in 2-space the **distance *d*** between the points $P_1\left(x_1, y_1\right)$ and $P_2(x_2,y_2)$ is

$$d = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$$

# Distance Formula in Three-Dimension

Given the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ the **distance *d*** between the points $P_1$ and $P_2$ is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

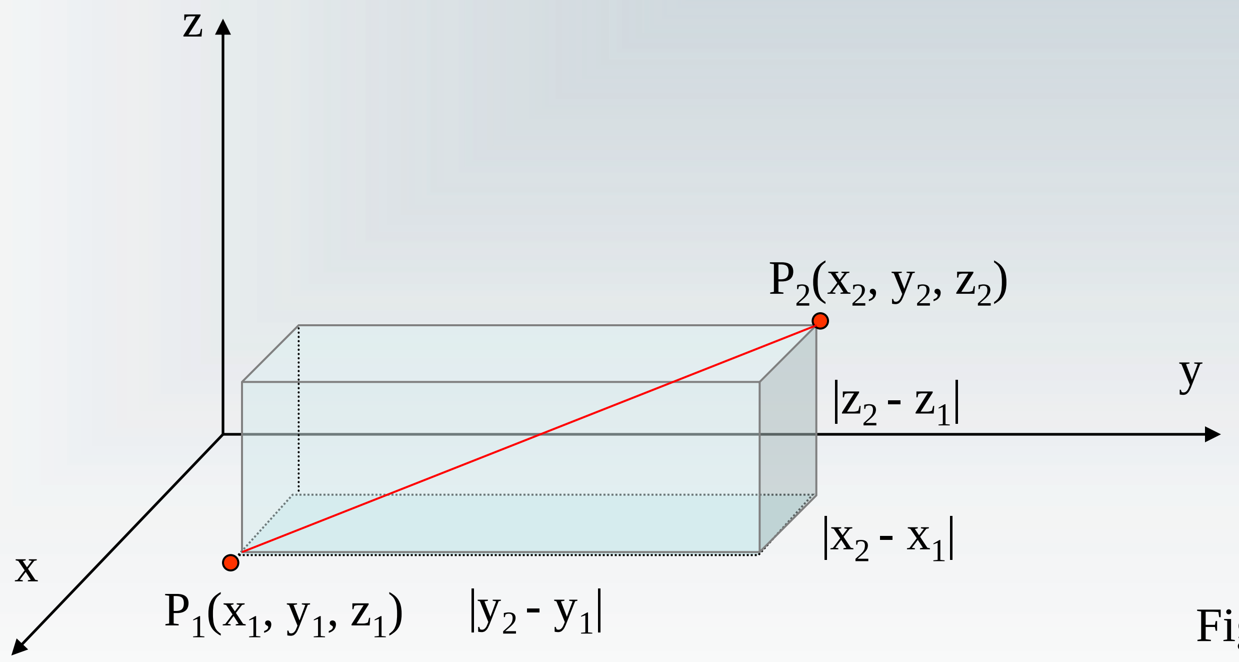



Figure 4

# Example 2

The distance between the points $P(2, 3, -5)$ and $Q(-1,4,2)$ is

$$d = \sqrt{(-1-2)^2 + (4-3)^2 + (2-(-5))^2}$$

$$= \sqrt{9+1+49}$$

$$= \sqrt{59}$$

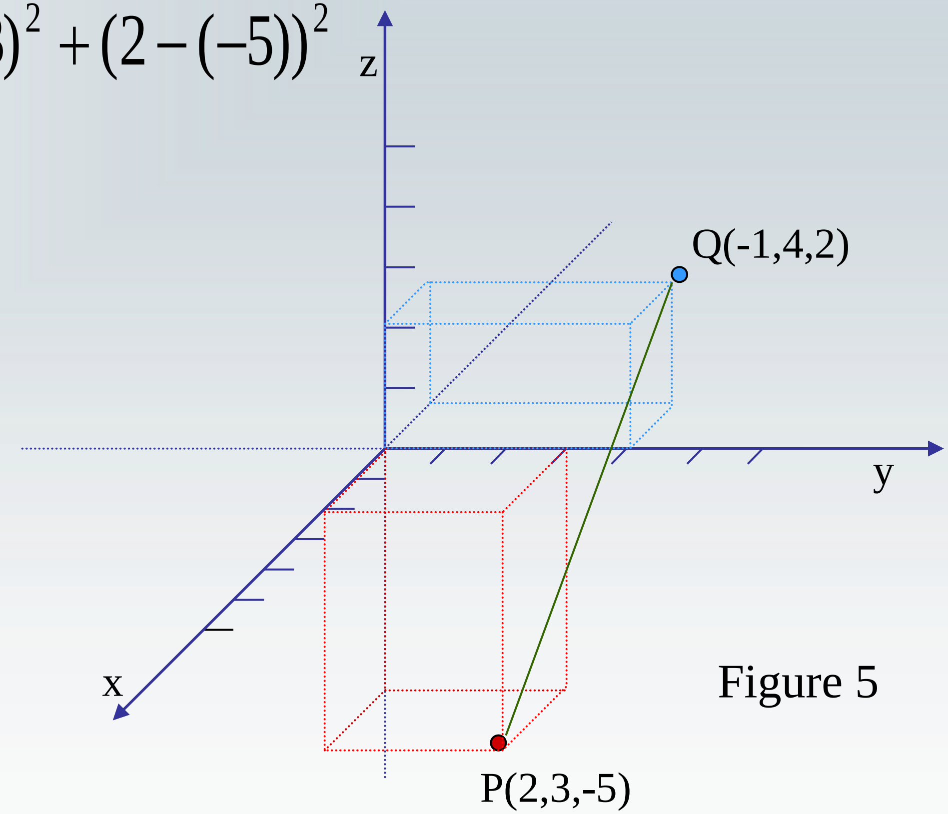



Figure 5

# Exercises

1. Show that (4, 5, 2), (1, 7, 3) and (2, 4, 5) are the vertices of an equilateral triangle.
2. Show that (2, 1, 6), (4, 7, 9) and (8, 5, -6) are the vertices of a right triangle.
3. Find the distance from the point (-5, 2, -3) to the
   - (a) $xy$ – plane
   - (b) $yz$ – plane
   - (c) $x$ – axis
   - (d) $z$ – axis.

# 2. Vectors

- ***Vector*** is a quantity that has both ***magnitude*** and ***direction***.
- A vector is often represented by an arrow or a directed line segment.
- The length represents the magnitude of the vector and the arrow points in the direction of the vector .

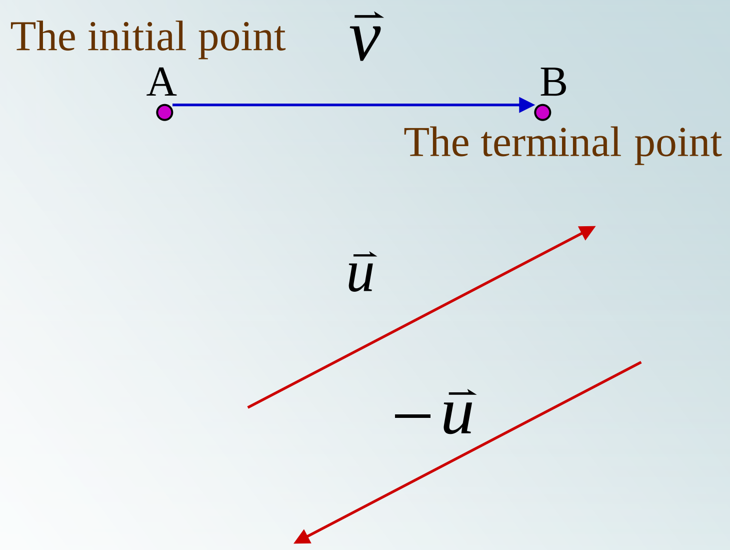



Figure 6

A vector can be represented by printing a letter in bold-face (***v***) or by putting an arrow above the letter ( $\vec{v}$)

# Vector Definition

- **Definition 1a** A two-dimensional vector is an ordered pair $\boldsymbol{u} = \langle u_1, u_2 \rangle$ of real numbers. The numbers $u_1$, and $u_2$ are called the ***components of* vector *u*.**

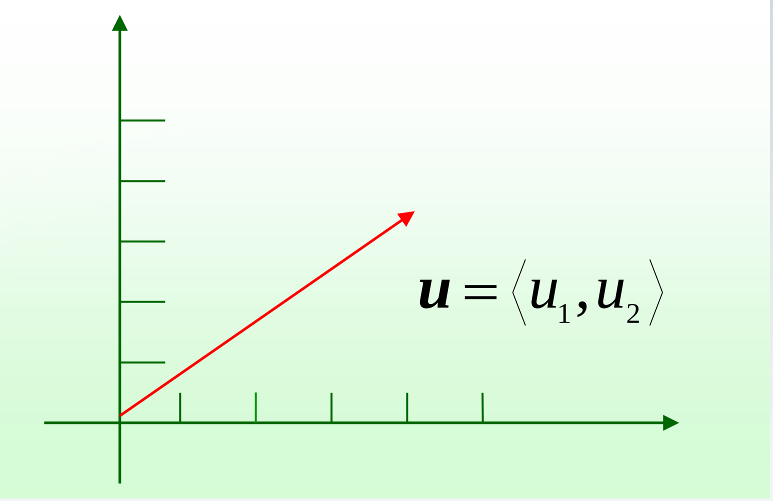



Figure 8

# Vector Definition

**Definition 1b** A three-dimensional vector is an ordered triple $\boldsymbol{u} = \langle u_1, u_2, u_3 \rangle$ of real numbers. The numbers $u_1$, $u_2$ and $u_3$ are called the ***components of u.***

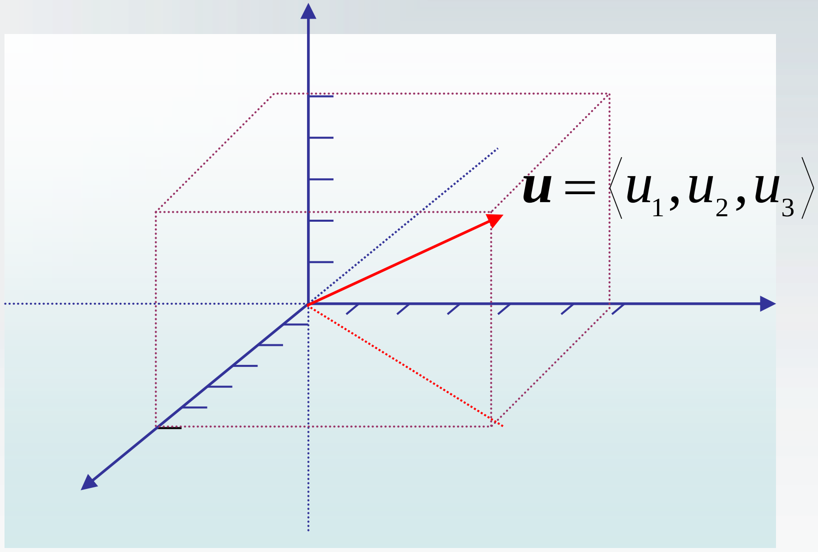
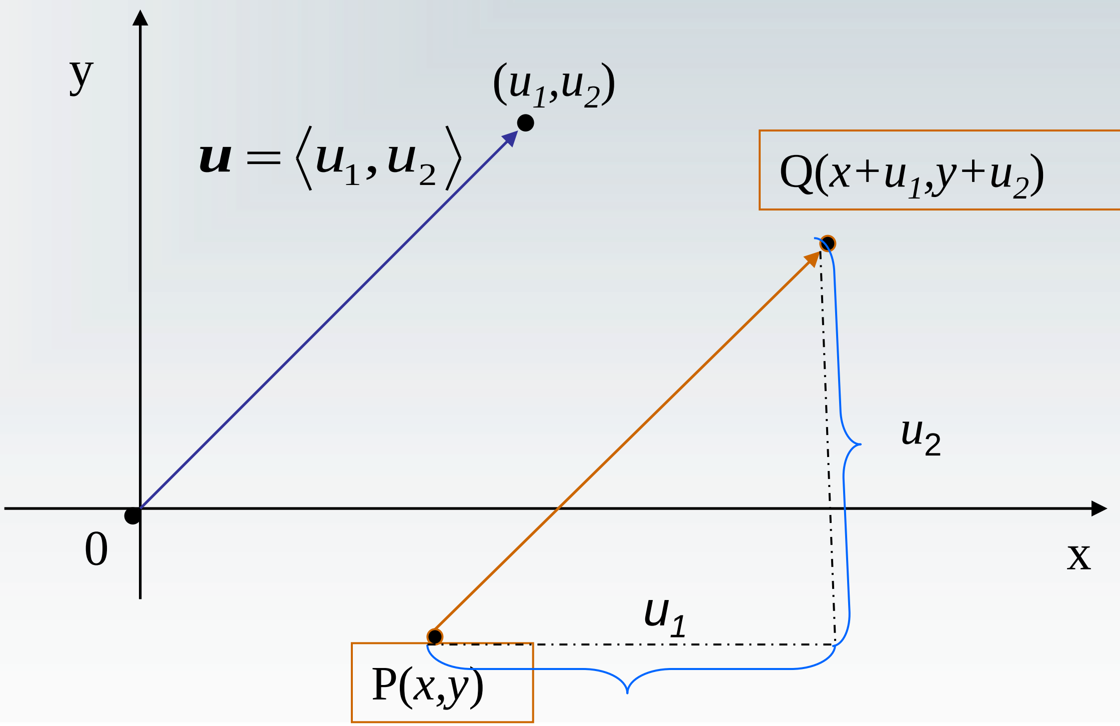



Figure 9

# Representation of the Vector

A representation of the vector $\boldsymbol{u} = \langle u_1, u_2 \rangle$ is a directed line segment $\overrightarrow{PQ}$ from any point $P(x, y)$ to the point $Q(x+u_1 , y+u_2)$

Figure 10



# Position Vector in $R^2$

The position vector $u$ of a point $P(u_1,u_2)$ in $R^2$ (2-space) is the vector $u = \langle u_1,u_2 \rangle$ whose initial point is the origin O and whose terminal point is $P$.

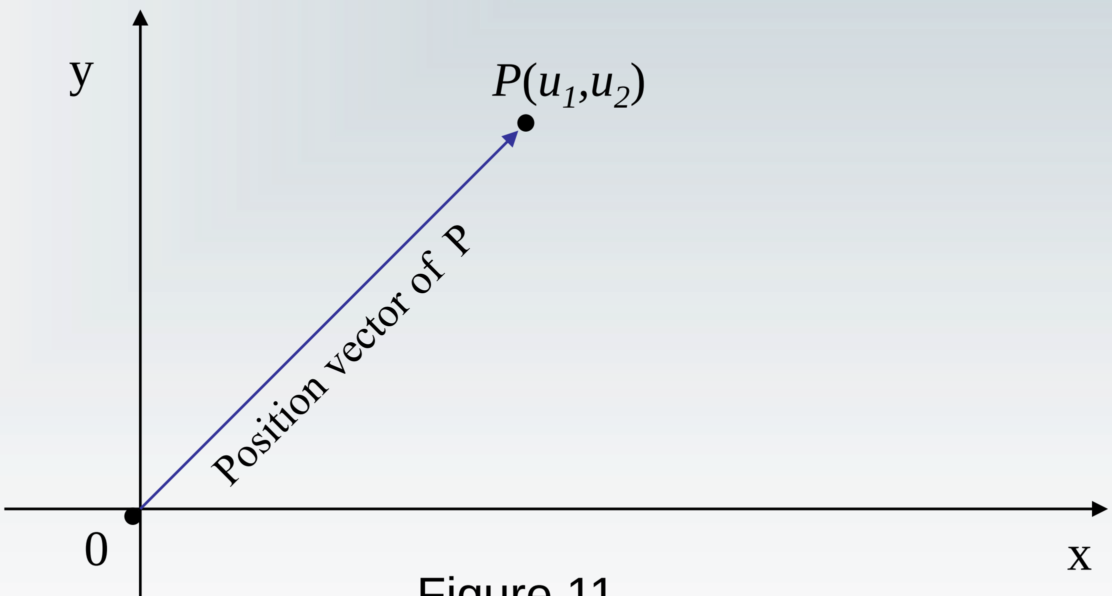



Figure 11

# Position Vector in $R^3$

The position vector of a point $P(x_1,y_1,z_1)$ in 3-space is the vector $\overrightarrow{OP} = \boldsymbol{u} = \langle x_1,y_1,z_1 \rangle$ whose initial point is the origin O and whose terminal point is $P$.

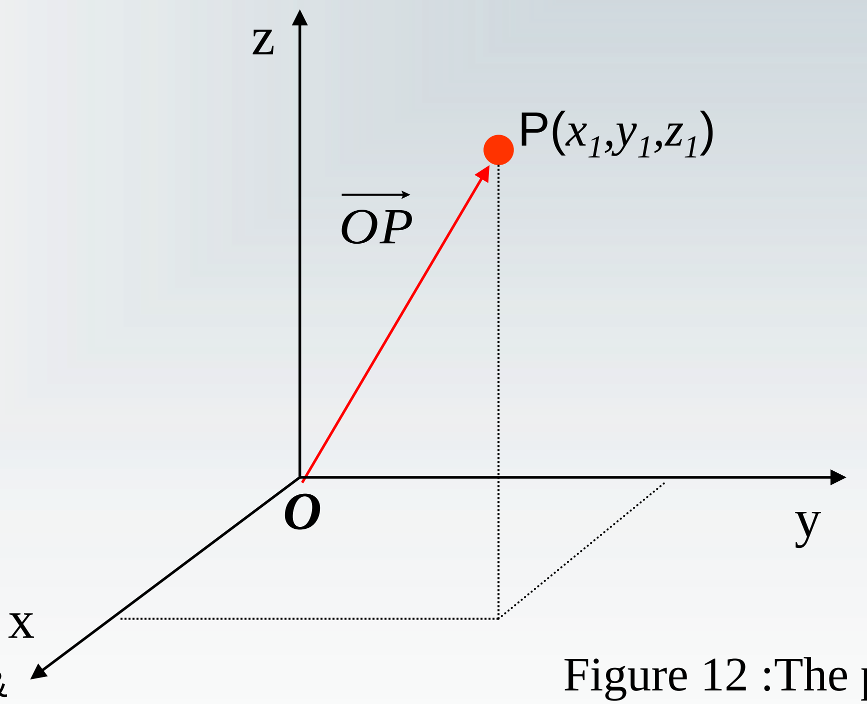



Figure 12 :The position vector

# Equivalent of Vector

$\vec{u} = \langle 2\ ,\ 3 \rangle$

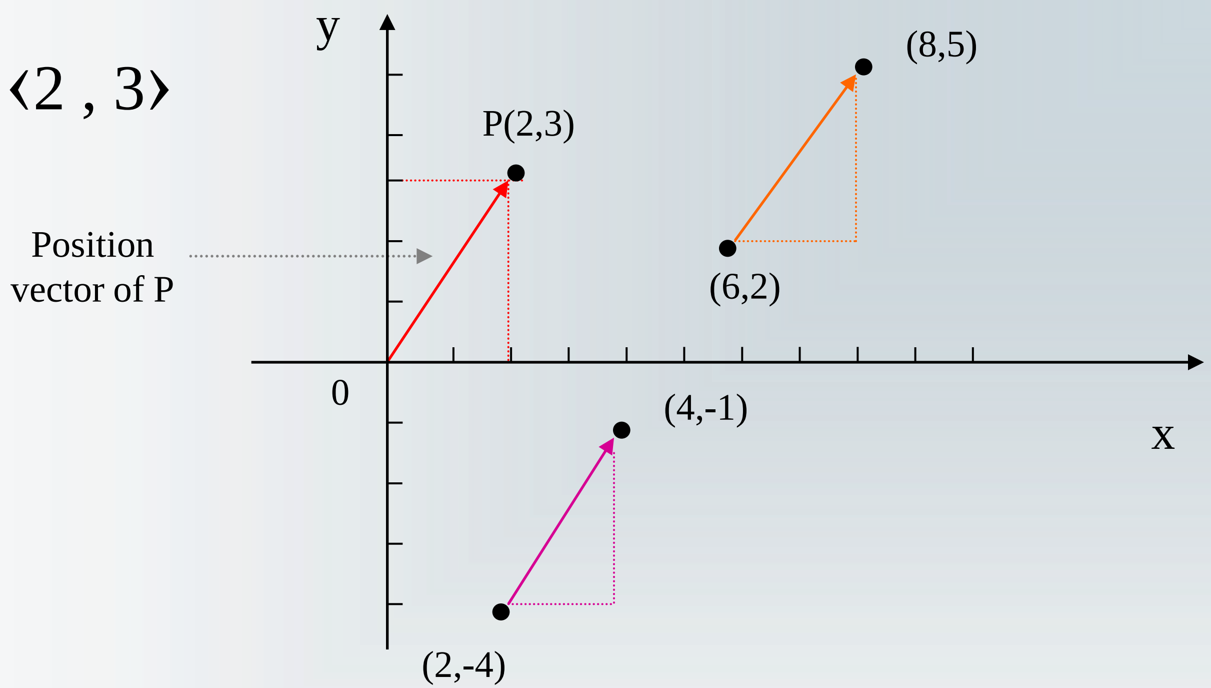



Figure 13

All vectors in Figure 13 are equivalent in the sense that they have the same length and point in the same direction even though they are in different locations.

**Definition 2** Given the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, the vector $\boldsymbol{u}$ or $\overrightarrow{PQ}$ is

$$\overrightarrow{PQ} = \boldsymbol{u} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

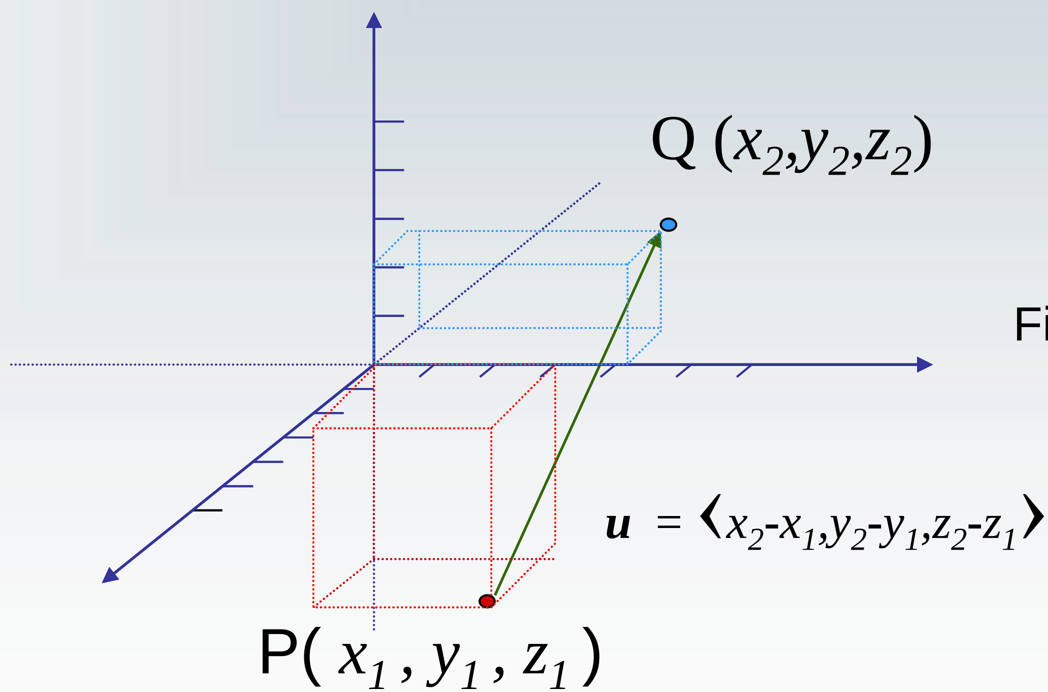



Figure 14

# Example 3

Find the vector represented by the directed line segment with initial point $P(0, -1, 3)$ and terminal point $Q(5, -4, -9)$.

**Solution**

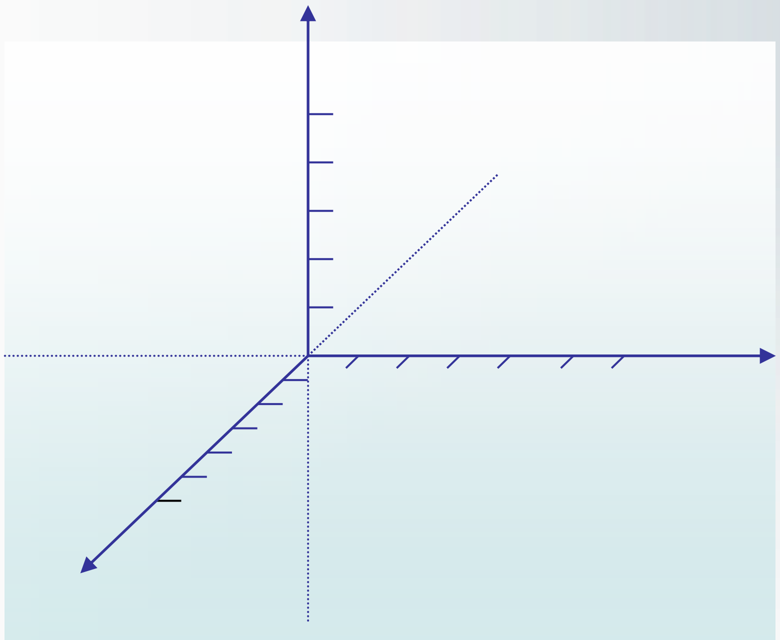


Figure 15

By Definition 2 the vector corresponding to $\overrightarrow{PQ}$ is

$$\boldsymbol{u} = \langle 5-0, -4-(-1), -9-3 \rangle$$

$$= \langle 5, -3, -12 \rangle$$

# Norm of Vector

Magnitude, length or norm of the vector is denoted by $\| \boldsymbol{u} \|$ .

**Definition 3a** The length of the two dimensional vector $\boldsymbol{u} = \langle u_1, u_2 \rangle$ is

$$\| \boldsymbol{u} \| = \sqrt{u_1^2 + u_2^2}$$

# Norm of Vector

**Definition 3b** The length of the three-dimensional vector $\boldsymbol{u} = \langle u_1, u_2, u_3 \rangle$ is

$$\| \boldsymbol{u} \| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

## Example 4

The components of a vector $\boldsymbol{u}$ with initial point $P(2,-1,4)$ and terminal point $Q(7,5,-8)$ are

$$\boldsymbol{u} = \langle 7-2,\ 5-(-1),\ -8-4 \rangle$$

$$= \langle 5,\ 6,\ -12 \rangle$$

The length of the vector $\boldsymbol{u}$ is

$$\| \boldsymbol{u} \| = \sqrt{5^2 + 6^2 + (-12)^2}$$

$$= \sqrt{205}$$

# Zero vector and Unit vector

The **zero vector** in 2-space and 3-space are

$\mathbf{O} = \langle 0, 0 \rangle$ and $\mathbf{O} = \langle 0, 0, 0 \rangle$

The zero vector has *no specific direction* .

A vector that has magnitude 1 is called **unit vector**.

# Arithmetic Operations on Vectors

**Vector addition**

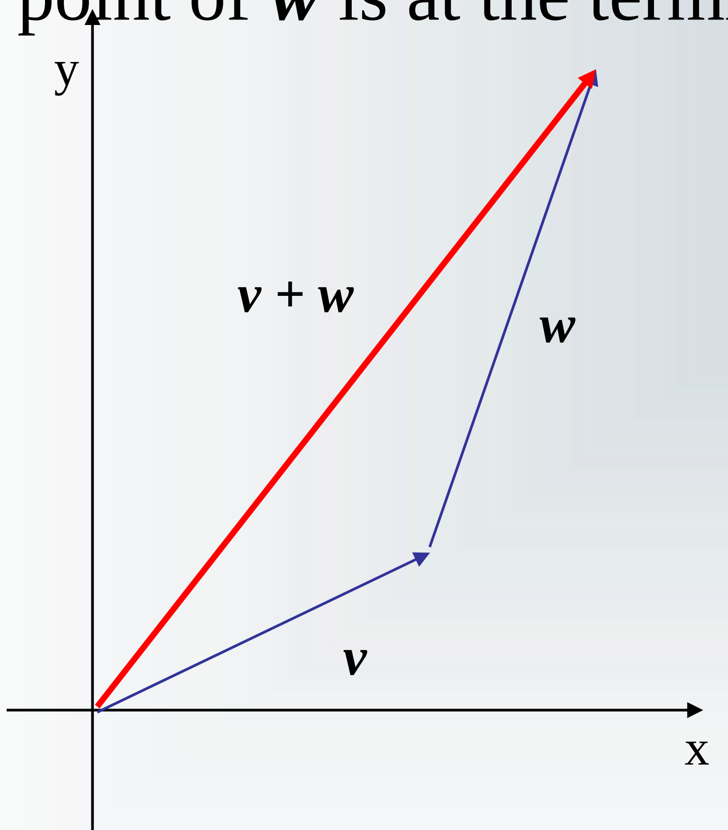
If $\boldsymbol{u} = \langle u_1, u_2 \rangle$ and $\boldsymbol{v} = \langle v_1, v_2 \rangle$ are vectors in 2-space, then

$$\boldsymbol{u} + \boldsymbol{v} = \langle u_1 + v_1 , u_2 + v_2 \rangle$$

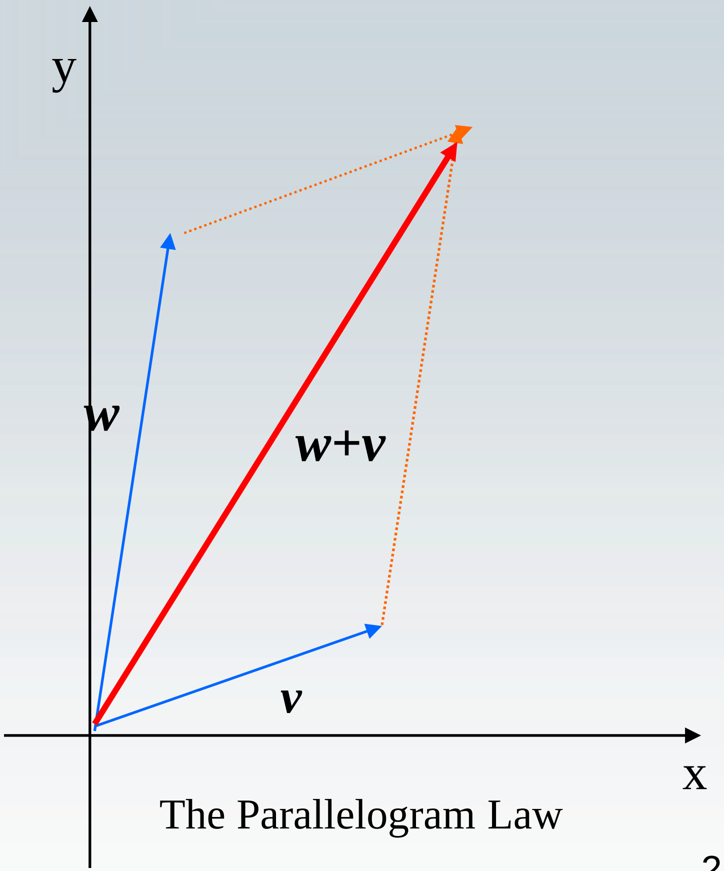
Similarly, for three-dimensional vectors if $\boldsymbol{u} = \langle u_1, u_2, u_3 \rangle$ and $\boldsymbol{v} = \langle v_1, v_2, v_3 \rangle$, then

$$\boldsymbol{u} + \boldsymbol{v} = \langle u_1, u_2, u_3 \rangle + \langle v_1, v_2, v_3 \rangle$$
$$= \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$$

If ***v*** and ***w*** are vectors, then the **sum, *v* + *w*** is the vector from the initial point of ***v*** to the terminal point of ***w*** when the vectors are positioned so the initial point of ***w*** is at the terminal point of ***v*** .




The Triangle Law



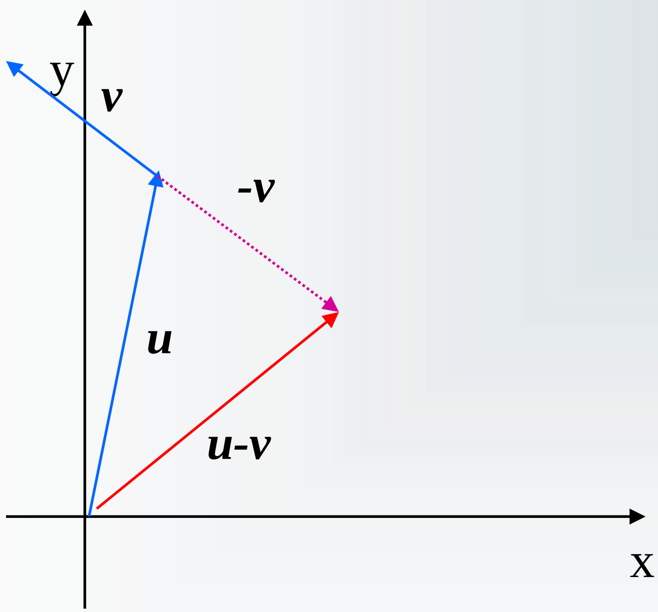

The Parallelogram Law

Figure 16

# Vector Subtraction

If $\boldsymbol{u} = \langle u_1, u_2 \rangle$ and $\boldsymbol{v} = \langle v_1, v_2 \rangle$, then the vector

$$\boldsymbol{u} - \boldsymbol{v} = \boldsymbol{u} + (-\boldsymbol{v})$$
$$= \langle u_1 - v_1, u_2 - v_2 \rangle$$




Similarly, in 3- space

$$\boldsymbol{u} - \boldsymbol{v} = \boldsymbol{u} + (-\boldsymbol{v})$$
$$= \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle$$

Figure 17

# Basic Properties of Vector Addition

## Theorems

For any vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ and any scalars $k$ and $l$, the following relationships hold:

(a) $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$

(b) $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$

(c) $\boldsymbol{u} + \mathbf{0} = \mathbf{0} + \boldsymbol{u} = \boldsymbol{u}$ (*additive identity*)

(d) $\boldsymbol{u} + (-\boldsymbol{u}) = \mathbf{0}$ (*additive inverse*)

# Scalar Multiplication of a Vector

If $\boldsymbol{u}$ is a nonzero vector and $k$ is any scalar (real number), then the product $k\boldsymbol{u}$ is obtained by multiplying each component of $\boldsymbol{u}$ by $k$ :

$$k\boldsymbol{u} = < k\boldsymbol{u}_1, k\boldsymbol{u}_2, k\boldsymbol{u}_3 >$$

Geometrically, the product $k\boldsymbol{u}$ has $k$ times the length of $\boldsymbol{u}$ and the same direction as $\boldsymbol{u}$ if $k > 0$ and opposite to that of $\boldsymbol{u}$ if $k < 0$.

# Example 5

Let $\boldsymbol{u} = \langle 4,0,1 \rangle$ and $\boldsymbol{v} = \langle 2,-5,1/3 \rangle$, find $-\boldsymbol{u}$, $7\boldsymbol{u}$, $\boldsymbol{u}+\boldsymbol{v}$, and $2(\boldsymbol{u}-\boldsymbol{v})$.

**Solution**

(a) $-\boldsymbol{u} = \langle -4, 0, -1 \rangle$

(b) $7\boldsymbol{u} = \langle 28, 0, 7 \rangle$

(c) $\boldsymbol{u} + \boldsymbol{v} = \langle 6, -5, 4/3 \rangle$

(d) $2(\boldsymbol{u}-\boldsymbol{v}) = 2\langle 4, 0, 1 \rangle - 2\langle 2,-5,1/3 \rangle$

$$= \langle 4,10,4/3 \rangle$$

# Unit Vectors

- A vector of length 1 is called a ***unit vector*****.** In an *xy*-coordinate system the unit vectors along the *x*-and *y*-axes are denoted by ***i*** and ***j***, respectively; $\boldsymbol{i} = \langle 1, 0\rangle$ $\boldsymbol{j} = \langle 0, 1\rangle$
- and in an *xyz*-coordinate system the unit vectors along the *x*-, *y*- and *z*- axes are denoted by ***i***, ***j*** and ***k***, respectively.

$$\boldsymbol{i} = \langle 1, 0, 0\rangle \qquad \boldsymbol{j} = \langle 0, 1, 0\rangle \qquad \boldsymbol{k} = \langle 0, 0, 1\rangle$$

# Alternative Vector Representation

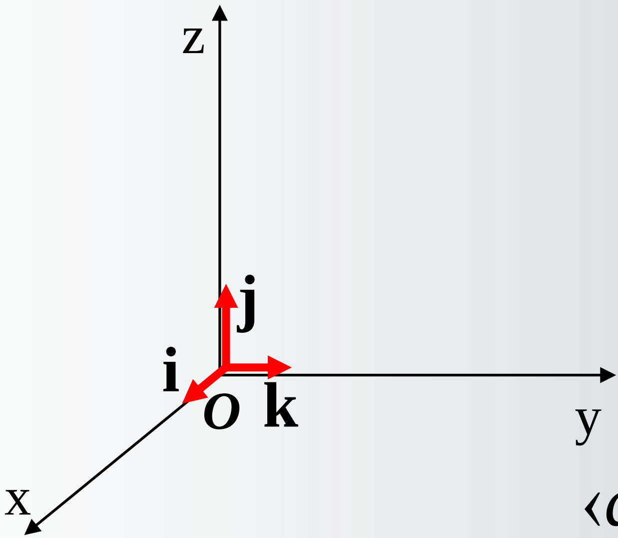
Another popular representation of vectors is

$$\boldsymbol{u} = \langle u_1, u_2, u_3 \rangle = u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k}$$

where ***i***, ***j***, ***k*** are the **standard unit vectors** in the positive directions of the axes in a Cartesian coordinate system, i.e.

$$\boldsymbol{i} = \langle 1, 0, 0 \rangle \qquad \boldsymbol{j} = \langle 0, 1, 0 \rangle \qquad \boldsymbol{k} = \langle 0, 0, 1 \rangle$$

# The *i*, *j*, and *k* Vectors




Any vector **a** = ‹*a*, *b*, *c*› in 3-space can be expressed as a linear combination of ***i***, ***j***, and ***k***.

$$\langle a, b, c\rangle = \langle a, 0, 0\rangle + \langle 0, \boldsymbol{b}, 0\rangle + \langle 0, 0, c\rangle$$

$$= a\langle 1, 0, 0\rangle + b\langle 0, 1, 0\rangle + c\langle 0, 0, 1\rangle$$

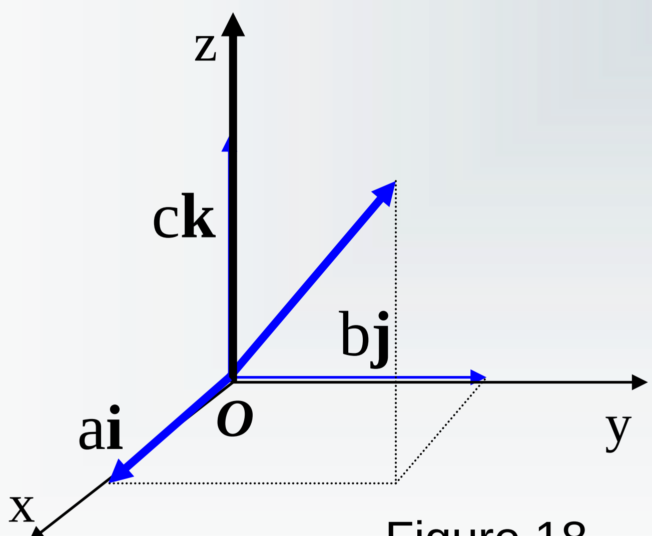



Figure 18

That is, $\mathbf{a} = a\boldsymbol{i}+b\boldsymbol{j}+c\boldsymbol{k}$

# Perpendicular, Orthogonal, Normal

The terms “perpendicular”, “orthogonal “and “normal” are all commonly used to describe geometric objects that meet at right angles.

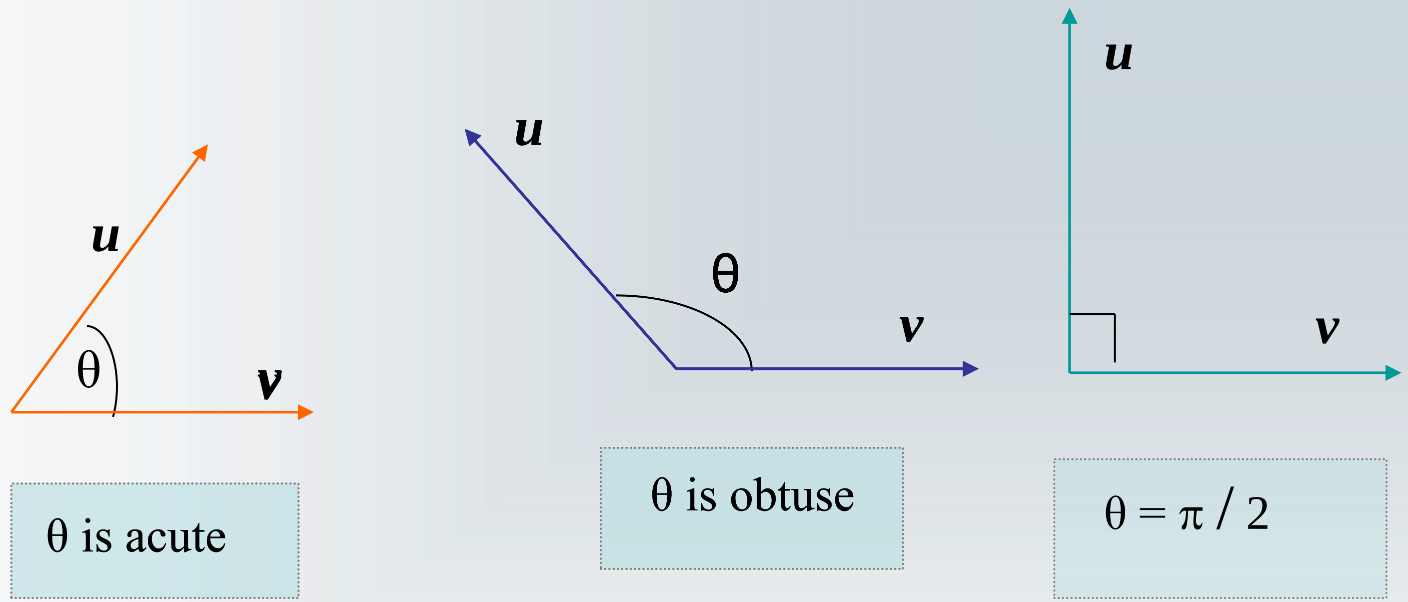
For consistency, we will say that

- two vectors are orthogonal
- a vector is normal to a plane, and
- two planes are perpendicular.

# Angle Between Vectors

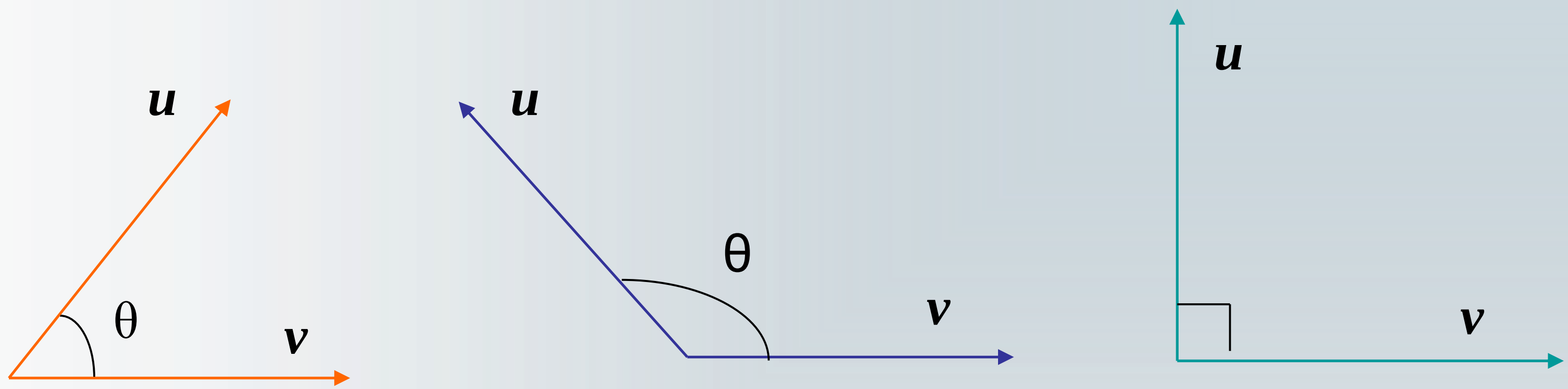
- Suppose that ***u*** and ***v*** are nonzero vectors in 2-space or 3- space that are positioned so their initial points coincide. We define the angle between ***u*** and ***v*** to be the angle θ determined by the vectors that satisfies the condition $0 \leq \theta \leq \pi$ (θ is the smaller angle between the vectors when they are drawn the same initial point).

# Angle Between Vectors




•Figure 19

$$\cos\theta = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|} \quad \text{or} \quad \boldsymbol{u}\cdot\boldsymbol{v} = \|\boldsymbol{u}\|\|\boldsymbol{v}\|\cos\theta$$




θ is acute
if and only if
$\boldsymbol{u} \bullet \boldsymbol{v} > 0$

θ is obtuse
if and only
if $\boldsymbol{u} \bullet \boldsymbol{v} < 0$

$\theta = \pi / 2$
if and only if
$\boldsymbol{u} \bullet \boldsymbol{v} = 0$

•Figure 20

# Example 6

If $\boldsymbol{a} = <1,-2,3>$, $\boldsymbol{b} = <-3,4,2>$, $\boldsymbol{c} = <3,6,3>$, then

$$\boldsymbol{a} \bullet \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$= (1)(-3)+(-2)(4)+(3)(2) = -5$$

$$\boldsymbol{b} \bullet \boldsymbol{c} = (-3)(3)+(4)(6)+(2)(3) = 21$$

$$\boldsymbol{a} \bullet \boldsymbol{c} = (1)(3)+(-2)(6)+(3)(3) = 0$$

Therefore, ***a*** and ***b*** make an obtuse angle, ***b*** and ***c*** make an acute angle, and ***a*** and ***c*** are perpendicular. The vectors ***a*** and ***c*** are **orthogonal vectors**.