20. Improper Integrals

Improper Integral

- A definite integral with an infinite interval of integration.
- A definite integral whose integrand becomes infinite within the interval of integration.

20.1) Integrals over Infinite Intervals

Definition

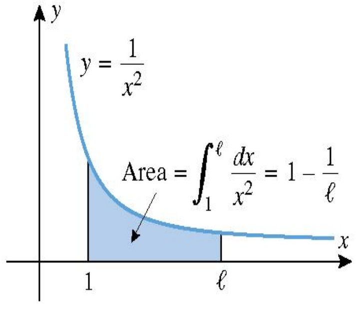
The *improper integral of f over the interval* $[a, +\infty)$ is defined as

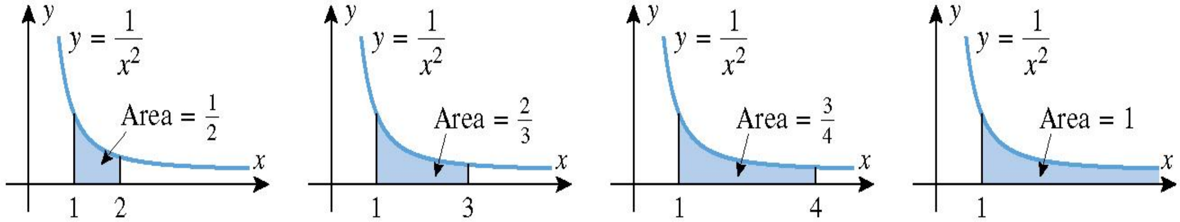
$$\int_a^{+\infty} f(x)\,dx = \lim_{l\to+\infty}\int_a^{l} f(x)\,dx$$

In the case where the limit exists, the improper integral is said to ***converge***, and the limit is defined to be the value of the integral. In the case where the limit does not exist, the improper integral is said to ***diverge***, and it is not assigned a value.

Consider the following integral $\int_1^{+\infty}\frac{dx}{x^2}$.

$$\int_1^{+\infty}\frac{dx}{x^2} = \lim_{l\to+\infty}\int_1^{l}\frac{dx}{x^2} = \lim_{l\to+\infty}\left(1-\frac{1}{l}\right) = 1$$





Definition

The *improper integral of f over the interval (–∞, b]* is defined as

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{k\to-\infty}\int_{k}^{b} f(x)\,dx$$

The integral is said to ***converge*** if the limit exists and ***diverge*** if it does not. The ***improper integral of f over the interval* (–∞, +∞)** is defined as

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{+\infty} f(x)\,dx$$

where c is any real number. The improper integral is said to converge if both terms **converge** and **diverge** if either term diverges.

Exercises: Is each integral convergent or divergent? If it is convergent, then evaluate it.

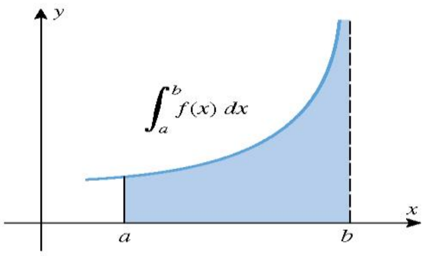
1. $\int_{1}^{\infty}\frac{1}{x}dx$ 2. $\int_{-\infty}^{2}e^{-x}dx$ 3. $\int_{-\infty}^{\infty}\frac{1}{x}dx$

20.2) Integrals whose integrand becomes infinite within the interval of integration

Definition

If f is continuous on the interval $[a, b]$, except for an infinite discontinuity at b, then the ***i****mproper integral of f over the interval [a, b]* is defined as

$$\int_{a}^{b} f(x)\,dx = \lim_{l\to b^-}\int_{a}^{l} f(x)\,dx$$



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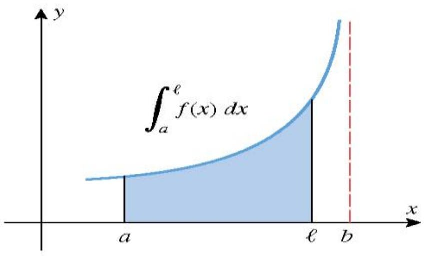


(b)

In the case where the limit exists, the improper integral is said to ***converge***, and the limit is defined to be the value of the integral. In the case where the limit does not exist, the improper integral is said to ***diverge*****,** and it is not assigned a value.

Definition

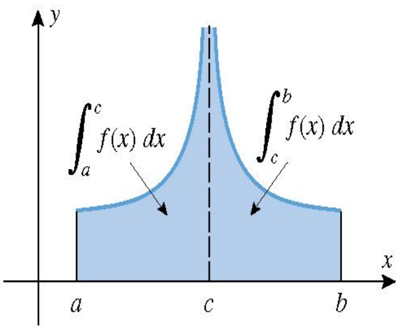
If f is continuous on the interval $[a, b]$, except for an infinite discontinuity at a, then the *improper integral of f over the interval* $[a, b]$ is defined as

$$\int_a^b f(x)\,dx = \lim_{k\to a^+}\int_k^b f(x)\,dx$$

The integral is said to ***converge*** if the limit exists and ***diverge*** if it does not. If f is continuous on the interval $[a, b]$, except for an infinite discontinuity at a number c in (a, b), then the *improper integral of f over the interval* $[a, b]$ is defined as

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

The improper integral is said to converge if both terms **converge** and **diverge** if either term diverges.



Exercises: Is each integral convergent or divergent? If it is convergent, then evaluate it.

4. $\displaystyle\int_{-1}^{0}\frac{1}{x}\,dx$
5. $\displaystyle\int_{1}^{2}\frac{1}{\sqrt{x-1}}\,dx$
6. $\displaystyle\int_{0}^{4}\frac{1}{(x-2)^2}\,dx$