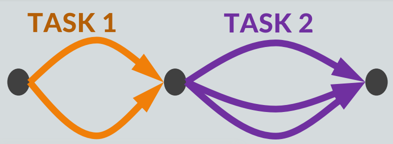
PERMUTATIONS
COMBINATIONS

Counting and Combinatorics

ways to select (arrange/order) or group elements from a set to form subsets with or without replacement (repetition)

Apply the sum rule for disjoint cases

SUM $n_1 + n_2 + \ldots + n_k$ ways



Tasks that require sequentially steps

product $n_1 \times n_2 \times \ldots \times n_k$ ways

When overcounting occurs, use **subtraction** or **division** rules to remove them
Use the **complement** rule when unwanted cases are fewer & less complicated

	REPETITION allowed	NO REPEAT
PERMUTATIONs Arrangement, Order matters	n^r	${}_{n}P_{r} = \frac{n!}{(n-r)!}$
COMBINATIONs Grouping/Selection, No ordering	${}_{n+r-1}C_{r} = \frac{(n+r-1)!}{r!(n-1)!}$	${}_{n}C_{r} = \frac{n!}{(n-r)!r!}$

RECALL

Given n=7 letters of English alphabet, choose r=3

A B C D E F G

Order matter
No repeat

Choose in order, each letter can be chosen only once (no repeat)

$\underline{7} \times \underline{6} \times \underline{5} = \frac{7!}{(7-3)!} = {}_7P_3 = {}_nP_r$

r-permutations of n elements

When r=n, nPr = nPn = n!

NEXT

Use division rule to go from permutations to combinations

Given n=7 letters of English alphabet, choose r=3

A B C D E F G

Order NOT matter
No repeat

Number of permutations = $7 \times 6 \times 5 = 7!/4! = {}_7\mathbf{P}_3$
Overcounting factor *(how many are counted, but they are, in fact, non-distinct)*

Apply division rule => # of ways =

WORKED EXAMPLES

There are 15 students in the class. Permutation vs. combination

❍How many ways to choose a team of 3 tutors (ordering not matter)?

❍How many ways to choose 3 tutors: one will be responsible for science, one for mathematics, and the other programming?

WORKED EXAMPLES

- There are 15 balls: 4 red, 5 blue, and 6 green. We draw three and at least one is red. We put the balls back, then draw another four and at most one is green. In how many ways can this be done?

PRACTICE PROBLEMS

- An urn contains 16 balls, of which 7 are red and the other 9 blue. Your friend takes out 8 balls and gets exactly 3 red and 5 blue. Then, without replacement, you take 3 balls and get all blue. How many ways can this happen?

PRACTICE PROBLEMS

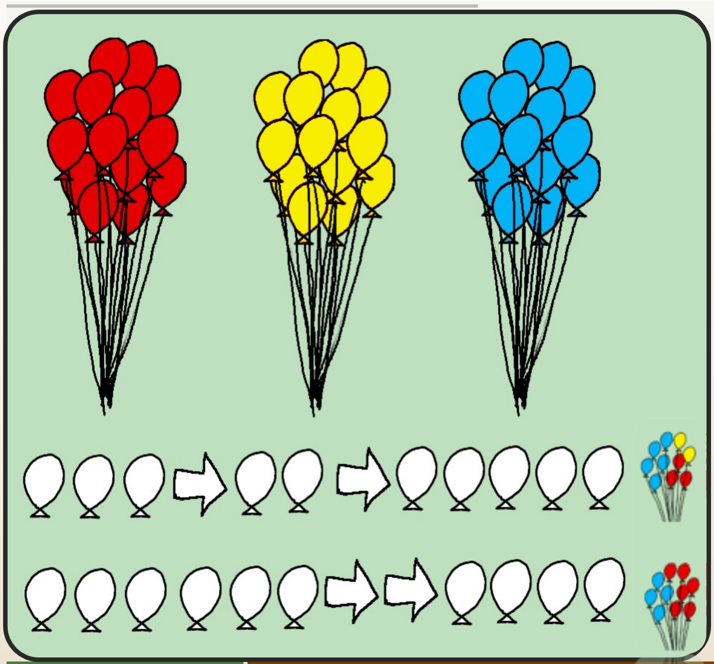
In how many ways can a committee, consisting of one chairman, one secretary, one treasurer and four ordinary members be chosen from eight persons?
(Committees with different chairmen, secretaries, treasurers count as different committees)

The number of ways to select r out of n elements

	REPETITION allowed	NO REPEAT
PERMUTATION Arrangement Order matters	n^r	${}_n\mathrm{P}_r = \frac{n!}{(n-r)!}$
COMBINATION Grouping/Selection No ordering	${}_{n+r-1}\mathrm{C}_r = \frac{(n+r-1)!}{r!(n-1)!}$	${}_n\mathrm{C}_r = \frac{n!}{(n-r)!r!}$

Remembering these formula is one thing. You will, however, **do a lot better** if you **understand** & are able to **apply** them effectively. They can be derived easily using basic counting rules: the rule of sum, the rule of product, and the rule of division.

Combination with repetitions

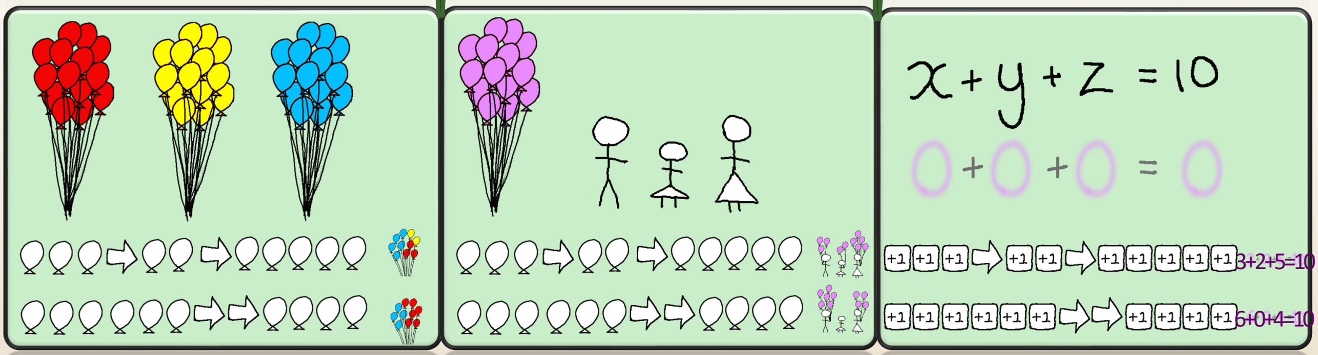


A man sells red, yellow, and blue identical balloons. Buying 10, how many different collections of balloons can be made?

How many ways to choose r=10 from r+n-1=12 positions?
(r is the # of balloons, n is #colors)

https://blogs.adelaide.edu.au/maths-learning/2015/10/17/a-story-of-stars-and-bars/

Picking 10 balloons from a selection of 3 colors. Unlimited stock of each color.	Distribute 10 balloons to 3 kids. Putting 10 identical balls into 3 baskets.	How many integer solutions does the equation $x_1 + \ldots + x_n = r$, where $x_i \geq 0$ has?
Stars and Strips	**Balls and Urns**	**Integer Solutions**
*** | ** | *****	●●● | ●● | ●●●●●	$x_1 + x_2 + x_3 = r$



WORKED EXAMPLES

- How many non-negative integer solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 13$?

WORKED EXAMPLES

- How many integer solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 13$ with $x_1 > 1$, $x_2 > 1$, $x_3 > 3$, $x_4 \geq 0$?

PRACTICE PROBLEMS

How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 21$, where x_i, i = 1, 2, 3, 4, 5, is a nonnegative integer such that

- $x_1 \geq 1$ and $x_2 = 4$
- $x_i > 1$ for all i
- $0 \leq x_1 \leq 10$

PRACTICE PROBLEMS

- We distribute 15 identical tennis balls among 6 children. Find the number of all possible distributions.

- Constraint: each child receives at least one ball.

PRACTICE PROBLEMS

- The college food plan allows a student to choose three pieces of fruit each day. The fruits available are apples, bananas, peaches, pears, and plums. For how many days can a student make a different selection?