Logic and Computer Design Fundamentals

Chapter 2 – Combinational Logic Circuits

Part 4 – Standard Form of Algebraic Representation

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Overview

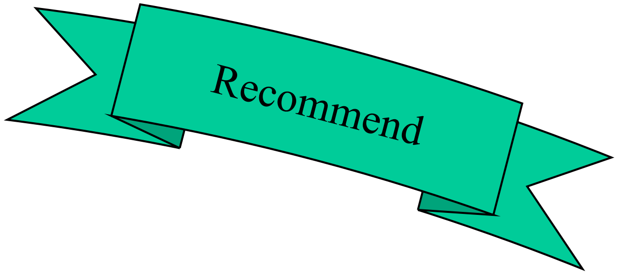
- **Recall Part 3 – Algebraic Manipulation**
  - **Review of algebraic manipulation**
  - **Additional Trick for applying all identities**
- **Part 4 – Standard Form of Algebraic Representation**
  - **Minterms and Maxterms**
  - **Index Representation of Minterms and Maxterms**
  - **Sum-of-Minterm (SOM) Representations**
  - **Product-of-Maxterm (POM) Representations**

Recall Part 3: Algebraic Manipulation

How to simplify the Boolean algebra?

Algebraic Manipulation: Basic Identities

An algebraic structure defined on a set of at least two elements, together with three traditional binary operators: Or, And, Not (denoted +, · , ‾) that satisfies the following basic identities:



1.	$X + 0 = X$	2.	$X \cdot 1 = X$	
3.	$X + 1 = 1$	4.	$X \cdot 0 = 0$	
5.	$X + X = X$	6.	$X \cdot X = X$	
7.	$X + \overline{X} = 1$	8.	$X \cdot \overline{X} = 0$	
9.	$\overline{\overline{X}} = X$			
10.	$X + Y = Y + X$	11.	$XY = YX$	Commutative
12.	$(X + Y) + Z = X + (Y + Z)$	13.	$(XY)Z = X(YZ)$	Associative
14.	$X(Y + Z) = XY + XZ$	15.	$X + YZ = (X + Y)(X + Z)$	Distributive
16.	$\overline{X + Y} = \overline{X} \cdot \overline{Y}$	17.	$\overline{X \cdot Y} = \overline{X} + \overline{Y}$	DeMorgan's

Review: Algebraic Manipulation

- To consider a simplification of the expression by applying some of the identities:

$$F = \overline{X}YZ + \overline{X}Y\overline{Z} + XZ$$

$$= \overline{X}Y(Z+\overline{Z}) + XZ \qquad \text{by identity 14}$$

$$= \overline{X}Y(1) + XZ \qquad \text{by identity 7}$$

$$= \overline{X}Y + XZ \qquad \text{by identity 2}$$

…Simplify to contain the smallest number of **literals** (result variables)

Useful Trick!

Recommend

- First, determine the most shared (or common) variables:

$$F = \overline{Z} + WXY\overline{Z} + WXY + XZ$$

From this expression, we have X, $\overline{Z}$ and WX are shared (or common) variables.

We choose WX to be the first determination,

Because WX is the most shared variables.

So, you will apply distributive identity as follows:

$$F = \overline{Z} + WX(Y\overline{Z} + Y) + XZ$$

Useful Trick!

Recommend

- Second, try to eliminate the inverted variables:

$$F = A\overline{B}BC + \overline{A}ABC + BC$$

From this expression, we can remove $\overline{B}B$ and $\overline{A}A$ Because we can apply identity 8 ($\overline{X}X = 0$) to eliminate the first and second variable set.

So, the solution is

$$F = BC$$

Useful Trick! Distributive Pattern

Recommend

Distributive identities 14 and 15 are most frequently used:

<u>Identity 14</u>: $X(Y+Z) = XY + XZ$

$X \cdot Y$

Pattern

$$X\,(\,Y + Z\,) \quad = (X \cdot Y) + (X \cdot Z)$$

$X \cdot Z$

How to apply this trick?

given algebra is $F = \overline{A}BC + ABC$

<u>Solution</u>

Step1: Extract shared variables → BC

Step2: Determine functions, you will get → $BC\,(\overline{A}+A)$

Useful Trick! Distributive Pattern

Recommend

Identity 15: $X+(YZ) = (X+Y)(X+Z)$

$X+Y$

Pattern

$$X + (Y \cdot Z) \quad = (X + Y) \cdot (X + Z)$$

$X + Z$

How to apply this trick?

given algebra is $F = (\overline{A}+BC)(A+BC)$

Solution

Step1: Extract shared variables → BC

Step2: Determine functions, you will get → $BC+ (\overline{A} \cdot A)$

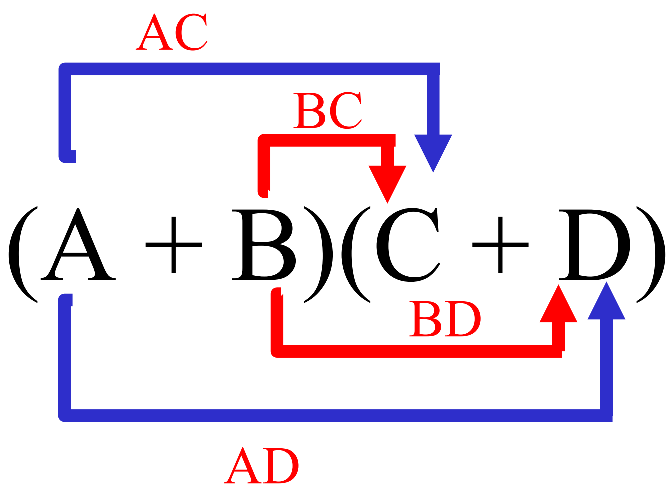
Useful Trick! Distributive Pattern

Recommend

<u>Applied Trick for Identity 15</u>:

$$(A+B)(C+D) \;=\; ?$$

Pattern



$$= AC+AD+BC+BD$$

Example

$$F=(A+C)(AD+A\overline{D})+AC+C$$

Solution

$$=(A+C)A(D+\overline{D})+AC+C \qquad \text{identity 14}$$

$$=(A+C)A+AC+C \qquad \text{identity 7}$$

$$=AA+AC+AC+C \qquad \text{identity 14}$$

$$=A(1+C+C)+C \quad \text{or } A+C(A+A+1) \qquad \text{identity 14}$$

$$=A+C \qquad \text{identity 3}$$

Part 4

Standard Form of Algebraic Representation

Related Topics

- Minterms and Maxterms
- Index Representation of Minterms and Maxterms
- Sum-of-Minterm (SOM) Representations
- Product-of-Maxterm (POM) Representations

Minterms

- **Minterms are AND terms with every variable present in either true or complemented form.**
- **Given that each binary variable may appear normal (e.g., x) or complemented (e.g., $\overline{x}$), there are 2^n minterms for n variables.**
- **Example: Two variables (X and Y) produce 2 x 2 = 4 combinations:**
  - XY (both normal)
  - $X\overline{Y}$ (X normal, Y complemented)
  - $\overline{X}Y$ (X complemented, Y normal)
  - $\overline{X}\,\overline{Y}$ (both complemented)
- **Thus there are four minterms of two variables.**

Maxterms

- **<u>Maxterms</u> are OR terms with every variable in true or complemented form.**
- **Given that each binary variable may appear normal (e.g., x) or complemented (e.g., $\overline{x}$), there are 2^n maxterms for n variables.**
- **<u>Example:</u> Two variables (X and Y) produce 2 x 2 = 4 combinations:**

  $\mathbf{X + Y}$ **(both normal)**

  $\mathbf{X + \overline{Y}}$ **(x normal, y complemented)**

  $\mathbf{\overline{X} + Y}$ **(x complemented, y normal)**

  $\mathbf{\overline{X} + \overline{Y}}$ **(both complemented)**

Index of Maxterms and Minterms

Recommend

- **Examples: Two variable minterms and maxterms.**

Index	Minterm	Maxterm
0	$\overline{x}\,\overline{y}$	$x + y$
1	$\overline{x}\,y$	$x + \overline{y}$
2	$x\,\overline{y}$	$\overline{x} + y$
3	$x\,y$	$\overline{x} + \overline{y}$

- **The index above is important for describing which variables in the terms are true and which are complemented.**

Purpose of the Index

- The <u>index</u> for the minterm or maxterm, expressed as a binary number, is used to determine whether the variable is shown in the true form or complemented form.
- For Minterms (AND):
  - "1" (or T) means the variable is "Not Complemented"
  - "0" (or F) means the variable is "Complemented".
- For Maxterms (OR):
  - "0" (or F) means the variable is "Not Complemented"
  - "1" (or T) means the variable is "Complemented".

Index Example for Three Variables

- Example: (for three variables)
- Assume the variables are called X, Y, and Z.
- The standard order is X, then Y, then Z.
- The <u>Index 0</u> (base 10) = 000 (base 2) for three variables. All three variables are complemented for <u>minterm 0</u> ($\overline{X}, \overline{Y}, \overline{Z}$) and no variables are complemented for <u>Maxterm 0</u> (X,Y,Z).
  - Minterm 0, called m_0 is $\overline{X}\overline{Y}\overline{Z}$.
  - Maxterm 0, called M_0 is (X + Y + Z).
  - Minterm 6 ? $XY\overline{Z}$
  - Maxterm 6 ? $\overline{X}+\overline{Y}+Z$

Index Examples – Four Variables

Recommend

Index i	Binary Pattern	Minterm m_i	Maxterm M_i	
0	0000	$\bar{a}\,\bar{b}\,\bar{c}\,\bar{d}$	$a+b+c+d$	
1	0001	$\bar{a}\,\bar{b}\,\bar{c}\,d$	?	$a+b+c+\bar{d}$
3	0011	?	$a+b+\bar{c}+\bar{d}$	$\bar{a}\ \bar{b}\ c\ d$
5	0101	$\bar{a}\,b\,\bar{c}\,d$	$a+\bar{b}+c+\bar{d}$	
7	0111	?	$a+\bar{b}+\bar{c}+\bar{d}$	$\bar{a}\ b\ c\ d$
10	1010	$a\,\bar{b}\,c\,\bar{d}$	$\bar{a}+b+\bar{c}+d$	
13	1101	$a\,b\,\bar{c}\,d$	?	$\bar{a}+\bar{b}+c+\bar{d}$
15	1111	$a\,b\,c\,d$	$\bar{a}+\bar{b}+\bar{c}+\bar{d}$	

Minterm and Maxterm Relationship

- **Review: DeMorgan's Theorem**
  $\overline{x \cdot y} = \overline{x} + \overline{y}$ **and** $\overline{x + y} = \overline{x} \cdot \overline{y}$
- **Two-variable example:**
  $M_2 = \overline{x} + y$ **and** $m_2 = x \cdot \overline{y}$
  Thus M_2 is the complement of m_2.
- **Since DeMorgan's Theorem holds for *n* variables, the above holds for terms of *n* variables**
- **giving:**
  $$M_i = \overline{m_i} \text{ and } m_i = \overline{M_i}$$
  Thus M_i is the complement of m_i.

Function Tables for 2 variables

- **Minterms of 2 variables**

x y	m_0	m_1	m_2	m_3	
0 0	1	0	0	0	$\overline{X}\,\overline{Y}$
0 1	0	1	0	0	$\overline{X}Y$
1 0	0	0	1	0	$X\overline{Y}$
1 1	0	0	0	1	XY

Maxterms of 2 variables

x y	M_0	M_1	M_2	M_3	
0 0	0	1	1	1	$X+Y$
0 1	1	0	1	1	$X+\overline{Y}$
1 0	1	1	0	1	$\overline{X}+Y$
1 1	1	1	1	0	$\overline{X}+\overline{Y}$

- **Each column in the maxterm function table is the complement of the column in the minterm function table since M_i is the complement of m_i.**

Minterms for 3 Variables

Recommend

X	Y	Z	Product term	Symbol	m_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7
0	0	0	$\bar{X}\bar{Y}\bar{Z}$	m_0	1	0	0	0	0	0	0	0
0	0	1	$\bar{X}\bar{Y}Z$	m_1	0	1	0	0	0	0	0	0
0	1	0	$\bar{X}Y\bar{Z}$	m_2	0	0	1	0	0	0	0	0
0	1	1	$\bar{X}YZ$	m_3	0	0	0	1	0	0	0	0
1	0	0	$X\bar{Y}\bar{Z}$	m_4	0	0	0	0	1	0	0	0
1	0	1	$X\bar{Y}Z$	m_5	0	0	0	0	0	1	0	0
1	1	0	$XY\bar{Z}$	m_6	0	0	0	0	0	0	1	0
1	1	1	XYZ	m_7	0	0	0	0	0	0	0	1

Maxterms for 3 Variables

Recommend

X	Y	Z	Product term	Symbol	M_0	M_1	M_2	M_3	M_4	M_5	M_6	M_7
0	0	0	$X+Y+Z$	M_0	0	1	1	1	1	1	1	1
0	0	1	$X+Y+\overline{Z}$	M_1	1	0	1	1	1	1	1	1
0	1	0	$X+\overline{Y}+Z$	M_2	1	1	0	1	1	1	1	1
0	1	1	$X+\overline{Y}+\overline{Z}$	M_3	1	1	1	0	1	1	1	1
1	0	0	$\overline{X}+Y+Z$	M_4	1	1	1	1	0	1	1	1
1	0	1	$\overline{X}+Y+\overline{Z}$	M_5	1	1	1	1	1	0	1	1
1	1	0	$\overline{X}+\overline{Y}+Z$	M_6	1	1	1	1	1	1	0	1
1	1	1	$\overline{X}+\overline{Y}+\overline{Z}$	M_7	1	1	1	1	1	1	1	0

Minterm Function Example

Recommend

- **Example:** $\boxed{\text{Find } F_1 = m_1 + m_4 + m_7}$
- $F_1 = \overline{x}\ \overline{y}\ z + x\ \overline{y}\ \overline{z} + x\ y\ z$ Solution

Method

x y z	index	m_1	+	m_4	+	m_7	$= F_1$
0 0 0	0	0	+	0	+	0	= 0
0 0 1	1	1	+	0	+	0	= 1
0 1 0	2	0	+	0	+	0	= 0
0 1 1	3	0	+	0	+	0	= 0
1 0 0	4	0	+	1	+	0	= 1
1 0 1	5	0	+	0	+	0	= 0
1 1 0	6	0	+	0	+	0	= 0
1 1 1	7	0	+	0	+	1	= 1

Maxterm Function Example

- **Example: Implement F1 in maxterms:**

$$\boxed{F_1 = M_0 \cdot M_2 \cdot M_3 \cdot M_5 \cdot M_6}$$

$$F_1 = (x + y + z) \cdot (x + \overline{y} + z) \cdot (x + \overline{y} + \overline{z}) \cdot (\overline{x} + y + \overline{z}) \cdot (\overline{x} + \overline{y} + z)$$

x y z	i	$M_0 \cdot M_2 \cdot M_3 \cdot M_5 \cdot M_6 = F1$
0 0 0	0	$0 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 0$
0 0 1	1	$1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 1$
0 1 0	2	$1 \cdot 0 \cdot 1 \cdot 1 \cdot 1 = 0$
0 1 1	3	$1 \cdot 1 \cdot 0 \cdot 1 \cdot 1 = 0$
1 0 0	4	$1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 1$
1 0 1	5	$1 \cdot 1 \cdot 1 \cdot 0 \cdot 1 = 0$
1 1 0	6	$1 \cdot 1 \cdot 1 \cdot 1 \cdot 0 = 0$
1 1 1	7	$1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 1$

Canonical Sum of Minterms

- **Any Boolean function can be expressed as a Sum of Minterms (SOM).**
  - **For the function table, the minterms used are the terms corresponding to the 1's**
  - **For expressions, expand all terms first to explicitly list all minterms. Do this by "ANDing" any term missing a variable v with a term $(v + \bar{v})$.**
- **Example: Implement $F = x + \bar{x}\,\bar{y}$ as a sum of minterms.**

Recommend

First expand terms: $F = x(y + \bar{y}) + \bar{x}\,\bar{y}$

Then distribute terms: $F = xy + x\bar{y} + \bar{x}\,\bar{y}$

Express as sum of minterms: $F = m_3 + m_2 + m_0$

Shorthand SOM Form

Recommend

- **From the previous example, we started with:**

  $F = X + \overline{X}\,\overline{Y}$

- **We ended up with:**

  $\mathbf{F = m_0+m_2+m_3}$

- **This can be denoted in the formal shorthand:**

  $\mathbf{F(X,Y) = \sum_m(0,2,3)}$

- **Note that we explicitly show the standard variables in order and drop the “m” designators.**

Canonical Product of Maxterms

- Any Boolean Function can be expressed as a **Product of Maxterms (POM)**.
  - For the function table, the maxterms used are the terms corresponding to the 0's.
  - For an expression, expand all terms first to explicitly list all maxterms. Do this by first applying the second distributive law , "ORing" terms missing variable v with a term equal to $v \cdot \overline{v}$ and then applying the distributive law again.
- Example: Convert to product of maxterms:

Step 1

$$f(x, y, z) = x + \overline{x}\,\overline{y}$$

Apply the distributive law:

Step 2

$$x + \overline{x}\,\overline{y} = (x + \overline{x})(x + \overline{y}) = 1 \cdot (x + \overline{y}) = x + \overline{y}$$

Add missing variable z:

Step 3

$$x + \overline{y} + z \cdot \overline{z} = (x + \overline{y} + z)(x + \overline{y} + \overline{z})$$

Solution

Express as POM: $f = M_2 \cdot M_3$ or $\Pi_M(2,3)$

Shorthand POM Form

Recommend

- **From the previous example, we started with:**

  $F(X,Y,Z) = X + \overline{X}\,\overline{Y}$

- **We ended up with:**

  $\mathbf{F = M_2 + M_3}$

- **This can be denoted in the formal shorthand:**

  $\mathbf{F(X,Y,Z) = \Pi_M(2,3)}$

- **Note that we explicitly show the standard variables in order and drop the "M" designators.**

Standard Forms

- **<u>Standard Sum-of-Products (SOP) form:</u>** **equations are written as an OR of AND terms**
- **<u>Standard Product-of-Sums (POS) form:</u>** **equations are written as an AND of OR terms**
- **Examples:**
  - **SOP:** $A\,B\,C + \overline{A}\,\overline{B}\,C + B$ **SOM**
  - **POS:** $(A + B)\cdot(A + \overline{B} + \overline{C})\cdot C$ **POM**
- **These "mixed" forms are <u>neither SOP nor POS</u>**
  - $(A\,B + C)\,(A + C)$
  - $A\,B\,\overline{C} + A\,C\,(A + B)$

SOP → SOM **POS → POM**

Useful Trick!

Recommend

- In Mathematics

  $\sum$ represents the process of adding,
  for example, 1+2+3+4

- In Digital System (Boolean function)

  $\sum$ represents the summation of AND
  for example, $A+(A \cdot B)+(B \cdot C)$

It represents the sum-of-product (SOP)

Useful Trick!

Recommend

- Identity 14 represents Sum-of-Product (SOP)
- Identity 15 represents Product-of-Sum (POS)

10.	$X+Y=Y+X$	11.	$XY=YX$	Commutative
12.	$(X+Y)+Z=X+(Y+Z)$	13.	$(XY)Z=X(YZ)$	Associative
14.	$X(Y+Z)=XY+XZ$	15.	$X+YZ=(X+Y)(X+Z)$	Distributive
16.	$\overline{X+Y}=\overline{X}\cdot\overline{Y}$	17.	$\overline{X\cdot Y}=\overline{X}+\overline{Y}$	DeMorgan's

SOP

POS

So, if the given expression is SOP form, you can apply identity 15 to convert into POS form!!

Example 1 (without missing var.)

- Given $F(X,Y,Z) = \overline{X}(\overline{Y}\overline{Z}+Y\overline{Z}) + X(Y\overline{Z}+YZ)$

Convert the above expression into SOP and list the Minterms of F

No Missing Variables

Solution

$$= \overline{X}\overline{Y}\overline{Z}+\overline{X}Y\overline{Z}+XY\overline{Z}+XYZ$$ SOP Form

Then, determine the index of SOM

$$= m_0+m_2+m_6+m_7$$ Apply Index of SOP

Thus, the solution is

$$= \sum_{\mathbf{m}}\mathbf{(0,2,6,7)}$$ List of Minterms

Example 2 (with missing var.)

- Given $F(A,B) = A+\bar{A}B$

Convert the above expression into SOP and list the Minterms of F.

Solution

Assign the missing variables

$$=A(B+\bar{B})+\bar{A}B$$

Missing Variables B

$$= AB + A\bar{B} + \bar{A}B$$

SOP Form

Then, determine the index of SOM

$$= m_3+m_2+m_1$$

Apply Index of SOP

Thus, the solution is

$$= \sum_{m}(1,2,3)$$

List of Minterms

Example 3

Given $F(A,B,C) = (AC+B)\overline{C}+(B+C)A$

Answer the following questions:

1) List the Minterms of function F

2) Convert function F into the sum-of-minterms (SOM) algebraic form.

3) Simplify function F to expression with minimal literals

Example 3 (SOP vs POS)

Given $F(A,B,C) = (AC+B)\overline{C}+(B+C)A$

$= AC\overline{C}+B\overline{C}+AB+AC$

$= B\overline{C}+AB+AC$

Missing Variables

$= (A+\overline{A})B\overline{C} + AB(C+\overline{C})+AC(B+\overline{B})$

$= AB\overline{C}+\overline{A}B\overline{C}+ABC+AB\overline{C}+ABC+A\overline{B}C$

$= AB\overline{C}+\overline{A}B\overline{C}+ABC+A\overline{B}C$

Answer 2

$= m6+m2+m7+m5$

Thus, the Minterms of F is

$\sum_m(2,5,6,7)$

Answer 1

Example 3 (SOP vs POS)

$$= AB\bar{C}+\bar{A}B\bar{C}+ABC+A\bar{B}C$$

$$= AB(C+\bar{C}) + \bar{A}B\bar{C}+A\bar{B}C$$

$$= AB + \bar{A}B\bar{C}+A\bar{B}C$$

$$= A(B+\bar{B}C)+\bar{A}B\bar{C}$$

Answer 3