Inverse of a Matrix

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Using Inverses to Solve Linear Systems

Inverse of a Matrix

- Let A be a **square matrix** of size n.
- A **square matrix** A^{-1} of size n such that

$$A^{-1}A = AA^{-1} = I_n$$

  is called the **inverse of A**.
- Not every matrix has an inverse.
  - A square matrix that **has** an inverse is said to be **nonsingular**.
  - A square matrix that **does not have** an inverse is said to be **singular**.

Example: A Nonsingular Matrix

- The matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **has** a matrix $A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$ as its **inverse**.
- This can be demonstrated by **multiplying them**:

$$AA^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$A^{-1}A = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Example: A Singular Matrix

- The matrix $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ **does not have an inverse.**
- If B had an inverse given by $B^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where

  a, b, c, and d are some appropriate numbers, then **by definition** of an **inverse** we would have $BB^{-1} = I$.
- That is

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

implying that $0 = 1$, which is **impossible**!

Finding the Inverse of a Square Matrix by Using Row Operations

- Given the $n \times n$ matrix A:
  1. Adjoin the $n \times n$ identity matrix I to obtain the **augmented matrix** $[A \mid I]$.
  2. Use a sequence of **row operations** to **reduce** $[A \mid I]$ to the form $[I \mid B]$ if possible.
- Then the matrix B is the **inverse** of A.

Example 1

- **Find the inverse of the matrix** $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Solution

- **We form the augmented matrix**

Example 2

- **Find the inverse of the matrix** $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

Solution

- **We form the augmented matrix**

Example 2

- **Find the inverse of the matrix** $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

Solution

- **We form the augmented matrix**

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

Example 2

- **Find the inverse of the matrix** $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

Solution

- **We form the augmented matrix:** $[A \mid I]$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

Example 2

- **Find the inverse of the matrix** $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

Solution

- **Use the Gauss-Jordan elimination method to reduce it to the form $[I \mid B]$:**

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & -1 \\ 0 & 1 & 0 & -4 & 2 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\underbrace{\qquad\qquad}_{I_n} \qquad \underbrace{\qquad\qquad}_{B=A^{-1}}$$

Example 2

- **Find the inverse of the matrix** $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

Solution

- **Thus, the inverse of A is the matrix**

$$A^{-1} = \begin{bmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

Using Inverses to Solve Linear Systems

- If $AX = B$ is a linear system of n equations in n unknowns and if A^{-1} exists, then

$$X = A^{-1}B$$

is the unique solution of the system.

Example 3

- **Solve the system of linear equations by using inverse matrix**

$$
\begin{aligned}
2x + y + z &= 1 \\
3x + 2y + z &= 2 \\
2x + y + 2z &= -1
\end{aligned}
$$

Solution

- **Write the system of equations in the form** $AX = B$ **where**

$$
A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}
$$

Example 3

- **Solve the system of linear equations by using inverse matrix**

$$\begin{aligned} 2x + y + z &= 1 \\ 3x + 2y + z &= 2 \\ 2x + y + 2z &= -1 \end{aligned}$$

<u>Solution</u>

- **Write the system of equations in the form** $AX = B$ **where**

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$[\,A \mid I\,] \rightarrow [\,I \mid A^{-1}]$

$AX = B$

$A^{-1}AX = A^{-1}B$

$X = A^{-1}B$

Example 3

- **Solve the system of linear equations by using inverse matrix**

$$\begin{aligned} 2x + \ \ y + \ \ z &= \ \ 1 \\ 3x + 2y + \ \ z &= \ \ 2 \\ 2x + y + 2z &= -1 \end{aligned}$$

Solution

- **Find the inverse matrix of** A**:**

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

Example 3

- **Solve the system of linear equations by using inverse matrix**

$$\begin{aligned} 2x + y + z &= 1 \\ 3x + 2y + z &= 2 \\ 2x + y + 2z &= -1 \end{aligned}$$

Solution

- **Finally, we write the matrix equation $X = A^{-1}B$ and multiply:**

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Example 3

- Solve the system of linear equations by using inverse matrix

$$\begin{aligned} 2x + y + z &= 1 \\ 3x + 2y + z &= 2 \\ 2x + y + 2z &= -1 \end{aligned}$$

Solution

- Finally, we write the matrix equation $X = A^{-1}B$ and multiply:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (3)(1) + (-1)(2) + (-1)(-1) \\ (-4)(1) + (2)(2) + (1)(-1) \\ (-1)(1) + (0)(2) + (1)(-1) \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

- Thus, the solution is $x = 2$, $y = -1$, and $z = -2$.

Example 4

- **Solve the system of linear equations by using Gauss-Jordan Elimination menthod**

$$\begin{aligned} 2x + y + z &= 1 \\ 3x + 2y + z &= 2 \\ 2x + y + 2z &= -1 \end{aligned}$$

Solution

Find the Inverse of a Matrix by using Determinant

A Formula for the Inverse of a 2 × 2 Matrix

- Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- Suppose $D = ad - bc$ is **not** equal to **zero**.
- Then A^{-1} exists and is given by

$$A^{-1} = \frac{1}{D}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example 1

- **Find the inverse of** $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$

Solution

- **Compute the determinant**

$$D = ad - bc = (1)(4) - (2)(3) = 4 - 6 = -2$$

- **Compute A^{-1}**

$$A^{-1}=\frac{1}{D}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}=\frac{1}{-2}\begin{bmatrix}4 & -2\\ -3 & 1\end{bmatrix}=\begin{bmatrix}-2 & 1\\ \frac{3}{2} & -\frac{1}{2}\end{bmatrix}$$

Example 2

- **Find the inverse of**

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

Solution

- **Compute the determinant ?**

- **Compute A^{-1} ?**

Inverse of a Matrix Using Its Adjoint

Theorem 4: Inverse of a Matrix Using Its Adjoint.
If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)}\, adj(\boldsymbol{A}) = \frac{1}{\det(A)}\, [\mathrm{C}_{\mathrm{ij}}]^{\mathrm{T}}$$

Adjoint of a Matrix

- **Theorem 3**: If $\boldsymbol{A}$ is any n x n matrix and C_{ij} is the *cofactor* of a_{ij} then the matrix $[C_{ij}]$ is called the *matrix of cofactors* from $\boldsymbol{A}$. The transpose of this matrix is called the *adjoint* of $\boldsymbol{A}$ and is denoted by $adj(\boldsymbol{A})$.

$$adj(\boldsymbol{A}) = [C_{ij}]^T$$