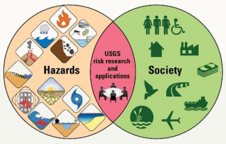
SETS AND SUBSETS

Set Theory in Discrete Mathematics

Definitions of sets and set notations



- A **set** is any **well-defined** collection of objects

  can decide if a given object belongs to the set or not

- Objects contained in a set are called **elements** or **members** of a set
  - The objects can be numbers, letters, words, symbols, relations, or functions. A set can also contains different mixes of elements, including another set.

- Set labels are **uppercase** letters → **A = {a, b, c, d, e}**
- Element labels are **lowercase** letters
- Use **curly brackets or braces** to enclose elements of a set

A = {1, 2, 3} **B** = {C++, Java, Python} **C** = {@, #, $, &} **D** = {x, {y}, {2,3}, 10}

https://www.smartickmethod.com/blog/math/logic/sets-and-subsets/; https://www.askiitians.com/iit-jee-algebra/set-relations-functions/set-theory/finite-and-infinite-sets/

Membership of a set and the empty set

$\in$: is an element of (it has a shape like an "E" as in element)

$\notin$: is not an element of

If $A = \{1, 3, 5, 7\}$ then $1 \in A$ but $2 \notin A$

{ } or $\varnothing$: denotes an **empty set**. An empty set has no element.

Order of elements is not important	**Repeated** elements can be ignored
$\{1,2,3\} = \{2,1,3\} = \{3,2,1\} = \{3,1,2\}$	$\{1,2,3,1,2,2\} = \{1,1,1,2,3\} = \{1,2,3\}$

Describing a set

The roster method: list all members between curly brackets or braces

Set builder notation: specifying properties of its elements

$S = \{x : P(x)\}$ A set S of all elements x such that P(x)

$S = \{x \mid P(x)\}$ A set S is a collection of all x for which P(x) is true

$S = \{x \mid x$ is ___ $\}$ A set S is composed of elements x that satisfies ___

PRACTICE PROBLEMS

Write the following sets in roster format.

- A = { x | x is a solution to the quadratic equation $x^2 - 11x = -30$ }

Know your algebra

- B = { (x, y) | x, y are positive integers less than 5, x + y is odd }

Other than a single number, elements of a set can be pairs or any n-tuples.

PRACTICE PROBLEMS

- Given **A** = {1, b, {4}, {x, y, z}}, fill in the blank with either ∈ or ∉

{x, y, z} ☐ **A**   x ☐ **A**   4 ☐ **A**   1 ☐ **A**   {1, b} ☐ **A**

A member of a set can be another set

- Use a set builder notation to represent the set **B** = {2, 4, 6, 8, 10}.

Common Sets

$\mathbb{Z}^+$ Positive integers

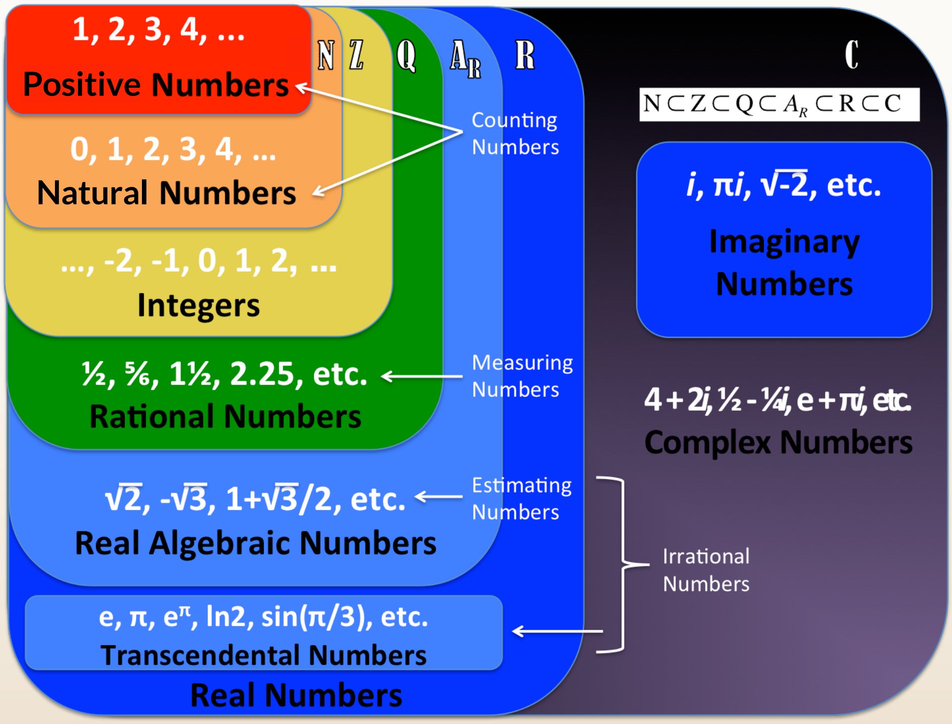
$\mathbb{N}$ Natural numbers

$\mathbb{Z}^-$ Negative integers

$\mathbb{Z}$ Integers

$\mathbb{Q}$ Rational numbers

$\mathbb{R}$ Real numbers



Is zero a natural number?

No consensus! Some consider "zero" a natural number, others do not.

In mathematics, the natural numbers are those used for counting (as in "there are six coins on the table") and ordering (as in "this is the third largest city in the country"). In common mathematical terminology, words colloquially used for counting are "cardinal numbers", and words used for ordering are "ordinal numbers". The natural numbers can, at times, appear as a convenient set of codes (labels or "names"); that is, as what linguists call nominal numbers, forgoing many or all of the properties of being a number in a mathematical sense. The set of natural numbers is often denoted by the symbol $\mathbb{N}$.

Some definitions, including the standard ISO 80000-2, begin the natural numbers with 0, corresponding to the non-negative integers 0, 1, 2, 3, ... (collectively denoted by the symbol $\mathbb{N}_0$), whereas others start with 1, corresponding to the positive integers 1, 2, 3, ... (collectively denoted by the symbol $\mathbb{N}_1$. Both definitions are acknowledged whenever convenient, and there is no general consensus on whether zero should be included as the natural numbers.

Texts that exclude zero from the natural numbers sometimes refer to the natural numbers together with zero as the whole numbers, while in other writings, that term is used instead for the integers (including negative integers).

In this course, we follow the standard ISO 80000-2

Why we consider "zero" a natural number

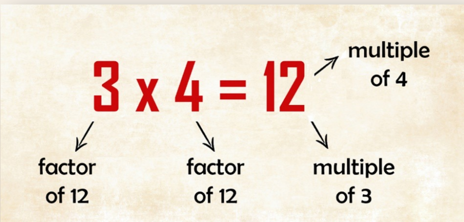
Defining **natural numbers** $\mathbb{N}$ as a set of cardinals of finite sets, i.e. **the set of the number of elements in any finite set**.

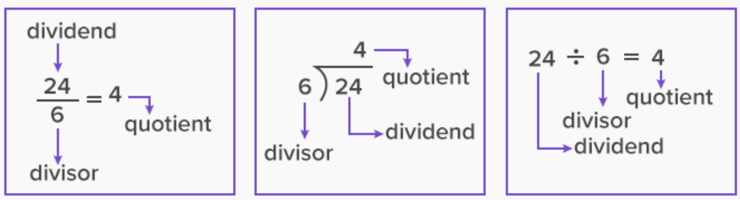
{}	has **0** element
{a}	has **1** element
{a, b}	has **2** elements
{a, b, c}	has **3** elements

Thus, by this definition, zero is a natural number

In set theory, we count zero. It is the number of element in the empty set. In other words, the cardinality of the empty set is zero.

y is a **multiple** of x	y is a product of x and a nonzero integers $\exists k \in \mathbb{Z}, k \neq 0, y = kx$, think of y as the times table of x
x is a **factor** of y	x divides y exactly or evenly, leaving no remainder $\exists k \in \mathbb{Z}, k \neq 0, y/x = k$ or $y = kx$, that is, y mod $x = 0$
$x \mid y$ y is **divisible** by x	x divides y exactly or evenly when $x \mid y$, x (also k) is a factor of y, and y is a multiple of x
x is a **prime** number	$x \in \mathbb{N}, x > 0$, x has only 2 factors, it can only be divided by 1 and itself





https://www.splashlearn.com/math-vocabulary/division/divisor; https://mathoriginal.com/number-theory/multiples-and-factors/?doing_wp_cron=1697341697.6014020442962646484375

WORKED EXAMPLES

Which of the following describes the set **A** = {-15, -30, -45, -60, ...} ?

- ❐ $\{x \in \mathbb{Z}^- \mid x \text{ divisible by } 15\}$
- ❐ $\{x \in \mathbb{Z}^- \mid x \text{ is a multiple of } 15\}$
- ❐ $\{x = -15y \mid y \in \mathbb{Z}^+\}$
- ❐ all of the options

Know the terms - divisible and multiple

Which of the following describes the set **B** = {1, 4, 9, 16, 25, 36, 49, ...}

- ❐ $\{x^2 \mid x \in \mathbb{N}\}$
- ❐ $\{x \in \mathbb{Z} \mid x \text{ is a square number}\}$
- ❐ $\{x = y^2 \mid y \in \mathbb{Z}^-\}$
- ❐ all of the options

PRACTICE PROBLEMS

- List all elements of the set (i.e. write the given set in the roster format), **A** = $\{x^3 \mid x \in \mathbb{Z}^- \text{ and } x > -100\}$

Know what a factor is

- Which of the following element is **NOT** a member of the set **B** = $\{ (x, y) \mid x \in \mathbb{R}, (x^2 - 2)(x^2 - 4) = 0 \text{ and } y \in \mathbb{N}, y \text{ is a factor of } 18\}$?

❐ $(\sqrt{2}, 9)$ ❐ (-2, 2) ❐ (2, 6) ❐ all are members

Cardinality of a set

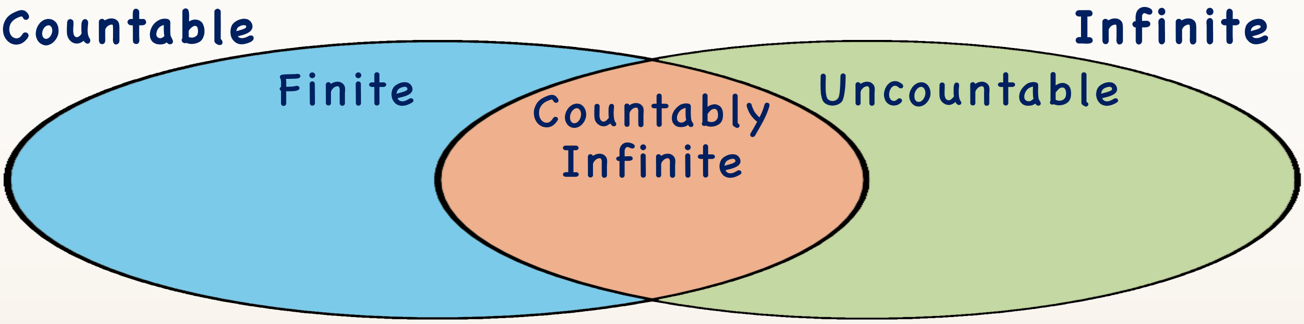
- The **cardinality** of a set is the number of elements in the set
- The cardinality of a set A is denoted by n(A) or |A|

If $A = \{1, \{3\}, \{5,7\}\}$ then $n(A) = |A| =$ ___

If $B = \{x \in \mathbb{N} \mid 9 < x < 10\}$ then $n(B) = |B| =$ ___

If $C = \{(x, y) \mid x \text{ is a solution to } x^2 - 5x + 6 = 0 \text{ and } y \in \mathbb{Z}^-, y > -3\}$
then $n(C) = |C| =$ ___

	COUNTABILITY	FINITENESS	
Countability and finiteness of a set	Can you count the elements, that is, can you list the elements in the set?	Can you tell how many elements there are in the set?	
A = {1, {1}, {2, 3}, 5, 8, 6}			
B = {0, 1, 4, 9, 16, 25, …}			
C = $\{x \mid x \in \mathbb{R}, 0 < x < 1\}$			



Categorize the following sets, put each in the correct region in the diagram

A = $\{x \mid x \in \mathbb{Z}^+, x \leq 50\}$

B = $\{2^q \mid q \in \mathbb{Q}^+\}$

C = $\{\ \} = \varnothing$

Know your prime numbers

D = {x | x is a multiple of 3}

E = {x | x is irrational}

F = {x | x is prime}

G = a set of all points in a plane

H = {x| x is a real number and $(x^2 - 2)(x + 6) = 0$}

PRACTICE PROBLEMS

Identify each set as finite, countably infinite, or uncountable.

- A = $\{ x \mid x \in \mathbb{R}$ and $x^2 + 41x + 41 = 0\}$

  hint: $x = \left(-b \pm \sqrt{b^2 - 4ac}\right)/2a$

- B = $\{ x = m/n \mid m, n \in \mathbb{Z}^+$ and $n > 4\}$

- C = $\{ x \mid x \in \mathbb{R}$ and $x^2 + 3x + 2 \neq 0\}$

- D = $\{(x, y, z) \mid x \in \mathbb{Z}, y \in \mathbb{R}^+, z \in \mathbb{Z}^+\}$

Set equality and set equivalence for finite sets

- Two sets are **equal** when they contain the **same elements**

  $C = \{1, 2, 3\}$

  $D = \{2, 2, 1, 2, 3, 3\}$

  $E = \{x \in \mathbb{Z}^+ \mid x^2 < 12\}$

A finite set A is **equivalent** to a finite set B if $n(A) = n(B)$, i.e. $|A| = |B|$, that is, the sets A and B have the **same cardinal numbers**

Know vowels and consonants

A is the set of vowels in the word CODING

B is the set of solutions to the equation $(2x-1)(x+6) = 0$

PRACTICE PROBLEMS

- Let $A = \{1, 2, 3, 4, 5\}$. Which of the following sets are equal to A and which are equivalent to A?

  $B = \{x \mid x$ is an integer and $x^2 \leq 25\}$

  $C = \{x \mid x$ is a positive integer and $x^2 \leq 25\}$

  $D = \{x \mid x$ is a negative integer and $x^2 \leq 25\}$

  $E = \{x \mid x$ is a positive rational number and $x \leq 5\}$

PRACTICE PROBLEMS

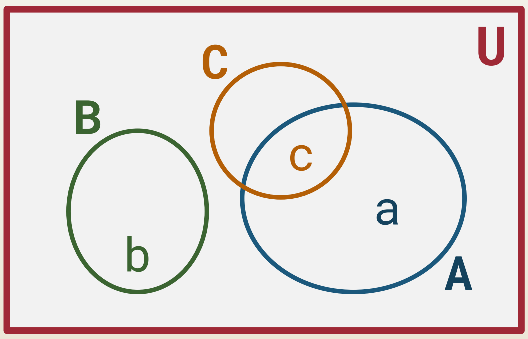
Choose all that apply: which statement is true about the set
$A = \{ p \mid p \text{ is a prime number that divides 70 exactly (no remainder)} \}$

- ❑ The cardinality of A is 4
- ❑ A is equal to the set $B = \{ 7, 7, 2, 5, 5, 5, 1 \}$
- ❑ $|A| = |D|$ where $D = \{x \in \mathbb{Z}^- \mid -42 < 7x \leq -21\}$
- ❑ A is equivalent to the set of negative odd integers greater than -10
- ❑ A is equivalent to the set of solutions to the equation $x^3 + 5x^2 - 4x - 20 = 0$

Venn diagrams

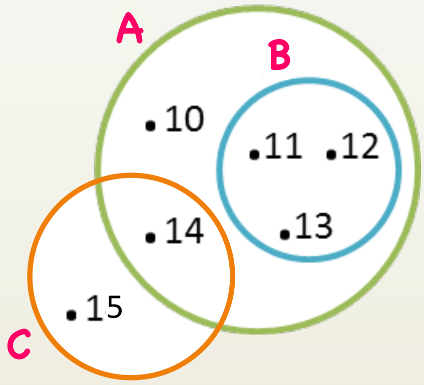
- Named after a British logician John Venn
- Graphical depiction of the relationship of multiple sets
- Does not represent the individual elements of the sets, rather it implies their existence

- A **circle** is used for a general set
- A **rectangle** is used for the **universal** set U (it is a collection of everything)



Subsets

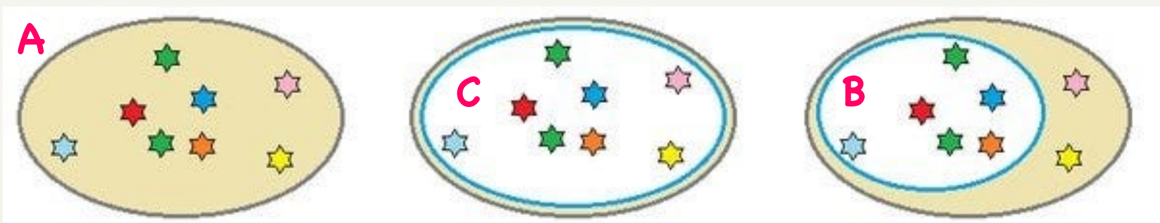
- B is a subset of A
  - if every element of B is also in A
  - if every element of B is contained in A
  - if every member of B is also a member of A



$\subseteq$ B is **a subset** of A when B is a set of some elements of A

$\nsubseteq$ C **not** a subset of A if there is an element $x \in$ C such that $x \notin$ A

Proper subsets vs. Subsets



- Both B and C are subsets of A, that is, **B ⊆ A** and **C ⊆ A**
- A **proper subset** contains some, but not all, elements of another set.
- **B IS a proper subset of A** and is denoted **B ⊂ A**
- **C is NOT a proper subset of A**, written as **C ⊄ A**

This is because C contains all elements of A, i.e., C = A

List all subsets of a set A = {1,2,3}

Subset	List all possible combinations of elements …

The **power set** of A, *P*(A), is the set of all subsets of A.

The **cardinality of the power set** of the set A is $|P(A)| = 2^{|A|}$

$\varnothing \subseteq A$

$A \subseteq A$

WORKED EXAMPLES

- A is a set with a cardinality of 4. which of the following is not true?
  - ❒ Number of proper subsets of A = 15
  - ❒ | P(A) | = 16
  - ❒ Number of non-empty proper subsets of A =14
  - ❒ all are true

- Find the set of smallest cardinality that contains the following sets as subsets: {a, b, c}, {a, d, e, f}, {b, c, e, g}

PRACTICE PROBLEMS

- How many subsets of A= {1, 2, 3, 4} contain the element 2 but not 3?

- The cardinality of the smallest set that contains the sets {{1}}, {1,2}, {{1,2}} as subsets is

- The number of subsets in set A is 192 more than number of subsets in set B. How many elements are there in sets A and B?

PRACTICE PROBLEMS

- Let **A** = {2, 4, 6, 8, 10}, **B** = $\{x \in \mathbb{Z}^+ \mid x$ is even and $x < 12\}$

  C = {2, 4, 6}, **D** = $\{x \in \mathbb{N} \mid x$ is a multiple of $2\}$, and **E** = {{2,4}, 6}

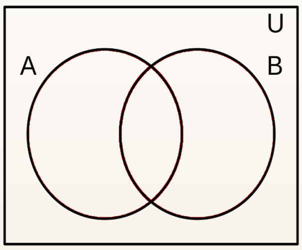
❐ $B \subseteq A$	❐ $C \subseteq A$	❐ $D \subseteq A$	❐ $\{2,4\} \subseteq E$
❐ $B \subset A$	❐ $C \subset A$	❐ $\{4,6\} \subset E$	❐ $\{2,4\} \subset C$
❐ $4 \in A$	❐ $\{4\} \in A$	❐ $\{4,6\} \in P(C)$	❐ $\{2,4\} \in C$

OPERATIONS ON SETS

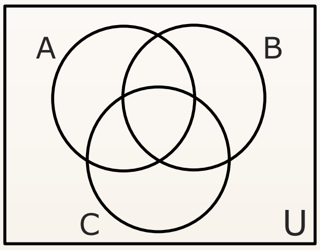
Set Theory in Discrete Mathematics

union in A or in B

$A \cup B$

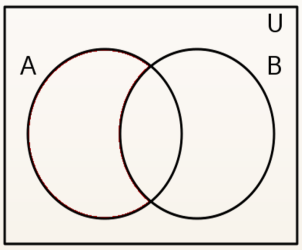


$A \cup B \cup C$

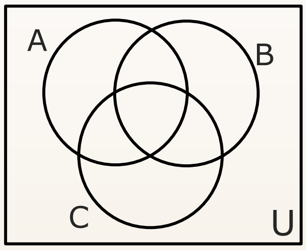


intersection in both A and B

$A \cap B$



$A \cap B \cap C$



$A = \{1, 2, 3\}$

$B = \{2, 3, 4, 5\}$

$U = \{1, 2, 3, 4, 5, 6, 7, 8\}$

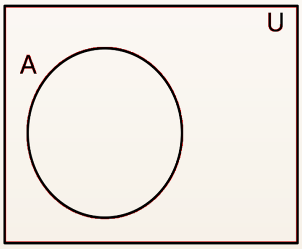
$A \cup B =$

$A \cap B =$

complement

not in A

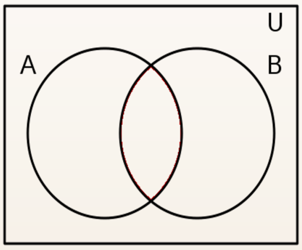
$\bar{A}$



difference

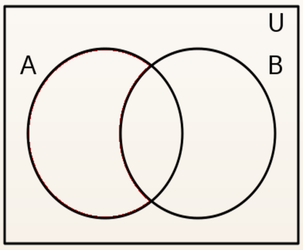
in A not in B

A – B



in B not in A

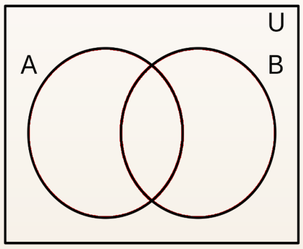
B – A



symmetric difference

in A or B not both

$A \oplus B$



$\bar{A}$ =

A – B =

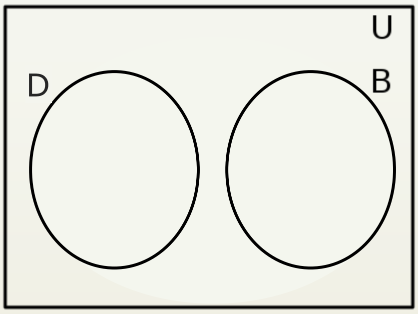
B – A =

$A \oplus B$ =

U = {1, 2, 3, 4, 5, 6, 7, 8} A = {1, 2, 3} B = {2, 3, 4, 5}

Intersection – Disjoint Set

- A **disjoint set** is a set where the intersection result is the empty set
- In this example, sets B and D are disjoint



A = {1, 2, 3}

B = {2, 3, 4, 5}

U = {1, 2, 3, 4, 5, 6, 7, 8}

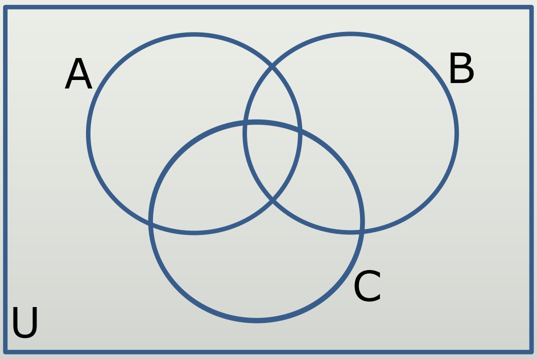
D = {7, 8}

$B \cap D$ =

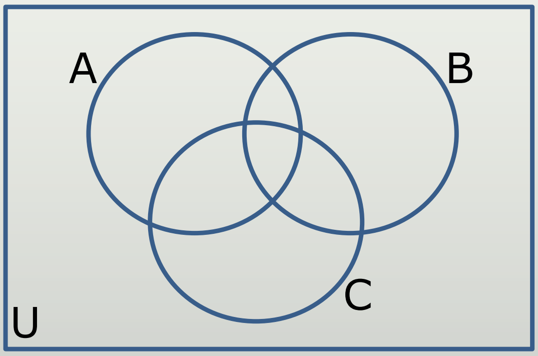
WORKED EXAMPLES

Shade a Venn diagram to represent the set.

- $(\overline{A} \cap \overline{C}) \cup B$



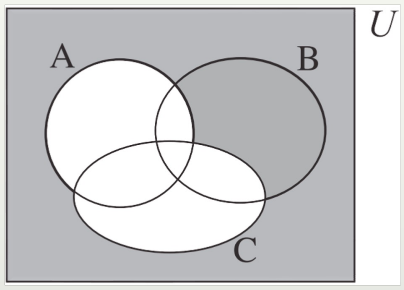
- $(A \cup B) - (A \cap B) - C$



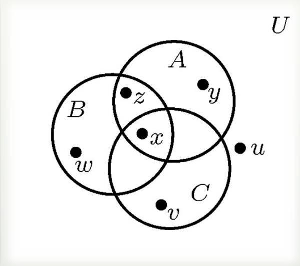
PRACTICE PROBLEMS

The shaded (gray) region represents

- ❒ $(A \cup C) \cap B$
- ❒ $B \cup (A \cap C)$
- ❒ $\overline{A} \cup \overline{C}$
- ❒ $\overline{A \cup C}$



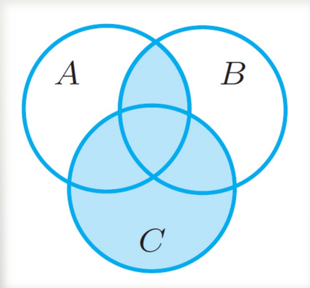
PRACTICE PROBLEMS



Identify the following as true or false.

- $y \in A \cap B$
- $w \in B \cap C$
- $x \in B \cup C$
- $u \notin C$

Describe the shaded region shown on the right using unions and intersections of the sets A, B, and C. (Several descriptions are possible.)



PRACTICE PROBLEMS

Let the universal set be $U = \{a, b, c, d, e, f, g, h, k\}$,

$A = \{a, b, c, g\}$, $B = \{d, e, f, g\}$, $C = \{a, c, f\}$, and $D = \{f, h, k\}$, compute

- $\overline{A \cup B}$
- $A \oplus C$
- $(A \cap B) - (B \cap D)$
- $C \cap \varnothing$

If $A \cup B = A \cup C$, must $B = C$? Explain.

IDENTITY

Identity does not change the value of the operand

$a + 0 = a$
$a \times 1 = a$

INVERSE

Inverse changes the number to the identity

$a + (-a) = 0$
$a \times (1/a) = 1$

COMMUTATIVE

Changing the order of the operands

$a + b = b + a$
$a \times b = b \times a$

properties in mathematics VOCAB

ASSOCIATIVE

Changing the group of the operands

$a + (b + c) = (a + b) + c$
$a \times (b \times c) = (a \times b) \times c$

DISTRIBUTIVE

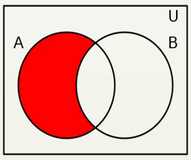
This involves two operators, e.g. combining addition & multiplication

$a \times (b + c) = (a \times b) + (a \times c)$

Algebraic properties of set operations

Identity Laws $A \cup \varnothing =$ $A \cap U =$	**Complement Laws** $A \cup \overline{A} =$ $A \cap \overline{A} =$
Idempotent Laws unchanged $A \cup A =$ $A \cap A =$	**Double Complementation** $\overline{\overline{A}} =$
Domination Laws $A \cup U =$ $A \cap \varnothing =$	**Complement of $\varnothing$ and U sets** $\overline{\varnothing} =$ $\overline{U} =$

Algebraic properties of set operations

Commutative properties order $A \cup B = B \cup A$ $A \cap B = B \cap A$	**Difference Law** $A - B = A \cap \overline{B}$ 
Associative properties GROUP $A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	**De Morgan's Law** negation $\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$
Distributive properties $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	**Absorption Laws** $A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$

Show that two sets are equal

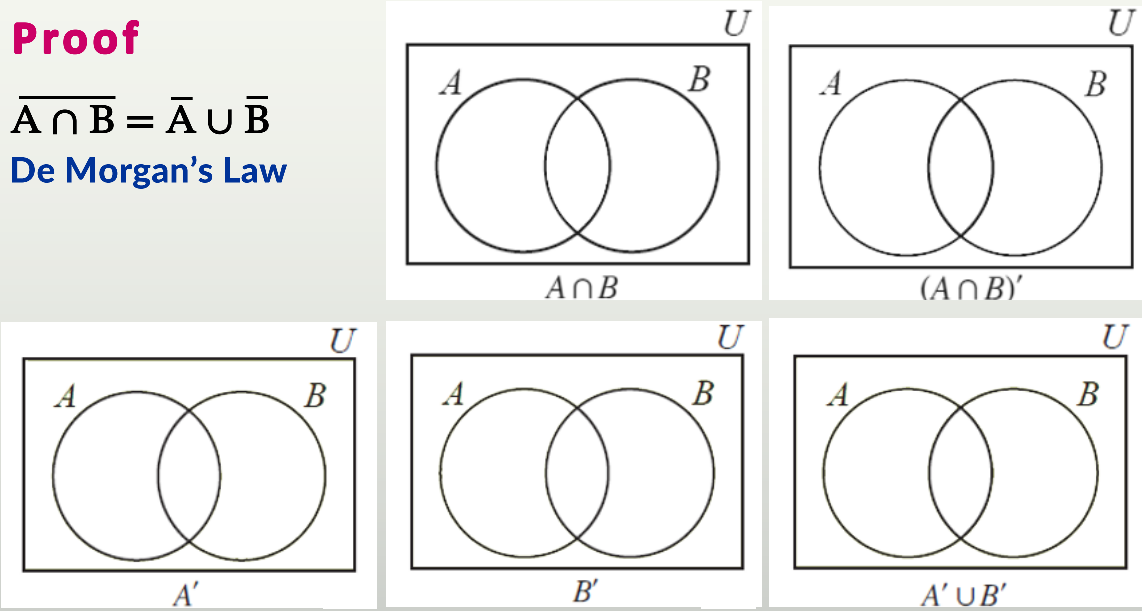
- Use Venn diagrams
- Apply sequences of common equivalences (algebraic properties)

WORKED EXAMPLES

Proof

$\overline{A \cap B} = \bar{A} \cup \bar{B}$

De Morgan's Law



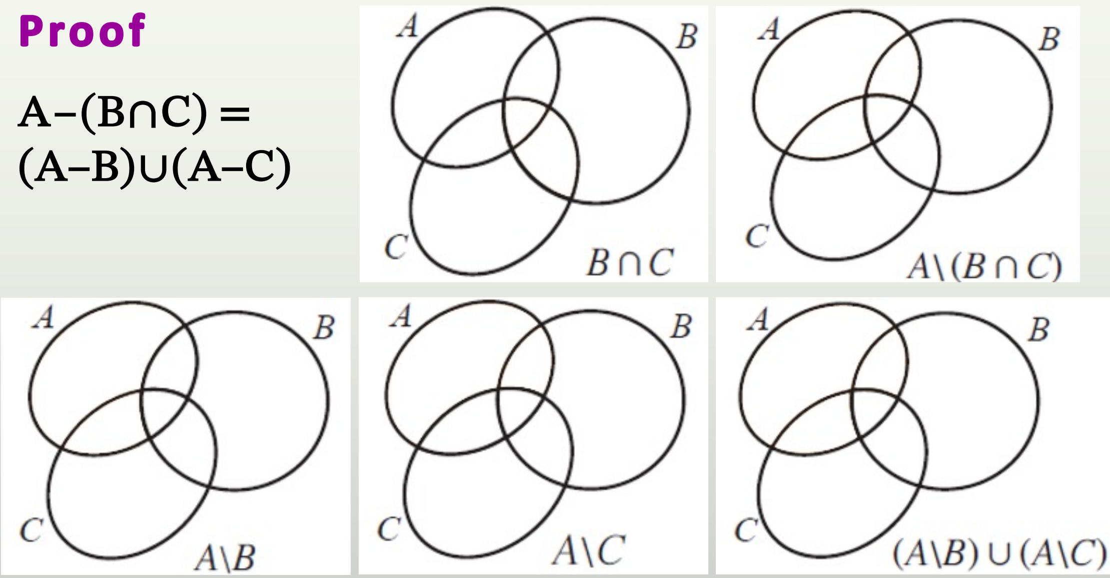
$\overline{A \cap B}$

$\bar{A} \cup \bar{B}$

PRACTICE PROBLEMS

Proof

$A-(B\cap C) =$
$(A-B)\cup(A-C)$



$A-(B\cap C)$

$(A-B)$
$\cup$
$(A-C)$

WORKED EXAMPLES

- Use set algebra to prove that $A \cap (B - C) = (A \cap B) - C$

PRACTICE PROBLEMS

- Use set algebra to prove that $((\overline{B - A}) \cap A) - \overline{A} = A$