Mahidol University
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Wisdom of the Land

Logic and Computer Design Fundamentals

Chapter 2 – Combinational Logic Circuits

Part 5 – Circuit Optimization

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Overview

- **Part 5 – Circuit Optimization**
  - **Map Manipulation**
  - **Simplification using Karnaugh Map Technique**

Map Manipulation

- Using Map called Karnaugh Map (or K-Map) is a simple simplification technique.
- The K-map method of solving the logical expressions is referred to as the graphical technique of simplifying Boolean expressions.

Karnaugh Maps (K-map)

- A K-map is a collection of squares (or Cubes)
  - Each square represents a Minterm
  - The collection of squares is a graphical representation of a Boolean function
  - Adjacent squares differ in the value of one variable
  - Alternative algebraic expressions for the same function are derived by recognizing patterns of squares
- The K-map can be viewed as
  - A reorganized version of the truth table
  - A topologically-warped Venn diagram as used to visualize sets in algebra of sets

K-Map Template (using Gray Code)

- Each cell within a K-map has a definite placed-value (or variable value) which is obtained by using an encoding technique known as Gray code.

Gray code

What is Gray Code?

Gray Code is an ordering of the binary numerical system such that two successive values differ in only one bit (binary digit).

Decimal	Binary	Gray Code
0	000	000
1	001	001
2	010	011
3	011	010
4	100	110
5	101	111
6	110	101
7	111	100

What is Gray Code?

Binary Code

0	0	0	0
0	0	0	1
0	0	1	0
0	0	1	1
0	1	0	0
0	1	0	1
0	1	1	0
0	1	1	1
1	0	0	0
1	0	0	1
1	0	1	0
1	0	1	1
1	1	0	0
1	1	0	1
1	1	1	0
1	1	1	1

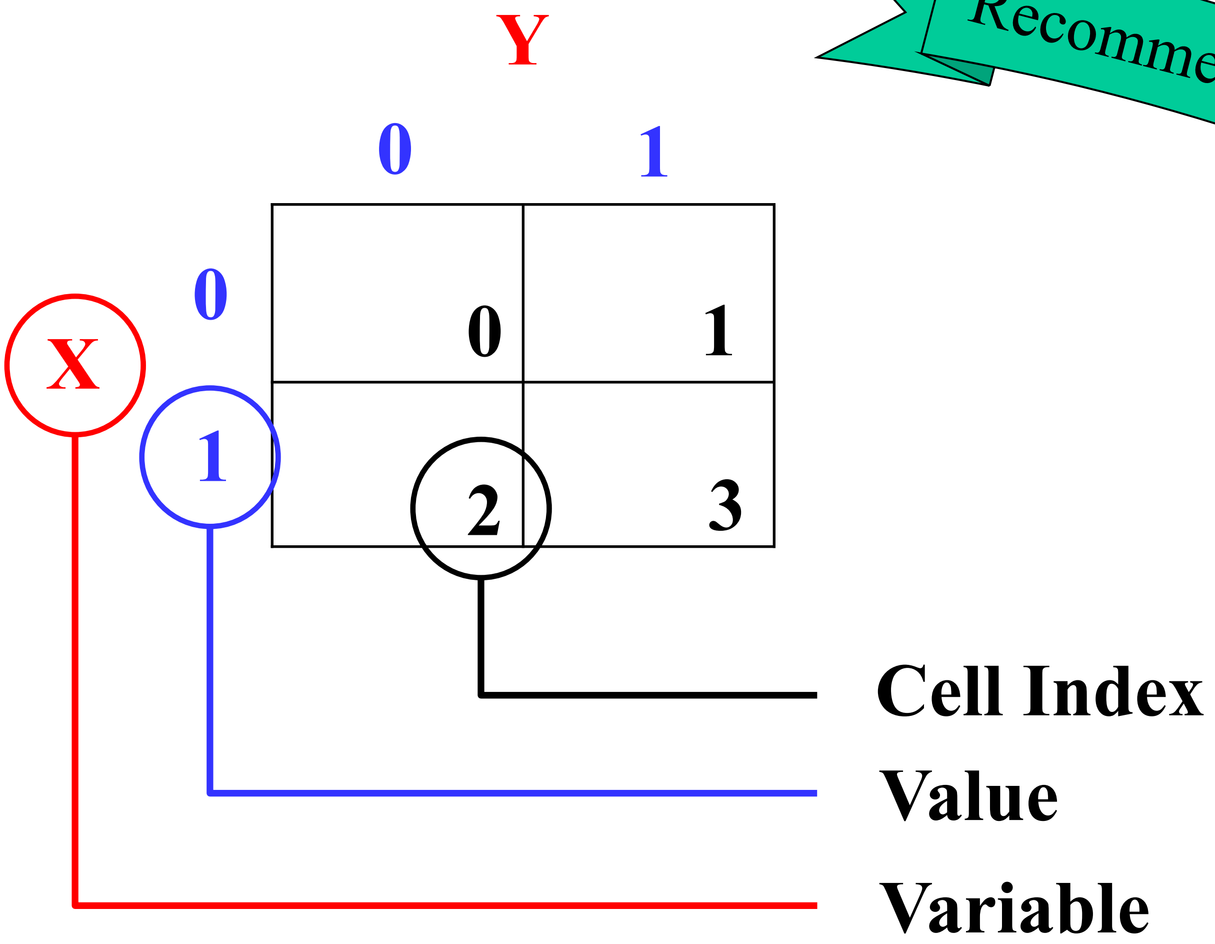
Binary equivalent gray code

Gray Code

0	0	0	0
0	0	0	1
0	0	1	1
0	0	1	0
0	1	1	0
0	1	1	1
0	1	0	1
0	1	0	0
1	1	0	0
1	1	0	1
1	1	1	1
1	1	1	0
1	0	1	0
1	0	1	1
1	0	0	1
1	0	0	0

Only 1-bit changes between each state transition

K-Map Template (Two Variables)



Two Variable Maps

X \ Y	0	1
0	00 $\overline{X}\,\overline{Y}$ 0	01 $\overline{X}Y$ 1
1	10 $X\overline{Y}$ 2	11 XY 3

Assigned values and variables to K-Map

K-Map and Truth Tables

- The K-Map is just a different form of the truth table.
- Example – Two variable function:
  - We choose a,b,c and d from the set {0,1} to implement a particular function, F(x,y).

Function Table

Input Values (x,y)	Function Value F(x,y)
0 0	a
0 1	b
1 0	c
1 1	d

K-Map

	y = 0	y = 1
x = 0	a	b
x = 1	c	d

How to draw K-Map?

Recommend

Assign Boolean expression into K-Map

$$F(X,Y) = \bar{X}Y + XY$$

X \ Y	0	1
0	0	1
1	0	1

How to draw K-Map?

$$F(X,Y) = XY + X\bar{Y} + \bar{X}Y$$

X \ Y	0	1
0		1
1	1	1

$$F(X,Y) = X$$

X \ Y	0	1
0		
1	1	1

How to draw K-Map?

$F(X,Y) = \sum_m (0, 1, 3)$

X \ Y	0	1
0	1	1
1		1

$F(X,Y) = \sum_m (1, 2)$

X \ Y	0	1
0		1
1	1	

K-Map Function Representation

- **Example: F(x,y) = x**

F = x	y = 0	y = 1
x = 0	0	0
x = 1	1	1

- **For function F(x,y), the two adjacent cells containing 1's can be combined using the Minimization Theorem:**

$$F(x, y) = \boxed{x\,\overline{y}} + \boxed{x\,y} = x$$

2 Cells in K-Map = 1 literal (output)

K-Map Function Representation

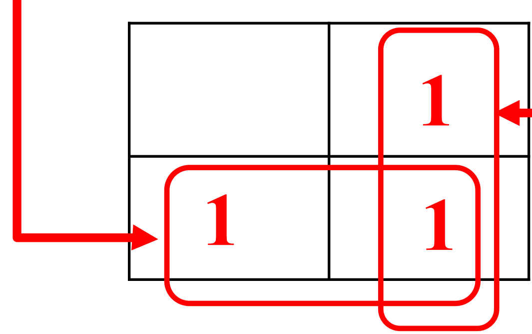
- **Example: G(x,y) = x + y**

G = x+y	y = 0	y = 1
x = 0	0	1
x = 1	1	1

- **For G(x,y), two pairs of adjacent cells containing 1's can be combined using the Minimization Theorem:**

Duplicate x y

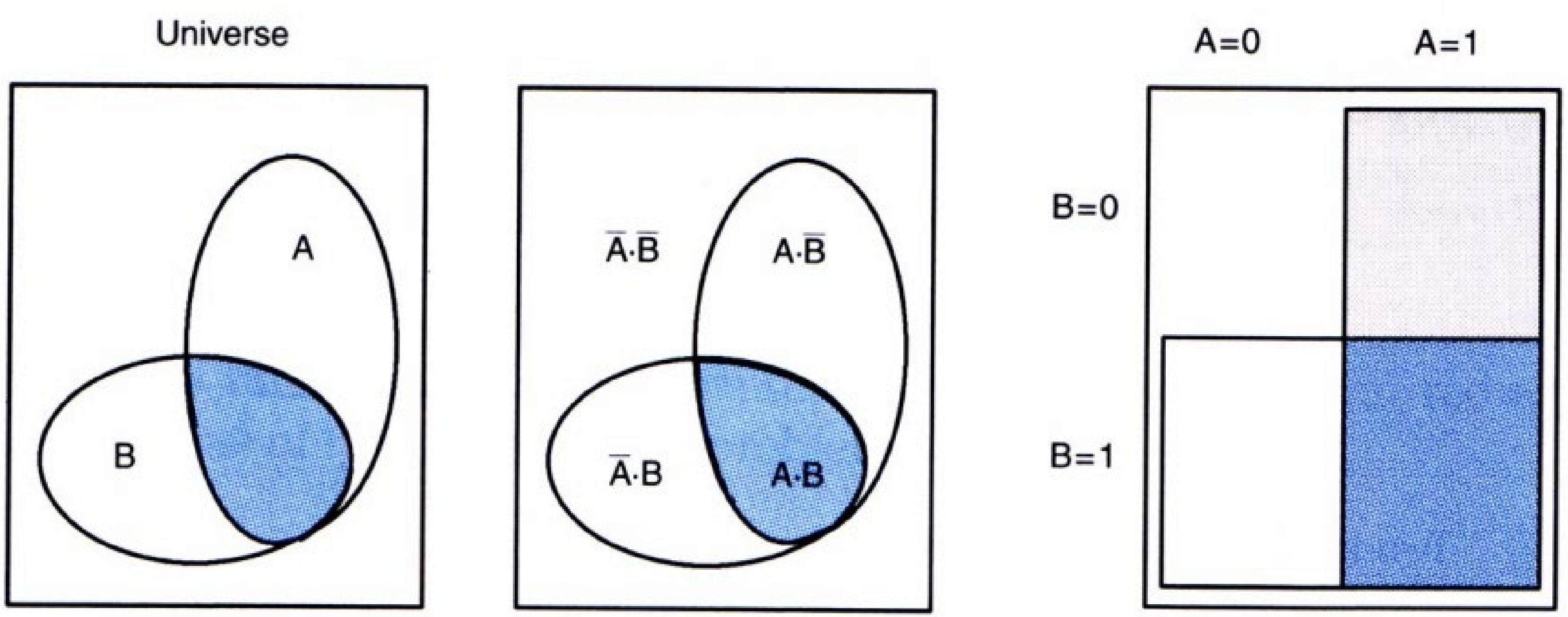
$$G(x,y) = \boxed{\left(x\,\bar{y} + x\,y\right)} + \boxed{\left(xy + \bar{x}\,y\right)} = x + y$$



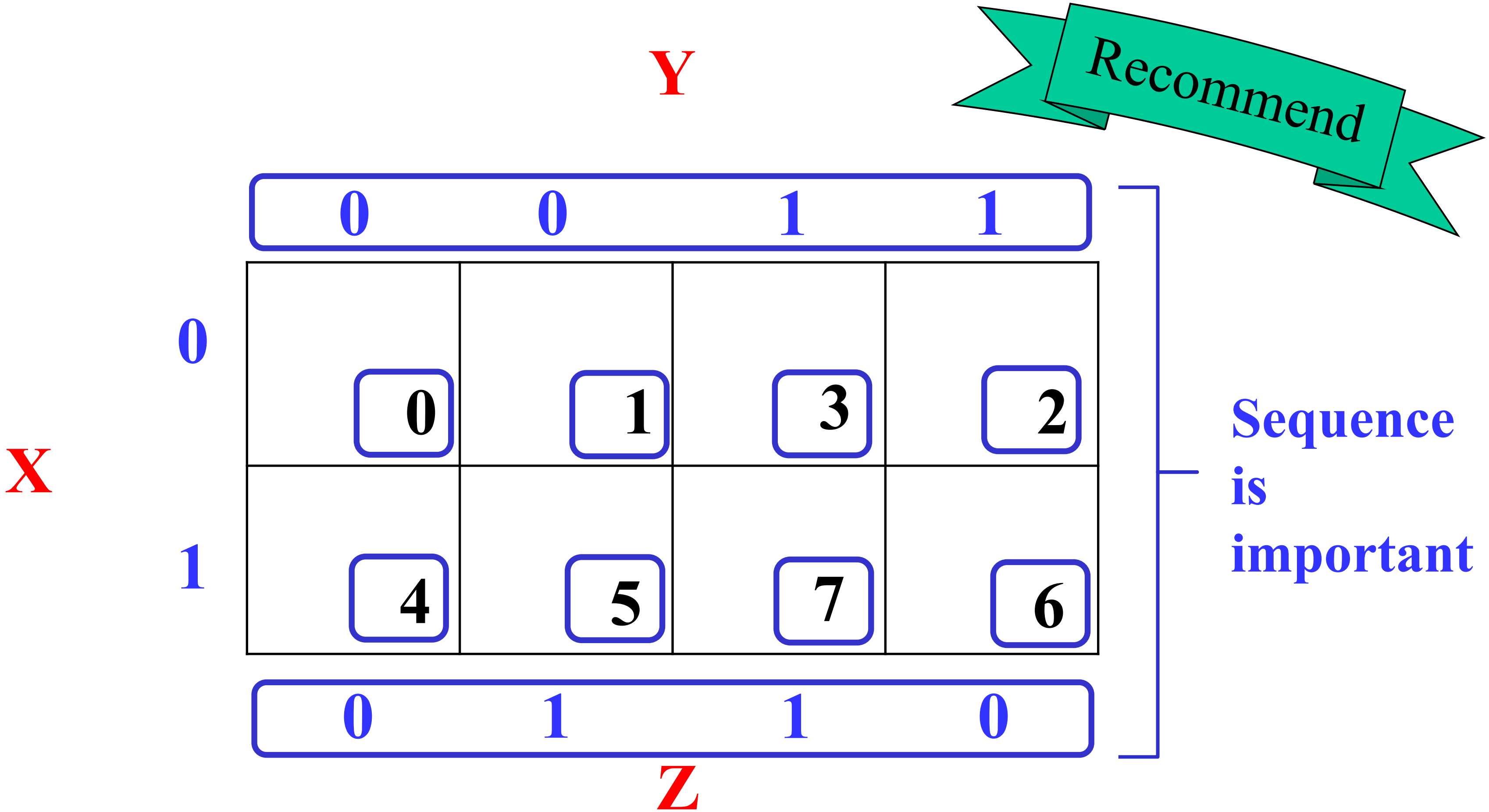
Venn Diagram vs K-Map

Venn Diagram represents over-lapping area between 2 sets

K-Map represents over-lapping area between 2 Adjacent



K-Map Template (Three Variables)



K-Map Template (Three Variables)

Recommend

Y

		0	0	1	1
X	0	000 $\overline{X}\,\overline{Y}\,\overline{Z}$ 0	001 $\overline{X}\,\overline{Y}Z$ 1	011 $\overline{X}YZ$ 3	010 $\overline{X}Y\overline{Z}$ 2
	1	100 $X\overline{Y}\,\overline{Z}$ 4	101 $X\overline{Y}Z$ 5	111 XYZ 7	110 $XY\overline{Z}$ 6
		0	1	1	0

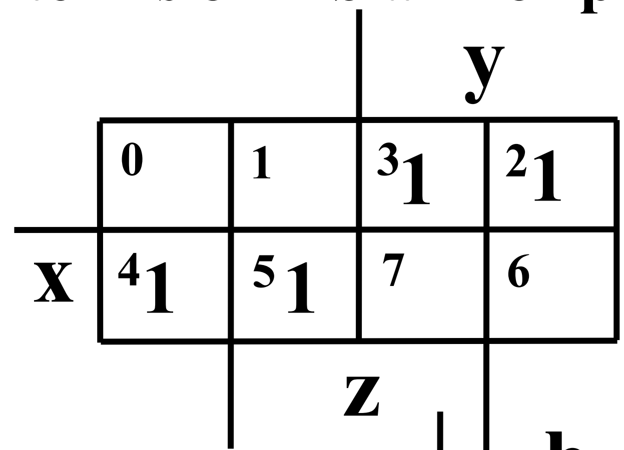
Z

Assigned values and variables to K-Map

Example Functions

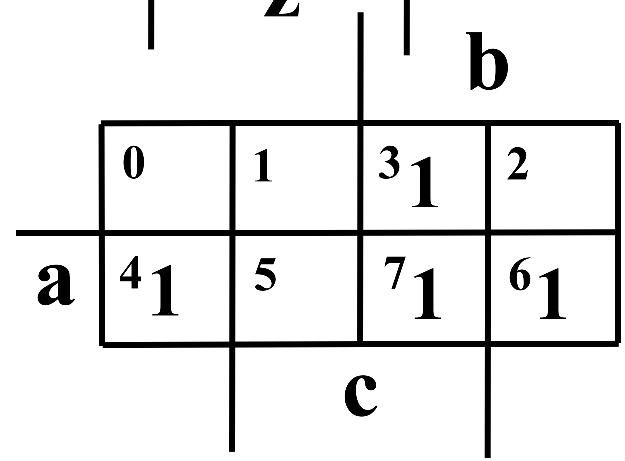
- By convention, we represent the minterms of F by a "1" in the map and leave the minterms of $\overline{F}$ blank or put "0"
- Example:

  $F(x, y, z) = \Sigma m(2,3,4,5)$



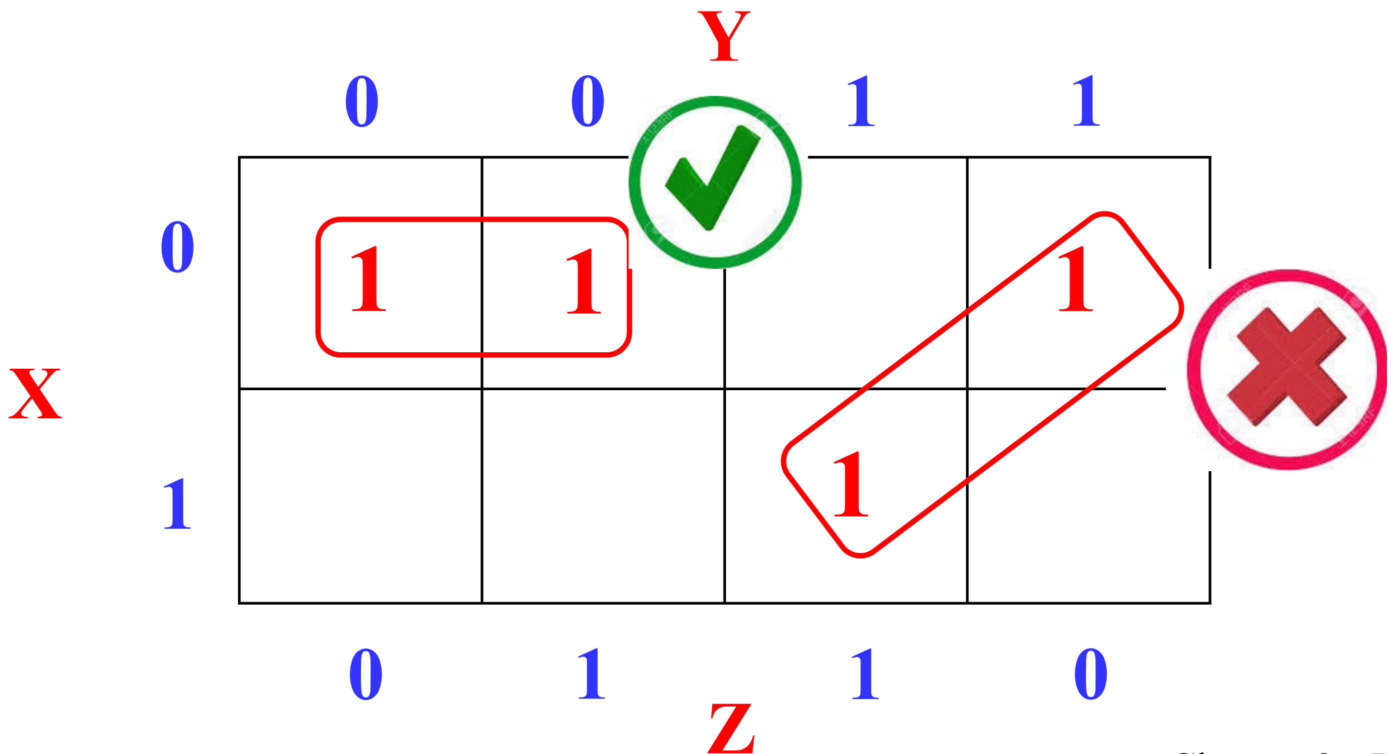
- Example:

  $G(a, b, c) = \Sigma m(3,4,6,7)$



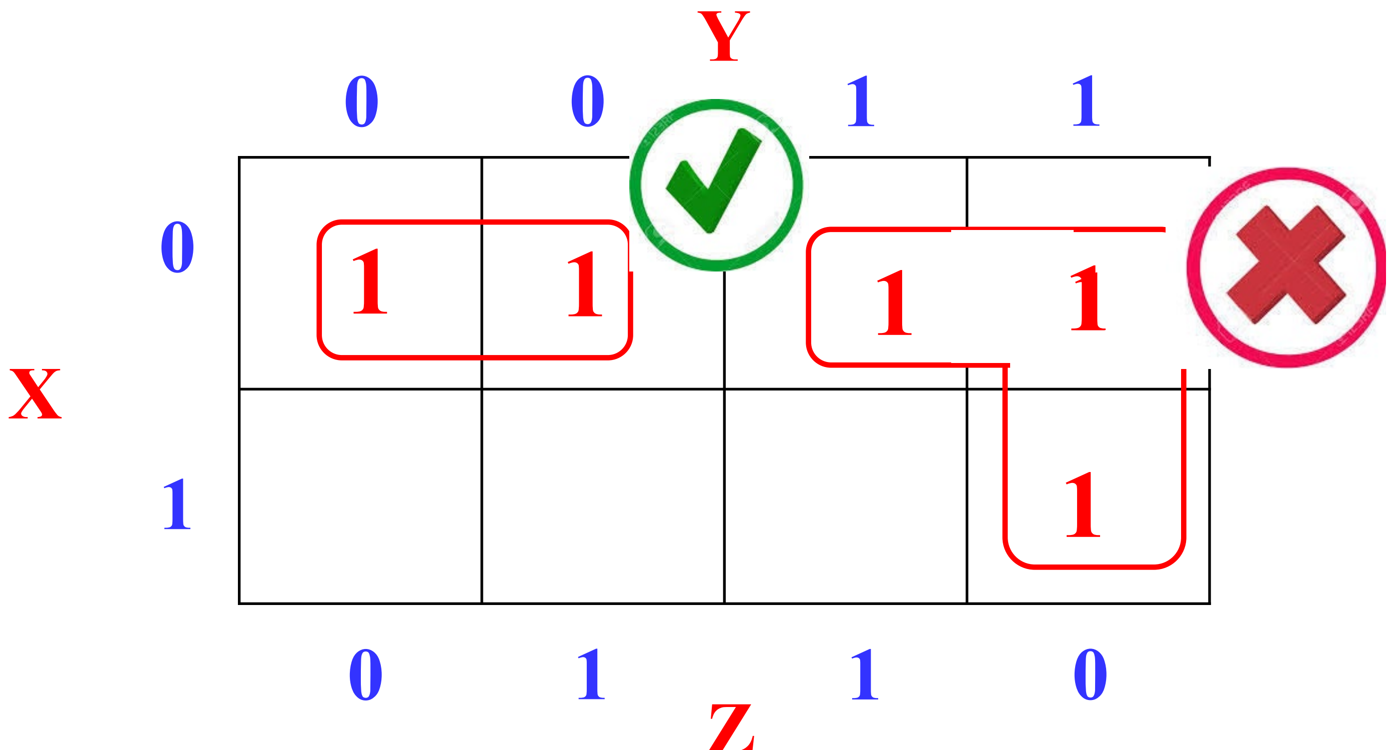
Determination of Adjacent Cell

- **if the binary value for an <u>index</u> differs in one bit position, the minterms are <u>adjacent</u> on the K-Map**



Determination of Adjacent Cell

- Number of cell for Adjacent on the K-Map must equal to 2^n
- Thus, Adjacent must be 1, 2, 4, 8, etc.



Determination of Adjacent Cell

Combining Squares

- By combining squares, we reduce number of literals in a product term, reducing the literal cost

- On a 3-variable K-Map:
  - One square represents a minterm with three variables
  - Two adjacent squares represent a product term with two variables
  - Four “adjacent” terms represent a product term with one variable
  - Eight “adjacent” terms is the function of all ones (no variables) = 1.

Prime Implicants

- Implicant: A product term that has non-empty intersection with on-set F and does not intersect with off-set R
- Prime Implicant: An implicant that is not a proper subset of any other implicant i.e. it is not completely covered by any single implicant

Y

CD \ AB	00	01	11	10
00	1	0	0	1
01	0	1	0	1
11	1	1	0	0
10	1	1	0	1

Q: Is this a prime implicant?

A. Yes

B. No

Maximum Adjacent Cells=2

Prime Implicants

- Implicant: A product term that has non-empty intersection with on-set F and does not intersect with off-set R
- Prime Implicant: An implicant that is not a proper subset of any other implicant i.e. it is not completely covered by any single implicant

Y

CD \ AB	00	01	11	10
00	1	0	0	1
01	0	1	0	1
11	1	1	0	0
10	1	1	0	1

Q: How about this one? Is it a prime implicant?

A. Yes
B. No

Maximum Adjacent Cells?

Prime Implicants

- Implicant: A product term that has non-empty intersection with on-set F and does not intersect with off-set R
- Prime Implicant: An implicant that is not a proper subset of any other implicant i.e. it is not completely covered by any single implicant

Y

CD \ AB	00	01	11	10
00	1	0	0	1
01	0	1	0	1
11	1	1	0	0
10	1	1	0	1

Q: Is the red group a prime implicant?

A. Yes
B. No: Because it is covered by a larger group

Maximum Adjacent Cells=4

Example: Combining Squares

- **Example: Let** $F = \Sigma m(2,3,6,7)$

			y	y
	0	1	3 **1**	2 **1**
x	4	5	7 **1**	6 **1**
		z	z	

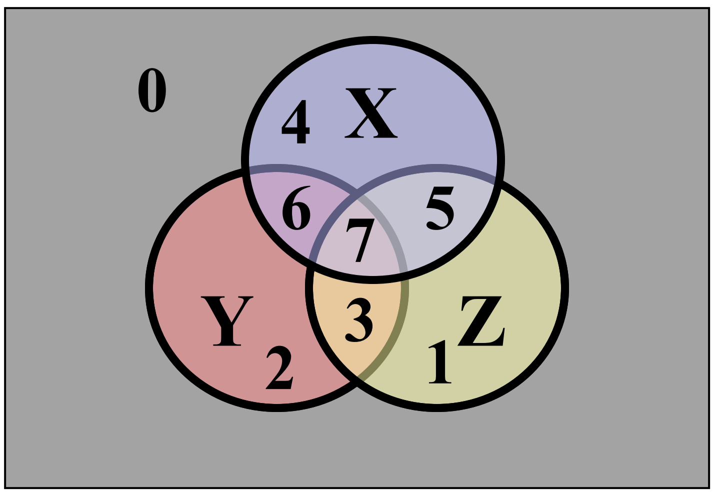
- **Applying the Minimization Theorem three times:**

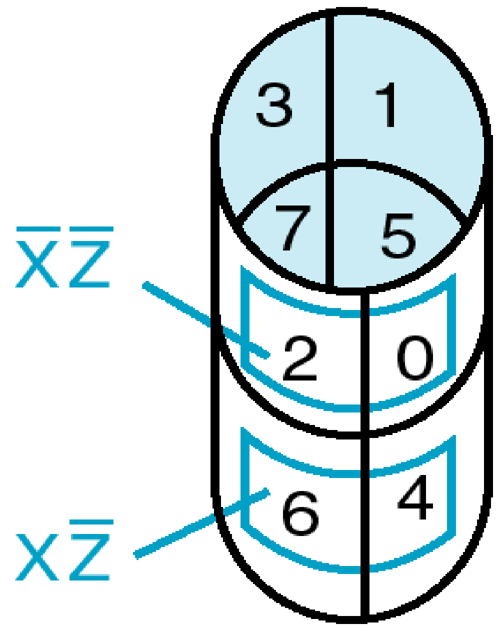
$$\begin{aligned} F(x,y,z) &= \bar{x}\,y\,z + x\,y\,z + \bar{x}\,y\,\bar{z} + x\,y\,\bar{z} \\ &= yz + y\bar{z} \\ &= y \end{aligned}$$

- **Thus the four terms that form a 2 × 2 square correspond to the term "y".**

Three-Variable Maps

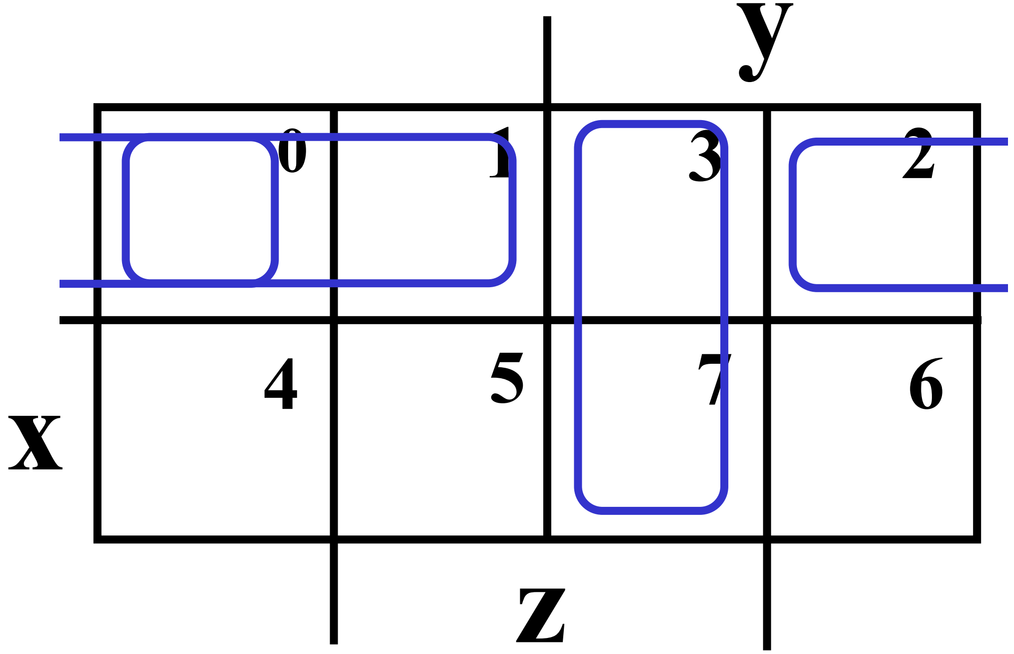
- Topological warps of 3-variable K-maps that show *all* adjacencies:
  - Venn Diagram
  - Cylinder





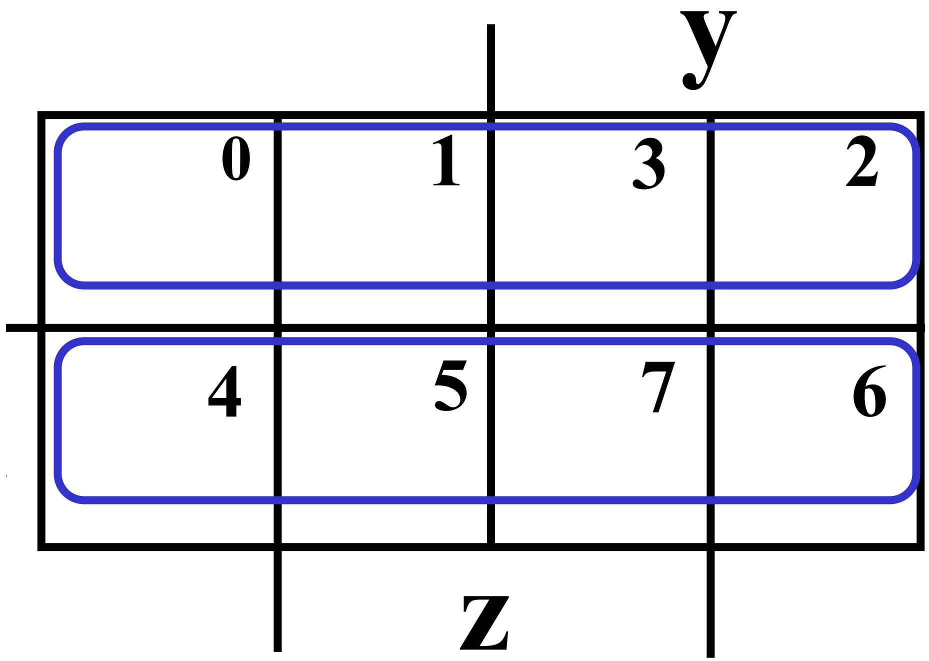
Three-Variable Maps

- **Example Shapes of 2-cell Adjacent:**



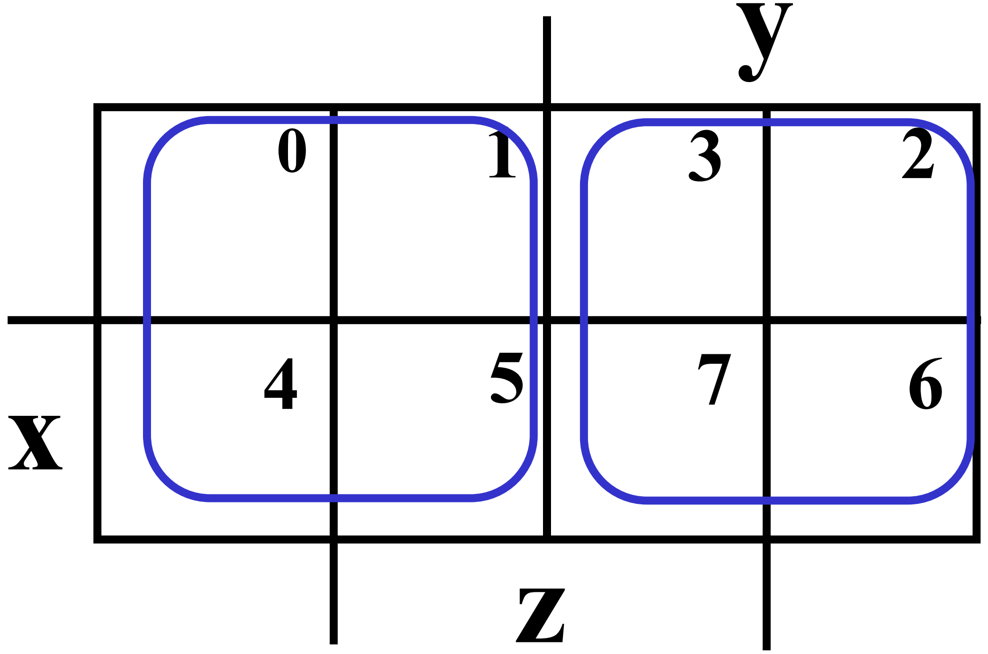
Three-Variable Maps

- **Example Shapes of 4-cell Adjacent:**



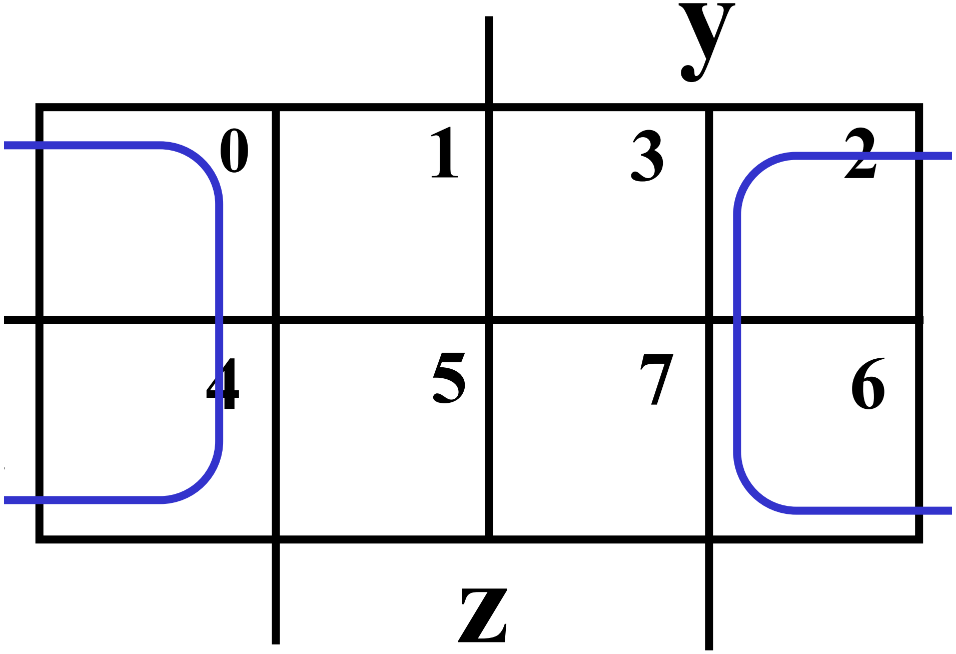
Three-Variable Maps

- **Example Shapes of 4-cell Adjacent:**



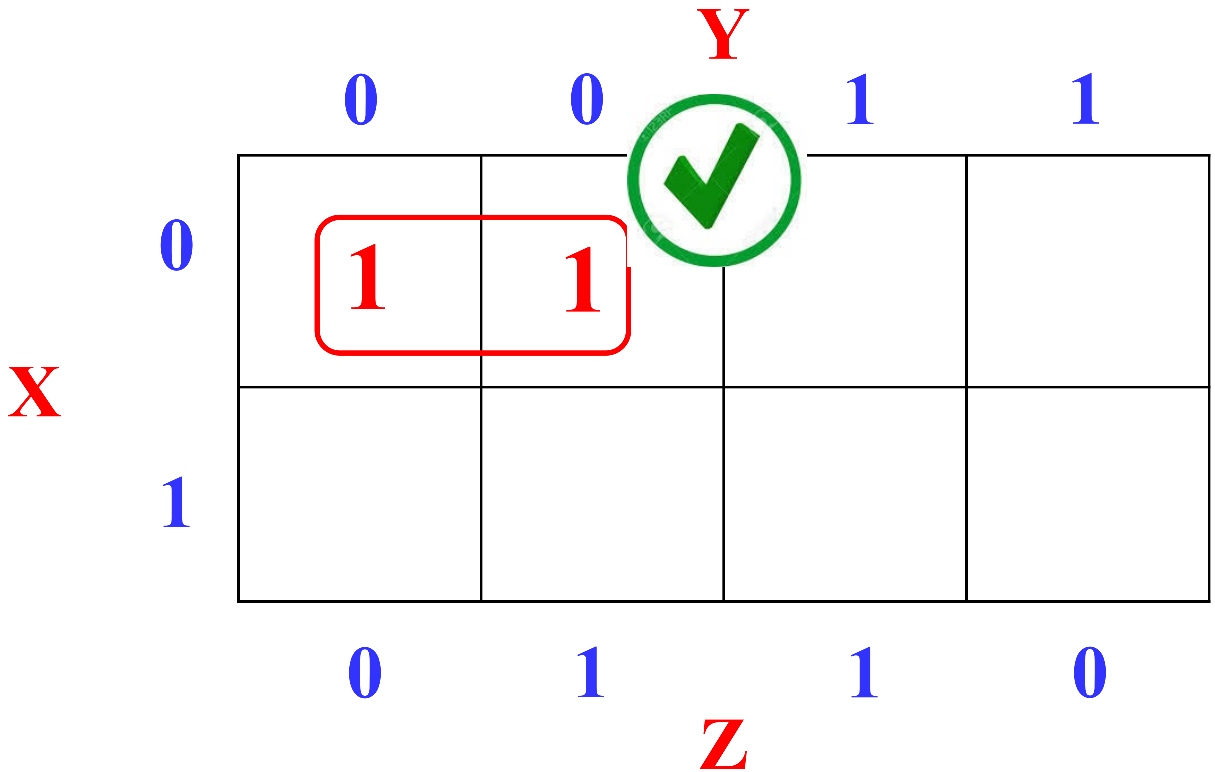
Three-Variable Maps

- **Example Shapes of 4-cell Adjacent:**



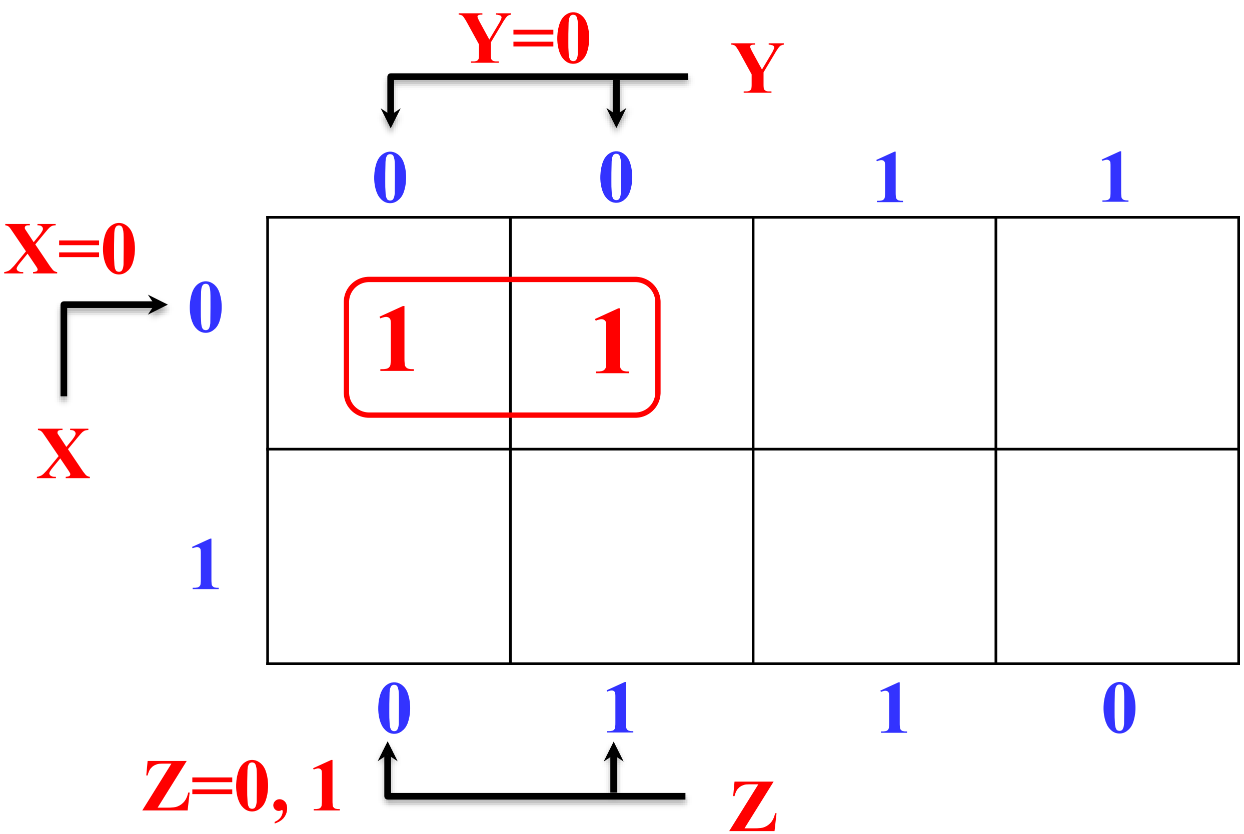
Simplification using K-Map

- Step 1: Determine Adjacent Cells



Simplification using K-Map

- Step 2: Verify each variable (one by one),
  - 2.1: Adjacent covers either 0 or 1, keep this variable
  - 2.2: Adjacent covers both 0 and 1, remove this variable



X=0

Y=0

Z=0,1

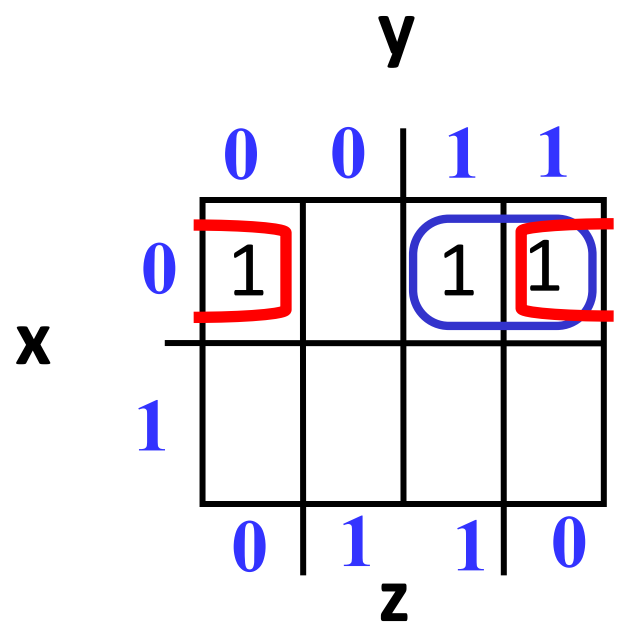
Thus, remove Z

Solution

$F=\overline{X}\,\overline{Y}$

Example 1

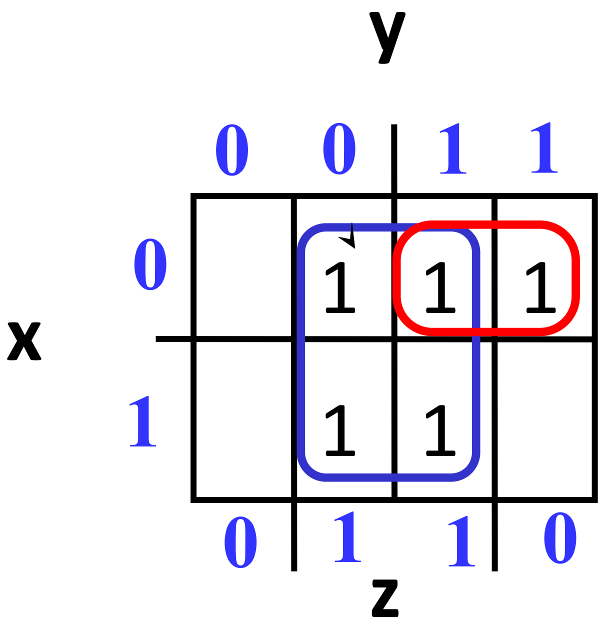
- **Example: Simplify** $F(X,Y,Z) = \bar{X}\bar{Y}\bar{Z}+\bar{X}YZ+\bar{X}Y$



$$F(x, y, z) = \bar{X}Y + \bar{X}\bar{Z}$$

Example 2

- **Example: Simplify** $F(x, y, z) = \Sigma m(1,2,3,5,7)$



$$F(x, y, z) = Z + \overline{X}Y$$

Four Variable Terms

- Four variable maps can have rectangles corresponding to:
  - A single 1 = 4 variables, (i.e. Minterm)
  - Two 1s = 3 variables,
  - Four 1s = 2 variables
  - Eight 1s = 1 variable,
  - Sixteen 1s = zero variables (i.e. Constant "1")

K-Map Template (Four Variables)

Recommend

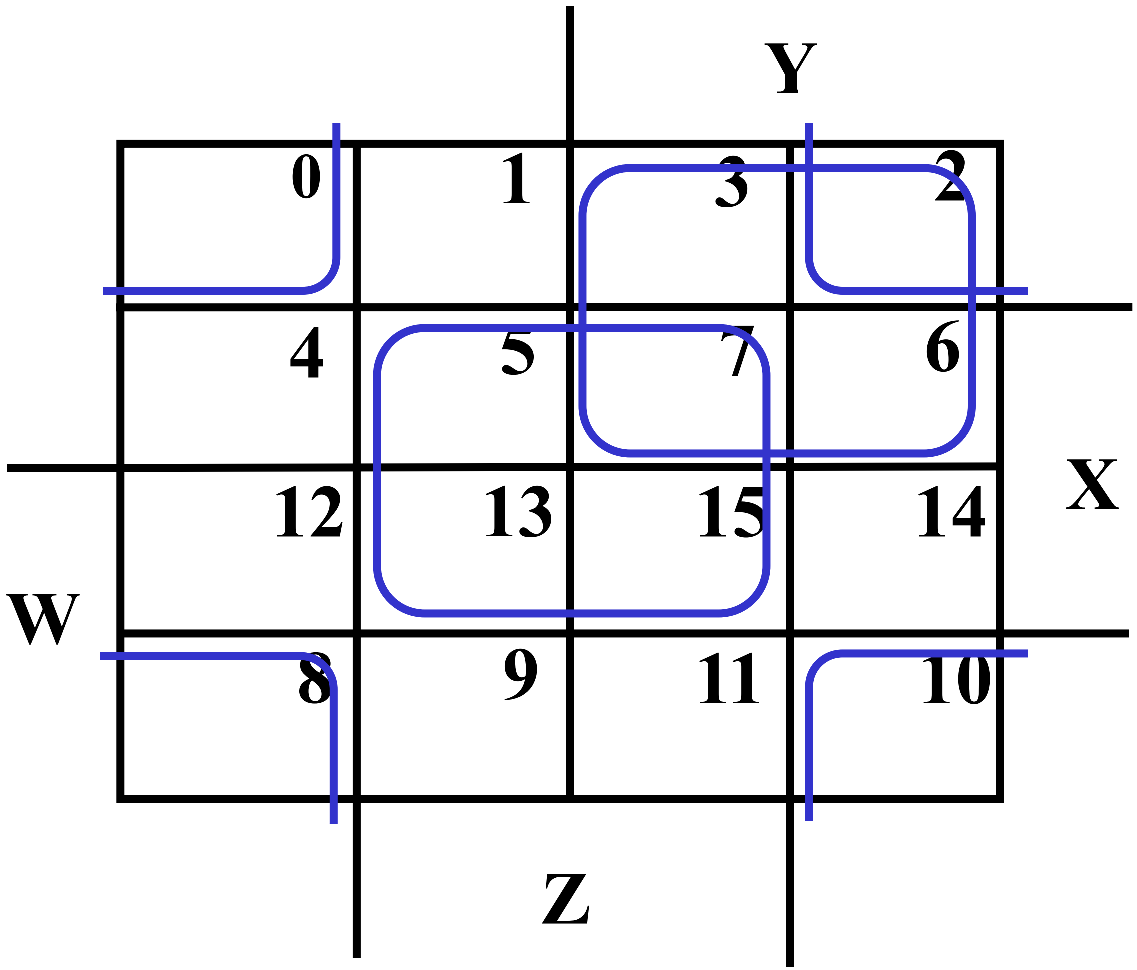
Y

		0	0	1	1		
	0	0	1	3	2	0	
	0	4	5	7	6	1	
W	1	12	13	15	14	1	X
	1	8	9	11	10	0	
		0	1	1	0		

Z

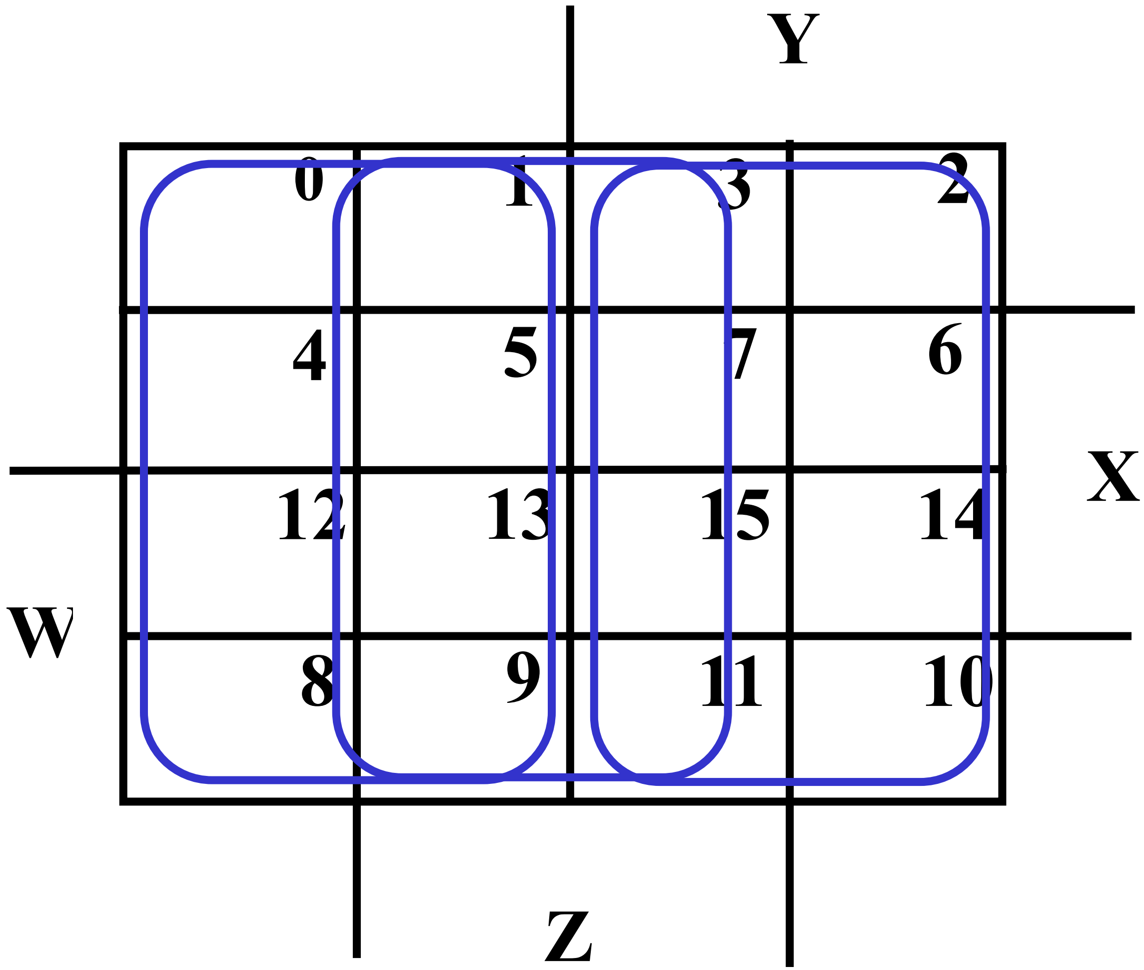
Four-Variable Maps

- **Example Shapes of Rectangles (4 Adjacent):**



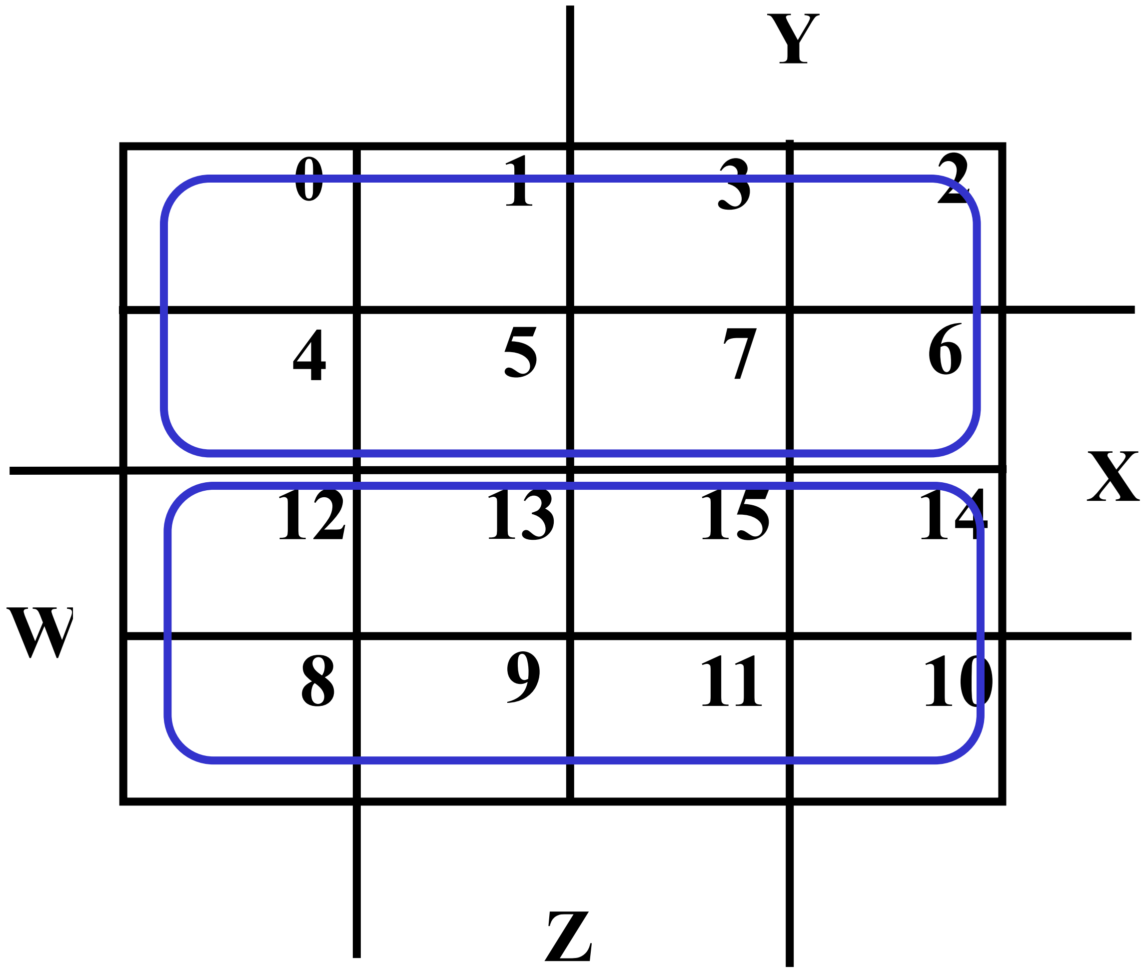
Four-Variable Maps

- **Example Shapes of Rectangles (8 Adjacent):**



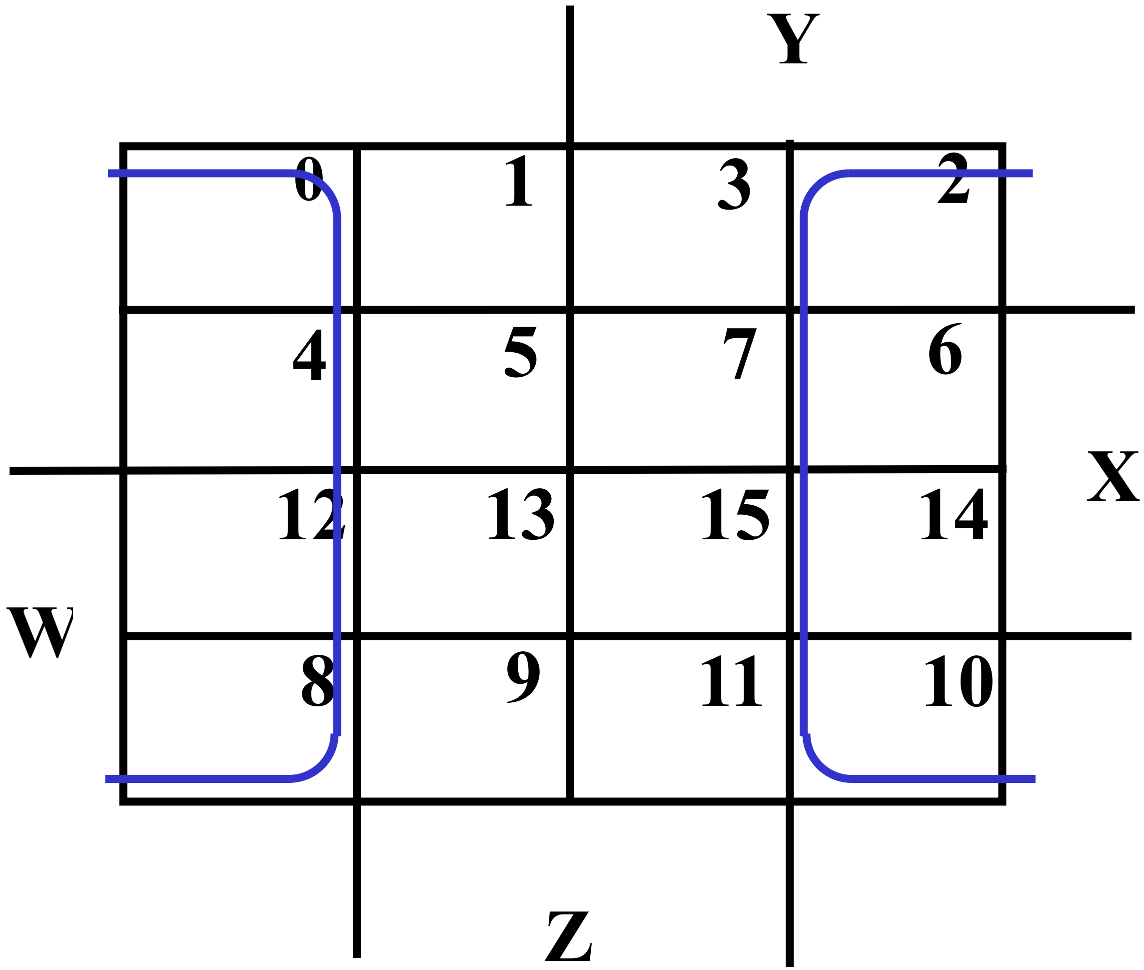
Four-Variable Maps

- **Example Shapes of Rectangles (8 Adjacent):**



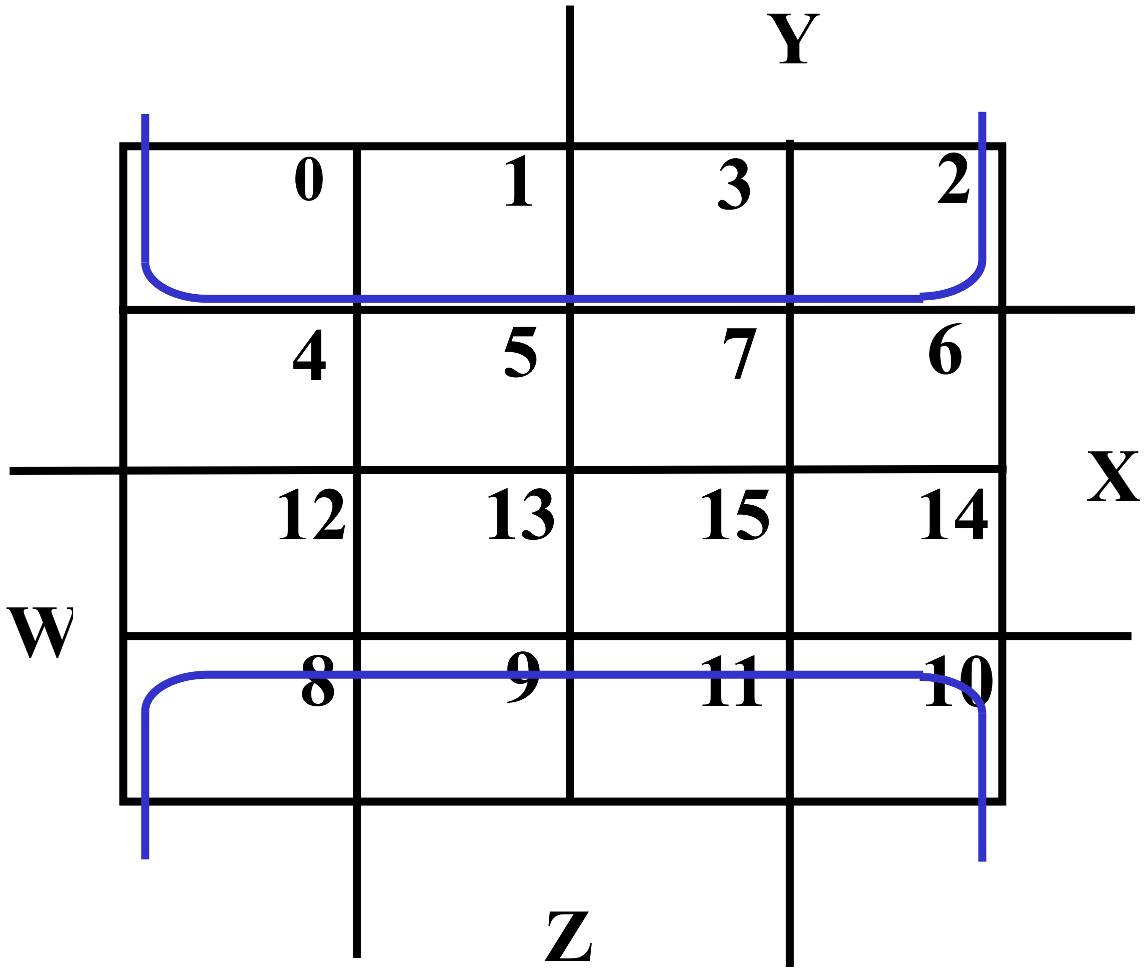
Four-Variable Maps

- **Example Shapes of Rectangles (8 Adjacent):**



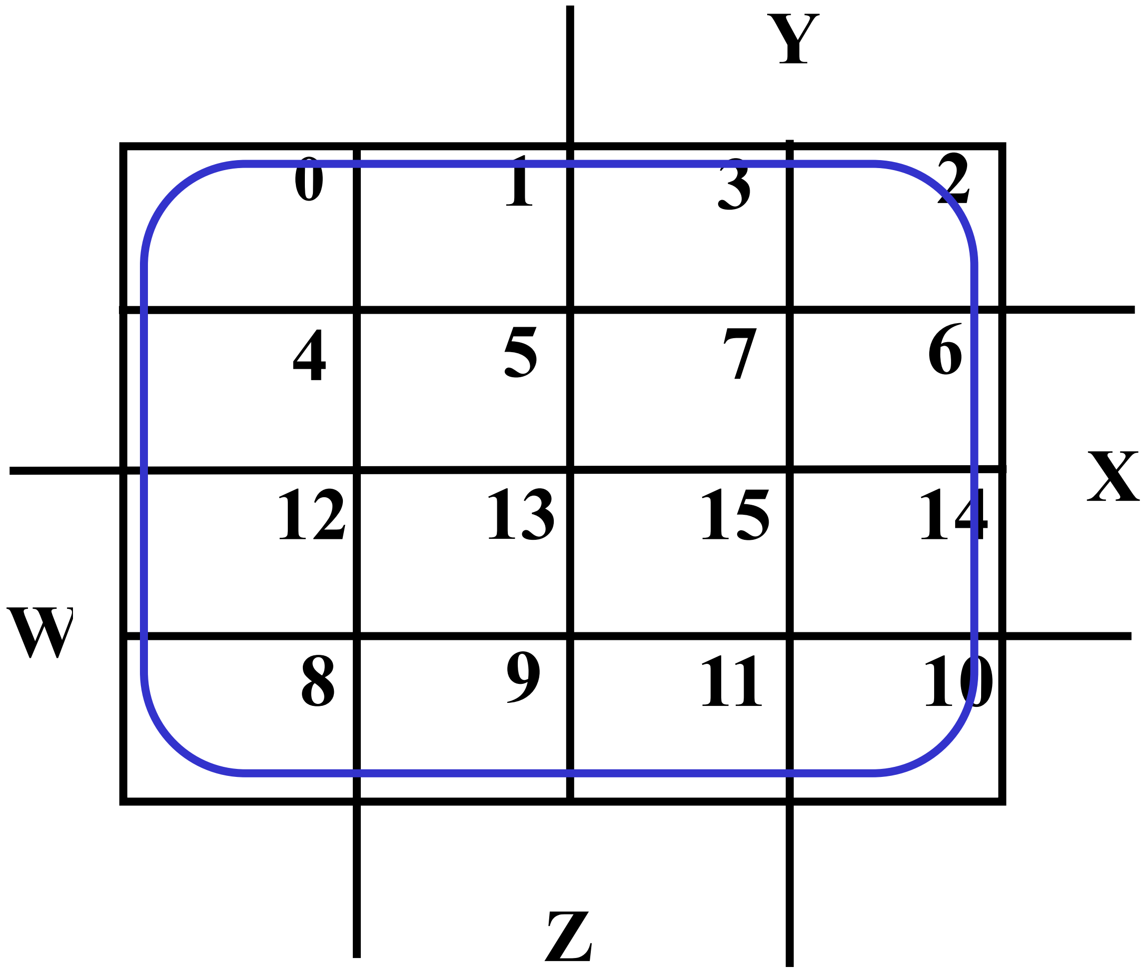
Four-Variable Maps

- **Example Shapes of Rectangles (8 Adjacent):**



Four-Variable Maps

- **Example Shapes of Rectangles (16 Adjacent):**



Five Variable or More K-Maps

- **For five variable problems, we use *two adjacent K-maps*. It becomes harder to visualize adjacent minterms for selecting PIs. You can extend the problem to six variables by using four K-Maps.**