CHAPTER 3: VECTORS IN 2–SPACE AND 3–SPACE

1. Let $\mathbf{u} = (1,7)$, $\mathbf{v} = (3,-2)$. Compute the following.

   (*a*) $\mathbf{u}+\mathbf{v}$ (*b*) $2\mathbf{u}-3\mathbf{v}$ (*c*) $\|\mathbf{u}-\mathbf{v}\|$

   (*d*) $\|\mathbf{u}\|-\|\mathbf{v}\|$ (*e*) $2\mathbf{u}\cdot\mathbf{v}$ (*f*) $\mathbf{u}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{u}$

2. Let $\mathbf{u} = (1,0,-2)$, $\mathbf{v} = (3,2,4)$, $\mathbf{w} = (2,2,-1)$. Compute the following.

   (*a*) $\mathbf{u}-\mathbf{v}$ (*b*) $2\mathbf{u}+\mathbf{v}$ (*c*) $3\mathbf{u}+\mathbf{v}-\mathbf{w}$

   (*d*) $\mathbf{u}\cdot\mathbf{v}$ (*e*) $\|\mathbf{u}+\mathbf{v}-2\mathbf{w}\|$ (*f*) $\|\mathbf{u}+\mathbf{v}\|-\|2\mathbf{w}\|$

3. Let $\mathbf{u} = (1,1,3)$, $\mathbf{v} = (2,0,1)$, $\mathbf{w} = (0,-1,5)$. Compute the following.

   (a) $2\mathbf{u}+\mathbf{w}$ (b) $2(\mathbf{u}+\mathbf{v}) - 3(\mathbf{v} - 2\mathbf{w})$ (c) $(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$

   (d) $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$ (*e*) $\|\mathbf{v}\|(\mathbf{u}\cdot\mathbf{w})$ (*f*) $\|\mathbf{u}\times(\mathbf{v}+\mathbf{w})\|$

4. Let $\mathbf{u} = (1,-1,1)$, $\mathbf{v} = (2,1,2)$, $\mathbf{w} = (1,0,-3)$. Compute the following.

   (a) $\mathbf{u}-2\mathbf{v}+\mathbf{w}$ (*b*) $\mathbf{w}\cdot\mathbf{u}-\|5\mathbf{v}\|$ (*c*) $(2\mathbf{u}-\mathbf{v})\times\mathbf{w}$

   (d) $(\mathbf{u}\times\mathbf{v})\times\mathbf{w}$ (*e*) $(\mathbf{u}\times\mathbf{w})\times(\mathbf{u}\times\mathbf{v})$ (*f*) $(\mathbf{v}\times\mathbf{w})\cdot\mathbf{u}$

5. Let $\mathbf{u} = (4,2,-1)$, $\mathbf{v} = (3,1,1)$, $\mathbf{w} = (0,2,1)$. Compute the following.

   (a) $2\mathbf{v}-3\mathbf{w}-\mathbf{u}$ (*b*) $(\mathbf{u}\cdot\mathbf{w})\|\mathbf{u}\times\mathbf{w}\|$ (*c*) $\mathbf{w}\cdot(5\mathbf{v}-\mathbf{u})$

   (d) $\mathbf{v}\cdot(\mathbf{w}\times\mathbf{u})$ (*e*) $\mathbf{v}\times(\mathbf{u}\times\mathbf{w})$ (*f*) $\mathbf{v}\times(\mathbf{u}\times 2\mathbf{u})$

6. Find the terminal point Q of a vetor $\mathbf{u}$ with initial point $P(4,-3,6)$ such that

   (a) $\mathbf{u}$ has the same direction as $\mathbf{v} = (1,1,7)$

   (b) $\mathbf{u}$ is oppositely directed to $\mathbf{v} = (1,1,7)$.

7. Find the terminal point Q of a vector $\mathbf{u}$ with initial point $P(-2,6,5)$ such that

   (a) $\mathbf{u}$ has the same direction as $\mathbf{v} = (4,0,3)$

   (b) $\mathbf{u}$ is oppositely directed to $\mathbf{v} = (4,0,3)$.

8. Find the initial point P of a vector $\mathbf{u}$ with terminal point $Q(3,5,5)$ such that

   (a) $\mathbf{u}$ has the same direction as $\mathbf{v} = (6,-2,1)$

   (b) $\mathbf{u}$ is oppositely directed to $\mathbf{v} = (6,-2,1)$.

9. Find the initial point P of a vector $\mathbf{u}$ with terminal point $Q(2,0,2)$ such that

   (a) $\mathbf{u}$ has the same direction as $\mathbf{v} = (5,4,-1)$

   (b) $\mathbf{u}$ is oppositely directed to $\mathbf{v} = (5,4,-1)$.

10. Find the distance between P and Q

(a) $P(1,7)$, $Q(2,3)$ (b) $P(-1,4)$, $Q(3,-5)$

(c) $P(1,0,6)$, $Q(4,3,-2)$ (d) $P(7,-4,5)$, $Q(8,-2,-3)$

11. Find the distance between R and S.

(a) $R(6,1)$, $S(1,-11)$ (b) $R(4,0)$, $S(0,-8)$

(c) $R(4,3,-8)$, $S(7,7,4)$ (d) $R(3,4,1)$, $S(4,3,4)$

12. Find the vector component of $\mathbf{u}$ parallel to $\mathbf{a}$.

(a) $\mathbf{u} = (6,5)$, $\mathbf{a} = (-3,4)$ (b) $\mathbf{u} = (1,2)$, $\mathbf{a} = (7,-3)$

(c) $\mathbf{u} = (3,0,2)$, $\mathbf{a} = (1,1,3)$ (d) $\mathbf{u} = (1,0,7)$, $\mathbf{a} = (0,5,0)$

13. Find the vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$.

(a) $\mathbf{u} = (4,1)$, $\mathbf{a} = (2,-3)$ (b) $\mathbf{u} = (5,-2)$, $\mathbf{a} = (1,1)$

(c) $\mathbf{u} = (1,5,-5)$, $\mathbf{a} = (1,2,3)$ (d) $\mathbf{u} = (4,-1-1)$, $\mathbf{a} = (1,-2,4)$

14. Consider the points $A(2,6)$, $B(3,7)$, $C(3,8)$. Compute the following.

(a) $\cos \angle ABC$ (b) $\cos \angle BAC$ (c) $\cos \angle ACB$

15. Consider the points $L(-1,3,5)$, $M(1,4,3)$, $N(7,2,6)$.

(a) $\cos \angle MLN$ (b) $\cos \angle MNL$ (c) $\cos \angle LMN$

16. Calculate the distance between the given point and line using the formula

$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2+b^2}}.$$

(a) $x - 2y + 3 = 0$; $(2,1)$

(b) $5x + 12y - 2 = 0$; $(2,-1)$

(c) $15x = 8y + 4$; $(1,1)$

(d) $y = \frac{3}{4}x + 3$; $(0,2)$

17. Consider the points $P(1,1,1)$, $Q(2,-1,3)$, $R(3,-1,4)$. Compute the following.

(a) $\sin \angle QPR$ (b) $\sin \angle PQR$ (c) $\sin \angle PRQ$

18. Find the area of the triangle PQR.

(a) $P(1,3,2), Q(2,3,1), R(2,2,3)$ (b) $P(3,-3,1), Q(1,-3,2), R(5,-2,-1)$

(c) $P(3,0,-1), Q(2,2,2), R(4,2,3)$ (d) $P(1,1,-1), Q(1,2,0), R(2,0,1)$

19. Find the area of the triangle determined by $\mathbf{u}$ and $\mathbf{v}$.

(a) $\mathbf{u} = (1,2,4), \mathbf{v} = (3,1,2)$ (b) $\mathbf{u} = (1,-1,2), \mathbf{v} = (2,0,3)$

(c) $\mathbf{u} = (6,1,0), \mathbf{v} = (2,1,-2)$ (d) $\mathbf{u} = (-3,0,1), \mathbf{v} = (0,1,2)$

20. Find the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$.

(a) $\mathbf{u} = (1, 0, -2), \mathbf{v} = (0, 3, 2)$ (b) $\mathbf{u} = (2, 0, 2), \mathbf{v} = (0, 2, 1)$

(c) $\mathbf{u} = (1, 4, 1)$, $\mathbf{v} = (3, 2, 2)$ (d) $\mathbf{u} = (0, 1, 1), \mathbf{v} = (1, 0, 2)$

21. Consider the parallelopiped determined by $\mathbf{u} = (1, -1, 0)$, $\mathbf{v} = (1, 0, -1)$, $\mathbf{w} = (3, -1, 2)$. Compute the following.

(a) the volume of the parallelopiped
(b) the area of the face determined by $\mathbf{u}$ and $\mathbf{v}$
(c) the cosine of the angle between $\mathbf{v}$ and the plane containing the face determined by $\mathbf{u}$ and $\mathbf{w}$.

22. Consider the parallelopiped determined by $\mathbf{u} = (2, 0, 5)$, $\mathbf{v} = (1, -1, 0)$, $\mathbf{w} = (1, 3, 8)$. Compute the following.

(a) the volume of the parallelopiped
(b) the area of the face determined by $\mathbf{v}$ and $\mathbf{w}$
(c) the sine of the angle between $\mathbf{w}$ and the plane containing the face determined by $\mathbf{u}$ and $\mathbf{v}$.

23. Find a parametric equation for the line passing through P and parallel to $\mathbf{n}$.

(a) $P(5, 1, 2)$; $\mathbf{n} = (1, 2, 6)$ (b) $P(-1, 2, 3)$; $\mathbf{n} = (1, 1, 1)$

(c) $P(0, 0, 0)$; $\mathbf{n} = (2, 2, 5)$ (d) $P(6, 2, 6)$; $\mathbf{n} = (1, 0, 0)$

24. Find a symmetric equation for the line passing through P and parallel to $\mathbf{n}$.

(a) $P(1, 3, 5)$; $\mathbf{n} = (2, 2, 7)$ (b) $P(2, -6, 3)$; $\mathbf{n} = (1, -1, 3)$

(c) $P(4, 5, 0)$; $\mathbf{n} = (\frac{1}{2}, \frac{1}{4}, -1)$ (d) $P(0, 0, 8)$; $\mathbf{n} = (1, 2, \sqrt{2})$

25. Determine whether the lines are parallel.

(a) $\begin{aligned} x &= 2 + t \\ y &= 1 + 2t \\ z &= 1 + t \end{aligned}$ and $\begin{aligned} x &= 4 + t \\ y &= 2 + 2t \\ z &= 4 + t \end{aligned}$

(b) $\begin{aligned} x &= 5 - t \\ y &= 6 + t \\ z &= -1 - 2t \end{aligned}$ and $\begin{aligned} x &= 3 + t \\ y &= -t \\ z &= 5 + 2t \end{aligned}$

(c) $\begin{aligned} x &= 4t \\ y &= 7 \\ z &= 1 - t \end{aligned}$ and $\begin{aligned} x &= 6 + 4t \\ y &= 6 + 7t \\ z &= -t \end{aligned}$

(d) $\begin{aligned} x &= -8 + 1/2t \\ y &= 5 + 1/3t \\ z &= 3 - t \end{aligned}$ and $\begin{aligned} x &= -3t \\ y &= 11 - 2t \\ z &= -6t \end{aligned}$

26. Determine whether the planes are parallel.

(a) $4x+y-7z = 1$ and $4x+y-7z = 0$ (b) $-x+2y-z = 1$ and $3x+6y+3z = 5$

(c) $6x = 21y - 3z$ and $2x - 7y + z = 11$ (d) $2z = 3 - x$ and $2z - x = 4$

27. Determine whether the planes are perpendicular.

(a) $2x - 3y + z = 0$ and $4x - 6y + 2z = 3$ (b) $x - 3y + 5z = 2$ and $4x + 3y + z = 0$

(c) $2x = 3z$ and $y = 7$ (d) $2y = 5x + z - 9$ and $x = -4y - 2z$

28. Determine whether the line and plane are parallel.

(a) $x = 2 + 4t, y = 1 - 3t, z = t; x + y - z = 5$ (b) $x = 3, y = -4t, z = 1 + 5t; x + y + z = 9$

(c) $x = 1 + \frac{1}{2}t, y = 1 - \frac{1}{3}t, z = 2 + \frac{1}{6}t; 6x + 6y = 6z + 1$ (d) $x = t, y = 2, z = 3 + 9t; 11y = 15$

29. Determine whether the line and plane are perpendicular.

(a) $x = 3 + 2t, y = 14t, z = 1 + 12t; x + 7y = 2 + 6z$

(b) $x = 2t, y = 1 + \sqrt{10}t, z = 3 - \sqrt{6}t; 3z = \sqrt{6}x + \sqrt{15}y + 9$

(c) $x = 5 - 2t, y = 0, z = 3t; \frac{1}{2}x + \frac{1}{3}z = 19$

(d) $x = 4t + 5, y = 2t + 1, z = 2; 4x + 2y + 2z = 7$

30. Give the point of intersection, if any, of the two lines.

(a) $\begin{array}{rcrcl} x & = & -9 & + & 5t \\ y & = & 1 & + & t \\ z & = & 10 & - & 4t \end{array}$ and $\begin{array}{rcrcl} x & = & -2 & - & 3t \\ y & = & 5 & + & 2t \\ z & = & 5 & + & 3t \end{array}$

(b) $\begin{array}{rcrcl} x & = & \frac{13}{2} & + & t \\ y & = & \frac{3}{2} & + & 3t \\ z & = & -1 & & \end{array}$ and $\begin{array}{rcrcl} x & = & 6 & & \\ y & = & 4 & - & 2t \\ z & = & -11 & + & 5t \end{array}$

(c) $\begin{array}{rcrcl} x & = & 10 & + & t \\ y & = & 2 & + & 3t \\ z & = & 6 & + & t \end{array}$ and $\begin{array}{rcrcl} x & = & 7 & + & t \\ y & = & 7 & - & 4t \\ z & = & -3 & + & 4t \end{array}$

(d) $\begin{array}{rcrcl} x & = & 3 & - & t \\ y & = & 2t & & \\ z & = & 1 & + & t \end{array}$ and $\begin{array}{rcrcl} x & = & 4 & + & 3t \\ y & = & 1 & - & 2t \\ z & = & 5 & - & 5t \end{array}$

31. Give the point of intersection, if any, of the line and plane.

(a) $x = 4 + 2t, y = 7 - t, z = 3t; 2x - y - z = 7$

(b) $x = 2 + 2t, y = 5, z = \frac{1}{2} - t; 3x + 5z = 10$

(c) $x = t - 3, y = 2t, z = t + 5; x - 4y - z = 0$

(d) $x = 1 - t, y = 3 + 2t, z = -2 + 3t; x + z = y + 4$

32. Find a parametric equation for the line of intersection of the planes.

(a) $x + 2y - z = 1$, $2x - y + 3z = 7$ (b) $3x + y + z = 5$, $y - 7z = 1$

(c) $5x+y-7z=2$, $4x+2y+z=4$

(d) $2x+3y+z=11$, $4x+9y-z=31$

33. Find a parametric equation for the line passing through the given points.

(a) $P(1,6,3)$, $Q(2,7,1)$

(b) $P(4,1,\frac{2}{3})$, $Q(8,0,\frac{5}{3})$

(c) $P(0,-1,1)$, $Q(2,-1,2)$

(d) $P(2,2,3)$, $Q(-1,-1,0)$

34. Find a symmetric equation for the line passing through the given points.

(a) $P(3,-5,1)$, $Q(2,2,2)$

(b) $P(4,1,6)$, $Q(3,2,3)$

(c) $P(4,3,\frac{-5}{2})$, $Q(\frac{9}{2},0,\frac{-3}{2})$

(d) $P(0,0,0)$, $Q(1,7,7)$

35. Find a point–normal form of the equation for the plane passing through P and having $\mathbf{n}$ as normal.

(a) $P(1,-1,5)$, $\mathbf{n}=(1,2,3)$

(b) $P(0,2,0)$, $\mathbf{n}=(\frac{4}{13},\frac{-3}{13},\frac{12}{13})$

(c) $P(2,6,-3)$, $\mathbf{n}=(1,1,1)$

(d) $P(4,1,-7)$, $\mathbf{n}=(\frac{1}{4},\frac{-1}{2},\frac{5}{4})$

36. Find a general form of the equation for the plane passing through P and having $\mathbf{n}$ as normal.

(a) $P(2,3,4)$, $\mathbf{n}=(6,1,6)$

(b) $P(-7,2,3)$, $\mathbf{n}=(0,0,1)$

(c) $P(8,-8,2)$, $\mathbf{n}=(\frac{-2}{3},\frac{1}{3},\frac{2}{3})$

(d) $P(0,0,0)$, $\mathbf{n}=(2,5,5)$

37. Find a point–normal form of the equation for the plane passing through the given points.

(a) $P(3,1,1),Q(1,6,7),R(4,2,2)$

(b) $P(3,4,4),Q(0,1,-11),R(1,3,-17)$

(c) $P(0,2,-1),Q(\frac{3}{2},\frac{5}{2},0),R(\frac{-1}{2},1,\frac{3}{2})$

(d) $P(7,2,7),Q(9,1,9),R(8,-1,6)$

38. Find a general foarm of the equation for the plane passing through the given points.

(a) $P(8,-5,2),Q(6,-6,1),R(10,0,0)$

(b) $P(1,5,4),Q(2,6,7),R(-1,5,0)$

(c) $P(4,4,3),Q(10,4,-1),R(-1,8,12)$

(d) $P(\frac{1}{3},0,\frac{-2}{3}),Q(\frac{5}{3},1,\frac{5}{3}),R(\frac{4}{3},\frac{2}{3},1)$

39. Find an equation for the plane through $P(1,1,3)$ that is perpendicular to the line $x=2-3t$, $y=1+t$, $z=2t$.

40. Find an equation for the plane through $P(2,7,-1)$ that is parallel to the plane $4x-y+3z=3$.

41. Find an equation for the plane that contains the line $x=3+t$, $y=5$, $z=5+2t$, and is perpendicular to the plane $x+y+z=4$.

42. Find an equation for the plane through $P(1,4,4)$ that contains the line of intersection of the planes $x-y+3z=5$ and $2x+2y+7z=0$.

43. Find an equation for the line through $P(2,-3,0)$ that is parallel to the planes $2x+2y+z=2$ and $x-3y=5$.

44. Find an equation for the plane through $P(-1,7,4)$ that is perpendicular to the planes $3x+y-z=5$ and $11x+2y+3z=0$.

45. Find an equation for the plane through $P(4,5,3)$ and $Q(6,3,-2)$ that is perpendicular to the plane $2x - y - z = 8$.

46. Find an equation for the plane through $P(1,1,7)$ that contains the line $x = 2t+4$, $y = 4t-1$, $z = t+1$.

47. Find an equation for the plane containing the line $x = 3 + 6t$, $y = 4$, $z = t$ and that is parallel to the line of intersection of the planes $2x + y + z = 1$ and $x - 2y + 3z = 2$.

48. Find an equation for the plane parallel to the line $x = 1 - 2t$, $y = 2 + 3t$, $z = 1 + 2t$ and that contains the line of intersection of the planes $x + 3y - 7z = 2$ and $3x + 11y - 17z = 2$.

49. Find an equation for the plane each of whose points is equidistant from P and Q.

(a) $P(1,4,\ 3)$, $Q(-1,6,5)$.

(b) $P(5,0,-2)$, $Q(-5,1,1)$.

50. Calculate the distance between the given point and plane using the formula

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

(a) $2x - 3y + 6z + 4 = 0$; $(1,1,-1)$

(b) $z = 2x + 2y + 8$; $(11,-2,3)$

(c) $4x - z = 5$; $(10,7,6)$

(d) $3x - 12y + 4z + 1 = 0$; $(4,1,1)$

WRITING QUESTIONS

51. How do vectors differ from simple numerical quantities such as length and mass?

52. With the help of a labeled diagram, translate the Pythagorean Theorem into a statement about vectors.

53. If $\mathbf{u}$ and $\mathbf{v}$ are vectors in 3–space, describe the geometric meaning of the following:

$$\text{if} \quad \mathbf{x}\cdot\mathbf{u} = 0 \quad \text{and} \quad \mathbf{x}\cdot\mathbf{v} = 0$$

$$\text{then} \quad \mathbf{x}\cdot(c_1\mathbf{u} + c_2\mathbf{v}) = 0 \quad \text{for all real numbers} \quad c_1 \quad \text{and} \quad c_2.$$

54. Describe the geometric significance of the cross–product.

55. Discuss the relative merits of the parametric and symmetric forms of the line equation.

56. Describe in words the procedure you would use to determine whether two lines (given in parametric form) are parallel, intersecting or skew.

57. Suppose you were carried by time machine to Alexandria, Egypt in 300 B.C. and had the opportunity to meet Euclid, the geometer. Assuming he could speak English, what sorts of things would you say or do to try to convince him that vectors are a good tool for studying geometry?