Logic and Computer Design Fundamentals

Chapter 2 – Combinational Logic Circuits

Part 3 – Boolean Algebra and Algebraic Manipulation

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Overview

- **Part 3–Boolean Algebra**
  - **Boolean Algebra**
  - **Algebraic Manipulation**

I. Boolean Algebra

Boolean Algebra

Boolean algebra is the branch of algebra in which the values of the variables are the truth values (true and false), usually denoted 1 and 0, respectively.

Boolean Algebra

- Boolean algebra is the theoretical foundation for digital system.
- Boolean algebra is a mathematical system with a set of elements, a set of operators and a number of unproved hypothesis.
- Boolean algebra can have only two values, 0 and 1. The Boolean 0 and 1 do not represent actual numbers, instead a state of voltage.
- Boolean algebra is used as a tool for the analysis and design of logic circuit.

Boolean Function Evaluation

using Truth Table

$F1 = xy\bar{z}$

$F2 = x + \bar{y}z$

$F3 = \bar{x}\bar{y}z + \bar{x}\,y\,\bar{z} + x\bar{z}$

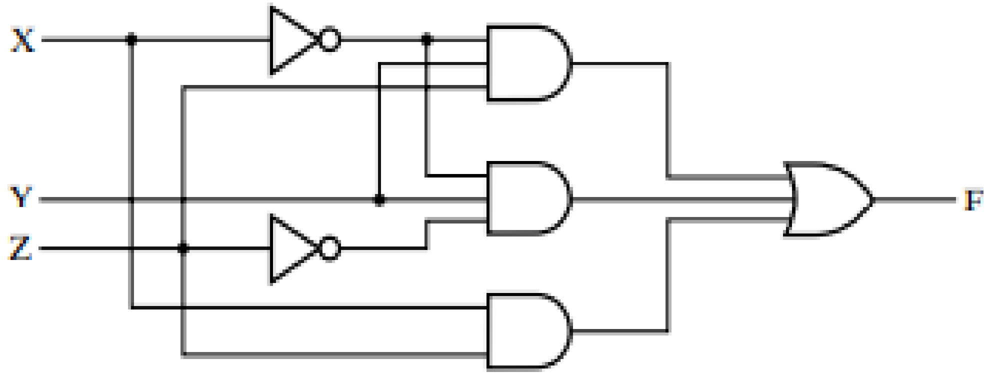
$F4 = \bar{x}y + x\,z$

x	y	z	F1	F2	F3	F4
0	0	0	0	0	0	0
0	0	1	0	1	0	0
0	1	0	0	0	1	1
0	1	1	0	0	1	1
1	0	0	0	1	0	0
1	0	1	0	1	1	1
1	1	0	1	1	0	0
1	1	1	0	1	1	1

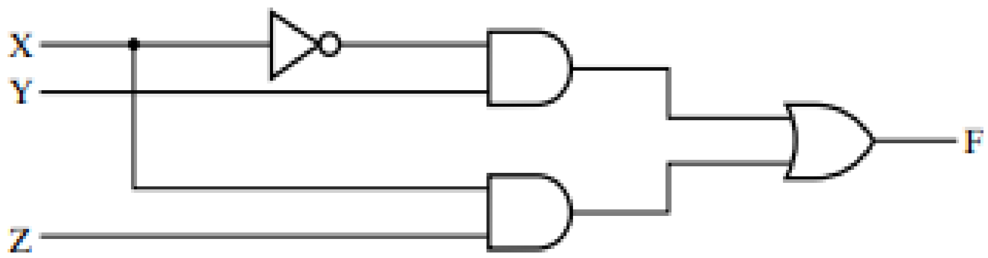
Finally, we found that F3 is equal to F4 (F3=F4)

Boolean Function Evaluation

Equivalent Circuit Diagram



(a) $F = \overline{X}YZ + \overline{X}Y\overline{Z} + XZ$



(b) $F = \overline{X}Y + XZ$

II. Algebraic Manipulation

How to simplify the Boolean algebra?

Algebraic Manipulation

- Algebraic Manipulation is a useful tool for Simplifying the digital circuit.
- Why do we need to do it?

  Answer: Simpler mean cheaper, smaller, and faster.

- For example

  $F3=\overline{X}YZ + \overline{X}Y\overline{Z} + XZ$

  is equal to …

  $F4=\overline{X}Y + XZ$

How to simplify? …go to next page

Algebraic Manipulation: Basic Identities

An algebraic structure defined on a set of at least two elements, together with three traditional binary operators: Or, And, Not (denoted +, · , ‾) that satisfies the following basic identities:

Recommend

1.	$X + 0 = X$	2.	$X \cdot 1 = X$
3.	$X + 1 = 1$	4.	$X \cdot 0 = 0$
5.	$X + X = X$	6.	$X \cdot X = X$
7.	$X + \overline{X} = 1$	8.	$X \cdot \overline{X} = 0$
9.	$\overline{\overline{X}} = X$		

10.	$X + Y = Y + X$	11.	$XY = YX$	Commutative
12.	$(X + Y) + Z = X + (Y + Z)$	13.	$(XY)Z = X(YZ)$	Associative
14.	$X(Y + Z) = XY + XZ$	15.	$X + YZ = (X + Y)(X + Z)$	Distributive
16.	$\overline{X + Y} = \overline{X} \cdot \overline{Y}$	17.	$\overline{X \cdot Y} = \overline{X} + \overline{Y}$	DeMorgan's

Some Properties of Identities

- **The identities above are organized into pairs. These pairs have names as follows:**

1-4 Existence of 0 and 1	**5-6 Idempotence**
7-8 Existence of complement	**9 Involution**
10-11 Commutative Laws	**12-13 Associative Laws**
14-15 Distributive Laws	**16-17 DeMorgan's Laws**

Useful Trick! Distributive Pattern

Recommend

Distributive identities 14 and 15 are most frequently used:

Identity 14: $X(Y+Z) = XY + XZ$

Pattern

$X \cdot Y$

$$X\,(\,Y + Z\,) \quad = (X \cdot Y) + (X \cdot Z)$$

$X \cdot Z$

How to apply this trick?

given algebra is $F = \overline{A}BC + ABC$

Solution

Step1: Extract shared variables → BC

Step2: Determine functions, you will get → $BC\,(\overline{A}+A)$

Useful Trick! Distributive Pattern

Recommend

<u>Identity 15</u>: $X+(YZ) = (X+Y)(X+Z)$

$X+Y$

Pattern

$$X + (Y \cdot Z) \quad = (X + Y) \cdot (X + Z)$$

$X + Z$

How to apply this trick?

given algebra is $F = (\overline{A}+BC)(A+BC)$

<u>Solution</u>

Step1: Extract shared variables → BC

Step2: Determine functions, you will get → $BC + (\overline{A} \cdot A)$

Algebraic Manipulation

Useful Trick! How to apply?

- Step1: To determine Distributive Identity by checking the most shared (or common) variables.

For example:

$$F = \overline{X}Y Z + \overline{X}Y\overline{Z} + XZ$$

So, $\overline{X}Y$ is the most shared variables

$$= \overline{X}Y(Z+\overline{Z}) + XZ$$

Algebraic Manipulation
Useful Trick! How to apply?

- Step2: To consider a simplification of the expression by applying some of the identities until getting final literals:

$$F = \overline{X}YZ + \overline{X}Y\overline{Z} + XZ$$

$$= \overline{X}Y(Z+\overline{Z}) + XZ \quad \text{Distributive, identity 14}$$

$$= \overline{X}Y(1) + XZ \quad \text{identity 7}$$

$$= \overline{X}Y + XZ \quad \text{identity 2}$$

Final simplification → 4 Literals

…Simplify to contain the smallest number of **literals** (result variables)

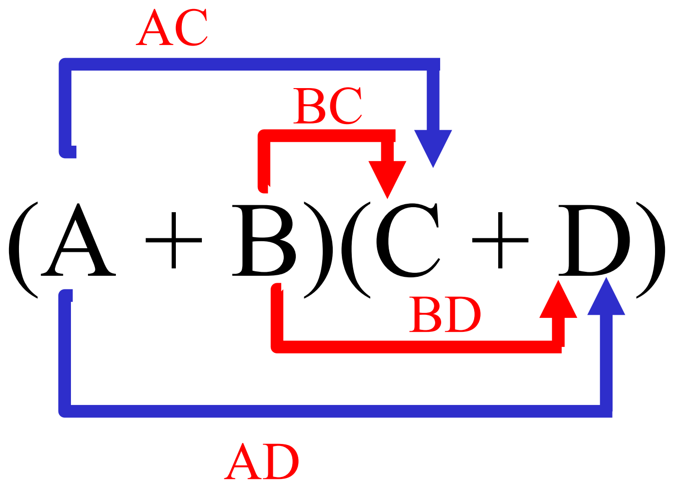
Useful Trick! Distributive Pattern

Recommend

Applied Trick for Identity 15:

$(A+B)(C+D) \;\; = ?$

Pattern



$= AC+AD+BC+BD$

Proof of Simplification

$$\mathbf{x \cdot y + \bar{x} \cdot y = y}$$

Proof

$$(X \cdot Y) + (\overline{X} \cdot Y) = Y$$

$$(X + \overline{X}) \cdot Y = Y \qquad \text{identity 14}$$

$$(1) \cdot Y = Y \qquad \text{identity 7}$$

$$Y = Y \qquad \text{identity 2}$$

Proof of Simplification

$$\left(x+y\right)\left(\overline{x}+y\right)=y$$

Proof

$$(X + Y)(\overline{X} + Y) \quad = Y$$

$$(X \cdot \overline{X}) + Y \quad = Y \qquad \text{identity 15}$$

$$(0) + Y \quad = Y \qquad \text{identity 8}$$

$$Y \quad = Y \qquad \text{identity 1}$$

Useful Theorems

Recommend

- $x \cdot y + \overline{x} \cdot y = y$ $\quad (x + y)(\overline{x} + y) = y$ **Minimization**
- $x + x \cdot y = x$ $\quad x \cdot (x + y) = x$ **Absorption**
- $x + \overline{x} \cdot y = x + y$ $\quad x \cdot (\overline{x} + y) = x \cdot y$ **Simplification**
- $x \cdot y + \overline{x} \cdot z + y \cdot z = x \cdot y + \overline{x} \cdot z$ **Consensus**

  $(x + y) \cdot (\overline{x} + z) \cdot (y + z) = (x + y) \cdot (\overline{x} + z)$
- $\overline{x + y} = \overline{x} \cdot \overline{y}$ $\quad \overline{x \cdot y} = \overline{x} + \overline{y}$ **DeMorgan's Laws**

Boolean Algebraic Proof

Our primary reason for doing proofs is to learn:

- Careful and efficient use of the identities and theorems of Boolean algebra, and
- How to choose the appropriate identity or theorem to apply to make forward progress, in order to simplify the Boolean algebra.

Useful Theorems: Absorption

Proof: $X + XY = X$ (Absorption theorem)

Solution:

$$\begin{aligned} X + XY &= X \cdot 1 + XY && X \cdot 1 = X\text{, identity 2} \\ &= X \cdot (1 + Y) && \text{Distributive, identity 15} \\ &= X \cdot 1 && Y + 1 = 1\text{, identity 3} \\ &= X && X \cdot 1 = X\text{, identity 2} \end{aligned}$$

Useful Theorems: Simplification

Proof: $X + \overline{X}Y = X + Y$ (Simplification theorem)

Solution:

$$X + \overline{X}Y = (X+\overline{X})(X+Y)$$ Distributive, identity 15

$$= 1 \cdot (X + Y)$$ $X+\overline{X}=1$, identity 7

$$= X + Y$$ $X \cdot 1 = X$, identity 2

Useful Trick! Consensus Pattern

$$(X \cdot Y) + (\overline{X} \cdot Z) + (Y \cdot Z) = (X \cdot Y) + (\overline{X} \cdot Z)$$

Consensus Pattern

Recommend

$$(X \cdot Y) + (\overline{X} \cdot Z) + \cancel{(Y \cdot Z)}$$

Conclusion

$$(X \cdot Y) + (\overline{X} \cdot Z)$$

Useful Trick! Consensus Pattern

$$(X+Y)\cdot(\overline{X}+Z)\cdot(Y+Z) = (X+Y)\cdot(\overline{X}+Z)$$

Recommend

Consensus Pattern

$$(X+Y)\cdot(\overline{X}+Z)\cdot\cancel{(Y+Z)}$$

Conclusion

$$(X+Y)\cdot(\overline{X}+Z)$$

Useful Theorems: DeMorgan

Truth Table to verify DeMorgan's Theorem

$$\overline{(X + Y)} = \overline{X} \cdot \overline{Y}$$

(a) X	Y	$X + Y$	$\overline{X + Y}$
0	0	0	1
0	1	1	0
1	0	1	0
1	1	1	0

(b) X	Y	$\overline{X}$	$\overline{Y}$	$\overline{X} \cdot \overline{Y}$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	0	0	0

Boolean Function Evaluation using Truth table

Useful Trick! DeMorgan Pattern

Recommend

DeMorgan, Identity 16

$$\overline{(X+Y)} \neq \overline{X}+\overline{Y}$$

$$\boxed{\overline{(X+Y)} = \overline{X}\cdot\overline{Y}}$$

DeMorgan, Identity 17

$$\overline{(X\cdot Y)} \neq \overline{X}\cdot\overline{Y}$$

$$\boxed{\overline{(X\cdot Y)} = \overline{X}+\overline{Y}}$$

Example: Algebraic Proof

- **(A+B)(A+CD) = A+BCD**

Proof Steps

(A+B)(A+CD)	**= AA+ACD+BA+BCD**	**identity 15** see slide 14
	= A+ACD+BA+BCD	**identity 6**
	= A(1+CD+B)+BCD	**identity 14**
	= A(1)+ BCD	**identity 3**
	= A + BCD	**identity 2**

Example: Algebraic Proof

- **$(A+B)(\overline{A}+C) = AC + \overline{A}B$**

Proof Steps

$$(A+B)(\overline{A}+C)$$

$= A\overline{A} + AC + B\overline{A} + BC$	**by distributive**
$= A\overline{A} + AC + \overline{A}B + BC$	**Re-ordering**
$= 0 + AC + \overline{A}B + BC$	**by identity 8**
$= AC + \overline{A}B$	***

***** The redundant term eliminated in the last step by the consensus theorem**

Expression Simplification

- **An application of Boolean algebra**
- **Simplify to contain the smallest number of <u>literals</u> (complemented and uncomplemented variables):**

$$A\,B + \overline{A}\,C\,D + \overline{A}\,B\,D + \overline{A}\,C\,\overline{D} + A\,B\,C\,D$$

$$= AB + ABCD + \overline{A}\,C\,D + \overline{A}\,C\,\overline{D} + \overline{A}\,B\,D$$

Absorption

$$= \boxed{AB\,(1 + CD) +} \overline{A}\,C\,(D + \overline{D}) + \overline{A}\,B\,D$$

$$= AB + \overline{A}\,C + \overline{A}\,B\,D = B\boxed{(A + \overline{A}D)} + \overline{A}C$$

Simplification

$$= B\,(A + D) + \overline{A}\,C$$

Final result contain 5 literals

Complementing Functions

- Use DeMorgan's Theorem to complement a function:
  1. Interchange AND and OR operators
  2. Complement each constant value and literal
- Question: Complement $F = \overline{x}y\overline{z} + x\overline{y}\overline{z}$

Solution: $\overline{F} = (x + \overline{y} + z)(\overline{x} + y + z)$

how to prove?

Complementing Functions

Initial algebra is $F = \overline{X}Y\overline{Z} + X\overline{Y}\overline{Z}$

Solve the complementing function of F which is equal to

$$\overline{F} = \overline{\overline{X}Y\overline{Z} + X\overline{Y}\overline{Z}}$$

Solution!!!

$$= (\overline{\overline{X}}+\overline{Y}+\overline{\overline{Z}})(\overline{X}+\overline{\overline{Y}}+\overline{\overline{Z}})$$

$$= (X+\overline{Y}+Z)(\overline{X}+Y+Z)$$