2D, 3D Vectors

Dot Product and Cross Product

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Dot Product

11.3.1 DEFINITION If $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ are vectors in 2-space, then the ***dot product*** of $\mathbf{u}$ and $\mathbf{v}$ is written as $\mathbf{u} \cdot \mathbf{v}$ and is defined as

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2$$

Similarly, if $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ are vectors in 3-space, then their dot product is defined as

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Properties of Dot Product

11.3.2 THEOREM *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in 2- or 3-space and* k *is a scalar, then:*

(*a*) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

(*b*) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

(*c*) $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v})$

(*d*) $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$

(*e*) $\mathbf{0} \cdot \mathbf{v} = 0$

Magnitude, Length, **Norm** of *V*: $||V|| = \sqrt{v1^2 + v2^2 + v3^2}$
Zero Vector : **0 =** *0i+0j+0k* **= <0, 0, 0>**
Unit Vector is a vector of length 1, e.g., standard unit vectors ***i*** =<1,0,0>, ***j***=<0,1,0>, ***k***=<0,0,1>
Vector Representation **U** = <u_1, u_2, u_3> = $u_1 i + u_2 j + u_3 k$

11.3.3 THEOREM *If* **u** *and* **v** *are nonzero vectors in 2-space or 3-space, and if* θ *is the angle between them, then*

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} \tag{2}$$

Dot Product: $u \cdot v = \|u\|\ \|v\| \cos\theta$

Orthogonal Projections

$$\mathrm{proj}_{\mathbf{b}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b}$$

Cross Product

11.4.2 DEFINITION If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ are vectors in 3-space, then the *cross product* $\mathbf{u} \times \mathbf{v}$ is the vector defined by

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k} \tag{3}$$

or, equivalently,

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2)\mathbf{i} - (u_1 v_3 - u_3 v_1)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k} \tag{4}$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Properties of Cross Product

11.4.3 THEOREM *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are any vectors in 3-space and* k *is any scalar, then:*

(*a*) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

(*b*) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$

(*c*) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$

(*d*) $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$

(*e*) $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$

(*f*) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

11.4.5 THEOREM *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be nonzero vectors in 3-space, and let* θ *be the angle between these vectors when they are positioned so their initial points coincide.*

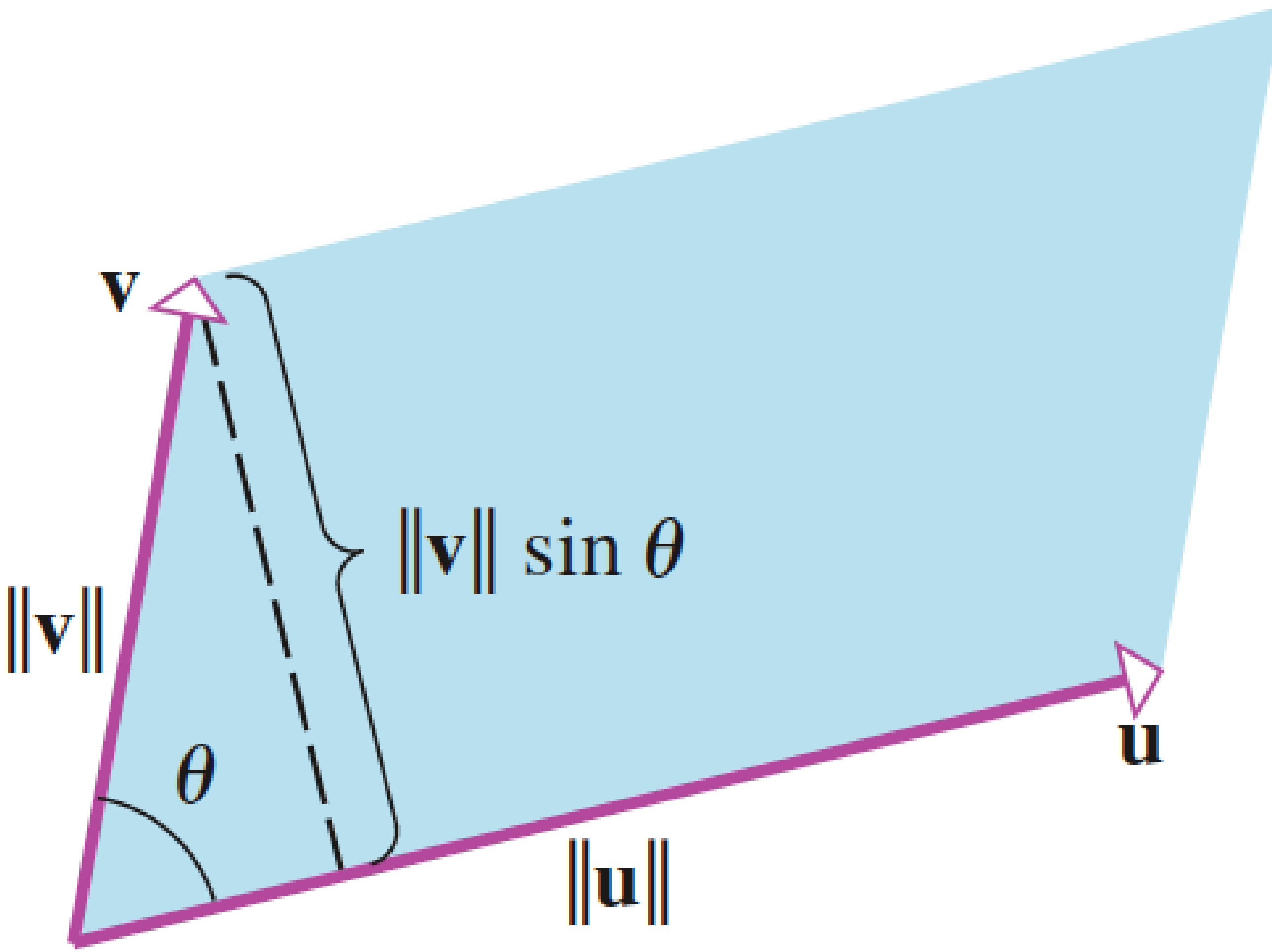
(*a*) $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$

(*b*) *The area* A *of the parallelogram that has* $\mathbf{u}$ *and* $\mathbf{v}$ *as adjacent sides is*

$$A = \|\mathbf{u} \times \mathbf{v}\| \tag{8}$$

(*c*) $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ *if and only if* $\mathbf{u}$ *and* $\mathbf{v}$ *are parallel vectors, that is, if and only if they are scalar multiples of one another.*

Area of the parallelogram



Area of the parallelogram = $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\| \sin\theta$

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

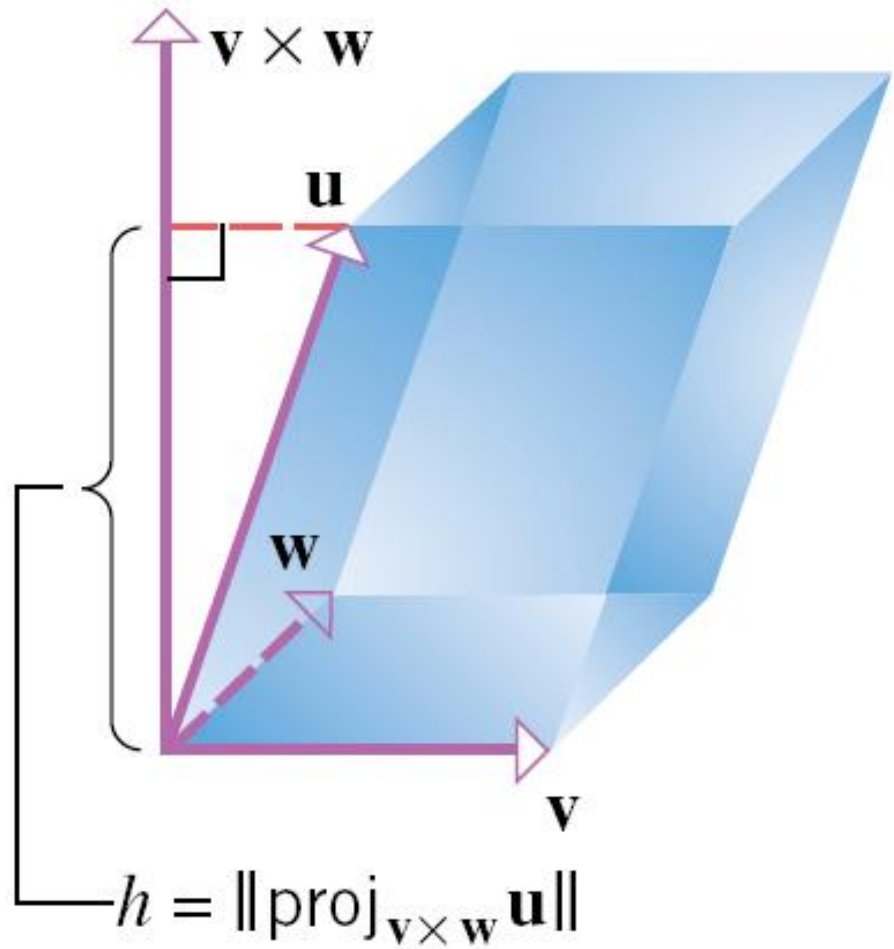
11.4.6 THEOREM *Let* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *be nonzero vectors in 3-space.*

(*a*) *The volume* V *of the parallelepiped that has* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *as adjacent edges is*

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| \tag{10}$$

(*b*) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 0$ *if and only if* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *lie in the same plane.*

Volume of the parallelepiped



Volume of the parallelepiped = $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$