SETS AND SUBSETS

Set Theory in Discrete Mathematics

Learning objectives! Know what you will learn today

Self-Reflection! Rate levels of your understanding

❍ **Checklist of key topics. Keep catching up with the course.**

- ☐ Set notations, memberships, and the cardinality of a set
- ☐ Countability and finiteness of sets, equal and equivalent sets
- ☐ Venn diagrams for sets, subsets, proper subsets, and power sets
- ☐ Bit String, a sequence of binary digits, computer representations of sets
- ☐ Operations on sets and algebraic properties of set operations

Confident	Got it	Okay	Fuzzy	Not a clue

Definitions of sets and set notations

- A **set** is any **well-defined** collection of objects

  can decide if a given object belongs to the set or not

- Objects contained in a set are called **elements** or **members** of a set
  - The objects can be numbers, letters, words, symbols, relations, or functions. A set can also contains different mixes of elements, including another set.

- Set labels are **uppercase** letters
- Element labels are **lowercase** letters

A = {a, b, c, d, e}

- Use **curly brackets or braces** to enclose elements of a set

A = {1, 2, 3} **B** = {C++, Java, Python} **C** = {@, #, $, &} **D** = {x, {y}, {2,3}, 10}

https://www.smartickmethod.com/blog/math/logic/sets-and-subsets/; https://www.askiitians.com/iit-jee-algebra/set-relations-functions/set-theory/finite-and-infinite-sets/

Membership of a set and the empty set

$\in$: is an element of (it has a shape like an "E" as in element)

$\notin$: is not an element of

If A = {1, 3, 5, 7} then $1 \in A$ but $2 \notin A$

{ } or $\varnothing$: denotes an **empty set**. An empty set has no element.

Order of elements is not important

{1,2,3} = {2,1,3} = {3,2,1} = {3,1,2}

Repeated elements can be ignored

{1,2,3,1,2,2} = {1,1,1,2,3} = {1,2,3}

WORKED EXAMPLES

- Given **A** = {1, b, {4}, {x, y, z}}, fill in the blank with either $\in$ or $\notin$

{x, y, z} ☐ **A**   x ☐ **A**   4 ☐ **A**   1 ☐ **A**   {1, b} ☐ **A**

Describing a set

The roster method: list all members between curly brackets or braces

Set builder notation: specifying properties of its elements

S = {x : P(x)}	A set S of all elements x such that P(x)
S = {x | P(x)}	A set S is a collection of all x for which P(x) is true
S = {x | x is ___ }	A set S is composed of elements x that satisfies ___

WORKED EXAMPLES

Write the following sets in roster format.

- A = { x | x is a solution to the quadratic equation $x^2 - 11x = -30$ }

Know your algebra

- B = { (x, y) | x, y are positive integers less than 5, x + y is odd }

Other than a single number, elements of a set can be pairs or any n-tuples.

PRACTICE PROBLEMS

- Use a set builder notation to represent the set **B** = {2, 4, 6, 8, 10}.

Common Sets

$\mathbb{Z}^+$ Positive integers

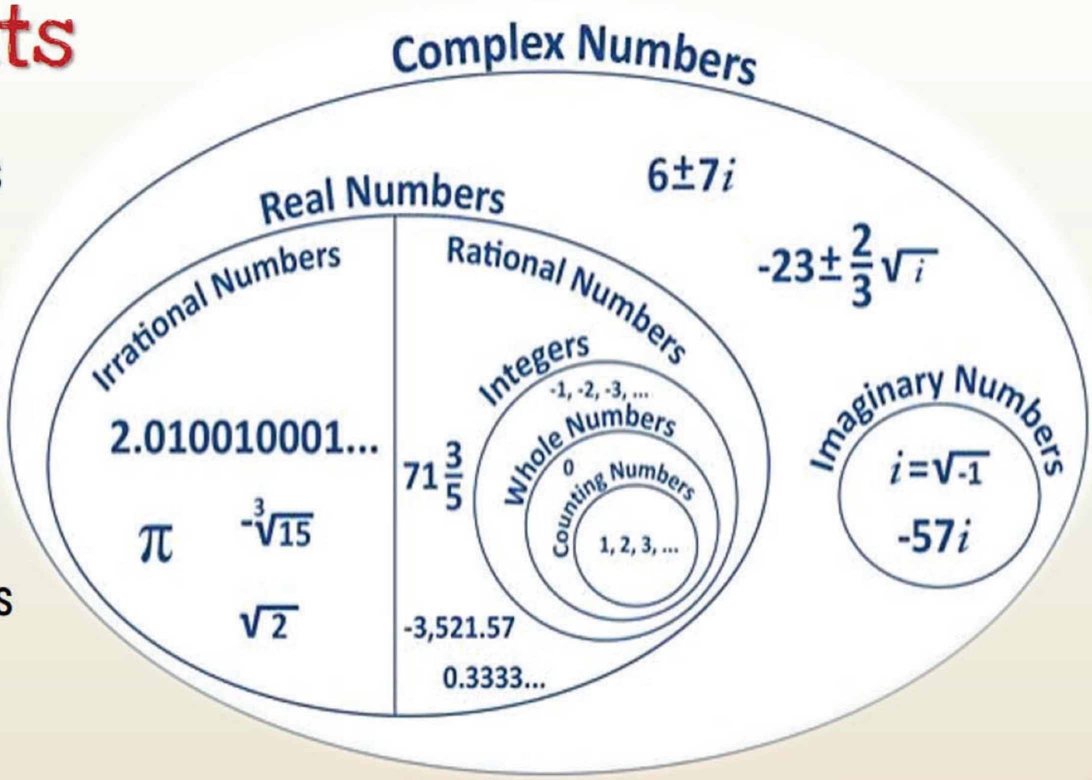
$\mathbb{N}$ Natural numbers

$\mathbb{Z}^-$ Negative integers

$\mathbb{Z}$ Integers

$\mathbb{Q}$ Rational numbers

$\mathbb{R}$ Real numbers



https://kaiserscience.wordpress.com/2018/03/21/uses-of-imaginary-numbers/

Is zero a natural number?

No consensus! Some consider "zero" a natural number, others do not.

In mathematics, the natural numbers are those used for counting (as in "there are six coins on the table") and ordering (as in "this is the third largest city in the country"). In common mathematical terminology, words colloquially used for counting are "cardinal numbers", and words used for ordering are "ordinal numbers". The natural numbers can, at times, appear as a convenient set of codes (labels or "names"); that is, as what linguists call nominal numbers, forgoing many or all of the properties of being a number in a mathematical sense. The set of natural numbers is often denoted by the symbol $\mathbb{N}$.

Some definitions, including the standard ISO 80000-2, begin the natural numbers with 0, corresponding to the non-negative integers 0, 1, 2, 3, ... (collectively denoted by the symbol $\mathbb{N}_0$), whereas others start with 1, corresponding to the positive integers 1, 2, 3, ... (collectively denoted by the symbol $\mathbb{N}_1$. Both definitions are acknowledged whenever convenient, and there is no general consensus on whether zero should be included as the natural numbers.

Texts that exclude zero from the natural numbers sometimes refer to the natural numbers together with zero as the whole numbers, while in other writings, that term is used instead for the integers (including negative integers).

Wikipedia
The Free Encyclopedia

In this course, we follow the standard ISO 80000-2

Why we consider "zero" a natural number

Defining **natural numbers** $\mathbb{N}$ as a set of cardinals of finite sets, i.e. **the set of the number of elements in any finite set**.

$\{\}$ has **0** element

$\{a\}$ has **1** element

$\{a, b\}$ has **2** elements

$\{a, b, c\}$ has **3** elements

Thus, by this definition, zero is a natural number

In set theory, we count zero. It is the number of element in the empty set. In other words, the cardinality of the empty set is zero.

WORKED EXAMPLES

Which of the following describes the set **A** = {-15, -30, -45, -60, ...} ?

- ❒ $\{x \in \mathbb{Z}^- \mid x \text{ divisible by } 15\}$
- ❒ $\{x \in \mathbb{Z}^- \mid x \text{ is a multiple of } 15\}$
- ❒ $\{x = -15y \mid y \in \mathbb{Z}^+\}$
- ❒ all of the options

Know the terms - divisible and multiple

Which of the following describes the set **B** = {1, 4, 9, 16, 25, 36, 49, ...}

- ❒ $\{x^2 \mid x \in \mathbb{N}\}$
- ❒ $\{x \in \mathbb{Z} \mid x \text{ is a square number}\}$
- ❒ $\{x = y^2 \mid y \in \mathbb{Z}^-\}$
- ❒ all of the options

PRACTICE PROBLEMS

- List all elements of the set (i.e. write the given set in the roster format), **A** = $\{x^3 \mid x \in \mathbb{Z}^- \text{ and } x > -100\}$

Know what a factor is

- Which of the following element is **NOT** a member of the set **B** = $\{ (x, y) \mid x \in \mathbb{R}, (x^2 - 2)(x^2 - 4) = 0 \text{ and } y \in \mathbb{N}, y \text{ is a factor of } 18\}$?

  - ❒ $(\sqrt{2}, 9)$
  - ❒ $(-2, 2)$
  - ❒ $(2, 6)$
  - ❒ all are members

PRACTICE PROBLEMS

- Describe using set builder notation the set **A** = {0, 4, 8, 12, 16, ...}

- Describe using set builder notation the set **B** = {13, 15, 26, 30, 39, 45, ...}

Cardinality of a set

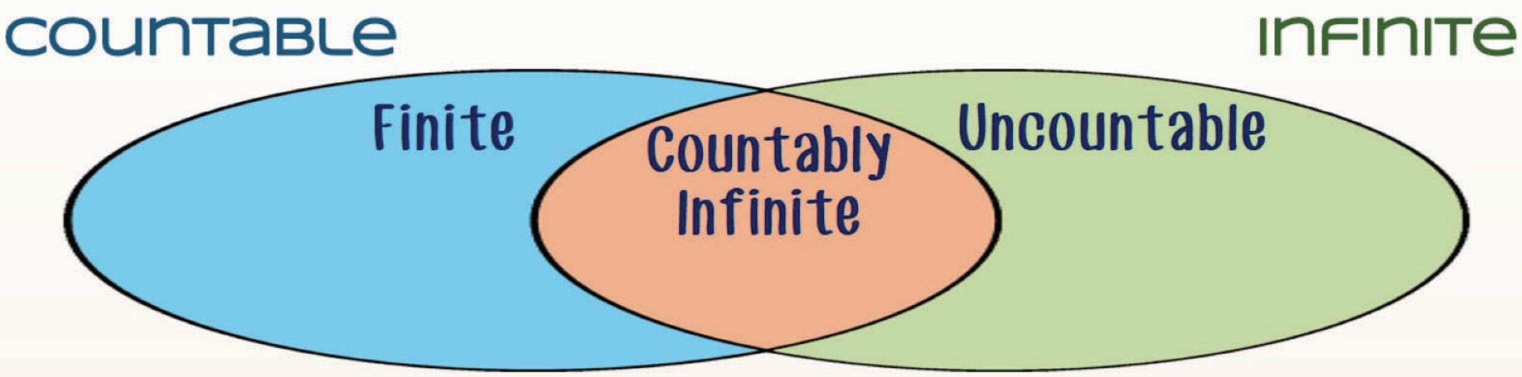
- The **cardinality** of a set is the number of elements in the set
- The cardinality of a set A is denoted by n(A) or |A|

If $A = \{1, \{3\}, \{5,7\}\}$ then $n(A) = |A| =$ ___

If $B = \{x \in \mathbb{N} \mid 9 < x < 10\}$ then $n(B) = |B| =$ ___

If $C = \{(x, y) \mid x \text{ is a solution to } x^2 - 5x + 6 = 0 \text{ and } y \in \mathbb{Z}^-, y > -3\}$
then $n(C) = |C| =$ ___

	COUNTABILITY	FINITENESS	
Countability and finiteness of a set	Can you count the elements, that is, can you list the elements in the set?	Can you tell how many elements there are in the set?	
$A = \{1, \{1\}, \{2, 3\}, 5, 8, 8\}$			
$B = \{0, 1, 4, 9, 16, 25, \ldots\}$			
$C = \{x \mid x \in \mathbb{R}, 0 < x < 1\}$			



- Categorize the following sets, put each in the correct region in the diagram

A = $\{x \mid x \in \mathbb{Z}^+, x \leq 50\}$ **B** = $\{2^q \mid q \in \mathbb{Q}^+\}$ **C** = $\{\ \} = \varnothing$

Know your prime numbers

D = $\{x \mid x$ is a multiple of $3\}$ **E** = $\{x \mid x$ is irrational$\}$ **F** = $\{x \mid x$ is prime$\}$

G = a set of all points in a plane **H** = $\{x \mid x$ is a real number and $(x^2 - 2)(x + 6) = 0\}$

PRACTICE PROBLEMS

Identify each set as finite, countably infinite, or uncountable.

- A = $\{ x \mid x \in \mathbb{R}$ and $x^2 + 41x + 41 = 0\}$

  hint: $x = \left(-b \pm \sqrt{b^2 - 4ac}\right)/2a$

- B = $\{ x = m/n \mid m, n \in \mathbb{Z}^+$ and $n > 4\}$

- C = $\{(x, y, z) \mid x \in \mathbb{Z}, y \in \mathbb{R}^+, z \in \mathbb{Z}^+\}$

- D = $\{ x \mid x \in \mathbb{R}$ and $x^2 + 3x + 2 \neq 0\}$

Ch1.3, Q.24

PRACTICE PROBLEMS

Identify each set as finite, countably infinite, or uncountable.

- $E = \{x \mid x \in \mathbb{R} \text{ and } x^2 = -25\}$
- F is the set of solutions to the equation $x^3 - x^2 - 9x + 9 = 0$
- $G = \{ x \mid x$ is a string over an alphabet set $\{a, b\}\}$

  G is a set of strings that are made-up of the alphabets 'a' and 'b'.

  G = {a, b, aa, ab, ba, bb, aaa, aab, aba, abb, baa, bab, bba, bbb, ...}

Set equality and set equivalence for finite sets

- Two sets are **equal** when they contain the **same elements**

  $C = \{1, 2, 3\}$

  $D = \{2, 2, 1, 2, 3, 3\}$

  $E = \{x \in \mathbb{Z}^+ \mid x^2 < 12\}$

  C, D, and E are equal sets

- A finite set A is **equivalent** to a finite set B if n(A) = n(B), i.e. |A| = |B|, that is, the sets A and B have the **same cardinal numbers**

  A is the set of vowels in the word CODING

  B is the set of solutions to the equation $(2x-1)(x+6) = 0$

  A and B are equivalent but not equal

WORKED EXAMPLES

- Let A = {1, 2, 3, 4, 5}. Which of the following sets are equal to A and which are equivalent to A?

  B = {x | x is an integer and $x^2 \le 25$}

  C = {x | x is a positive integer and $x^2 \le 25$}

  D = {x | x is a negative integer and $x^2 \le 25$}

  E = {x | x is a positive rational number and $x \le 5$}

PRACTICE PROBLEMS

Which of the following sets are equal

- A = {2n + 1 : n is an integer}
- B = {..., -5, -3, -1, 1, 3, 5, ...}
- C = {$x \in \mathbb{N}$: x is a multiple of 2}

True or False:

- Two equal sets may or may not be equivalent.

PRACTICE PROBLEMS

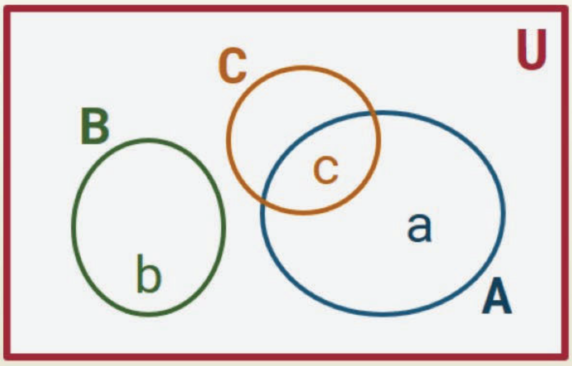
Choose all that apply: which statement is true about the set
A = { p | p is a prime number that divides 70 exactly (no remainder) }

In another word, p is a factor of 70.
Def. A factor is a number that divides another number, leaving no remainder.

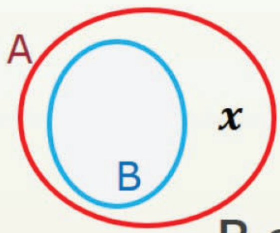
- ❑ The cardinality of A is 4
- ❑ A is equal to the set B = { 7, 7, 2, 5, 5, 5, 1 }
- ❑ $|A| = |D|$ where $D = \{x \in \mathbb{Z}^- \mid -42 < 7x \leq -21\}$
- ❑ A is equivalent to the set of negative odd integers greater than -10
- ❑ A is equivalent to the set of solutions to the equation $x^3 + 5x^2 - 4x - 20 = 0$

Venn diagrams

- Named after a British logician John Venn
- Graphical depiction of the relationship of multiple sets
- Does not represent the individual elements of the sets, rather it implies their existence
- A **circle** is used for a general set
- A **rectangle** is used for the **universal** set U (it is a collection of everything)



Subsets

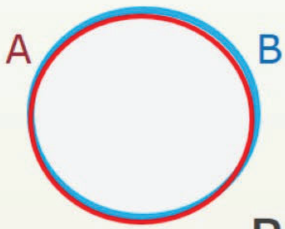


$B \subset A$

Proper subset

B is contained in A and $B \neq A$

ex. $\{1,2\} \subset \{1,2,3\}$

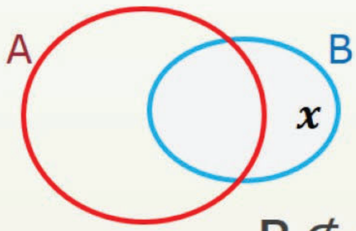


$B \subseteq A$

B and A may be equal

B is a ~~proper~~ **subset** of A

ex. $\{1,2,3\} \subseteq \{1,2,3\}$



$B \not\subseteq A$

B in not contained in A

because x in B but not in A

ex. $\{1,2,4\} \not\subseteq \{1,2,3\}$

$\subseteq$ B is **a subset** of A → every element of B is also an element of A

$\not\subseteq$ B **not** a subset of A → there is an element $x \in B$ such that $x \notin A$

WORKED EXAMPLES

Let A = {1, {2, 3}, 4}. Identify each of the following as true or false.

- ❒ $3 \in A$
- ❒ $\{2, 3\} \in A$
- ❒ $\{4\} \in A$
- ❒ $\{1, 4\} \subseteq A$
- ❒ $\{2, 3\} \subseteq A$
- ❒ $\{1, 2, 3\} \subseteq A$

Ch1.1, Q.5

Subsets and power set

- Let the set $A = \{a,b,c\}$

Know how to list all subsets

- Subsets of A are $\varnothing, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}$

LISTING ALL POSSIBLE SUBSETS OF A SET A ...

Be systematic!!

⇨ List the subset with 0 element (this is the empty set)
⇨ Then, subsets with 1 element
⇨ Then, subsets with 2 elements
⇨ ⋮
⇨ Lastly, the subset with |A| elements (this is the set A itself)

REMINDER

A subset B of any set A is a set of some elements of A

Subsets and power set

- Let the set $A = \{a,b,c\}$
- Subsets of A are $\varnothing, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}$

- $P(A) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$
- $|P(A)| = 2^{|A|} = 2^3 = 8$

The **power set** of A, *P*(A), is the set of all subsets of A.

The **cardinality of the power set** of the set A is $|P(A)| = 2^{|A|}$

$\varnothing \subseteq A$

$A \subseteq A$

WORKED EXAMPLES

- What is the power set of (i) an empty set and (ii) a set of an empty set?

- A is a set with a cardinality of 4. which of the following is not true?
  - ❒ Number of proper subsets of A = 15
  - ❒ | P(A) | = 16
  - ❒ Number of non-empty proper subsets of A =14
  - ❒ all are true

PRACTICE PROBLEMS

- $A = \{a, b, c\}$. Which of the following is not true? Choose all that apply.
  - ❒ $\{c\} \in A$
  - ❒ $\{a\} \subseteq A$
  - ❒ $b \in A$
  - ❒ $\{\} \subseteq A$
  - ❒ $\{a, b\} \in A$
  - ❒ $\{b, c\} \subseteq A$
  - ❒ $b \subseteq A$
  - ❒ $A \subset A$

- Let A be any set, $B = \{A, \{A\}\}$. Identify each of the following as true or false.
  - ❒ $A \subseteq B$
  - ❒ $\{A\} \in P(B)$
  - ❒ $\{\{A\}\} \subseteq B$
  - ❒ $\{\} \notin P(B)$

PRACTICE PROBLEMS

- How many subsets of A= {1, 2, 3, 4} contain the element 2 but not 3?
- Find the set of smallest cardinality that contains the following sets as subsets: {a, b, c}, {a, d, e, f}, {b, c, e, g}
- The number of subsets in set A is 192 more than number of subsets in set B. How many elements are there in sets A and B?

OPERATIONS ON SETS

Set Theory in Discrete Mathematics

Computer representation of sets

- Though sets are unordered, storing its elements in an unordered fashion would not be practical. Computations on sets would be time consuming.
- First, specify **an arbitrary ordering** of the elements of the universal set U, for instance $a_1, a_2, \ldots, a_n$.

A sequence of bits,
i.e. binary digits 0 and 1

- Represent a subset A of U with the **bit string** of length n, where the ith bit in this string is 1 if a_i belongs to the set A and is 0 if a_i does not belong to A.

Bit String Representation of Sets

Write elements of U in some (sorted) order, U = {1, 2, 3, 4, 5, 6, 7, 8}

To define a set A ⊆ U, A = {2, 4, 5, 6, 8}, write a sequence of 0s and 1s, corresponding to each member of U

A = __ __ __ __ __ __ __ __

TRY ME!

- Write the bit string that represent the following sets:
- B ⊆ U, B = {x | x is prime number}

  B = __ __ __ __ __ __ __ __
- C ⊆ U, C = {x | 1 ≤ x < 20}

  C = __ __ __ __ __ __ __ __

PRACTICE PROBLEMS

- Given U = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}
  What is the bit string representation of a set A = {4, 3, 3, 5, 2, 3, 3}?

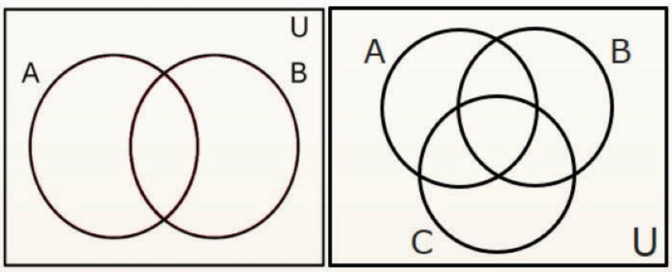
  If the bit string of a set B is **10 1001 1011**, list element of B?

- Given U = {a, b, e, g, h, m, n, r, s, w} and the set C = **01 1101 0110**
  What is the set C in roster form?

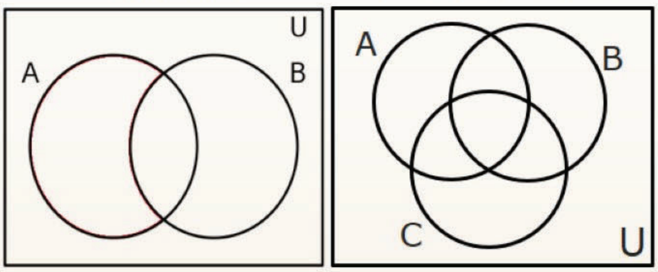
union IN A OR IN B

intersection IN BOTH A AND B

A ∪ B A ∪ B ∪ C



A ∩ B A ∩ B ∩ C



A = {1, 2, 3} =

B = {2, 3, 4, 5} =

U = {1, 2, 3, 4, 5, 6, 7, 8}

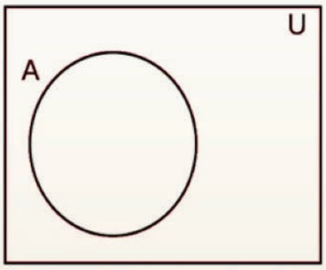
A ∪ B =

A ∩ B =

complement

NOT IN A

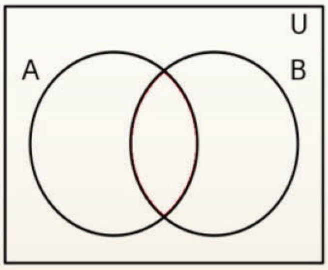
$\bar{A}$



difference

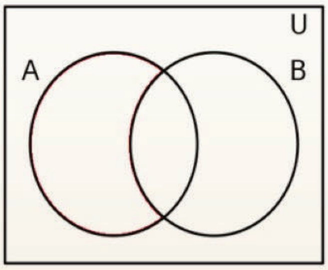
IN A NOT IN B

$A - B$



IN B NOT IN A

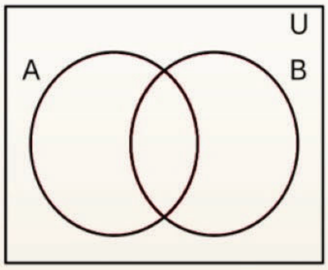
$B - A$



symmetric difference

IN A OR B NOT BOTH

$A \oplus B$



$\bar{A}$ =

$A - B$ =

$B - A$ =

$A \oplus B$ =

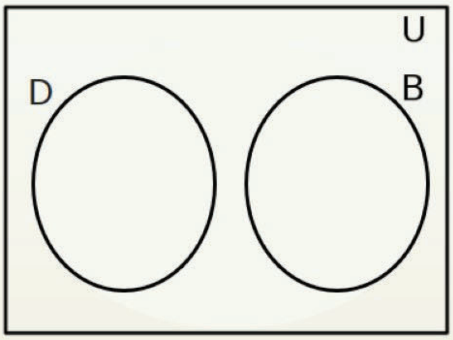
U = {1, 2, 3, 4, 5, 6, 7, 8}

A = {1, 2, 3}

B = {2, 3, 4, 5}

Intersection – Disjoint Set

- A **disjoint set** is a set where the intersection result is the empty set
- In this example, sets B and D are disjoint



A = {1, 2, 3}

B = {2, 3, 4, 5}

U = {1, 2, 3, 4, 5, 6, 7, 8}

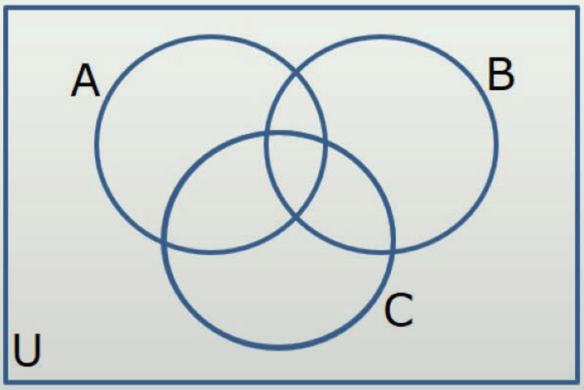
D = {7, 8}

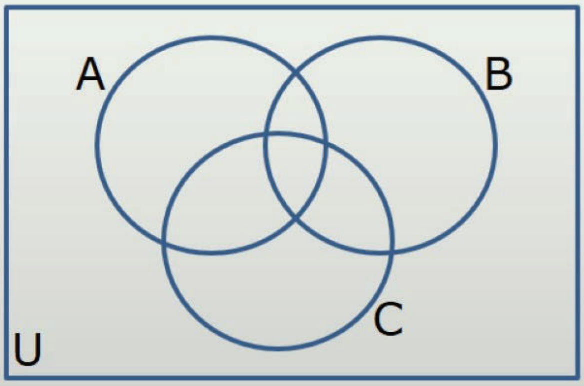
$B \cap D$ = { } = 00000000

WORKED EXAMPLES

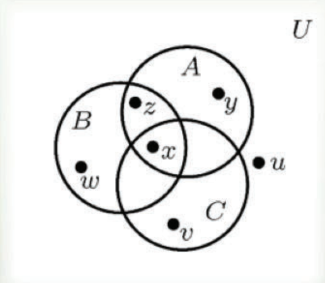
Shade a Venn diagram to represent the set.

- $(\overline{A} \cap \overline{C}) \cup B$
- $(A \cup B) - (A \cap B) - C$





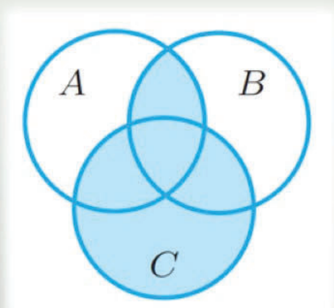
PRACTICE PROBLEMS



Identify the following as true or false.

- $y \in A \cap B$
- $w \in B \cap C$
- $x \in B \cup C$
- $u \notin C$

Describe the shaded region shown on the right using unions and intersections of the sets A, B, and C. (Several descriptions are possible.)



Ch1.2, Q.12,14

PRACTICE PROBLEMS

Let the universal set be U = {a, b, c, d, e, f, g, h, k},

A = {a, b, c, g}, B = {d, e, f, g}, C = {a, c, f}, and D = {f, h, k}, compute

- $\overline{A \cup B}$
- $A \oplus C$
- $(A \cap B) - (B \cap D)$
- $C \cap \varnothing$

If $A \cup B = A \cup C$, must $B = C$? Explain.

Ch1.2, Q.1i,2j,3e

WORKED EXAMPLES

Let U = {11, 12, 13, 14, 15, 16, 17, 18}, B = {11, 12, 14, 15},
C = {x | x is divisible by 3}, and D = {x | x is odd and x is more than 15}

- Find the sequences of length 8 that correspond to sets B, C, and D

B =	**C =**	**D =**

- Write a bit string for $B \cup C$, $(B - C) \oplus D$, and $C \cap D$

$B \cup C$ =	**$(B - C) \oplus D$ =**	**$C \cap D$ =**

PRACTICE PROBLEMS

Let U = {b, d, e, g, h, k, m, n}, B = {b}, C = {d, g, m, n}, and D = {d, k, n}

- Write the bit strings that correspond to sets B, C, and D

- Use bit strings to represent $(B \cup C) - D$, $C \cup D$, and $C \cap D$

Ch1.3, Q.27

PROPERTIES OF SET OPERATIONS

Set Theory in Discrete Mathematics

IDENTITY

Identity does not change the value of the operand

$$a + 0 = a$$
$$a \times 1 = a$$

INVERSE

Inverse changes the number to the identity

$$a + (-a) = 0$$
$$a \times (1/a) = 1$$

COMMUTATIVE

Changing the order of the operands

$$a + b = b + a$$
$$a \times b = b \times a$$

properties in mathematics VOCAB

ASSOCIATIVE

Changing the group of the operands

$$a + (b + c) = (a + b) + c$$
$$a \times (b \times c) = (a \times b) \times c$$

DISTRIBUTIVE

This involves two operators, e.g. combining addition & multiplication

$$a \times (b + c) = (a \times b) + (a \times c)$$

Algebraic properties of set operations

Identity Laws $A \cup \varnothing =$ $A \cap U =$	**Complement Laws** $A \cup \overline{A} =$ $A \cap \overline{A} =$
Idempotent Laws UNCHANGED $A \cup A =$ $A \cap A =$	**Double Complementation** $\overline{\overline{A}} =$
Domination Laws $A \cup U =$ $A \cap \varnothing =$	**Complement of $\varnothing$ and U sets** $\overline{\varnothing} =$ $\overline{U} =$

Algebraic properties of set operations

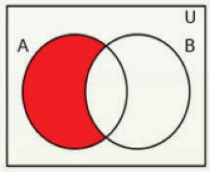
Commutative properties

ORDER

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

Difference Law

$$A - B = A \cap \overline{B}$$



Associative properties

GROUP

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

De Morgan's Law

NEGATION

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$
$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Distributive properties

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Absorption Laws

$$A \cup (A \cap B) = A$$
$$A \cap (A \cup B) = A$$

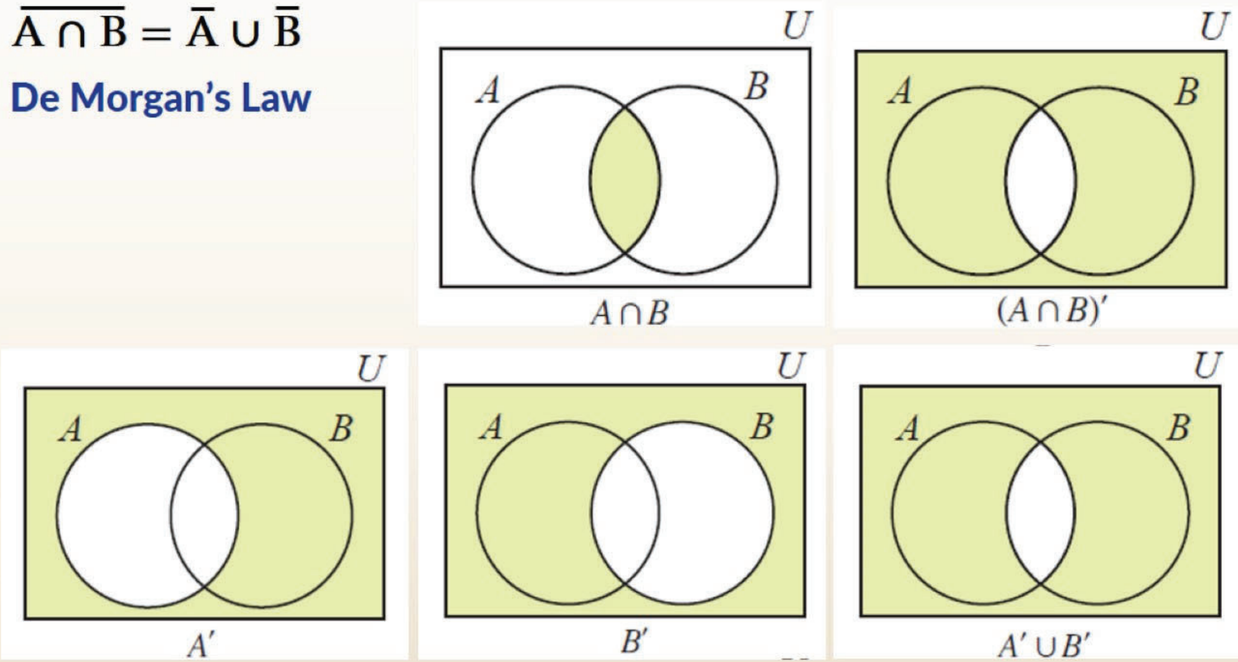
Proving or showing that two sets are equivalent

- Use Venn diagrams
- Apply sequences of common equivalences (algebraic properties)

Proof

$\overline{A \cap B} = \bar{A} \cup \bar{B}$

De Morgan's Law



$\overline{A \cap B}$

$\bar{A} \cup \bar{B}$

https://www.onlinemath4all.com/Proofs.html

PRACTICE PROBLEMS

- Show using Venn diagram that (A – B) ∪ (B – C) = (A ∪ B) – (B ∩ C)

PRACTICE PROBLEMS

- Use Venn diagram to show that if $C \subseteq A$ then $(A \cap B) \cup C = A \cap (B \cup C)$ and show also that when $C \not\subseteq A$ the equality does not hold.

WORKED EXAMPLES

- Use set algebra to prove that $A \cap (B - C) = (A \cap B) - C$

PRACTICE PROBLEMS

- Use set algebra to prove that $((\overline{B - A}) \cap A) - A = \overline{A}$

What's next?

DON'T FORGET!

A WEEKLY QUIZ

DON'T FORGET!

Reading
KBR, Rosen, Levin

Textbook exercises

HW - Practice Problems