



# Determinants



# Determinants

- Evaluating Determinants by Cofactor Expansion
- Evaluating Determinants by Arrow Technique
- Evaluating Determinants by Row Reduction



# Determinant Test for Invertibility

- **Theorem 1:** A square matrix  $A$  is invertible if and only if
$$\det(A) \neq 0$$
- **Theorem 2:** If  $A$  and  $B$  are matrices of the same size, then
$$\det(AB) = \det(A) \det(B)$$

# Determinants by Cofactor Expansion

**DEFINITION 2** If  $A$  is an  $n \times n$  matrix, then the number obtained by multiplying the entries in any row or column of  $A$  by the corresponding cofactors and adding the resulting products is called the *determinant of  $A$* , and the sums themselves are called *cofactor expansions of  $A$* . That is,

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \quad (5)$$

[cofactor expansion along the  $j$ th column]

and

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \quad (6)$$

[cofactor expansion along the  $i$ th row]

► **EXAMPLE 3** Cofactor Expansion Along the First Row

Find the determinant of the matrix

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

by cofactor expansion along the first row.

**Solution**

$$\begin{aligned} \det(A) &= \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 3 \begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - 1 \begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0 \begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix} \\ &= 3(-4) - (1)(-11) + 0 = -1 \end{aligned}$$

► **EXAMPLE 4** Cofactor Expansion Along the First Column

Let  $A$  be the matrix in Example 3, and evaluate  $\det(A)$  by cofactor expansion along the first column of  $A$ .

**Solution**

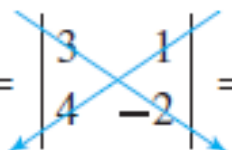
$$\begin{aligned} \det(A) &= \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 3 \begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - (-2) \begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} + 5 \begin{vmatrix} 1 & 0 \\ -4 & 3 \end{vmatrix} \\ &= 3(-4) - (-2)(-2) + 5(3) = -1 \end{aligned}$$

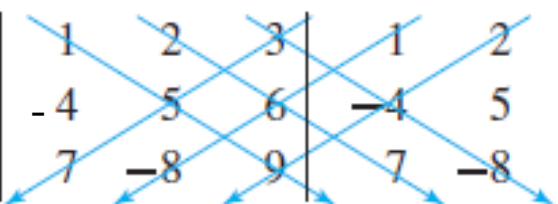
This agrees with the result obtained in Example 3.

# The Arrow Technique

(A technique for determinants of 2x2 and 3x3 matrices only)

► **EXAMPLE 7** A Technique for Evaluating 2 x 2 and 3 x 3 Determinants

$$\begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = (3)(-2) - (1)(4) = -10$$


$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} = [45 + 84 + 96] - [105 - 48 - 72] = 240$$


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# Row Reduction and Determinants

**THEOREM 2.2.3** Let  $A$  be an  $n \times n$  matrix.

- (a) If  $B$  is the matrix that results when a single row or single column of  $A$  is multiplied by a scalar  $k$ , then  $\det(B) = k \det(A)$ .
- (b) If  $B$  is the matrix that results when two rows or two columns of  $A$  are interchanged, then  $\det(B) = -\det(A)$ .
- (c) If  $B$  is the matrix that results when a multiple of one row of  $A$  is added to another row or when a multiple of one column is added to another column, then  $\det(B) = \det(A)$ .

Table 1

Relationship	Operation
$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = k\det(A)$	The first row of $A$ is multiplied by $k$ .
$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = -\det(A)$	The first and second rows of $A$ are interchanged.
$\begin{vmatrix} a_{11} + ka_{21} & a_{12} + ka_{22} & a_{13} + ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = \det(A)$	A multiple of the second row of $A$ is added to the first row.



# Properties of Determinants

- See the pdf file
- [DeterminantProperty.pdf](#)



# Exercises

1. Using Row Reduction to evaluate a determinant of

$$A = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}$$

2. Use determinant to decide whether A is invertible.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 4 & 6 \end{bmatrix}$$

3. Find the adjoint of

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

**4. Find inverse of matrices in Exercises 1, 2, 3.**

# Exercises

5. Decide whether  $A$  is invertible, and if so, use the adjoint method to find its inverse.

$$A = \begin{bmatrix} 2 & 5 & 5 \\ -1 & -1 & 0 \\ 2 & 4 & 3 \end{bmatrix}$$