COUNTING WITH SETS

Principle of Inclusion-Exclusion
The Cartesian Products of Sets

Counting and Combinatorics

Learning objectives! Know what you will learn today

Self-Reflection! Rate levels of your understanding

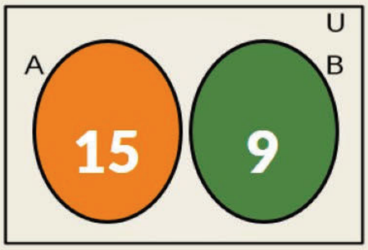
❍ Checklist of key topics. Keep catching up with the course.

- ☐ Counting with sets, PIE and Cartesian products
- ☐ Decompose a problem and combine them using the sum or product rules
- ☐ Avoid overcounting, apply the subtraction/complement or division rules
- ☐ When order does matter, permutation with and without repetitions
- ☐ When order does not matter, combination with and without repetitions

Confident	Got it	Okay	Fuzzy	Not a clue

The cardinality of $A \cup B$ when A and B are <u>disjoint</u>: $|A \cup B| = |A| + |B|$

$A \cap B = \varnothing$



Find the cardinality of the union of A and B.

Answer =>

The same question asked in different ways

- In a box, there are 15 orange and 9 green balls. How many balls are there in total?
- How many ways can we pick a ball from a box of 15 orange and 9 green balls (each ball is unique)?

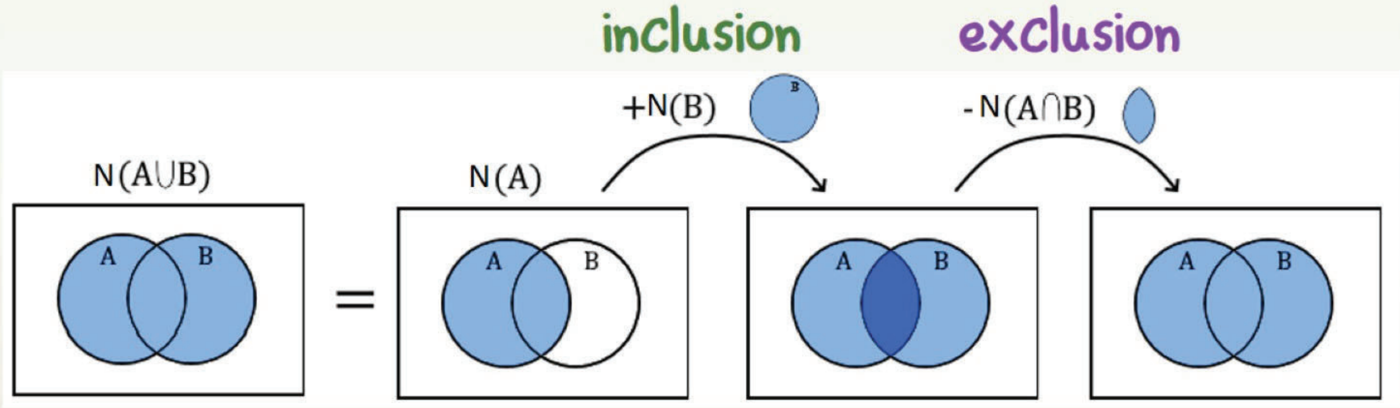
PROBLEM SOLVING

Find the cardinality of the set $A \cup B$

PRACTICE PROBLEMS

- A student can choose a project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. How many possible projects are there to choose from?

PIE: finding $|A \cup B|$ when the sets are not disjoint ...

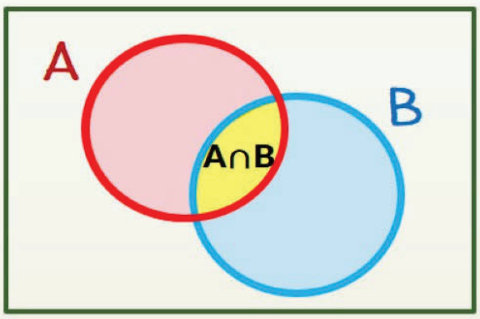


Principle of inclusion and exclusion

If A and B are finite sets, then $|A \cup B| = |A| + |B| - |A \cap B|$

https://www.usu.edu/math/schneit/StatsStuff/Probability/probability4.html

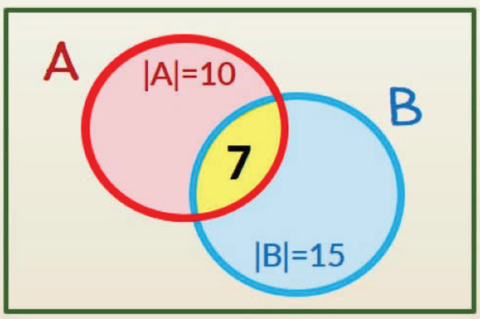
PIE: finding $|A \cup B|$ when the sets are not disjoint ...



Principle of inclusion and exclusion

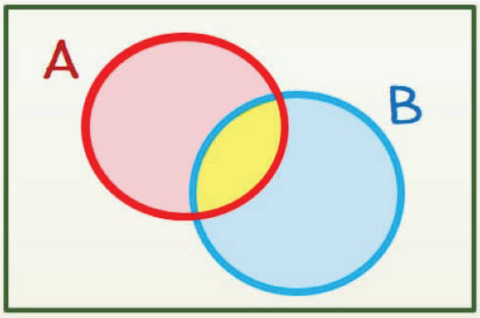
If A and B are finite sets, then

$|A \cup B| = |A| + |B| - |A \cap B|$



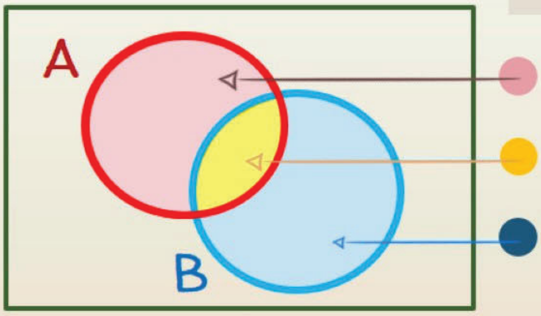
Given the sets as shown, find $|A \cup B|$

Another approach: when the sets are not disjoint ...



Regions in a Venn diagram are disjoint. Identify what each represents, fill in ones you're interested in, then add/subtract them to get what needed.

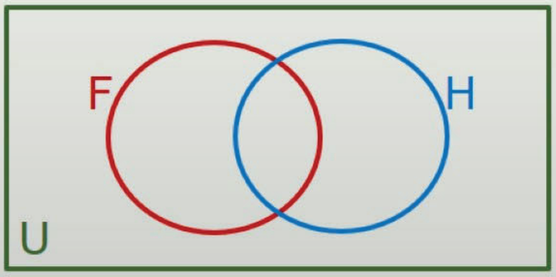
In a group of friends, 10 like aerobics, 15 prefer boxing, and 7 do both. How many friends enjoy exercises?



WORKED EXAMPLES

- In a class, there are
  8 students who play football and hockey
  7 students who do not play football or hockey
  13 students who play hockey
  19 students who play football.

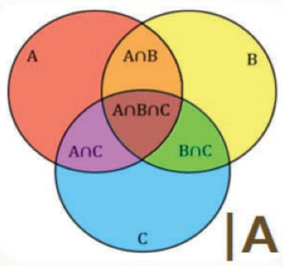
  How many students are there in the class?

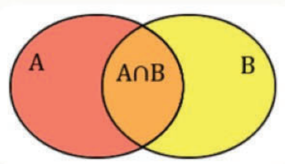


PRACTICE PROBLEMS

A computer company wants to hire 25 programmers to handle systems programming jobs and 40 programmers for applications programming. Of those hired, 10 will be expected to perform jobs of both types. How many programmers must be hired?

Ch1.2, Ex.9





Principle of inclusion-exclusion (PIE)

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

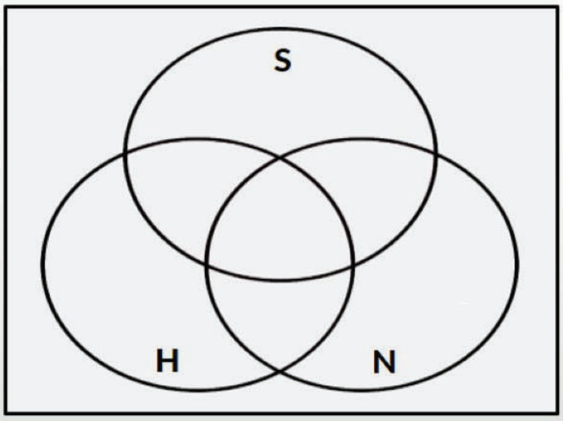
When there're more than two sets

- Draw a Venn diagram that represents a given scenario
- Determine what each (**disjoint**) region corresponds to
- Fill-in the values, usually from inside out
- Use variables when there are unknowns

PROBLEM SOLVING PRACTICE

WORKED EXAMPLES

- A company is hiring. Every candidate has at least one skill. Out of 75 candidates, 48 are software engineers, 35 hardware engineers, 42 network engineers, and 18 have skills in all three areas.
- If the company hires everyone with exactly two skills, how many candidates are hired?



PRACTICE PROBLEMS

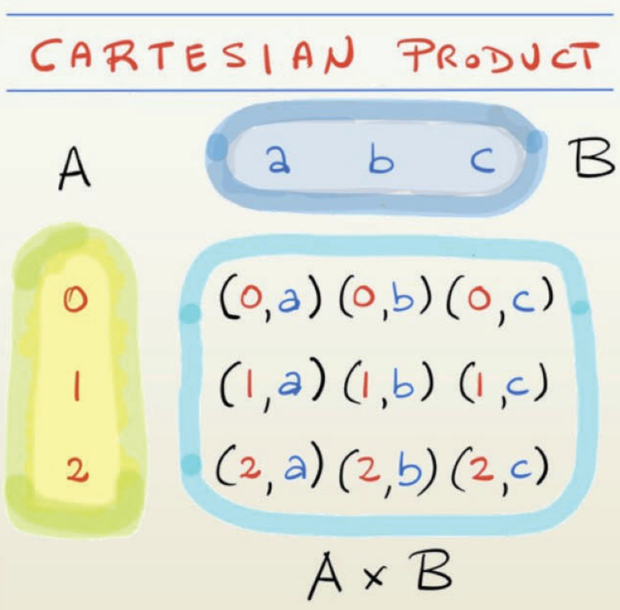
A survey was taken on methods of traveling to work. Each respondent was asked to check BUS, TRAIN, or CAR. More than one answer allowed. The results were: BUS 30, TRAIN 35, CAR 100, BUS & TRAIN 15, BUS & CAR 15, TRAIN & CAR 20, and all three methods 5 people. How many people completed a survey?

Ch1.2, Ex.10

PRACTICE PROBLEMS

- In a town, 85% speak Tamil, 40% English, 20% Hindi. Also, 32% speak Tamil & English, 13% Tamil & Hindi, and 10% English & Hindi, find the percentage of people who can speak all three languages.

Product sets or Cartesian products



The **cartesian product** of two sets A and B is the set of all **ordered pairs** whose first component comes from A and second component from B.

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

The cartesian product of a set with itself is abbreviated $A \times A$ or A^2

Cardinality: $|A \times B| = |A| \times |B|$

https://medium.com/not-zero/ordered-pairs-cartesian-product-and-binary-relations-809ad2a274ca

WORKED EXAMPLES

$A = \{2, 3\}$ and $B = \{x \in \mathbb{Z}^+ \mid x < 3\}$, identify which is $A \times B$, $B \times A$, A^2, and B^2

I. $\{(1, 2), (1, 3), (2, 2), (2, 3)\}$

II. $\{(2, 2), (2, 3), (3, 2), (3, 3)\}$

III. $\{(2, 1), (2, 2), (3, 1), (3, 2)\}$

IV. $\{(1, 1), (1, 2), (2, 1), (2, 2)\}$

- Let $C = \{x \in \mathbb{Z} \mid |x| < 3\}$, then what is $|A \times B \times C|$?

  the absolute value of x

The cartesian product is not commutative, in general, $A \times B \neq B \times A$

Summary — Cartesian products of several sets, ordered m-tuples

call this an ordered m-tuple

$$A_1 \times A_2 \times \cdots \times A_m = \{(a_1, a_2, \cdots, a_m) \mid a_i \in A_i, i = 1, 2, \cdots, m\}$$

The cardinality $|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \times |A_2| \times \cdots \times |A_m|$

Using Cartesian products in set builder notation

From last week practice problem

- Which of the following element is NOT a member of the set $A = \{ (x, y) \mid x \in \mathbb{R}, (x^2 - 2)(x^2 - 4) = 0 \text{ and } y \in \mathbb{N}, y \text{ is a factor of } 18\}$?

  - [ ] $(\sqrt{2}, 9)$
  - [ ] $(-2, 2)$
  - [ ] $(2, 6)$
  - [x] all are members

$A = \{ (x, y) \in \mathbb{R} \times \mathbb{N} \mid (x^2 - 2)(x^2 - 4) = 0 \text{ and } y \text{ is a factor of } 18\}$

PRACTICE PROBLEMS

- Find x and y if (x + 2, 4) = (5, 2x +y)

- Let $A = \{1, 2\}$, find $A \times A$ and A^3

  $A^3 = A \times A \times A$

- Determine the cartesian product $P(\{\&, \#\}) \times \{z\}$

  Recall, P is the power set

Two ordered pairs are equal if and only if the corresponding coordinates are equal: (a,b) = (c,d) $\Leftrightarrow$ a=c and b=d

PRACTICE PROBLEMS

the absolute value of y

P is the power set

- Let the set A = $\{(x, y) \in \mathbb{Z}^+ \times \mathbb{Z}^- \mid 2x^2 = 2 \text{ and } |y| < 4\}$ and B is the set of months in a year, how many elements are there in $P(A \times B)$, i.e. the power set of the Cartesian product $A \times B$? What about $P(A) \times B$?

|A| =

|B| =

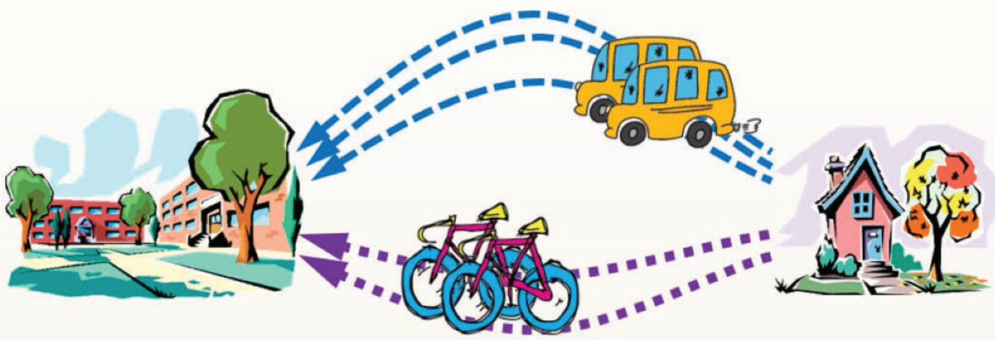
|P (A × B)| =

|P (A) × B| =

THE SUM AND PRODUCT RULES

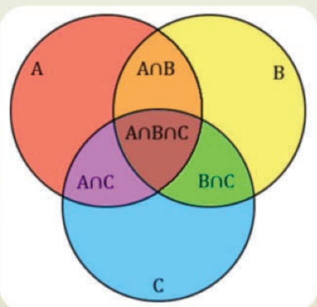
How many?

Counting and Combinatorics

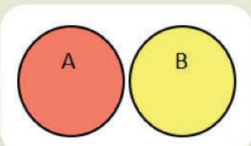


There are 3 bus services from home to school and 2 bike routes. How many ways for a student to go to school?

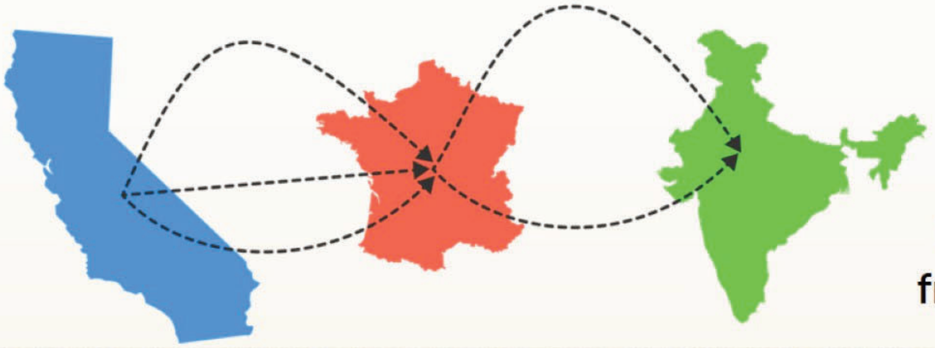
Additive Principle - The Sum Rule



Principle of inclusion-exclusion (**PIE**): solve a counting problem using sets, draw a Venn diagram and find the number of elements in each disjoint region

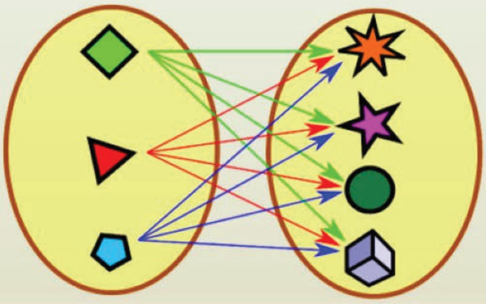


$$|A \cup B| = |A| + |B| - |A \cap B|$$



There are 3 flights from LAX-California to France, and 2 from France to India. How many ways can one fly from LAX to India?

Multiplicative Principle – The Product Rule



A Cartesian product $A \times B$ combine elements of A and B to produce new elements, which are **ordered pairs** (a, b) where $a \in A$ and $b \in B$

$$|A \times B| = |A| \times |B|$$

https://brilliant.org/problems/combi-2/

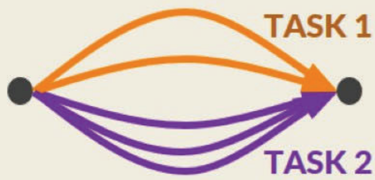
THE SUM RULE

If there are tasks $T_1, T_2, \ldots, T_k$ that can be done in $n_1, n_2, \ldots, n_k$ ways,

No tasks are done at the same time

then, we can do it in $n_1 + n_2 + \ldots + n_k$ **ways**

ADDITIVE PRINCIPLE



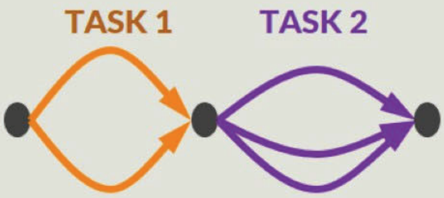
THE PRODUCT RULE

If there are tasks $T_1, T_2, \ldots, T_k$ that can be done in $n_1, n_2, \ldots, n_k$ ways,

Tasks are performed in sequence

then, we can do it in $n_1 \times n_2 \times \ldots \times n_k$ **ways**

MULTIPLICATIVE PRINCIPLE



WORKED EXAMPLES

- There are 12 boys in the class and 13 girls. How many possible representatives can the class select if it needs …

- only one representative, either a boy or a girl.

- a pair of representatives, one boy and one girl.

WORKED EXAMPLES

- Suppose statement labels in a programming language can be either a single letter or a letter followed by a digit. Find a number of possible labels?

Many problems require a combination of both sum & product rules

PRACTICE PROBLEMS

- A label consists of one English alphabet followed by three digits. If repetition of digits are allowed, how many distinct labels are there?

PRACTICE PROBLEMS

There are 3 routes between Atlanta and Athens, 4 routes between Athens and Augusta, and 2 routes between Atlanta and Augusta.

- How many ways are there to travel from Atlanta to Augusta?
- How many ways are there to travel from Athens to Atlanta?
- How many different ways can the round trip between Augusta and Athens be made if the trip does not go through Atlanta?
- How many different ways can the round trip between Augusta and Athens be made if the trip does not go through Atlanta and each route is used only once?

Ch3.1,Q.35,37

PERMUTATIONS COMBINATIONS

Counting and Combinatorics

How many lock combinations are there?

Applying the product rule with four subtasks, one for each digit.

- The lock has four-dial with 10 digits (0-9) on each
- **Repetition** of digits on any of the 4 dials is **allowed**
- The **order** in which the code is entered **matters**, switching them will not open the lock!

Given 7 letters, how many ways can they be rearranged?

Order matter
No repeat

One by one, in any order, pick a letter that is left and put it in place

Task 1: 1^{st} position, 7 ways
Task 2: 2^{nd} position,
Task 3: 3^{rd} position,
⋮ ⋮
Task 7: 7^{th} position,

— — — — — — —

Given n=7 letters of English alphabet, choose r=3

Order matter
No repeat

Choose in order, each letter can be chosen only once (no repeat)

— — —

WORKED EXAMPLES

- How many 4-digit pin number are there if the first digit must be a prime?

- How many 4-digit pin number can be made using digits 1-7, repetition of digits is not allowed, and the digit 4 is always there in the resulting 4-digit pin?

WORKED EXAMPLES

A homeowner has two collections. One consists of 6 photographs and the other 4 paintings. How many ways can he decorate a wall of his living room if ____

- he has a limited space and he must choose only one of the two collections?

- he displays every piece of his work, keeping all photographs together on one side of the wall and all painting on the other side?

PRACTICE PROBLEMS

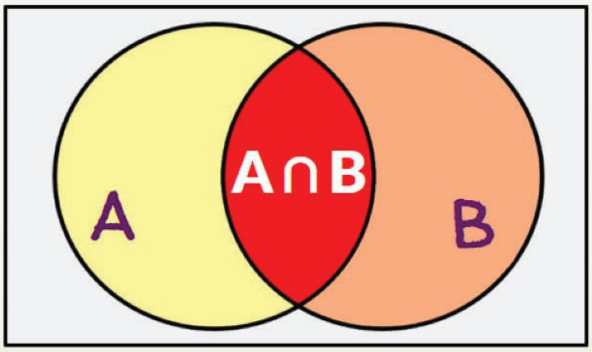
- There are 5 American, 4 British, and 3 Japanese, to be seated in row. How many ways are there to ... seat them so that all person of the same nationality sit together?
- ... seat 2 from each country so that representatives of the same nationality sit together?
- ... seat 2 representatives from each country?

AVOID OVERCOUNTING

The Subtraction Rule
The Division Rule
The Complement Rule

Counting and Combinatorics

The Subtraction Rule



Principle of inclusion and exclusion

If A and B are finite sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

inclusion (+) — exclusion (−)

We "overcount" the middle (the intersection region), so we need to "subtract" it out.

WORKED EXAMPLES

- How many 7-bit strings are there?

REMINDER A bit string is a sequence of bits, i.e. a sequence of binary digits 0 and 1. We can use it to represent sets in computing, e.g. when U is a set of positive integers less than 6, the odd ones are 10101.

WORKED EXAMPLES

- How many 7-bit strings have **exactly one zero**?

The Division Rule

- How many 7-bit strings have **exactly two zeros**?

The Complement Rule

- How many 7-bit strings have **at least one zero**?

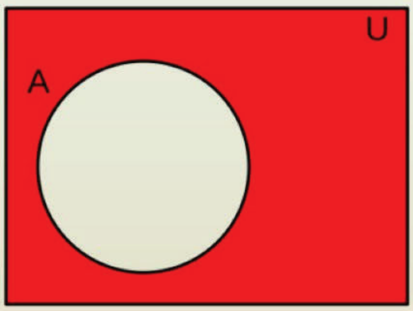
When it appears that a task can be done in n different ways, but it turns out that for each way, there are d equivalent ways of doing it, we conclude that there are n/d distinct ways of doing the task.

Recognize which of what we are counting are the same & how many are actually referring to the same element

The Division Rule & The Complement Rule

To get the cardinality of a set A, where |A| may be difficult to compute or A consists of many subcases, find the cardinality of the universe U and subtract from it the number of elements that is not in A.

$|A| = |U| - |\bar{A}|$ **wanted = universe – unwanted**



WORKED EXAMPLES

Use the complement rule to find the number of bit strings of length 8 that start with a 0 bit or do not end with the two bits of 00.

PRACTICE PROBLEMS

Use the complement rule to find the number of bit strings of length 8 that start with a 0 bit and do not end with the two bits of 00.

PRACTICE PROBLEMS

- A ternary string is a sequence of 0s, 1s, and 2s. How many length-8 ternary strings are palindromes? A palindrome is a string which reads the same from the front and back, for example, 21100112 and 12211221 are palindromes.

The number of ways to select r out of n elements

	REPETITION allowed	NO REPEAT
PERMUTATION Arrangement Order matters	n^r	${}_n\mathbf{P}_r = \frac{n!}{(n-r)!}$
COMBINATION Grouping/Selection No ordering	${}_{n+r-1}\mathbf{C}_r = \frac{(n+r-1)!}{r!(n-1)!}$	${}_n\mathbf{C}_r = \frac{n!}{(n-r)!r!}$

Remembering these formula is one thing. You will, however, do a lot better if you understand & are able to apply them effectively. They can be derived easily using basic counting rules: the rule of sum, the rule of product, and the rule of division.

RECALL

Given n=7 letters of English alphabet, choose r=3

A B C D E F G

Order matter
No repeat

Choose in order, each letter can be chosen only once (no repeat)

$$\underline{7}\times\underline{6}\times\underline{5} = \frac{7!}{(7-3)!} = {}_7P_3 = {}_nP_r$$

r-permutations of n elements

When r=n, $_nP_r = {}_nP_n = n!$

NEXT

Use division rule to go
from permutations to combinations

Given n=7 letters of English alphabet, choose r=3

A B C D E F G

Order NOT matter
No repeat

Number of permutations = $7 \times 6 \times 5 = 7!/4! = {}_7P_3$

Overcounting factor *(how many are counted, but they are, in fact, non-distinct)*

Apply division rule => # of ways =

WORKED EXAMPLES

- There are 15 students in the class. How many ways to choose ...
  - three tutors: one for math, one for science, and the other English?
  - a team of three tutors?

WORKED EXAMPLES

- There are 15 balls: 4 red, 5 blue, and 6 green. We draw three and at least one is red. We put the balls back, then draw another four and at most one is green. In how many ways can this be done?

PRACTICE PROBLEMS

- An urn contains 16 balls, of which 7 are red and the other 9 blue. Your friend takes out 8 balls and gets exactly 3 red and 5 blue. Then, without replacement, you take 3 balls and get all blue. How many ways can this happen?

PRACTICE PROBLEMS

In a box there are 5 white balls, 7 green balls, and 10 red balls. A man picks 12 balls from the box. How many ways in which

- exactly 5 red balls are drawn
- a red ball is not drawn
- he gets at least one white, one green, and 6 red balls

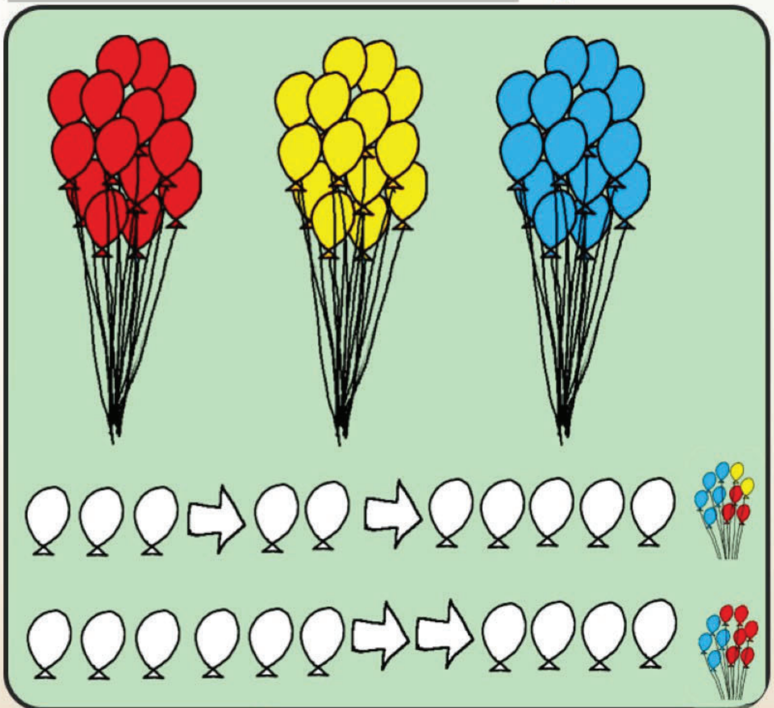
PRACTICE PROBLEMS

In how many ways can a committee, consisting of one chairman, one secretary, one treasurer and four ordinary members be chosen from eight persons?
(Committees with different chairmen, secretaries, treasurers count as different committees)

PRACTICE PROBLEMS

- An equal number of juniors and seniors are trying out for six spots on the university debating team. If the team must consist of at least 4 seniors, then how many different possible debating teams can result if 5 juniors try out?

Combination with repetitions



A man sells red, yellow, and blue identical balloons. Buying 10, how many different collections of balloons can be made?

How many ways to choose r=10 from r+n-1=12 positions?

(r is the # of balloons, n is #colors)

https://blogs.adelaide.edu.au/maths-learning/2015/10/17/a-story-of-stars-and-bars/

Picking 10 balloons from a selection of 3 colors. Unlimited stock of each color.

Stars and Strips

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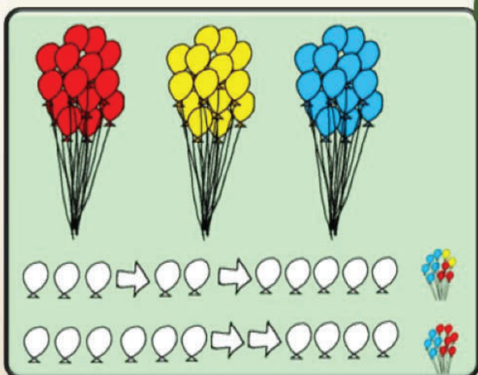
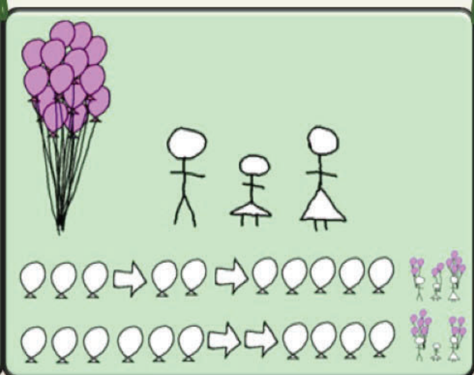


Distribute 10 balloons to 3 kids. Putting 10 identical balls into 3 baskets.

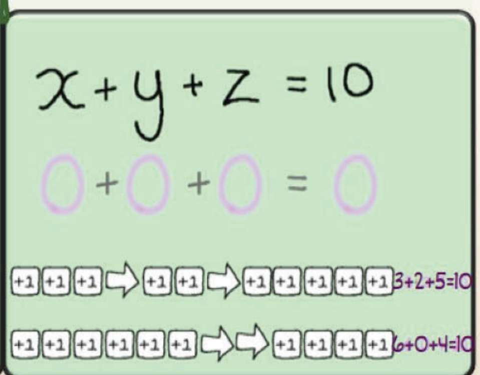
Balls and Urns



How many integer solutions does the equation $x_1 + \ldots + x_n = r$, where $x_i \geq 0$ has?

Integer Solutions

$$x_1 + x_2 + x_3 = r$$



WORKED EXAMPLES

- We distribute 15 identical tennis balls among 6 children. Find the number of all possible distributions.

- Constraint: each child receives at least one ball.

PRACTICE PROBLEMS

- How many integer solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 13$?

PRACTICE PROBLEMS

- How many integer solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 13$ with $x_1 > 1$, $x_2 > 1$, $x_3 > 3$, $x_4 \geq 0$?

PRACTICE PROBLEMS

- How many integer solutions are there to the system of equation $x_1 + x_2 + x_3 + x_4 + x_5 = 20$ and $x_1 + x_2 = 15$ where all $x_i \geq 0$?

PRACTICE PROBLEMS

- The college food plan allows a student to choose three pieces of fruit each day. The fruits available are apples, bananas, peaches, pears, and plums. For how many days can a student make a different selection?

Ch3.2,Q.18

What's next?