An Example of Thesis Formatting with $\LaTeX$

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Abstract

This short example thesis meets all of the formatting requirements in Mathematics and Computer Science.

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Thanks, Mom!

Contents

Chapter 1

Introduction

The study of *graph invariants*, properties of graphs that are preserved by isomorphism, is central to graph theory. Such properties are independent of the representation or labelling of graphs. There are many examples of interesting graph invariants, such as vertex-count, edge-count, chromatic number, crossing-number, maximum clique-size, and characteristic equation. However, none of these invariants is a *complete invariant* that is sufficient to distinguish non-isomorphic graphs. There is no known complete invariant that can be determined in polynomial-time; the existence of such an invariant would imply a polynomial-time test for graph isomorphism.

Topological graph theory, which examines how graphs can be imbedded in surfaces of different genus, may provide the insights needed to obtain a complete invariant that can be computed quickly. This thesis presents results on two topological invariants: the *maximum genus* and the *genus distribution* .

A graph has cellular imbeddings in surfaces of each genus between some minimum and some maximum. Progress has been made on understanding both minimum- and maximum-genus imbeddings. There is a polynomial-time algorithm to find imbeddings in surfaces of fixed genus [**?**], and maximum genus has been related to other graph invariants [**?**]. Chapter 2 presents a new polynomial-time algorithm that computes the maximum genus of a graph and finds a maximum-genus imbedding. This seems to be the first polynomial-time algorithm that computes an interesting imbedding and runs in time independent of the genus of the imbedding surface. The maximum-genus imbedding algorithm will also appear in [**?**].

Chapter 3 presents results on the genus distribution of a graph, an invariant that describes how many cellular imbeddings there are in surfaces of different genus. We completely calculate this distribution for two classes of graphs, *circular* and *Möbius ladders* and consider it as a tool for determining graph isomorphism. Genus distribution is not a complete invariant: there are an infinite number of pairs of non-isomorphic graphs with the same genus distribution. Experimental evidence also suggests that genus distribution is not a sufficiently strong invariant to be useful in testing the isomorphism of similar *strongly-regular* graphs.

1.1 Preliminaries about Graph Imbeddings

In topological graph theory, a *graph* is defined to be a (possibly) non-simplicial 1-complex. In other words, multiple adjacencies and self-loops are permitted. In this work, we consider only simplicial (simple) graphs. Any graph containing self-loops and multiple adjacencies can be transformed into a simplicial graph by inserting one or more vertices in the interior of these edges. Moreover, the resulting graph is homeomorphic to the original graph. This enables us to simplify the notation. We use the standard definitions relating to graphs (see, for example, Harary [**?**]). All the graphs we discuss will be connected and undirected.

1.1.1 Surfaces

Our terminology is compatible with that of Gross and Tucker [**?**] and of White [**?**].

The *topological spaces* of interest here are all homeomorphic to subspaces of E^3. A *homeomorphism* between two topological spaces is a continuous bijective mapping with a continuous inverse. A connected two-dimensional subspace is a *surface* if every point has a neighborhood that is homeomorphic to the closed unit disk. A surface S is *orientable* if it does not contain a Möbius band.

We deal only with compact orientable surfaces. Every such surface S is homeomorphic to a generalized torus. The number of handles is denoted $\gamma(S)$ and is called the *genus* of the surface. A sphere, for example, is a surface of genus 0, a torus is a surface of genus 1, and a 2-handled torus is a surface of genus 2.

1.1.2 Graph imbeddings and faces

Although a graph is an abstract combinatorial object, there is a topological representation of it: in Euclidean 3-space, we represent each vertex by a distinct point and each edge by a distinct curve between the two endpoints, where a *curve* means a homeomorphic image of the unit interval [0,1]. We require that the interior of an edge intersect no other edge or vertex of the graph. When referring to a graph in a topological setting, we mean such a representation.

An *imbedding* $G \to S$ of a graph G in the surface S is a continuous one-to-one mapping. The components of $S - G$ are called *regions*. If each region is homeomorphic to an open disk, the imbedding is *cellular*, and the regions are called *faces*. All our imbeddings are cellular. The set of faces of an imbedding is denoted F. The Euler polyhedral equation

$$|V| - |E| + |F| = 2 - 2\gamma(S) \tag{1.1}$$

holds for all cellular imbeddings.