Supplement: Dynamical bunching and density peaks in expanding Coulomb clouds

X. Xiang,* B. Zerbe,† Others TBA, and P. M. Duxbury‡
Department of Physics and Astronomy, Michigan State University
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ONE DIMENSIONAL DENSITY SLOPE DERIVATION

DERIVATION OF TIME-LOCATION RELATIONS

Starting with the relativistic expression for change in particle energy derived in the main text

$$E_{2\mathrm{D}}(t) - E(0) = \frac{qQ_{tot}\lambda_0}{2\pi\epsilon_0} ln\left(\frac{r}{r_0}\right) \tag{1}$$

$$E_{3\mathrm{D}}(t) - E(0) = \frac{qQ_{tot}Q_0}{4\pi\epsilon_0}\left(\frac{1}{r_0} - \frac{1}{r}\right) \tag{2}$$

we approximate the energy change with a change in non-relativistic kinetic energy

$$\frac{1}{2}mv_{2\mathrm{D}}^2 - \frac{1}{2}mv_0^2 = \frac{qQ_{tot}\lambda_0}{2\pi\epsilon_0} ln\left(\frac{r}{r_0}\right) \tag{3}$$

$$\frac{1}{2}mv_{3\mathrm{D}}^2 - \frac{1}{2}mv_0^2 = \frac{qQ_{tot}Q_0}{4\pi\epsilon_0}\left(\frac{1}{r_0} - \frac{1}{r}\right) \tag{4}$$

where v_0 is the initial velocity of the particle and $v_{[2,3]\mathrm{D}} = \frac{dr}{dt}$ are the velocity of the particle at time t in the two or one of the three dimensional models, respectively, with the appropriate definition of r. Solving these equations for the velocity at time t, we get

$$\frac{dr}{dt} = \sqrt{\frac{qQ_{tot}\lambda_0}{\pi m\epsilon_0} ln\left(\frac{r}{r_0}\right) + v_0^2} \tag{5}$$

$$\frac{dr}{dt} = \sqrt{\frac{qQ_{tot}Q_0}{2\pi m\epsilon_0}\left(\frac{1}{r_0} - \frac{1}{r}\right) + v_0^2} \tag{6}$$

where Eq. (5) and Eq. (6) represent the 2D and 3D formulations, respectively. Separating the variables and integrating, we obtian

$$t_{2\mathrm{D}} = \int_{r_0}^{r} \frac{d\tilde{r}}{\sqrt{\frac{qQ_{tot}\lambda_0}{\pi m\epsilon_0} ln\left(\frac{\tilde{r}}{r_0}\right) + v_0^2}} \tag{7}$$

$$t_{3\mathrm{D}} = \int_{r_0}^{r} \frac{d\tilde{r}}{\sqrt{\frac{qQ_{tot}Q_0}{2\pi m\epsilon_0}\left(\frac{1}{r_0} - \frac{1}{\tilde{r}}\right) + v_0^2}} \tag{8}$$

Defining $a = \frac{qQ_{tot}}{\pi m\epsilon_0}$ and $b = \frac{aQ_0}{2r_0} + v_0^2$, we rewrite Eq. (7) and Eq. (8) as

$$t_{2\mathrm{D}} = \int_{r_0}^{r} \frac{d\tilde{r}}{\sqrt{a\lambda_0 ln\left(\frac{\tilde{r}}{r_0}\right) + v_0^2}} \tag{9}$$

$$t_{3\mathrm{D}} = \int_{r_0}^{r} \frac{d\tilde{r}}{\sqrt{b - \frac{aQ_0}{2\tilde{r}}}} \tag{10}$$

2D integral solution

We solve the two dimensional integral first. Define $\tilde{u} = \sqrt{a\lambda_0 ln\left(\frac{\tilde{r}}{r_0}\right) + v_0^2}$. Solving this equation for $\tilde{r}$ in terms of $\tilde{u}$, we see that $\tilde{r} = r_0 e^{-\frac{v_0^2}{a\lambda_0}} e^{\frac{\tilde{u}^2}{a\lambda_0}}$. It is also straightforward to see that

$$\begin{aligned} d\tilde{u} &= \frac{1}{2}\frac{1}{\sqrt{a\lambda_0 ln\left(\frac{\tilde{r}}{r_0}\right) + v_0^2}}\frac{a\lambda_0}{\tilde{r}} d\tilde{r} \\ &= \frac{1}{\sqrt{a\lambda_0 ln\left(\frac{\tilde{r}}{r_0}\right) + v_0^2}}\frac{a\lambda_0}{2r_0} e^{\frac{v_0^2}{a\lambda_0}} e^{\frac{-\tilde{u}^2}{a\lambda_0}} d\tilde{r} \end{aligned}$$

Applying this change of coordinates to Eq. (9), we get

$$\begin{aligned} t_{2\mathrm{D}} &= \int_{u_0}^{u} \frac{2r_0}{a\lambda_0} e^{-\frac{v_0^2}{a\lambda_0}} e^{\frac{\tilde{u}^2}{a\lambda_0}} d\tilde{u} \\ &= \frac{2r_0}{a\lambda_0} e^{-\frac{v_0^2}{a\lambda_0}} \int_{u_0}^{u} e^{\frac{\tilde{u}^2}{a\lambda_0}} d\tilde{u} \\ &= \frac{2r_0}{\sqrt{a\lambda_0}} e^{-\frac{v_0^2}{a\lambda_0}} \int_{w_0}^{w} e^{\tilde{w}^2} d\tilde{w} \end{aligned} \tag{11}$$

where $u_0 = v_0$, $u = \sqrt{a\lambda_0 ln\left(\frac{r}{r_0}\right) + v_0^2}$, $\tilde{w} = \frac{\tilde{u}}{\sqrt{a\lambda_0}}$, $w = \sqrt{ln\left(\frac{r}{r_0}\right) + \frac{v_0^2}{a\lambda_0}}$, and $w_0 = \frac{v_0}{\sqrt{a\lambda_0}}$. The remaining integral, $\int_{w_0}^{w} e^{\tilde{w}^2} d\tilde{w}$ can be written in terms of the wellstudied Dawson function, $F(\cdot)$:

$$\begin{aligned}\int_{w_0}^{w} e^{\tilde{w}^2} d\tilde{w} &= \int_{0}^{w} e^{\tilde{w}^2} d\tilde{w} - \int_{0}^{w_0} e^{\tilde{w}^2} d\tilde{w} \\ &= e^{w^2} e^{-w^2} \int_{0}^{w} e^{\tilde{w}^2} d\tilde{w} - e^{w_0^2} e^{-w_0^2} \int_{0}^{w_0} e^{\tilde{w}^2} d\tilde{w} \\ &= e^{w^2} F(w) - e^{w_0^2} F(w_0) \\ &= \frac{r}{r_0} e^{\frac{v_0^2}{a\lambda_0}} F\left(\sqrt{ln\left(\frac{r}{r_0}\right) + \frac{v_0^2}{a\lambda_0}}\right) \\ &\qquad - e^{\frac{v_0^2}{a\lambda_0}} F\left(\frac{v_0}{\sqrt{a\lambda_0}}\right) \end{aligned} \tag{12}$$

Subbing Eq. (12) back into Eq. (11) gives us our time-position relation

$$t_{2\mathrm{D}} = \frac{2r}{\sqrt{a\lambda_0}} F\left(\sqrt{ln\left(\frac{r}{r_0}\right) + \frac{v_0^2}{a\lambda_0}}\right) - \frac{2r_0}{\sqrt{a\lambda_0}} F\left(\frac{v_0}{\sqrt{a\lambda_0}}\right) \tag{13}$$

3D integral solution

We now solve the three dimensional integral with an analogous approach. Define $\tilde{u} = \sqrt{b - \frac{aQ_0}{2\tilde{r}}}$ and solving for $\tilde{r}$ gives $\tilde{r} = \frac{aQ_0}{2(b-\tilde{u}^2)}$. Thus

$$\begin{aligned} d\tilde{u} &= \frac{1}{2} \frac{1}{\sqrt{b - \frac{aQ_0}{2\tilde{r}}}} \frac{aQ_0}{2\tilde{r}^2} d\tilde{r} \\ &= \frac{1}{\sqrt{b - \frac{aQ_0}{2\tilde{r}}}} \frac{(b-\tilde{u}^2)^2}{aQ_0} d\tilde{r} \end{aligned}$$

Applying this change of coordinates to Eq. (10) with $u_0 = v_0$ and $u = \sqrt{b - \frac{aQ_o}{2r}}$, we get

$$\begin{aligned} t_{3\mathrm{D}} &= \int_{u_0}^{u} \frac{aQ_0}{(b-\tilde{u}^2)^2} d\tilde{u} \\ &= aQ_0 \int_{u_0}^{u} \frac{1}{(b-\tilde{u}^2)^2} d\tilde{u} \\ &= aQ_0 \left(\frac{1}{2b^{3/2}} \tanh^{-1}\left(\frac{\tilde{u}}{\sqrt{b}}\right) + \frac{1}{2b} \frac{\tilde{u}}{b-\tilde{u}^2} \right) \Bigg|_{\tilde{u}=v_0}^{\tilde{u}=\sqrt{b-\frac{aQ_o}{2r}}} \\ &= aQ_0 \left(\frac{1}{2b^{3/2}} A + \frac{1}{2b} B \right) \end{aligned} \tag{14}$$

where the solution to the integral was obtained with Mathematica's online tool[Reference], where

$$\begin{aligned} A &= \tanh^{-1}\left(\frac{\tilde{u}}{\sqrt{b}}\right) \Bigg|_{\tilde{u}=v_0}^{\tilde{u}=\sqrt{b-\frac{aQ_o}{2r}}} \\ &= \tanh^{-1}\left(\sqrt{1 - \frac{aQ_o}{2br}}\right) - \tanh^{-1}\left(\frac{v_0}{\sqrt{b}}\right) \end{aligned} \tag{15}$$

and where

$$\begin{aligned} B &= \frac{\tilde{u}}{b-\tilde{u}^2} \Bigg|_{\tilde{u}=v_0}^{\tilde{u}=\sqrt{b-\frac{aQ_o}{2r}}} \\ &= \frac{\sqrt{b - \frac{aQ_o}{2r}}}{b - b + \frac{aQ_o}{2r}} - \frac{v_0}{b - v_0^2} \\ &= \frac{2r}{aQ_o} \sqrt{b - \frac{aQ_o}{2r}} - \frac{v_0}{\frac{aQ_o}{2r_0}} \\ &= \frac{2b}{aQ_0} \left(\frac{r}{\sqrt{b}} \sqrt{1 - \frac{aQ_o}{2br}} - \frac{r_0}{\sqrt{b}} \frac{v_0}{\sqrt{b}} \right) \end{aligned} \tag{16}$$

Substituting Eq. (15) and Eq. (16) back into Eq. (14) gives

$$\begin{aligned} t_{3\mathrm{D}} &= \frac{aQ_0}{2b^{3/2}} \left(\tanh^{-1}\left(\sqrt{1 - \frac{aQ_o}{2br}}\right) - \tanh^{-1}\left(\frac{v_0}{\sqrt{b}}\right) \right. \\ &\qquad \left. + \frac{r}{\sqrt{b}} \sqrt{1 - \frac{aQ_o}{2br}} - \frac{r_0}{\sqrt{b}} \frac{v_0}{\sqrt{b}} \right) \\ &= \frac{\alpha}{\sqrt{b}} \left(\tanh^{-1}\left(\sqrt{1 - \frac{1}{\tilde{r}}}\right) + \tilde{r}\sqrt{1 - \frac{1}{\tilde{r}}} \right. \\ &\qquad \left. - \tanh^{-1}\left(\sqrt{1 - \frac{1}{\tilde{r}_0}}\right) - \tilde{r}_0 \sqrt{1 - \frac{1}{\tilde{r}_0}} \right) \end{aligned} \tag{17}$$

where $\alpha = \frac{aQ_0}{2b} = \left(\frac{1}{r_0} + \frac{2}{aQ_0} v_0^2\right)^{-1} = \left(\frac{1}{r_0} + \frac{2\pi m \epsilon_0}{qQ_0} v_0^2\right)^{-1}$, $\tilde{r} = \frac{r}{\alpha}$, and $\tilde{r}_0 = \frac{r_0}{\alpha}$. Also, we used $\frac{v_0^2}{b} = 1 - \frac{aQ_0}{2br_0} = 1 - \frac{\alpha}{r_0} = 1 - \frac{1}{\tilde{r}_0}$.

DERIVATION OF SPATIAL DERIVATIVES WITH RESPECT TO INITIAL POSITION

As noted in the main text, much of the physics of distribution evolution in our models is captured in the term $\frac{dr}{dr_0}$. The general procedure to derive th expressions for this derivative is to take the derivative of Eq. (13) and Eq. (17). We do this mathematics here.

The derivative in two dimensions

We introduce the function $\tau_{2\mathrm{D}}(x) = \frac{2x}{\sqrt{a\lambda_0}} F\left(\sqrt{ln\left(\frac{x}{r_0}\right) + \frac{v_0^2}{a\lambda_0}}\right)$ so that $t_{2\mathrm{D}}$ may be written as $\tau_{2\mathrm{D}}(r) - \tau_{2\mathrm{D}}(r_0)$. We will begin by taking derivatives of pieces of $\tau_{2\mathrm{D}}(x)$ with respect to r_0. Note that

$$\begin{aligned} \frac{d}{dr_0} \frac{1}{\sqrt{\lambda_0}} &= -\frac{1}{2} \frac{1}{(\lambda_0)^{3/2}} \frac{d\lambda_0}{dr_0} \\ &= -\frac{1}{2\sqrt{\lambda_0}} \frac{d \ln(\lambda_0)}{dr_0} \end{aligned} \tag{18}$$

Likewise

$$\frac{d}{dr_0}\frac{1}{\lambda_0} = -\frac{1}{\lambda_0^2}\frac{d\lambda_0}{dr_0} \\ = -\frac{1}{\lambda_0}\frac{d\ln(\lambda_0)}{dr_0} \tag{19}$$

Denote $y = \sqrt{\ln\left(\frac{x}{r_0}\right) + \frac{v_0^2}{a\lambda_0}} = \sqrt{\ln(x) - \ln(r_0) + \frac{v_0^2}{a\lambda_0}}$, thus

$$\frac{dy}{dr_0} = \frac{1}{2y}\left(\frac{1}{x}\frac{dx}{dr_0} - \frac{1}{r_0} - \frac{v_0^2}{a\lambda_0}\frac{d\ln(\lambda_0)}{dr_0} + \frac{2v_0}{a\lambda_0}\frac{dv_0}{dr_0}\right) \tag{20}$$

The Dawson function has the property $\frac{d}{dy}F(y) = 1 - 2yF(y) = \left(\frac{1}{F(y)} - 2y\right)F(y)$, and with the chain rule this becomes $\frac{d}{dr_0}F(y) = \left(\frac{1}{F(y)} - 2y\right)\frac{dy}{dr_0}F(y)$. Using Eq. (20), this becomes

$$\frac{d}{dr_0}F(y) = \left(\frac{1}{F(y)} - 2y\right)\frac{F(y)}{2y}\times \\ \left(\frac{1}{x}\frac{dx}{dr_0} - \frac{1}{r_0} - \frac{v_0^2}{a\lambda_0}\frac{d\ln(\lambda_0)}{dr_0} + \frac{2v_0}{a\lambda_0}\frac{dv_0}{dr_0}\right) \\ = F(y)\left(\frac{1}{2yF(y)} - 1\right)\times \\ \left(\frac{1}{x}\frac{dx}{dr_0} - \frac{1}{r_0} - \frac{v_0^2}{a\lambda_0}\frac{d\ln(\lambda_0)}{dr_0} + \frac{2v_0}{a\lambda_0}\frac{dv_0}{dr_0}\right) \tag{21}$$

Putting this all together, we get

$$\frac{d\tau_{2D}}{dr_0} = \frac{\tau_{2D}}{x}\frac{dx}{dr_0} - \frac{\tau_{2D}}{2}\frac{d\ln(\lambda_0)}{dr_0} + \tau_{2D}\left(\frac{1}{2yF(y)} - 1\right)\times \\ \left(\frac{1}{x}\frac{dx}{dr_0} - \frac{1}{r_0} - \frac{v_0^2}{a\lambda_0}\frac{d\ln(\lambda_0)}{dr_0} + \frac{2v_0}{a\lambda_0}\frac{dv_0}{dr_0}\right) \\ = \frac{\tau_{2D}}{2xyF(y)}\frac{dx}{dr_0} + \frac{\tau_{2D}}{r_0}\left(1 - \frac{1}{2yF(y)}\right) \\ - \tau_{2D}\frac{d\ln(\lambda_0)}{dr_0}\left(\frac{1}{2} + \frac{v_0^2}{2a\lambda_0 yF(y)} - \frac{v_0^2}{a\lambda_0}\right) \\ + \frac{2v_0\tau_{2D}}{a\lambda_0}\frac{dv_0}{dr_0}\left(\frac{1}{2yF(y)} - 1\right) \\ = \frac{1}{\sqrt{a\lambda_0}}\left(\frac{1}{y}\frac{dx}{dr_0} + \frac{x}{r_0}\left(2F(y) - \frac{1}{y}\right)\right. \\ - x\frac{d\ln(\lambda_0)}{dr_0}\left(\left(1 - \frac{2v_0^2}{a\lambda_0}\right)F(y) + \frac{v_0^2}{a\lambda_0}\frac{1}{y}\right) \\ \left. - \frac{2xv_0}{a\lambda_0}\frac{dv_0}{dr_0}\left(2F(y) - \frac{1}{y}\right)\right) \\ = \frac{1}{\sqrt{a\lambda_0}}\left(\frac{1}{y}\frac{dx}{dr_0}\right. \\ + \left(\frac{x}{r_0} - \frac{2xv_0}{a\lambda_0}\frac{dv_0}{dr_0}\right)\left(2F(y) - \frac{1}{y}\right) \\ \left. - x\frac{d\ln(\lambda_0)}{dr_0}\left(\left(1 - \frac{2v_0^2}{a\lambda_0}\right)F(y) + \frac{v_0^2}{a\lambda_0}\frac{1}{y}\right)\right) \tag{22}$$

where τ_{2D} is shorthand for $\tau_{2D}(x) = \frac{2x}{\sqrt{a\lambda_0}}F(y(x))$ and it is understood that $y = y(x) = \sqrt{\ln\left(\frac{x}{r_0}\right) + \frac{v_0^2}{a\lambda_0}}$. For $x = r$, this becomes

$$\frac{d\tau_{2D}(r)}{dr_0} = \frac{1}{\sqrt{a\lambda_0}}\left(\frac{1}{y(r)}\frac{dr}{dr_0}\right. \\ + \frac{r}{r_0}\left(1 - \frac{2r_0v_0}{a\lambda_0}\frac{dv_0}{dr_0}\right)\left(2F(y(r)) - \frac{1}{y(r)}\right) \\ \left. - r\frac{d\ln(\lambda_0)}{dr_0}\left(\left(1 - \frac{2v_0^2}{a\lambda_0}\right)F(y(r)) + \frac{v_0^2}{a\lambda_0}\frac{1}{y(r)}\right)\right) \tag{23}$$

For $x = r_0$ and subbing $y(r_0) = \frac{v_0}{\sqrt{a\lambda_0}}$, we get

$$\frac{d\tau_{2D}(r_0)}{dr_0} = \frac{1}{v_0} + \left(1 - \frac{2r_0v_0}{a\lambda_0}\frac{dv_0}{dr_0}\right)\left(\frac{2}{\sqrt{a\lambda_0}}F\left(\frac{v_0}{\sqrt{a\lambda_0}}\right) - \frac{1}{v_0}\right) \\ - \frac{r_0}{\sqrt{a\lambda_0}}\frac{d\ln(\lambda_0)}{dr_0}\left(\left(1 - \frac{2v_0^2}{a\lambda_0}\right)F\left(\frac{v_0}{\sqrt{a\lambda_0}}\right) + \frac{v_0}{\sqrt{a\lambda_0}}\right) \\ = \left(1 - \frac{2r_0v_0}{a\lambda_0}\frac{dv_0}{dr_0}\right)\frac{2}{\sqrt{a\lambda_0}}F\left(\frac{v_0}{\sqrt{a\lambda_0}}\right) \\ + \frac{2r_0}{a\lambda_0}\frac{dv_0}{dr_0} \\ - \frac{r_0}{\sqrt{a\lambda_0}}\frac{d\ln(\lambda_0)}{dr_0}\left(\left(1 - \frac{2v_0^2}{a\lambda_0}\right)F\left(\frac{v_0}{\sqrt{a\lambda_0}}\right) + \frac{v_0}{\sqrt{a\lambda_0}}\right) \tag{24}$$

As in the one dimensional case, we are interested in the change in r with respect to r_0 at a specific time, so time is held constant in these derivatives. As a results

$$\begin{aligned} 0 &= \frac{dt_{2\mathrm{D}}}{dr_0} \\ &= \frac{d\tau_{2\mathrm{D}}(r)}{dr_0} - \frac{d\tau_{2\mathrm{D}}(r_0)}{dr_0} \end{aligned} \tag{25}$$

Since the first term on the right hand side of this equation is linear in $\frac{dr}{dr_0}$, we may solve for it

$$\begin{aligned} \frac{dr}{dr_0} = \frac{r}{r_0}\left(1 - \frac{2r_0 v_0}{a\lambda_0}\frac{dv_0}{dr_0}\right) \times \\ \left(1 - 2\left(F(y(r)) + \frac{r_0}{r}F\left(\frac{v_0}{\sqrt{a\lambda_0}}\right)\right)y(r)\right) \\ + r\frac{d\ln(\lambda_0)}{dr_0}\left(\frac{v_0^2}{a\lambda_0} + \frac{r_0}{r}\frac{v_0}{\sqrt{a\lambda_0}}y(r)\right. \\ \left. + \left(1 - \frac{2v_0^2}{a\lambda_0}\right)\left(F(y(r)) + \frac{r_0}{r}F\left(\frac{v_0}{\sqrt{a\lambda_0}}\right)\right)y(r)\right) \\ - \frac{2r_0}{\sqrt{a\lambda_0}}y(r)\frac{dv_0}{dr_0} \end{aligned} \tag{26}$$

In the special case of no initial velocity ($v_0 = 0$ everywhere) where $F(0) = 0$ and $y(r)$ simplifies to $\sqrt{\ln\left(\frac{r}{r_0}\right)}$, this expression simplifies greatly:

$$\begin{aligned} \frac{dr}{dr_0} &= \frac{r}{r_0}\left(1 - 2F\left(\sqrt{\ln\left(\frac{r}{r_0}\right)}\right)\right)\sqrt{\ln\left(\frac{r}{r_0}\right)} \\ &\quad + r\frac{d\ln(\lambda_0)}{dr_0}F\left(\sqrt{\ln\left(\frac{r}{r_0}\right)}\right)\sqrt{\ln\left(\frac{r}{r_0}\right)} \\ &= \frac{r}{r_0}\left(1 + \sqrt{\ln\left(\frac{r}{r_0}\right)}\left(r_0\frac{d\ln(\lambda_0)}{dr_0} - 2\right) \times \right. \\ &\quad \left. F\left(\sqrt{\ln\left(\frac{r}{r_0}\right)}\right)\right) \\ &= \frac{r}{r_0}\left(1 + D_{2\mathrm{D}}(r_0)2\sqrt{\ln\left(\frac{r}{r_0}\right)}F\left(\sqrt{\ln\left(\frac{r}{r_0}\right)}\right)\right) \end{aligned} \tag{27}$$

where $D_{2\mathrm{D}}(r_0) = \frac{r_0}{2}\frac{d\ln(\lambda_0)}{dr_0} - 1$. Using the fundamental theorem of calculus,

$$\begin{aligned} \frac{r_0}{2}\frac{d\ln(\lambda_0)}{dr_0} &= \frac{r_0}{2\lambda_0}\frac{d\lambda_0}{dr_0} \\ &= \frac{r_0}{2\lambda_0(r_0)}2\pi r_0\rho_0(r_0) \\ &= \frac{\rho_0(r_0)}{\frac{\lambda_0(r_0)}{\pi r_0^2}} \end{aligned} \tag{28}$$

where $\frac{\lambda_0(r_0)}{\pi r_0^2}$ can be interpreted as the average density in contained within the Gaussian surface determined by r_0. We call this $\tilde{\rho}_0(r_0)$ in the main text so that $D_{2\mathrm{D}}(r_0) = \frac{\rho_0(r_0)}{\tilde{\rho}_0(r_0)} - 1$.

The derivatives in three dimensions

Analogous to the two dimensional case, we introduce the function $\tau_{3\mathrm{D}}(x) = \frac{\alpha}{\sqrt{b}}\left(\tanh^{-1}\left(\sqrt{1-\frac{1}{x}}\right) + x\sqrt{1-\frac{1}{x}}\right)$ so that $t_{2\mathrm{D}}$ may be written as $\tau_{2\mathrm{D}}(\frac{r}{\alpha}) - \tau_{2\mathrm{D}}(\frac{r_0}{\alpha})$. We will again begin by taking derivatives of pieces of this equation:

$$\begin{aligned} \frac{db}{dr_0} &= \frac{d\left(\frac{qQ_0}{2\pi m\epsilon_0 r_0} + v_0^2\right)}{dr_0} \\ &= -\frac{qQ_0}{2\pi m\epsilon_0 r_0^2} + \frac{\alpha b}{Q_0 r_0}\frac{dQ_0}{dr_0} + 2v_0\frac{dv_0}{dr_0} \\ &= -2b\left(\frac{\alpha}{2r_0^2} - \frac{\alpha}{2r_0}\frac{d\ln(Q_0)}{dr_0} - \frac{v_0}{b}\frac{dv_0}{dr_0}\right) \end{aligned} \tag{29}$$

Here, Q_0 is $\int_0^{r_0} 4\pi\tilde{r}^2\rho_0(\tilde{r})d\tilde{r}$. Therefore $\frac{dQ_0}{dr_0}$ is $4\pi r_0^2\rho_0(r_0)$, respectively. For now, we will keep $\frac{dQ_0}{dr_0}$. Also

$$\begin{aligned} \frac{d\frac{1}{\alpha}}{dr_0} &= \frac{d\left(\frac{1}{r_0} + \frac{2\pi m\epsilon_0}{qQ_0}v_0^2\right)}{dr_0} \\ &= -\frac{1}{r_0^2} - \frac{2\pi m\epsilon_0}{qQ_0^2}v_0^2\frac{dQ_0}{dr_0} + 2\frac{2\pi m\epsilon_0}{qQ_0}v_0\frac{dv_0}{dr_0} \\ &= -\frac{1}{r_0^2} - \frac{2v_0^2}{aQ_0^2}\frac{dQ_0}{dr_0} + \frac{4v_0}{aQ_0}\frac{dv_0}{dr_0} \\ &= -\frac{1}{r_0^2} - \frac{v_0^2}{\alpha b}\frac{d\ln(Q_0)}{dr_0} + \frac{2v_0}{\alpha b}\frac{dv_0}{dr_0} \end{aligned} \tag{30}$$

where again $a = \frac{q}{\pi m\epsilon_0}$. Next

$$\begin{aligned} \frac{d\alpha}{dr_0} &= \frac{d\left(\frac{1}{r_0} + \frac{2\pi m\epsilon_0}{qQ_0}v_0^2\right)^{-1}}{dr_0} \\ &= -\alpha^2\frac{d\frac{1}{\alpha}}{dr_0} \\ &= \alpha\left(\frac{\alpha}{r_0^2} + \frac{v_0^2}{b}\frac{d\ln(Q_0)}{dr_0} - \frac{2v_0}{b}\frac{dv_0}{dr_0}\right) \end{aligned} \tag{31}$$

Looking at the x-dependent terms

$$\begin{aligned} \frac{d\sqrt{1-\frac{1}{x}}}{dr_0} &= \frac{1}{2\sqrt{1-\frac{1}{x}}}\frac{1}{x^2}\frac{dx}{dr_0} \\ &= \frac{1}{2x^2\sqrt{1-\frac{1}{x}}}\frac{dx}{dr_0} \end{aligned} \tag{32}$$

and

$$\frac{d\tanh^{-1}\left(\sqrt{1-\frac{1}{x}}\right)}{dr_0} = \frac{1}{1-\left(1-\frac{1}{x}\right)}\frac{d\sqrt{1-\frac{1}{x}}}{dr_0} = \frac{1}{2x\sqrt{1-\frac{1}{x}}}\frac{dx}{dr_0} \tag{33}$$

and

$$\frac{d\left(x\sqrt{1-\frac{1}{x}}\right)}{dr_0} = \sqrt{1-\frac{1}{x}}\frac{dx}{dr_0} + x\frac{1}{2x^2\sqrt{1-\frac{1}{x}}}\frac{dx}{dr_0} = \frac{2x-1}{2x\sqrt{1-\frac{1}{x}}}\frac{dx}{dr_0} \tag{34}$$

Putting this all together

$$\begin{aligned}\frac{d\tau_{3\mathrm{D}}}{dr_0} &= \frac{d\left(\frac{\alpha}{\sqrt{b}}\left(\tanh^{-1}\left(\sqrt{1-\frac{1}{x}}\right)+x\sqrt{1-\frac{1}{x}}\right)\right)}{dr_0}\\ &= \frac{\tau_{3\mathrm{D}}}{\alpha}\frac{d\alpha}{dr_0} - \frac{\tau_{3\mathrm{D}}}{2b}\frac{db}{dr_0}\\ &\quad + \frac{\alpha}{\sqrt{b}}\left(\frac{1}{2x\sqrt{1-\frac{1}{x}}}\frac{dx}{dr_0} + \frac{2x-1}{2x\sqrt{1-\frac{1}{x}}}\frac{dx}{dr_0}\right)\\ &= \tau_{3\mathrm{D}}\left(\frac{\alpha}{r_0^2} + \frac{v_0^2}{b}\frac{d\ln(Q_0)}{dr_0} - \frac{2v_0}{b}\frac{dv_0}{dr_0} + \frac{\alpha}{2r_0^2}\right.\\ &\quad\left. - \frac{\alpha}{2r_0}\frac{d\ln(Q_0)}{dr_0} - \frac{v_0}{b}\frac{dv_0}{dr_0}\right)\\ &\quad + \frac{\alpha}{\sqrt{b}}\frac{1}{\sqrt{1-\frac{1}{x}}}\frac{dx}{dr_0}\\ &= \tau_{3\mathrm{D}}\left(\frac{3\alpha}{2r_0^2} + \left(\frac{v_0^2}{b} - \frac{\alpha}{2r_0}\right)\frac{d\ln(Q_0)}{dr_0} - \frac{3v_0}{b}\frac{dv_0}{dr_0}\right)\\ &\quad + \frac{\alpha}{\sqrt{b}}\frac{1}{\sqrt{1-\frac{1}{x}}}\frac{dx}{dr_0}\end{aligned} \tag{35}$$

Again where it is understood that $\tau_{3\mathrm{D}}$ represents $\tau_{3\mathrm{D}}(x)$. In the case $x = \frac{r}{\alpha}$,

$$\begin{aligned}\frac{dx}{dr_0} &= \frac{1}{\alpha}\frac{dr}{dr_0} + r\frac{d\frac{1}{\alpha}}{dr_0}\\ &= \frac{1}{\alpha}\frac{dr}{dr_0} - \frac{r}{r_0^2} - \frac{v_0^2}{b}\frac{r}{\alpha}\frac{d\ln(Q_0)}{dr_0}\\ &\quad + \frac{2v_0}{b}\frac{r}{\alpha}\frac{dv_0}{dr_0}\end{aligned} \tag{36}$$

and in the case $x = \frac{r_0}{\alpha}$,

$$\begin{aligned}\frac{dx}{dr_0} &= \frac{1}{\alpha} + r_0\frac{d\frac{1}{\alpha}}{dr_0}\\ &= \frac{1}{\alpha} - \frac{1}{r_0} - \frac{v_0^2}{b}\frac{r_0}{\alpha}\frac{d\ln(Q_0)}{dr_0}\\ &\quad + \frac{2v_0}{b}\frac{r_0}{\alpha}\frac{dv_0}{dr_0}\end{aligned} \tag{37}$$

Again we use the observation that the spatial derivative of the time is 0. Therefore

$$\begin{aligned}0 &= \frac{dt_{3\mathrm{D}}}{dr_0}\\ &= \frac{d\tau_{2\mathrm{D}}(r)}{dr_0} - \frac{d\tau_{2\mathrm{D}}(r_0)}{dr_0}\\ &= t_{3\mathrm{D}}\left(\frac{3\alpha}{2r_0^2} + \left(\frac{v_0^2}{b} - \frac{\alpha}{2r_0}\right)\frac{d\ln(Q_0)}{dr_0} - \frac{3v_0}{b}\frac{dv_0}{dr_0}\right)\\ &\quad + \frac{\alpha}{\sqrt{b}}\frac{1}{\sqrt{1-\frac{\alpha}{r}}}\left(\frac{1}{\alpha}\frac{dr}{dr_0} - \frac{r}{r_0^2}\right.\\ &\quad\left. - \frac{v_0^2}{b}\frac{r}{\alpha}\frac{d\ln(Q_0)}{dr_0} + \frac{2v_0}{b}\frac{r}{\alpha}\frac{dv_0}{dr_0}\right)\\ &\quad - \frac{\alpha}{\sqrt{b}}\frac{1}{\sqrt{1-\frac{\alpha}{r_0}}}\left(\frac{1}{\alpha} - \frac{1}{r_0}\right.\\ &\quad\left. - \frac{v_0^2}{b}\frac{r_0}{\alpha}\frac{d\ln(Q_0)}{dr_0} + \frac{2v_0}{b}\frac{r_0}{\alpha}\frac{dv_0}{dr_0}\right)\\ &= t_{3\mathrm{D}}\left(\frac{3\alpha}{2r_0^2} + \left(\frac{v_0^2}{b} - \frac{\alpha}{2r_0}\right)\frac{d\ln(Q_0)}{dr_0} - \frac{3v_0}{b}\frac{dv_0}{dr_0}\right)\\ &\quad + \frac{1}{\sqrt{b}}\frac{1}{\sqrt{1-\frac{\alpha}{r}}}\left(\frac{dr}{dr_0} - \frac{\alpha r}{r_0^2}\right.\\ &\quad\left. - \frac{v_0^2}{b}r\frac{d\ln(Q_0)}{dr_0} + \frac{2v_0}{b}r\frac{dv_0}{dr_0}\right)\\ &\quad - \frac{1}{\sqrt{b}}\frac{1}{\sqrt{1-\frac{\alpha}{r_0}}}\left(1 - \frac{\alpha}{r_0}\right.\\ &\quad\left. - \frac{v_0^2}{b}r_0\frac{d\ln(Q_0)}{dr_0} + \frac{2v_0}{b}r_0\frac{dv_0}{dr_0}\right)\\ &= t_{3\mathrm{D}}\left(\frac{3\alpha}{2r_0^2} + \left(\frac{v_0^2}{b} - \frac{\alpha}{2r_0}\right)\frac{d\ln(Q_0)}{dr_0}\right)\\ &\quad + \frac{1}{\sqrt{b}}\frac{1}{\sqrt{1-\frac{\alpha}{r}}}\left(\frac{dr}{dr_0} - \frac{\alpha r}{r_0^2}\right.\\ &\quad\left. - \frac{v_0}{b}r\left(v_0\frac{d\ln(Q_0)}{dr_0} - 2\frac{dv_0}{dr_0}\right)\right)\\ &\quad - \frac{\sqrt{1-\frac{\alpha}{r_0}}}{\sqrt{b}} + \frac{v_0}{b}\frac{r_0}{\sqrt{b}\sqrt{1-\frac{\alpha}{r_0}}}\left(v_0\frac{d\ln(Q_0)}{dr_0} - 2\frac{dv_0}{dr_0}\right)\end{aligned} \tag{38}$$

Solving for $\frac{dr}{dr_0}$ we get

$$\frac{dr}{dr_0} = -t_{3D}\sqrt{b}\sqrt{1-\frac{\alpha}{r}}\left(\frac{3\alpha}{2r_0^2}+\left(\frac{v_0^2}{b}-\frac{\alpha}{2r_0}\right)\frac{d\ln(Q_0)}{dr_0}\right)$$
$$+\sqrt{1-\frac{\alpha}{r_0}}\sqrt{1-\frac{\alpha}{r}}+\frac{\alpha r}{r_0^2}$$
$$-\frac{v_0}{b}\frac{\sqrt{1-\frac{\alpha}{r}}}{\sqrt{1-\frac{\alpha}{r_0}}}r_0\left(v_0\frac{d\ln(Q_0)}{dr_0}-2\frac{dv_0}{dr_0}\right)$$
$$+\frac{v_0}{b}r\left(v_0\frac{d\ln(Q_0)}{dr_0}-2\frac{dv_0}{dr_0}\right)$$
$$=\sqrt{1-\frac{\alpha}{r}}\left[\sqrt{1-\frac{\alpha}{r_0}}+\frac{\alpha^2}{r_0^2}\left(\frac{\frac{r}{\alpha}}{\sqrt{1-\frac{\alpha}{r}}}-\frac{3\sqrt{b}t_{3D}}{2\alpha}\right)\right.$$
$$+\frac{v_0^2}{b}\left(\frac{r}{\alpha}-\frac{\sqrt{1-\frac{\alpha}{r}}}{\sqrt{1-\frac{\alpha}{r_0}}}\frac{r_0}{\alpha}-\frac{\sqrt{b}t_{3D}}{\alpha}\right)\alpha\frac{d\ln(Q_0)}{dr_0}$$
$$+\frac{\sqrt{b}t_{3D}}{2\alpha}\frac{\alpha}{r_0}\alpha\frac{d\ln(Q_0)}{dr_0}$$
$$\left.-\frac{2v_0^2}{b}\left(\frac{r}{\alpha}-\frac{\sqrt{1-\frac{\alpha}{r}}}{\sqrt{1-\frac{\alpha}{r_0}}}\frac{r_0}{\alpha}\right)\frac{\alpha}{v_0}\frac{dv_0}{dr_0}\right]$$
$$=\sqrt{1-\frac{\alpha}{r}}\left[\sqrt{1-\frac{\alpha}{r_0}}+\frac{\alpha^2}{r_0^2}\left(\frac{\frac{r}{\alpha}}{\sqrt{1-\frac{\alpha}{r}}}-\frac{3\sqrt{b}t_{3D}}{2\alpha}\right)\right.$$
$$+\left(1-\frac{\alpha}{r_0}\right)\left(\frac{r}{\alpha}-\frac{\sqrt{1-\frac{\alpha}{r}}}{\sqrt{1-\frac{\alpha}{r_0}}}\frac{r_0}{\alpha}-\frac{\sqrt{b}t_{3D}}{\alpha}\right)\alpha\frac{d\ln(Q_0)}{dr_0}$$
$$+\frac{\sqrt{b}t_{3D}}{2\alpha}\frac{\alpha}{r_0}\alpha\frac{d\ln(Q_0)}{dr_0}$$
$$\left.-2\left(1-\frac{\alpha}{r_0}\right)\left(\frac{r}{\alpha}-\frac{\sqrt{1-\frac{\alpha}{r}}}{\sqrt{1-\frac{\alpha}{r_0}}}\frac{r_0}{\alpha}\right)\alpha\frac{d\ln(v_0)}{dr_0}\right] \quad (39)$$

where we again used $\frac{v_0^2}{b}=1-\frac{\alpha}{r_0}$. Note, $\frac{r}{\alpha}$, $\frac{r_0}{\alpha}$, $\frac{v_0^2}{b}$, $\frac{\sqrt{b}t_{3D}}{\alpha}$, and $\alpha\frac{d\ln(Q_0)}{dr_0}$ are all dimensionless.

Analysis of the zero initial velocity case in 3D

In the special case of $v_0 = 0$, we have the following relations, we have $\alpha = r_0$, $b = \frac{qQ_{tot}Q_0}{2\pi m\epsilon_0 r_0}$, and $t_{3D} = \frac{r_0}{\sqrt{b}}\left(\tanh^{-1}\left(\sqrt{1-\frac{r_0}{r}}\right)+\frac{r}{r_0}\sqrt{1-\frac{r_0}{r}}\right)$. Thus Eq. (39) becomes

$$\frac{dr}{dr_0} = \frac{r}{r_0}+\sqrt{1-\frac{r_0}{r}}\left(\frac{r_0}{2}\frac{d\ln(Q_0)}{dr_0}-\frac{3}{2}\right)\frac{\sqrt{b}}{r_0}t_{3D}$$
$$=\frac{r}{r_0}+D_{3D}(r_0)\left(\sqrt{1-\frac{r_0}{r}}\tanh^{-1}\left(\sqrt{1-\frac{r_0}{r}}\right)\right.$$
$$\left.+\frac{r}{r_0}\left(1-\frac{r_0}{r}\right)\right)$$
$$=\frac{r}{r_0}\left(1+D_{3D}(r_0)\left(\frac{r_0}{r}\sqrt{1-\frac{r_0}{r}}\tanh^{-1}\left(\sqrt{1-\frac{r_0}{r}}\right)\right.\right.$$
$$\left.\left.+1-\frac{r_0}{r}\right)\right) \quad (40)$$

where $D_{3D}(r_0) = \frac{3}{2}\left(\frac{r_0}{3}\frac{d\ln(Q_0)}{dr_0}-1\right)$. Using the fundamental theorem of calculus,

$$\frac{r_0}{3}\frac{d\ln(Q_0)}{dr_0} = \frac{r_0}{3Q_0}\frac{dQ_0}{dr_0}$$
$$=\frac{r_0}{3Q_0(r_0)}4\pi r_0^2\rho_0(r_0)$$
$$=\frac{\rho_0(r_0)}{\frac{Q_0(r_0)}{\frac{4}{3}\pi r_0^3}} \quad (41)$$

where $\frac{Q_0(r_0)}{\frac{4}{3}\pi r_0^3}$ can be interpreted as the average density in contained within the Gaussian surface determined by r_0. We again call this $\tilde{\rho}_0(r_0)$ in the main text so that $D_{3D}(r_0) = \frac{3}{2}\left(\frac{\rho_0(r_0)}{\tilde{\rho}_0(r_0)}-1\right)$.

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† zerbe@msu.edu

‡ duxbury@msu.edu