Linear Regression: Extensions

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Announcements

Recap: Model Evaluation

- Accuracy of Model:
  - Residual Sum of Squares: RSS $\triangleq \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$.
  - Residual Standard Error: RSE $\triangleq \sqrt{\frac{1}{n-p-1} \sum_{i=1}^{n} (y_i - \hat{y}_i)^2}$.
  - R^2 statistics: $R^2 = \frac{\text{TSS}-\text{RSS}}{\text{TSS}} = \frac{\text{explained var.}}{\text{total var.}}$
  - Hypothesis test of the whole model:

$$H_0 : \beta_1 = \cdots = \beta_p = 0 \quad \text{vs.} \quad H_1 : \text{at least one } \beta_j \text{ is non-zero}$$

    Smaller p-value $\Rightarrow$ More evidence of relationship (against H_0).
    (Standard cutoff: 0.05, 0.01)

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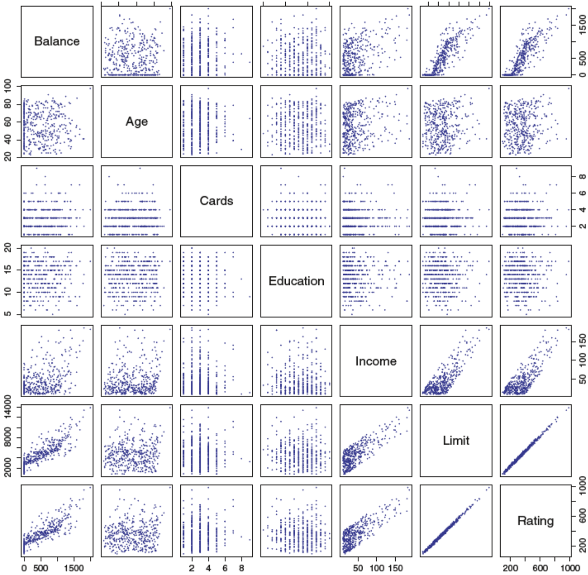
- Accuracy of Coefficient Estimates:
  - Standard Error (SE) of each estimate: SE$(\hat{\beta}_j)$
  - An estimate of $|\hat{\beta}_j - \beta_j|$
  - Can be used to construct confidence intervals.
  - Can be used to perform hypothesis tests

Today: Other considerations in linear regression

Reading: ISLR Section 3.3

Categorical Predictors



Categorical Predictors

(ISLR Section 3.3.1)

- Married: Yes, No
- Fuel Type: Diesel, Petrol, CNG

Predictors with 2 levels:

Often called dummy variables.

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$$x_{i2} = \begin{cases} 1 & \text{if } i\text{th person uses Diesel} \\ 0 & \text{otherwise} \end{cases}, \quad \text{and} \quad x_{i3} = \begin{cases} 1 & \text{if } i\text{th person uses Petrol} \\ 0 & \text{otherwise} \end{cases}$$

If $x_{i2} = x_{i3} = 0$, then fuel type is CNG.

Categorical Predictors

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Categorical Predictors

Predictors with 2 levels:

$$\text{balance}(i) = \beta_0 + \beta_1 \text{Married}(i) + \epsilon_i = \begin{cases} \beta_0 + \beta_1 + \epsilon_i & \text{if } i\text{th person is married} \\ \beta_0 + \epsilon_i & \text{if } i\text{th person is not married} \end{cases}$$

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Predictors with >2 levels

$$\begin{aligned} \text{balance}(i) &= \beta_0 + \beta_1 \text{Diesel}(i) + \beta_2 \text{Petrol}(i) + \epsilon_i \\ &= \begin{cases} \beta_0 + \beta_1 + \epsilon_i & \text{if } i\text{th person uses Diesel} \\ \beta_0 + \beta_2 + \epsilon_i & \text{if } i\text{th person uses Petrol} \\ \beta_0 + \epsilon_i & \text{if } i\text{th person uses CNG} \end{cases} \end{aligned}$$

Nonlinearity

🔥 **Kareem Carr** 🔥
@kareem_carr

God of Statistics: *creates linear regression* people can use you when stuff is linear

linear regression: what about when stuff isn't linear?

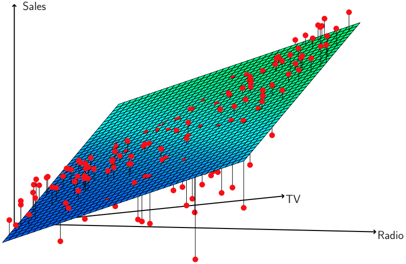
God of Statistics: shhh. we don't talk about that

8:59 PM · Jul 17, 2020 · Twitter for iPhone

46 Retweets **5** Quote Tweets **488** Likes

Nonlinearity: Advertising Example

TV	Radio	Newspaper	Sales
230.1	37.8	69.2	22.1
44.5	39.3	45.1	10.4
17.2	45.9	69.3	9.3
151.5	41.3	58.5	18.5
180.8	10.8	58.4	12.9
8.7	48.9	75.0	7.2
57.5	32.8	23.5	11.8
120.2	19.6	11.6	13.2
8.6	2.1	1.0	4.8
199.8	2.6	21.2	10.6
66.1	5.8	24.2	8.6
214.7	24.0	4.0	17.4
23.8	35.1	65.9	9.2
97.5	7.6	7.2	9.7
204.1	32.9	46.0	19.0



Assume a linear model:

$$\text{sales}(i) \approx \beta_0 + \beta_1 \times \text{TV}(i) + \beta_2 \times \text{radio}(i) + \beta_3 \times \text{newspaper}(i),$$

$$i = 1, 2, \ldots, n$$

Nonlinearity

(ISLR Section 3.3.2 and 3.3.3)

Original Model:

$$\text{sales} = \beta_0 + \beta_1 \times \text{TV} + \beta_2 \times \text{radio} + \epsilon$$

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Interaction terms:

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Transform the Data Table:

1	TV	radio	TV$\times$ radio
1	.83	.3	0.249
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	Coefficient	Std. error	t-statistic	p-value
`Intercept`	6.7502	0.248	27.23	< 0.0001
`TV`	0.0191	0.002	12.70	< 0.0001
`radio`	0.0289	0.009	3.24	0.0014
`TV`$\times$`radio`	0.0011	0.000	20.73	< 0.0001

- R^2 : 89.7% $\rightarrow$ 96.8%
- Applies to categorical predictors as well.

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Interpretation: A $1000 increase in TV results in increased sales of

$$19 + 1.1 \times \text{radio units}$$

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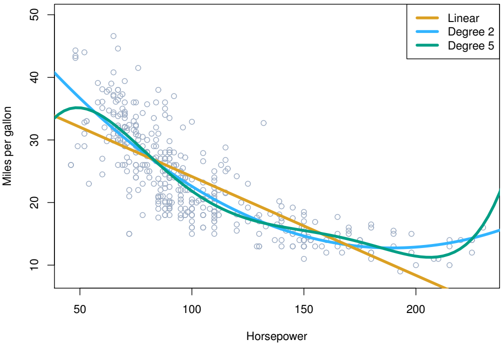
A \$1000 increase in radio results in increased sales of

A. $19 + 1.1 \times$ TV units.

B. $289 + 1.1 \times$ TV units

Nonlinearity

High order terms:



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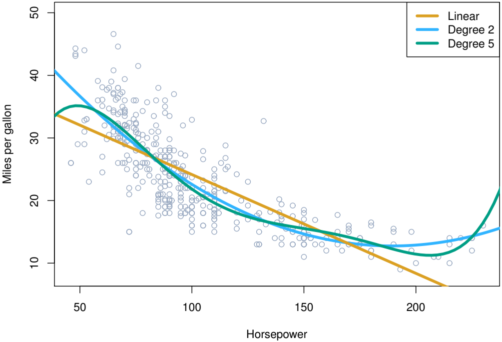

$$\text{mpg} = \beta_0 + \beta_1 \times \text{horsepower} + \beta_2 \times \text{horsepower}^2 + \epsilon$$

- R^2 : 60.6% $\rightarrow$ 68.8%

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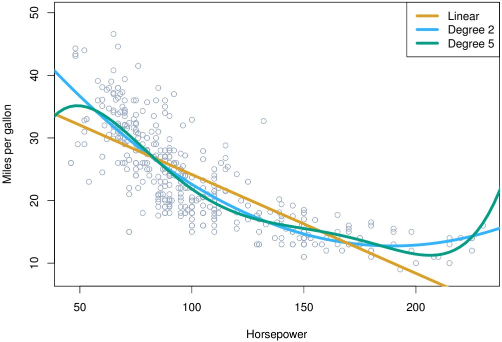

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- R^2 : 60.6% $\rightarrow$ 68.8%
- Can include higher order terms. Danger of overfitting!

Nonlinearity

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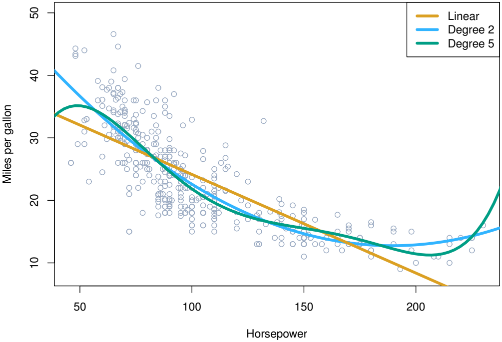

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How do we find the the coefficients of this model?

A. with linear algebra

B. with new techniques we haven't covered yet

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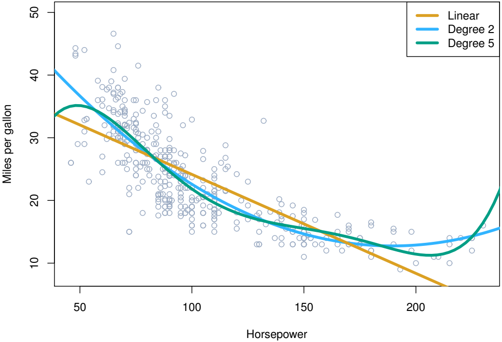

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Transform the Data Table:

1	horsepower	horsepower^2
1	120	14,400
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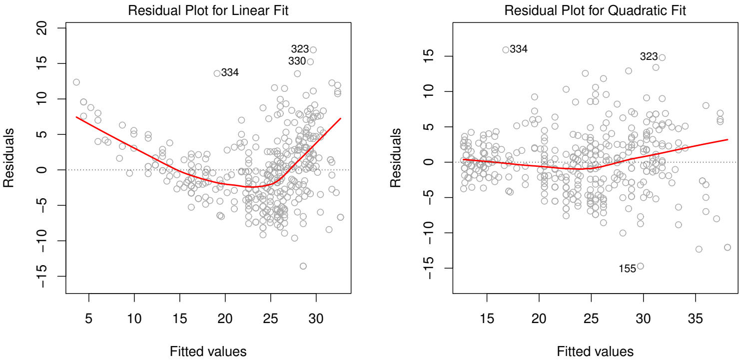

Nonlinearity: Summary

Interaction terms: $X_1X_2,\ X_2X_5X_6, \ldots$

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More on nonlinear methods later (ISLR Chap 7)

Assumptions on Errors

(ISLR Section 3.3.3)

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots \beta_p x_{ip} + \epsilon_i, \quad i = 1, 2, \ldots, n$$

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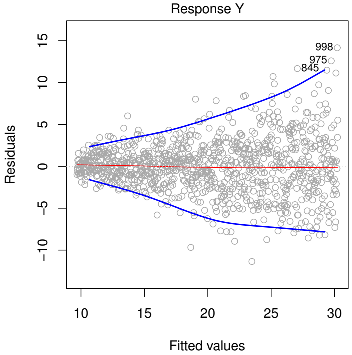
Some model evaluation techniques (p-values, conf. int.) are based on the following *additional* assumptions:

- ϵ_i is (approximately) mean-zero Gaussian.

Non-constant Error Variance (Heteroscedasticity)



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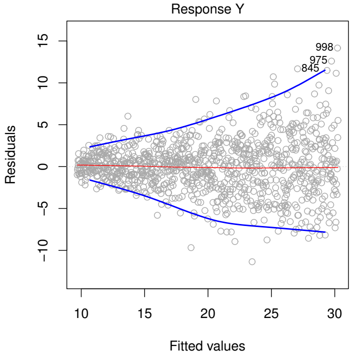

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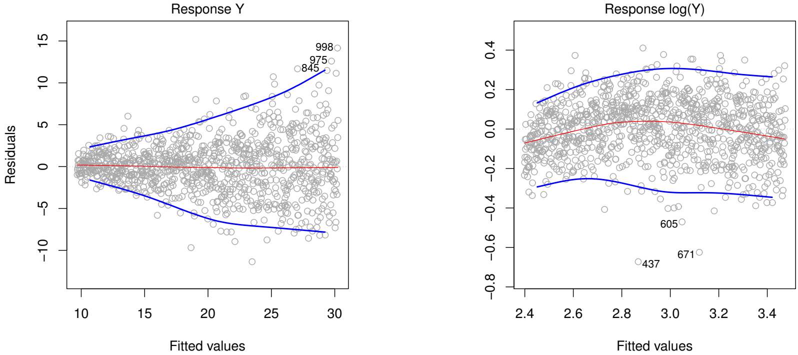

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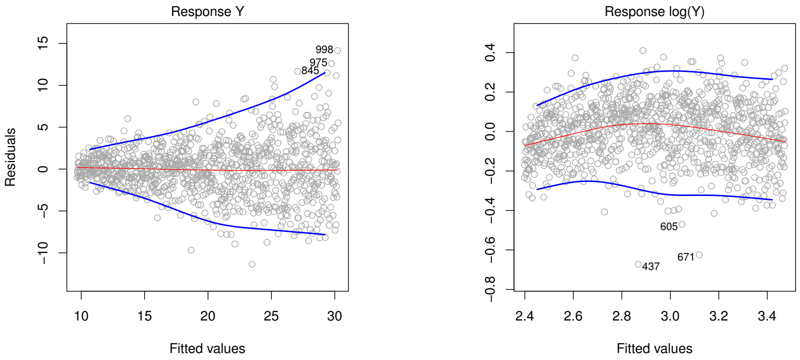
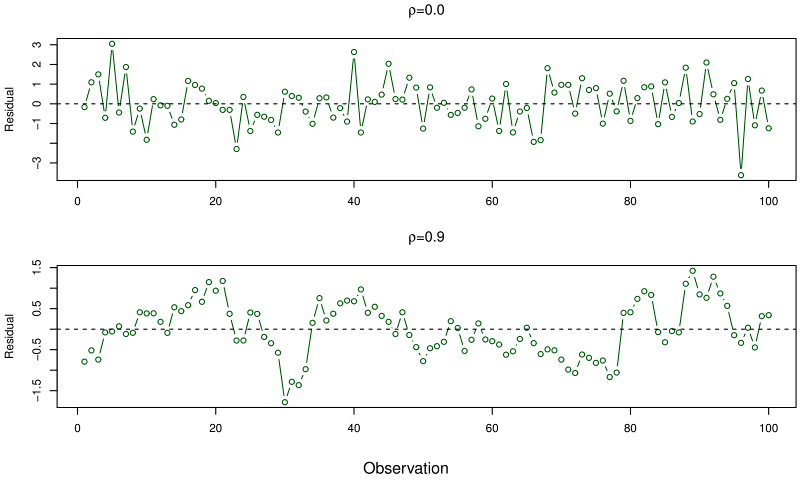


Possible solutions:

- Transformation of response. E.g. $\log Y$ or $\sqrt{Y}$
- Weighted least squares: weight $\propto \frac{1}{\mathrm{Var}(\epsilon_i)}$ (if $\mathrm{Var}(\epsilon_i)$ are known)

Correlated Errors



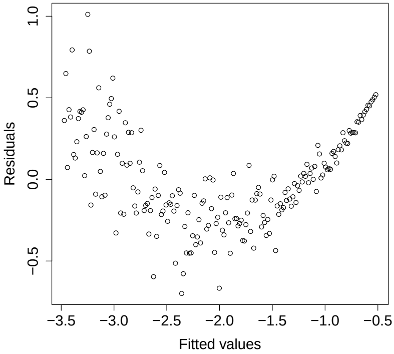
The residual plot on the right indicates that

A The errors have non-constant variance but the linear assumption is correct.

B The errors have non-constant variance and the linear assumption is wrong.

C The linear assumption is correct and the errors have constant variance.

D The linear assumption is wrong and the errors have constant variance.



Outliers

(ISLR Section 3.3.4)

A point $(x_{i1}, x_{i2}, \ldots, x_{ip}, y_i)$ with abnormal Y values

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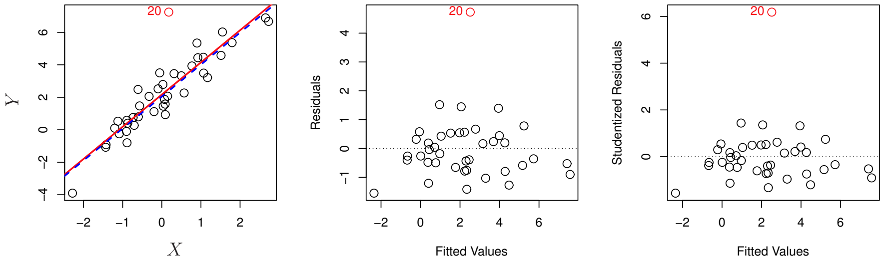

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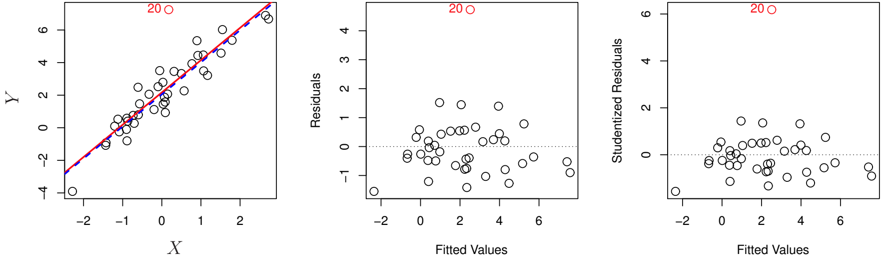

Detecting outliers: studentized residuals

High Leverage Points

(ISLR Section 3.3.5)

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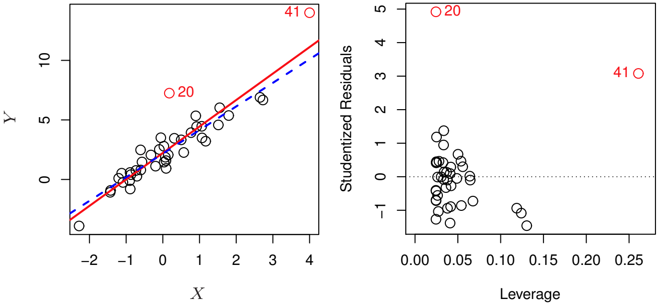

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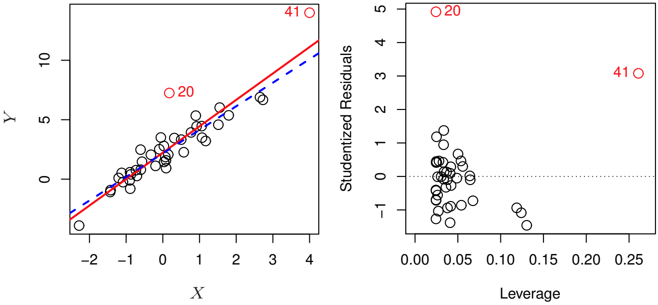

Detecting high leverage points: leverage statistic

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- Outliers: Detect by studentized residuals and leverage statistics

Some of the figures in this presentation are taken from “An Introduction to Statistical Learning, with applications in R” (Springer, 2013) with permission from the authors: G. James, D. Witten, T. Hastie and R. Tibshirani

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