Written assessment, January 16, 2025

Last name, First name ______________________________

Exercise 1 (value 8)

For each of n jobs the starting time on a computer equipped with a single processor has to be scheduled. The duration in minutes of each job $i \in N = \{1, 2, \ldots, n\}$ is t_i. The sequence of execution of the jobs follows the indices and is $1 - 2 - \cdots - n$ and cannot be changed. At most one job can be run at each time on the processor. Each job $i \in N$ has a given delivery time (in minutes) d_i. The delivery of jobs has to be on time, and a penalty of P euros has to be paid for each minute of early/late delivery, for each job. Write a Mixed Integer Linear Program to select the starting time (in minute) for each job, with the objective of minimizing the total penalty paid.

Exercise 2 (value 9)

Consider the following linear program:

$$
\begin{aligned}
\min\ & 3x_1 + 2x_2 + 4x_3 \\
s.t.\ & x_1 + 4x_2 \geq 6 \\
& 2x_1 - 3x_2 + 2x_3 \geq 4 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

- Write the dual
- Solve the dual using the graphical method (write the optimal solution and the objective value of the dual)
- Find the optimal solution of the primal using the complementary slackness conditions

Exercise 3 (value 8)

Consider a Knapsack problem with four objects characterized by the following profits $p = (1, 2, 3, 2)$ and weights $w = (100, 50, 80, 71)$, and with a container of capacity $c = 200$. Calculate the optimal solution using dynamic programming. Report all the steps of the algorithm together with the final solution and its value.

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Solution sketch

Exercise 1

x_i = starting time for job $i \in N$ (in minutes)

y_i = minutes of early/late completion time for job $i \in N$

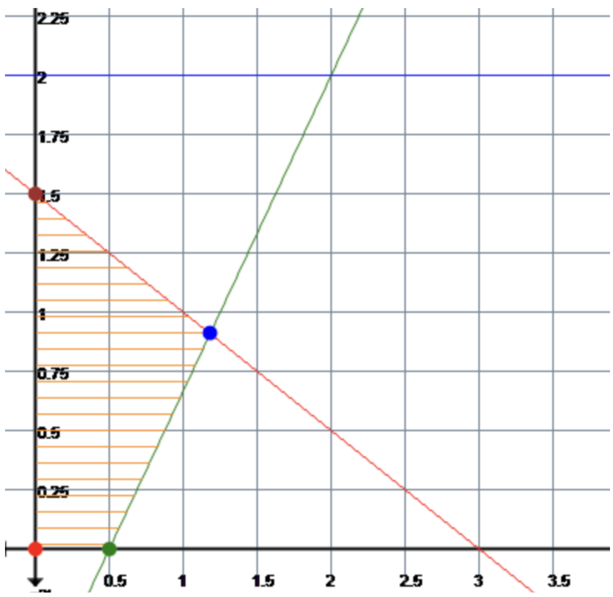
$$
\begin{aligned}
\min\ & P \sum_{i \in N} y_i \\
& y_i \geq (x_i + t_i) - d_i & i \in N \\
& y_i \geq d_i - (x_i + t_i) & i \in N \\
& x_i \geq x_{i-1} + t_{i-1} & i \in N \backslash \{n\} \\
& x_i \geq 0, \text{integer} & i \in N \\
& y_i \geq 0 & i \in N
\end{aligned}
$$

Exercise 2

The dual problem is:

$$
\begin{aligned}
\max\ & 6u_1 + 4u_2 \\
s.t.\ & u_1 + 2u_2 \leq 3 \\
& 4u_1 - 3u_2 \leq 2 \\
& 2u_2 \leq 4 \\
& u_1, u_2 \geq 0
\end{aligned}
$$

The graphical solution looks as follows:



The optimal solution of the dual is $u_1 = \frac{13}{11}, u_2 = \frac{10}{11}$ with value $z_D = \frac{118}{11}$.

The complementary slackness conditions are:

$$\begin{cases} (x_1 + 4x_2 - 6)u_1 = 0 \\ (2x_1 - 3x_2 + 2x_3 - 4)u_2 = 0 \\ (u_1 + 2u_2 - 3)x_1 = 0 \\ (4u_1 - 3u_2 - 2)x_2 = 0 \\ (2u_2 - 4)x_3 = 0 \end{cases} \quad \rightarrow \begin{cases} x_1 + 4x_2 = 6 \\ 2x_1 - 3x_2 + 2x_3 = 4 \\ x_3 = 0 \end{cases}$$

The optimal solution of the primal is $x_1 = \frac{34}{11}, x_2 = \frac{8}{11}, x_3 = 0$ with value $z_P = \frac{118}{11}$.

Exercise 3

$p = (1, 2, 3, 2)$ $w = (100, 50, 80, 71)$

$P = 1 + 2 + 3 + 2 = 8$

	0	1	2	3	4	5	6	7	8
f_0	0	∞	∞	∞	∞	∞	∞	∞	∞
f_1	0	100	∞	∞	∞	∞	∞	∞	∞
f_2	0	100	50	150	∞	∞	∞	∞	∞
f_3	0	100	50	80	180	130	230	∞	∞
f_4	0	100	50	80	121	$\boxed{130}$	230	201	301

	0	1	2	3	4	5	6	7	8
J_0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
J_1	$\emptyset$	$\{1\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
J_2	$\emptyset$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
J_3	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$	$\emptyset$	$\emptyset$
J_4	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{2,4\}$	$\boxed{\{2,3\}}$	$\{1,2,3\}$	$\{2,3,4\}$	$\{1,2,3,4\}$

The optimal solution $x = \{2, 3\}$ has profit 5 and weight 130.