7

LIMIT CYCLES

7.0 Introduction

A ***limit cycle*** is an isolated closed trajectory. *Isolated* means that neighboring trajectories are not closed; they spiral either toward or away from the limit cycle (Figure 7.0.1).

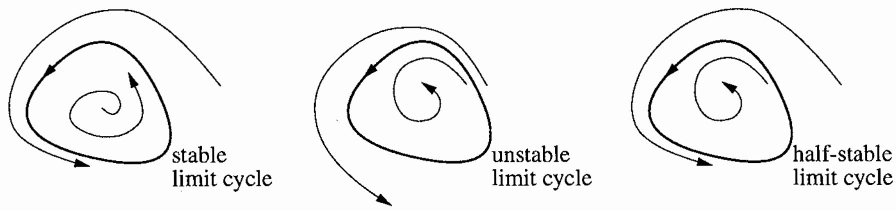

Figure 7.0.1

If all neighboring trajectories approach the limit cycle, we say the limit cycle is *stable* or *attracting*. Otherwise the limit cycle is *unstable*, or in exceptional cases, *half-stable*.

Stable limit cycles are very important scientifically—they model systems that exhibit self-sustained oscillations. In other words, these systems oscillate even in the absence of external periodic forcing. Of the countless examples that could be given, we mention only a few: the beating of a heart; the periodic firing of a pacemaker neuron; daily rhythms in human body temperature and hormone secretion; chemical reactions that oscillate spontaneously; and dangerous self-excited vibrations in bridges and airplane wings. In each case, there is a standard oscillation of some preferred period, waveform, and amplitude. If the system is perturbed slightly, it always returns to the standard cycle.

Limit cycles are inherently nonlinear phenomena; they can't occur in linear systems. Of course, a linear system $\dot{\mathbf{x}} = A\mathbf{x}$ can have closed orbits, but they won't be *isolated*; if $\mathbf{x}(t)$ is a periodic solution, then so is $c\mathbf{x}(t)$ for any constant $c \neq 0$. Hence $\mathbf{x}(t)$ is surrounded by a one-parameter family of closed orbits (Figure 7.0.2).

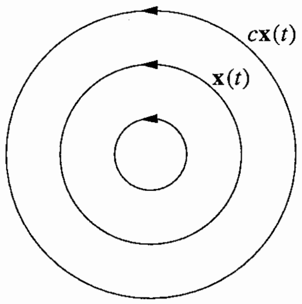

Figure 7.0.2

Consequently, the amplitude of a linear oscillation is set entirely by its initial conditions; any slight disturbance to the amplitude will persist forever. In contrast, limit cycle oscillations are determined by the structure of the system itself.

The next section presents two examples of systems with limit cycles. In the first case, the limit cycle is obvious by inspection, but normally it's difficult to tell whether a given system has a limit cycle, or indeed any closed orbits, from the governing equations alone. Sections 7.2–7.4 present some techniques for ruling out closed orbits or for proving their existence. The remainder of the chapter discusses analytical methods for approximating the shape and period of a closed orbit and for studying its stability.

7.1 Examples

It's straightforward to construct examples of limit cycles if we use polar coordinates.

EXAMPLE 7.1.1: A SIMPLE LIMIT CYCLE

Consider the system

$$\dot{r} = r(1 - r^2), \qquad \dot{\theta} = 1 \tag{1}$$

where $r \geq 0$. The radial and angular dynamics are uncoupled and so can be analyzed separately. Treating $\dot{r} = r(1 - r^2)$ as a vector field on the line, we see that $r^* = 0$ is an unstable fixed point and $r^* = 1$ is stable (Figure 7.1.1).

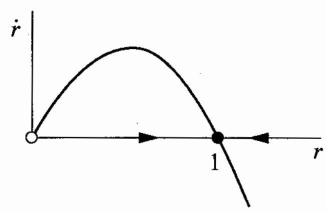

Figure 7.1.1

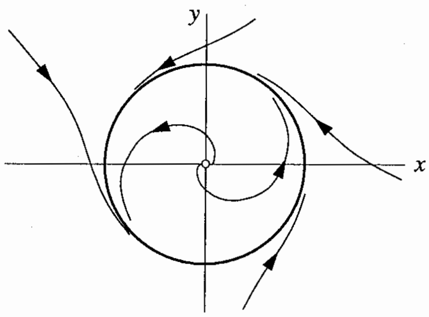

Figure 7.1.2

Hence, back in the phase plane, all trajectories (except $r^* = 0$) approach the unit circle $r^* = 1$ monotonically. Since the motion in the θ-direction is simply rotation at constant angular velocity, we see that all trajectories spiral asymptotically toward a limit cycle at $r = 1$ (Figure 7.1.2).

It is also instructive to plot solutions as functions of t. For instance, in Figure 7.1.3 we plot $x(t) = r(t)\cos\theta(t)$ for a trajectory starting outside the limit cycle. As expected, the solution settles down to a sinusoidal oscillation of constant amplitude, corresponding to the limit cycle solution $x(t) = \cos(t + \theta_0)$ of (1). ■

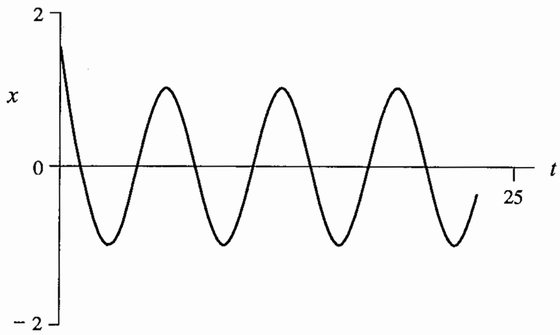

Figure 7.1.3

EXAMPLE 7.1.2: VAN DER POL OSCILLATOR

A less transparent example, but one that played a central role in the development of nonlinear dynamics, is given by the ***van der Pol equation***

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0 \tag{2}$$

where $\mu \geq 0$ is a parameter. Historically, this equation arose in connection with the nonlinear electrical circuits used in the first radios (see Exercise 7.1.6 for the circuit). Equation (2) looks like a simple harmonic oscillator, but with a ***nonlinear damping*** term $\mu(x^2 - 1)\dot{x}$. This term acts like ordinary positive damping for $|x| > 1$, but like *negative* damping for $|x| < 1$. In other words, it causes large-amplitude oscillations to decay, but it pumps them back up if they become too small.