4 Linearity in models

A vast quantity of different system dynamics can be found in both natural and social sciences. However, they all have several common characteristics which enable them to be described in fairly simple and, above all, very general mathematical terms. We are referring to the dynamic processes typical of systems close to a configuration of stable equilibrium. Such systems tend towards equilibrium by means of a process that, to a good approximation, can be described by a so-called linear model. In other words, we can approximate the description of the evolution of these systems towards equilibrium by means of a calculable evolutive trajectory, if we know the initial conditions, using a law of motion expressed in the form of an extremely simple differential equation: a linear differential equation.

Linear differential equations are equations in which the unknown function and its derivatives appear *only* summed together or multiplied by continuous functions of the independent variable. They are therefore the following type of equation:

$$x^{(n)} + a_{n-1}(t)y^{(n-1)} + a_{n-2}(t)x^{(n-2)} + \ldots + a_1(t)x' + a_0(t)x = b(t) \qquad (4.1)$$

where the $a_i(t)$ and $b(t)$ are continuous functions of the sole independent variable t (they might also be constants, which considerably simplifies the resolution of the problem) and the $x^{(i)}$ are the i-th derivatives of the unknown function $x(t)$.[14] The first member of (4.1) is a linear combination of

[14] Note that the presence of nonlinear functions in a model does not necessarily mean that the equation is nonlinear. Consider for example:

$$\dot{x} = \sin \omega t \text{ and } \dot{x} = \sin x$$

The first is a linear equation (non-autonomous, as we will see in chapter 5, because time *also* appears as an independent variable in the second member function, as well as being present in the derivative of the $x(t)$); on the contrary, the second equation (autonomous, because time does not appear explicitly) is nonlinear, as the sole unknown function $x(t)$ is the argument of the nonlinear function $\sin x$.

x and its subsequent derivatives, according to coefficients that are *only* functions of t (independent variable) and not of x (dependent variable, unknown function of t).[15]

The case of the homogeneous equation is particularly important, in which we have $b(t) = 0$. Homogeneous linear differential equations, in fact, have a very specific property, which makes the search for their general solution relatively simple: if in some way we are able to determine any two solutions, called $x_1(t)$ and $x_2(t)$, then we have that the sum of $x_1(t)$ and $x_2(t)$, even if multiplied by arbitrary coefficients c_1 and c_2, is still a solution to the given equation. In other words, any linear combination whatsoever $c_1x_1(t) + c_2x_2(t)$ of two functions that satisfy the equation is still a solution of the same: this property is known as the principle of superposition. Without entering into a discussion on the theory of the solutions to a linear equation and without being too rigorous, we can limit ourselves to observing that, by virtue of this principle, the general solution that describes the dynamics of a process according to a linear model (an equation) can be seen as a function that can be broken down into parts, all of which evolve individually over time. At each point in time, any linear combination of these parts provides a solution to the equation. The principle of superposition, therefore, expresses and summarizes the reductionist point of view (see chapter 3), which, therefore, closely links it to the concept of linearity.

A physical process that clearly exemplifies the above is that of constructive or destructive interference between low amplitude waves. For example, it can easily be observed that disturbances of a low amplitude spreading from different points on the surface of a liquid overlap simply by summing their amplitudes. It is easy to recognize a phenomenon of this type by observing the overlapping at a point of two waves, generated, for example by two stones thrown into two different points on the surface of the water. The equation that describes, in a first approximation, the motion of the point of the surface is a linear equation; the phenomenon of the interference between

[15] Equation (4.1) can also be condensed in the form $L(x) = b(t)$, having defined the linear differential operator $L(x)$ as:

$$L(x) = x^{(n)} + a_{n-1}(t)x^{(n-1)} + a_{n-2}(t)x^{(n-2)} + \ldots + a_1(t)x' + a_0(t)x$$

A fundamental property of $L(x)$ is expressed by the following relation, in which c_1 and c_2 are two constants:

$$L(c_1x_1(t) + c_2x_2(t)) = c_1L(x_1(t) + c_2L(x_2(t)) \qquad (4.1')$$

by virtue of which we can say that the operator $L(x)$ is a linear operator, which justifies calling equations like (4.1) linear equations.

the two waves is mathematically reflected in the fact that even though the two waves have, so as to speak, independent lives, if considered together they represent a new disturbance that develops in accordance with the same differential equation that governed the single waves. If, however, the waves on the surface of the water are of high amplitude, then the linearity of the description ceases to be an acceptable approximation. This happens, for example, with a wave on the surface of the sea that is moving towards the shore, which, when it becomes too high with relation to the distance between the surface and bottom, becomes a roller and breaks.

A second basic example of linear dynamic evolution can be seen in the growth of a sum of capital invested at a constant rate of interest where, to make things simpler from a mathematical point of view, we assume that the growth of the total amount occurs by means of a continuous process, namely that the interest is capitalized instantaneously.

A process of this type, where the interest continuously and instantaneously generates further interest, is commonly called compound interest. Simple considerations of a general nature lead us to define the differential equation that governs the system's dynamics as a function of time t:

$$\dot{x}(t) = k\,x(t) \tag{4.2}$$

where $x(t)$ represents the total amount and k is the growth rate.

Equation (4.2), solved, provides a simple exponential law:

$$x(t) = x(0)e^{kt} \tag{4.3}$$

where $x(0)$ indicates the initial capital.

This example regards the growth of a sum of capital, but extending the concept a little, the situation described could also be interpreted in other ways: the model, for example, could represent the growth of a population whose initial value is $x(0)$, and which increases at a constant growth rate k, in the presence of unlimited environmental resources, i.e. without any element that intervenes to slow down growth. Equation (4.3) expresses the continuous growth of a magnitude (capital, population or other) according to an exponential function whose limit, over time, is infinity.[16]

[16] The version of (4.3) in discrete form is a simple geometric progression and is the tendency to grow at a constant rate which, according to Thomas Malthus (1798), a population demonstrates in the presence of resources that grow directly in proportion to time (therefore not at a constant rate). We will return to Malthus' law in chapter 20, where, furthermore, we will also discuss the variation introduced by Pierre-François Verhulst to

In (4.3) the time dependency only affects the exponential, which contains the growth rate k; the value of the total amount at a certain point in time, moreover, is directly proportional to the initial value of the capital $x(0)$. Consequently, two separate amounts of capital invested at the same rate produce, after a certain time, two amounts of interest that, summed, are equal to the interest produced by an initial amount of capital equal to the sum of the two separate amounts of capital: it follows the same pattern as the linearity illustrated in (4.1′). This conclusion, in reality, could be possible for small sums of money; for large amounts of capital, on the other hand, other aspects may intervene that alter the linear mechanism described. It is unlikely, for instance, that an investor who has a large amount of capital to invest, would content himself with the same rate of interest obtained by those with small sums of capital, for which, presumably, lots of small amounts of capital invested separately would together obtain a lower amount of interest than that obtained by a single large amount of capital equal to the sum of the former. Alternatively, after a certain period of time, variations in market conditions may intervene that give rise to variations in interest rates or other aspects still.

The dynamics of a linear model are obviously not a mathematical abstraction made on the basis of the observation of the evolution of a system in certain specific conditions: linear abstraction *can* be acceptable (but not *necessarily*) in cases in which the magnitudes in question undergo ‘small’ variations, like the small disturbances to the surface of the water or like the effects that the low growth rates have on a small amount of capital over a short period of time. Similarly, the dynamics of any system that is brought towards a state of stable equilibrium by a return force can be considered linear, if we can assume that the intensity of the force is directly proportional to the distance of the system in question from the state of equilibrium. This, for example, is the case of a horizontal beam that supports a load at its centre: for small loads, the extent of the deformation that the beam undergoes is proportional to the load, whereas for heavy loads this is no longer true, to the extent that for too large a load, the beam will break. This is also the case of a particular set of oscillating systems such as, for example, the previously cited movement of a point on a surface of water, or that of a pendulum, when it makes small oscillations.

If the system’s distance from equilibrium is not ‘*small*’, then it is not realistic to assume that the return force is proportional to the distance, just as it is not realistic to assume that the growth of an amount of capital does

represent a more realistic case in which the limited nature of resources hinders the unlimited growth of the population.

not depend on its initial value, or that a population continues to grow according to Malthus' model even when it reaches high values, i.e. very high growth rates or very long periods of time. In cases such as these, we cannot talk of linearity: the linear model provides an unacceptable approximation of the dynamics in question.

We will return to the concept of linearity in Chapter 25, where we will extend our discussion on its meaning, its value and its limitations. In Chapter 5 we will apply the linear hypothesis to the description of the oscillation of a pendulum. Linearity is the simplest hypothesis to apply to oscillations, it was the first to be formulated and is the most studied, so much so that it has almost become a metaphor applicable to other areas of science, as we mentioned in Chapter 3. Occasionally, in fact, we also speak of periodic oscillations in economic and social systems, which often we attempt to describe as if they were real physical systems that are oscillating.[17] The discussion will occasionally touch on some technical elements, which are useful to clarify how we can construct and refine a model of a phenomenon that, although basic, is of fundamental importance in terms of the frequency with which it is used, through analogies, to interpret other more complicated phenomena.

[17] In this regard, and particularly in economics, it would be more correct to talk about the cyclicity of change, rather than periodicity in its strictest sense. It could be argued, in reality, if the alleged oscillations in historic series of data of economic systems were effectively comparable to those of a mechanical system, or if, when attempting to describe systems with extremely complicated evolutions over time and with hardly any repeat patterns, one wanted to establish an unjustified homology with the laws of mechanics. We already mentioned in Chapter 1 the risks that the direct application of the theories and the schemas of physics to social systems may involve in terms of the correct and effective interpretation of the phenomena of social sciences.

5 One of the most basic natural systems: the pendulum

The linear model (Model 1)

Let us start by illustrating the model of an oscillating pendulum, basing ourselves on the simplest assumption: a pendulum that makes small oscillations in obviously completely theoretical conditions, in which there is no form of friction.

Let us imagine that we strike the pendulum, initially still in a position of stable equilibrium, with a small force: it distances itself from equilibrium, but, as it does so, it starts to undergo the action of a restoring force, which increases with the size of the open angle with respect to the vertical (Figure 5.1). The speed of the pendulum decreases until it cancels itself out, when the restoring force becomes equal to the force applied initially that moved the pendulum from equilibrium.

The linear hypothesis is in the assumption that the restoring force and the size of the angle are directly proportional at any given moment during oscillation. This means, for example, that if we hit the pendulum with double the force, then the maximum amplitude of the oscillation, as long as it does not become 'too' large, doubles; in this way, we can assume that the oscillation consists of the sum of two oscillations caused by two equal initial forces, applied separately, that are summed together.

It should be borne in mind that the linear hypothesis, namely this direct proportionality between the action, the force initially applied, and the response, the amplitude of the oscillation, or between the restoring force and the oscillation angle, is more accurate the smaller the angle by which the pendulum distances itself from the vertical. Therefore, we consider the approximation acceptable only for 'small' angles. To be more precise, the linear hypothesis is valid within a limit in which the maximum amplitude of the oscillation tends to zero; if on the other hand the angles and the

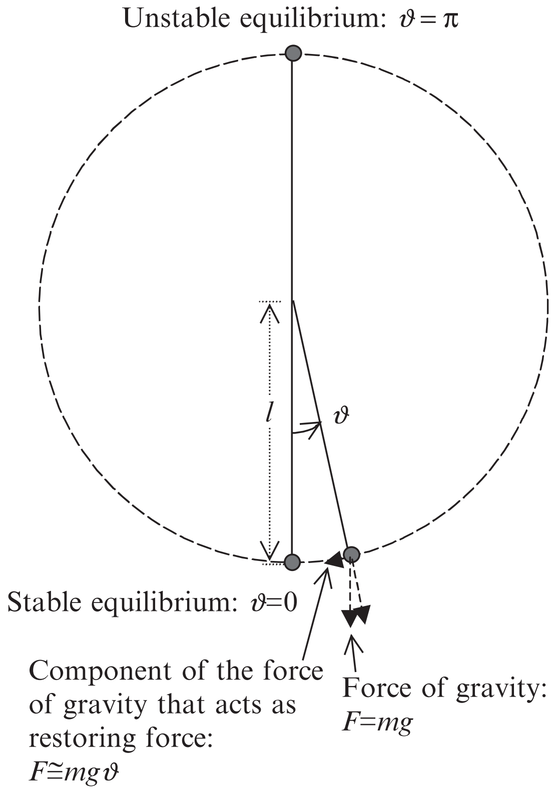

Figure 5.1 Diagram of a pendulum that makes small oscillations in the absence of friction.

restoring forces are both large, other aspects of the oscillatory phenomenon come into play which make the response disproportional to the action and which we will discuss in Chapter 7.

Assuming that the system is linear, i.e. that the action and effect are directly proportional, the equation that describes the dynamics of the pendulum can be extracted from Newton's second law of dynamics, $F = ma$, the first real example of effective mathematical modelling of a dynamic process in the history of science,[18] that we now rewrite as:

$$F = m\ddot{x}(t) \tag{5.1}$$

in which we use the symbol $\ddot{x}(t)$ to indicate the second derivative of $x(t)$ with respect to time, i.e. the acceleration. Equation (5.1) is an actual example of a second order differential equation, which can be either linear, if the force F is a linear function of x and/or of its first derivative or it does

[18] We are obviously setting aside the observations of Galileo Galilei on the motion of a pendulum and on the fall of bodies, which, however, were not formulated in terms of differential equations (introduced much later by Newton) and therefore did not lead to a real dynamical system, such as (5.1).

not depend on x, or nonlinear, if the force F is a nonlinear function of x and/or of its first derivative. In the linear case that we are about to discuss (the simplest model of oscillation), the pendulum is subjected to a restoring force that only depends on its position and that therefore, in a given position, is constant over time. The mathematical model of a linear oscillation of this type is known in mathematical physics as a 'harmonic oscillator'.

We therefore have an object with a mass m, whose barycentre distances l from a centre of rotation to which it is connected by a rigid rod of a negligible mass, as shown in Figure 5.1 $\vartheta(t)$ indicates the angle, a function of time, that the rod of the pendulum forms with respect to the vertical (position of stable equilibrium) and g indicates the acceleration of gravity. As we have said, the oscillations are assumed to be of a small amplitude, where by 'small' we mean that they are such that the arc described, whose maximum amplitude is ϑ_1, can be considered to have a negligible length with respect to the length of the pendulum l. This implies, and it is here that the linear hypothesis takes shape on a technical level, that one approximates $\sin(\vartheta) \cong \vartheta$. The law of dynamics (5.1), therefore, can be transformed as follows:

$$ml\ddot{\vartheta}(t) = -mg\vartheta(t) \tag{5.2}$$

The first member of (5.2) is none other than the term $m\ddot{x}(t)$ from Newton's equation (5.1): $l\ddot{\vartheta}(t)$ is obtained by deriving the length $l\vartheta(t)$ of the arc travelled by the mass m during oscillation twice with respect to time, expressing the angle $\vartheta(t)$ in radians (see Figure 5.1). The second member of (5.2), on the other hand, is the component of the force of gravity that acts in a perpendicular direction to the rod of the pendulum: it is the restoring force that pushes the pendulum towards the position of equilibrium and, therefore, is directed in the opposite direction to that of the growth of the angle ϑ, which is why it has a minus sign. The linear hypothesis lies really in the fact that, for small angles, the restoring force is assumed to be proportional to the amplitude of the angle.[19]

Equation (5.2) is a homogeneous linear differential equation with constant coefficients, whose integration leads to a combination of exponentials in a complex field, which translates into simple sinusoidal oscillatory dynamics. Simplifying and assuming $\omega^2 = g/l$, we obtain:

[19] In practice, if we limit ourselves to considering an angle of 1 degree as the maximum amplitude of the oscillation, assuming that the intensity of the restoring force is directly proportional to the angle of oscillation, the error that we introduce is around 1 per cent.

$$\vartheta(t) = \vartheta_0 \cos(\omega t) \tag{5.3}$$

where the constant of integration ϑ_0 indicates the initial amplitude of the oscillation, in the case that the pendulum is initially moved away from equilibrium and then left to oscillate. The solution of (5.2) is a periodic function $\vartheta(t)$, whose period T is constant, as can be obtained from (5.3) with some simple operations:

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$$

Therefore it represents a dynamics which is stable over time.

We will now attempt to improve our model of a pendulum, making the description of the phenomenon more plausible, on the basis of different, more realistic initial hypotheses, while remaining within linearity.

The linear model of a pendulum in the presence of friction (Model 2)

Let us ignore for the moment the hypothesis according to which there is no form of friction present in the motion of the pendulum and let us introduce an element into (5.2) that describes the effect of friction. In this case, taking into consideration that friction manifests itself as a force that opposes motion, the intensity of which, at low speeds, is proportional to the speed of the latter, we can correct equation (5.2) by inserting a term that is added to the restoring force. The speed with which the pendulum moves can be obtained by differentiating the expression $s(t) = l\vartheta(t)$ with respect to time, obtaining $\dot{s}(t) = l\dot{\vartheta}(t)$.

Therefore, (5.2), which is still Newton's renowned law, is now corrected by the addition of a term proportional to $l\dot{\vartheta}(t)$:

$$ml\ddot{\vartheta}(t) = -kl\dot{\vartheta}(t) - mg\vartheta(t) \tag{5.4}$$

The minus sign, in front of the first term of the second member, representing the force of friction, is justified by the fact that such force manifests itself by acting in the opposite direction to the peripheral speed $l\dot{\vartheta}(t)$, in the same way as the restoring force has a minus sign before it.

Equation (5.4), like (5.2), is a second order linear differential equation with constant coefficients. The solution of (5.4) depends on the values of the coefficients: for certain sets of values, the pendulum makes damped oscillations, whose amplitude decreases exponentially over time; for other sets of values, it can bring itself to the vertical position, the state of equilibrium, at a speed that decreases exponentially over time, without oscillating. Nil amplitude, i.e. when the pendulum is still, is a position of equilibrium that is called asymptotically stable (we will give an initial definition of the concept of asymptotic stability in Chapter 8; several aspects of the concept of asymptotic stability will then be examined in greater detail in Chapters 17–19).

We observe at this point, without going into the mathematical details, that the case of the forced pendulum also merits consideration: the motion of a pendulum subjected not only to the force of gravity and the force of friction, but also to another force which is a function of time. There are a great number of examples of oscillating systems that are similar to that of a forced pendulum: all of those in which a system is subjected to the action of a restoring force that makes it make periodic oscillations, but also undergoes the action of another external variable force. Consider the example of a forked branch of a tree, whose two parts (forks) are able to oscillate; if you make one oscillate and you observe the overall motion of the two parts: the stem of the fork, oscillating, acts on the second stem (fork), making it oscillate; it, in its turn, acts in return on the first fork. Basically you will observe that the motion of oscillation passes alternately but without interruption from one fork to another. This represents a system made up of two coupled pendulums.

We can also consider, as a second example of forced oscillation, the phenomenon of resonance: what we observe, for example in an oscillating system that receives a fixed intensity impulse at constant intervals, integer multiples of the period of oscillation.

Let us return to the model of the pendulum. The equation of the motion of the forced pendulum can generally be written in the following form:

$$ml\ddot{\vartheta}(t) = -kl\dot{\vartheta}(t) - mg\vartheta(t) + F(t) \tag{5.5}$$

In (5.5), the force $F(t)$ directly acts on the oscillating mass, summing itself algebraically to the return component of the pendulum due to the force of gravity and to friction. In Chapter 10, when we touch upon the chaotic pendulum (Model 4), we will write yet another form of the equation for a particular forced pendulum, in which the force $F(t)$ doesn't act directly on the oscillating mass, as in (5.5), but on the oscillation pivot.

Autonomous systems

With a system that makes forced oscillations of the type illustrated by (5.5), we meet our first example of a particular type of system, called non-autonomous. A system is non-autonomous if it is described by equations that contain a term in which a direct and explicit dependence on time manifests itself, like that caused by an external force $F(t)$ in (5.5), whereas the mechanism that determines the evolution of an autonomous system does not directly depend on time.

Let us clarify what we mean by explicit dependence on time with an example. The concept is important and far-reaching and does not only apply to a pendulum but to any system. It is true nevertheless that in a pendulum that makes small oscillations, the restoring force is not constant, but depends directly on the position of the pendulum itself and it is the latter that depends on time. If we held the pendulum still, out of equilibrium, with our hand, we would feel a constant restoring force (the weight component perpendicular to the rod). However, in the case of a non-autonomous system, like a forced pendulum, we would feel a variable restoring force, as if the pendulum was a live 'pulsating' being, due to the effect of the restoring force that now depends (commonly referred to as 'explicitly') on time. It is evident that the description of the motion of an autonomous system is, in general, much simpler than that of a non-autonomous system, because the equations used to describe it are generally simpler.

In an autonomous system, therefore, time does not appear directly among the variables in the equations that describe the dynamics of the system, which not only makes its analytical treatment much simpler, it also gives the system certain properties that characterize it in a particular way. If the forces in play do not depend explicitly on time, then every time that the system assumes a given configuration, namely a given speed at a given point in space, the dynamics that ensue immediately afterwards are always the same. Basically, the subsequent evolution at a given moment in time is determined with no uncertainty, as the forces, for that given configuration, are known and constant.[20] For example, a certain pendulum is subjected to

[20] By using a slightly stricter language and anticipating the contents of Chapter 8 regarding the phase space, we can rewrite the sentence in the text, saying that at a given point in the phase space the conditions are stationary. This means that, when passing at different times through the same point, the orbit always passes there 'in the same way', i.e. the trajectory has the same geometric characteristics, i.e. at that point it has the same tangent. Stationary orbits basically only exist in the case of an autonomous system, as it is only in this case that the conditions for the validity of the theorem of existence and uniqueness of the solutions of a

the same restoring force, every time that the angle that it forms with respect to the vertical position, by oscillating, assumes a given value.

Returning to equation (5.5), we observe that the solution obviously depends on the form of $F(t)$. Apart from specific cases in which the expression of $F(t)$ is particularly simple, such as the case in which $F(t)$ has a sinusoidal trend, as in the cited example of the two coupled pendulums, and the period of oscillation of the pendulum has a rational relationship to that of $F(t)$, in general the evolution of the pendulum can no longer be ascribable to regular periodic oscillations and cannot be calculated in exact terms, but only through numerical integrations.

It is evident that a description of the dynamics of a system as simple as that of a pendulum, that has some claim of resembling an observed situation, cannot be based on the simplified hypotheses presented here, unless as a mathematical abstraction. If we wanted to be more realistic, we would at least need to take into account the fact that the amplitude of the oscillations, although low, is not negligible. The linear model described, both in the form of (5.2) and in the form of (5.4) basically needs to be corrected, by introducing oscillation amplitudes that are not 'small'. Considering amplitudes that are not 'small', however, forces us to abandon the condition of linearity: a problem that warrants further examination. In Chapter 7, we will see how to alter the model of the pendulum, making it nonlinear; but first in Chapter 6, we will continue our exploration of the meaning and the limitations of linearity.

differential equation are satisfied. The equations that describe the oscillating systems that we are discussing, as we have said, are variants that originate from Newton's equation $F = ma$; therefore, if the system is autonomous, the forces in play only depend on the position (technically, they are referred to as conservative forces) and the system can be described in terms of energy conservation.