RICERCA OPERATIVA

Written assessment, February, 20, 2025

Last name, First name ______________________________

Exercise 1 (value 8)

The shortest path tree. Given a directed graph $G = (V, A)$, with a non-negative cost c_{ij} associated to each arc $(i, j) \in A$, write a MIP model to find a spanning tree T on G such that given a source node $s \in V$ there is a path in T, from s to all the remaining nodes of V. The objective is to minimize the sum of the costs incurred by each path (Hint: for each arc, use a variable counting how many paths use the arc and impose a balance of the paths at each node.).

Exercise 2 (value 8)

Consider the following linear program:

$$\begin{aligned}
\min\ & 8x_1 - 4x_2 + 8x_3 \\
s.t.\ & -2x_1 + 2x_2 - 2x_4 = 20 \\
& 6x_1 + 9x_2 - 3x_3 + 3x_5 = 60 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0
\end{aligned}$$

A given optimal solution is $x = (0, 10, 10, 0, 0)$ with value 120.

Calculate the range in which the r.h.s "20" of the first constraint can variate while the basis associated with the given optimal solution remains optimal.

Show all the calculations.

Exercise 3 (value 11)

Consider the following LP:

$$\begin{aligned}
\min\ & -35x_2 + 9x_4 \\
s.t.\ & -x_1 - 2x_2 + x_3 + x_4 \geq 2 \\
& x_1 - 3x_2 - 2x_3 \geq 1 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}$$

- Write the dual
- Solve the dual problem using the graphical method, clearly identifying each constraint and the gradient in the chart. Write the optimal dual solution and its cost
- Write the form of the optimal primal solution and its cost

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Solution sketch

Exercise 1

Variables

$$x_{ij} = \text{number of paths using arc } (i,j) \in A$$

$$\begin{aligned}
\min & \sum_{(i,j)\in A} c_{ij}x_{ij} \\
& \sum_{(j,i)\in A} x_{ji} - \sum_{(i,j)\in A} x_{ij} = 1 \quad i \in V\backslash\{s\} \\
& \sum_{(j,s)\in A} x_{js} - \sum_{(s,j)\in A} x_{sj} = 1 - n \\
& x_{ij} \in \mathcal{Z}_0^+ \quad (i,j) \in A
\end{aligned}$$

Exercise 2

$$B = \begin{bmatrix} +2 & 0 \\ 9 & -3 \end{bmatrix} \Rightarrow B^{-1} = \begin{bmatrix} +1/2 & 0 \\ +3/2 & -1/3 \end{bmatrix}$$

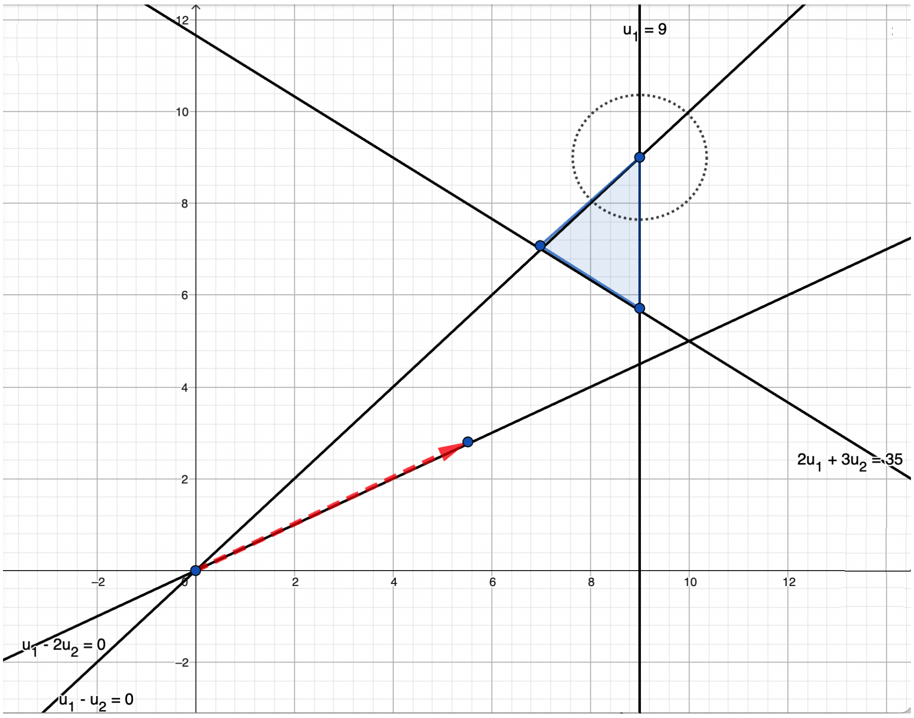
$$B^{-1}(b + \Delta b) \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} +1/2 & 0 \\ +3/2 & -1/3 \end{bmatrix} \begin{bmatrix} +20 + \Delta \\ 60 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 10{+}1/2\Delta \\ 10 + 3/2\Delta \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$$

$\Rightarrow \begin{cases} \Delta \geq -20 \\ \Delta \geq -20/3 \end{cases}$ $\Rightarrow$ The basis changes once b_1 deincreases of a quantity higher than 20/3.

The solution remains the same when $40/3 \leq b_1 \leq \infty$.

Exercise 3

$$\begin{aligned}
\max\ & 2u_1 + u_2 \\
s.t.\ & u_1 - u_2 \geq 0 \\
& 2u_1 + 3u_2 \geq 35 \\
& u_1 - 2u_2 \leq 0 \\
& u_1 \leq 9 \\
& u_1, u_2 \geq 0
\end{aligned}$$



The optimal dual solution is $u = (9, 9)$ with value 27.

$$\begin{cases} (-x_1 - 2x_2 + x_3 + x_4 - 2)u_1 = 0 \\ (x_1 - 3x_2 - 2x_3 - 1)u_2 = 0 \\ (u_1 - u_2)x_1 = 0 \\ (2u_1 + 3u_2 - 35)x_2 = 0 \\ (u_1 - 2u_2)x_3 = 0 \\ (u_1 - 9)x_4 = 0 \end{cases} \Rightarrow \begin{cases} -x_1 - 2x_2 + x_3 + x_4 = 2 \\ x_1 - 3x_2 - 2x_3 = 1 \\ . \\ x_2 = 0 \\ x_3 = 0 \\ . \end{cases} \Rightarrow \begin{cases} x_4 = 3 \\ x_1 = 1 \\ . \\ x_2 = 0 \\ x_3 = 0 \\ . \end{cases}$$

The optimal primal solution is $x = (1, 0, 0, 3)$ with value 27.