Table 5.1: Examples of important types of phase transitions in physical systems. When the transition is continuous/discontinuous one speaks of a second-/first-order phase transition. Note that most order parameters are non-intuitive. The superconducting state, notable for its ability to carry electrical current without dispersion, breaks what one calls the $U(1)$-gauge invariance of the normal (non-superconducting) metallic state

Transition	Type	Order parameter ϕ
Superconductivity	Second-order	U(1)-gauge
Magnetism	Mostly second-order	Magnetization
Ferroelectricum	Mostly second-order	Polarization
Bose–Einstein	Second-order	Amplitude of $k = 0$ state
Liquid–gas	First-order	Density

minimize its energy but a quantity called the *free energy* F, which differs from the energy by a term proportional to the entropy and to the temperature.[1]

Close to the transition temperature T_c the order parameter ϕ is small and one assumes within the Landau–Ginsburg model that the free energy density $f = F/V$,

$$f \;=\; f(T,\phi,h)\;,$$

can be expanded for a small order parameter ϕ and a small external field h:

$$f(T,\phi,h) \;=\; f_0(T,h) - h\,\phi + a\,\phi^2 + b\,\phi^4 + \ldots \tag{5.1}$$

where the parameters $a = a(T)$ and $b = b(T)$ are functions of the temperature T and of an external field h, e.g. a magnetic field for the case of magnetic systems. Note the linear coupling of the external field h to the order parameter in lowest order and that $b > 0$ (stability for large ϕ), compare Fig. 5.2.

Spontaneous Symmetry Breaking All odd terms $\sim \phi^{2n+1}$ vanish in the expansion (5.1). The reason is simple. The expression (5.1) is valid for all temperatures close to T_c and the disordered high-temperature state is invariant under the symmetry operation

$$f(T,\phi,h) \;=\; f(T,-\phi,-h), \qquad \phi \;\leftrightarrow\; -\phi, \qquad h \;\leftrightarrow\; -h\;.$$

This relation must therefore hold also for the exact Landau–Ginsburg functional. When the temperature is lowered the order parameter ϕ will acquire a finite expectation value. One speaks of a "spontaneous" breaking of the symmetry inherent to the system.

The Variational Approach The Landau–Ginsburg functional (5.1) expresses the value that the free-energy would have for all possible values of ϕ. The true physical state, which one calls the "thermodynamical stable state", is obtained by finding the minimal $f(T,\phi,h)$ for all possible values of ϕ:

$$\begin{aligned}\delta f \;&=\; \left(-h + 2a\phi + 4b\phi^3\right)\delta\phi \;=\; 0,\\ 0 \;&=\; -h + 2a\phi\; + 4b\phi^3\;,\end{aligned} \tag{5.2}$$

[1]Details can be found in any book on thermodynamics and phase transitions, e.g. Callen (1985), they are, however, not necessary for an understanding of the following discussions.

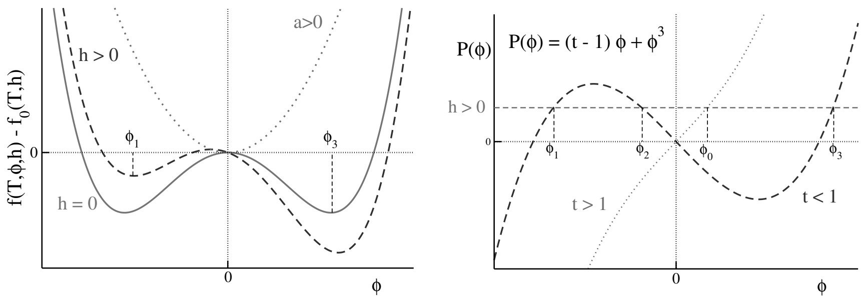

Figure 5.2: *Left*: The functional dependence of the Landau–Ginzburg free energy $f(T,\phi,h) - f_0(T,h) = -h\phi + a\phi^2 + b\phi^4$, with $a = (t-1)/2$. Plotted is the free energy for $a < 0$ and $h > 0$ (*dashed line*) and $h = 0$ (*full line*) and for $a > 0$ (*dotted line*). *Right*: Graphical solution of Eq. (5.9) for a non-vanishing field $h \neq 0$; ϕ_0 is the order parameter in the disordered phase ($t > 1$, *dotted line*), ϕ_1, ϕ_3 the stable solutions in the order phase ($t < 1$, *dashed line*) and ϕ_2 the unstable solution, compare the *left-hand* side illustration

where δf and $\delta\phi$ denote small variations of the free energy and of the order parameter, respectively. This solution corresponds to a minimum in the free energy if

$$\delta^2 f \;>\; 0, \qquad \delta^2 f \;=\; \left(2a + 12b\phi^2\right)(\delta\phi)^2 \,. \tag{5.3}$$

One also says that the solution is “locally stable”, since any change in ϕ from its optimal value would raise the free energy.

Solutions for $h = 0$ We consider first the case with no external field, $h = 0$. The solution of Eq. (5.2) is then

$$\phi \;=\; \left\{ \begin{array}{rcl} 0 & \text{for} & a > 0 \\ \pm\sqrt{-a/(2b)} & \text{for} & a < 0 \end{array} \right. \,. \tag{5.4}$$

The trivial solution $\phi = 0$ is stable,

$$\left(\delta^2 f\right)_{\phi=0} \;=\; 2a\,(\delta\phi)^2 \,, \tag{5.5}$$

if $a > 0$. The nontrivial solutions $\phi = \pm\sqrt{-a/(2b)}$ of Eq. (5.4) are stable,

$$\left(\delta^2 f\right)_{\phi\neq 0} \;=\; -4a\,(\delta\phi)^2 \,, \tag{5.6}$$

for $a < 0$. Graphically this is immediately evident, see Fig. 5.2. For $a > 0$ there is a single global minimum at $\phi = 0$, for $a < 0$ we have two symmetric minima.

Continuous Phase Transition We therefore find that the Ginsburg–Landau functional (5.1) describes continuous phase transitions when $a = a(T)$ changes sign at the critical temperature T_c. Expanding $a(T)$ for small $T - T_c$ we have

$$a(T) \;\sim\; T - T_c, \qquad a \;=\; a_0\,(t-1), \qquad t = T/T_c, \qquad a_0 > 0 \,,$$

where we have used $a(T_c) = 0$. For $T < T_c$ (ordered phase) the solution Eq. (5.4) then takes the form

$$\phi \;=\; \pm\sqrt{\frac{a_0}{2b}\,(1-t)}, \qquad t < 1, \qquad T < T_c \,. \tag{5.7}$$

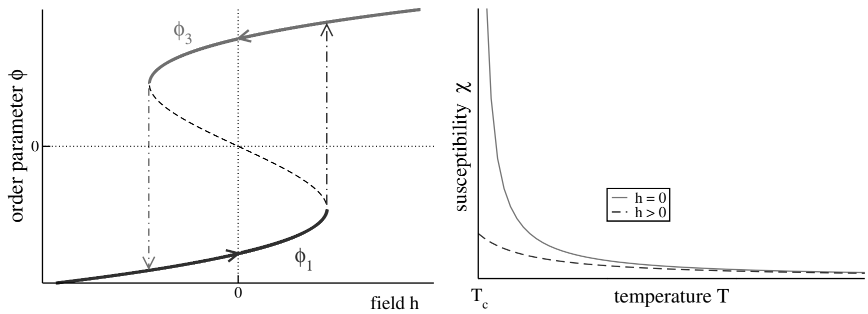

Figure 5.3: *Left*: Discontinuous phase transition and hysteresis in the Landau model. Plotted is the solution $\phi = \phi(h)$ of $h = (t-1)\phi + \phi^3$ in the ordered phase ($t < 1$) when changing the field h. *Right*: The susceptibility $\chi = \partial\phi/\partial h$ for $h = 0$ (*solid line*) and $h > 0$ (*dotted line*). The susceptibility divergence in the absence of an external field ($h = 0$), compare Eq. (5.11)

Simplification by Rescaling We can always rescale the order parameter ϕ, the external field h and the free energy density f such that $a_0 = 1/2$ and $b = 1/4$. We then have

$$a = \frac{t-1}{2}, \qquad f(T,\phi,h) - f_0(T,h) \;=\; -h\phi + \frac{t-1}{2}\phi^2 + \frac{1}{4}\phi^4$$

and

$$\phi \;=\; \pm\sqrt{1-t}, \qquad t \;=\; T/T_c \tag{5.8}$$

for the non-trivial solution Eq. (5.7).

Solutions for $h \neq 0$ The solutions of Eq. (5.2) are determined in rescaled form by

$$h \;=\; (t-1)\,\phi + \phi^3 \;\equiv\; P(\phi)\,, \tag{5.9}$$

see Fig. 5.2. In general one finds three solutions $\phi_1 < \phi_2 < \phi_3$. One can show (see the Exercises) that the intermediate solution is always locally instable and that ϕ_3 (ϕ_1) is globally stable for $h > 0$ ($h < 0$).

First-Order Phase Transition We note, see Fig. 5.2, that the solution ϕ_3 for $h > 0$ remains locally stable when we vary the external field slowly (adiabatically)

$$(h > 0) \;\to\; (h = 0) \;\to\; (h < 0)$$

in the ordered state $T < T_c$. At a certain critical field, see Fig. 5.3, the order parameter changes sign abruptly, jumping from the branch corresponding to $\phi_3 > 0$ to the branch $\phi_1 < 0$. One speaks of hysteresis, a phenomenon typical for first-order phase transitions.

Susceptibility When the system is disordered and approaches the phase transition from above, it has an increased sensitivity towards ordering under the influence of an external field h.