RICERCA OPERATIVA

Written assessment, July 10, 2024

Last name, First name ______________________________

Exercise 1 (value 8)

Machine Learning (ML) techniques require long computational time for training. Using more GPUs in parallel is sometime necessary to speed up the computation. Let $P = \{1, \ldots, n\}$ denote a set of computational tasks to be performed during the training of a ML method. Let $G = \{1, \ldots, g\}$ denote a set of available GPUs. Each task $i \in P$ has a running time p_i. The basic problem consists in assigning the tasks to the GPUs, by minimizing the difference between the total running time of the GPU with maximum load and the total running time of the GPU with minimum load. Write an LP model to assign the computational task while minimizing the above metrics.

In some cases the GPUs are associated to a sequence of numbers $(1, 2, \ldots, g)$ and is given a set of precedences $A \subset P \times P$. If a precedence $(i, j) \in A$ exists task j must be executed on a GPU with a number greater or equal to that of the GPU processing task i. Modify the LP model to satisfy also this constraint.

Exercise 2 (value 10)

They are given:

- the following LP

$$
\begin{aligned}
z = \min \quad & 2x_1 + 3x_2 + x_4 + 4x_5 \\
& x_1 + 2x_2 + x_3 + 4x_4 + 2x_5 = 10 \\
& 3x_1 - x_3 + 3x_4 + 6x_5 = 12 \\
& x_1, \ldots, x_5 \geq 0
\end{aligned}
$$

- the basis $B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$ and its inverse $B^{-1} = \begin{bmatrix} 0 & 1/3 \\ 1/2 & -1/6 \end{bmatrix}$
- the associated primal and dual solutions

Answer to the following questions without using the simplex algorithm and clearly justifying each answer

a. What are the values of the primal and dual variables ?

b. What is the value of of the primal and dual objective function ?

c. Is the dual solution feasible?

d. is the primal solution optimal?

Exercise 3 (value 9)

Given the following ILP

$$
\begin{aligned}
z = \max \quad & 3x_1 + 4x_2 \\
& 10x_1 + 10x_2 \leq 13 \\
& x_2 \leq 4 \\
& x_1, x_2 \geq 0, \text{integer}
\end{aligned}
$$

Solve the problem using the graphical method within the standard branch-and-bound **algorithm, without rounding down the upper bound values from the continuous relaxation**. Draw the decision tree and clearly show the solution of each problem.

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Solution sketch

Exercise 1

Variables

$$x_{ij} = \ 1 \text{ if task } i \in P \text{ is assigned to GPU } j \in G$$

$$\begin{array}{lll}
\min & zM - zm & \\
& \sum_{j \in G} x_{ij} = 1 & i \in P \\
& \sum_{i \in P} p_i x_{ij} \leq zM & j \in G \\
& \sum_{i \in P} p_i x_{ij} \geq zm & j \in G \\
& \sum_{k \in G} k x_{ik} \leq \sum_{k \in G} k x_{jk} & (i,j) \in A \\
& \sum_{h=1}^{k} x_{jh} \leq \sum_{h=1}^{k} x_{ih} & (i,j) \in A, k \in G (\text{alternative to the above}) \\
& x_{ij} \in \{0,1\} & i \in P, j \in G
\end{array}$$

Exercise 2

a.

$$x_B = B^{-1}b = \begin{bmatrix} 0 & 1/3 \\ 1/2 & -1/6 \end{bmatrix} \begin{bmatrix} 10 & 12 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$u^T = c_B^T B^{-1} = \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1/3 \\ 1/2 & -1/6 \end{bmatrix} = \begin{bmatrix} 3/2 & 1/6 \end{bmatrix}$$

b.

$$z_P = c_B x_b = \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = 17$$

$$z_D = u^T b = \begin{bmatrix} 3/2 & 1/6 \end{bmatrix} \begin{bmatrix} 10 & 12 \end{bmatrix} = 17$$

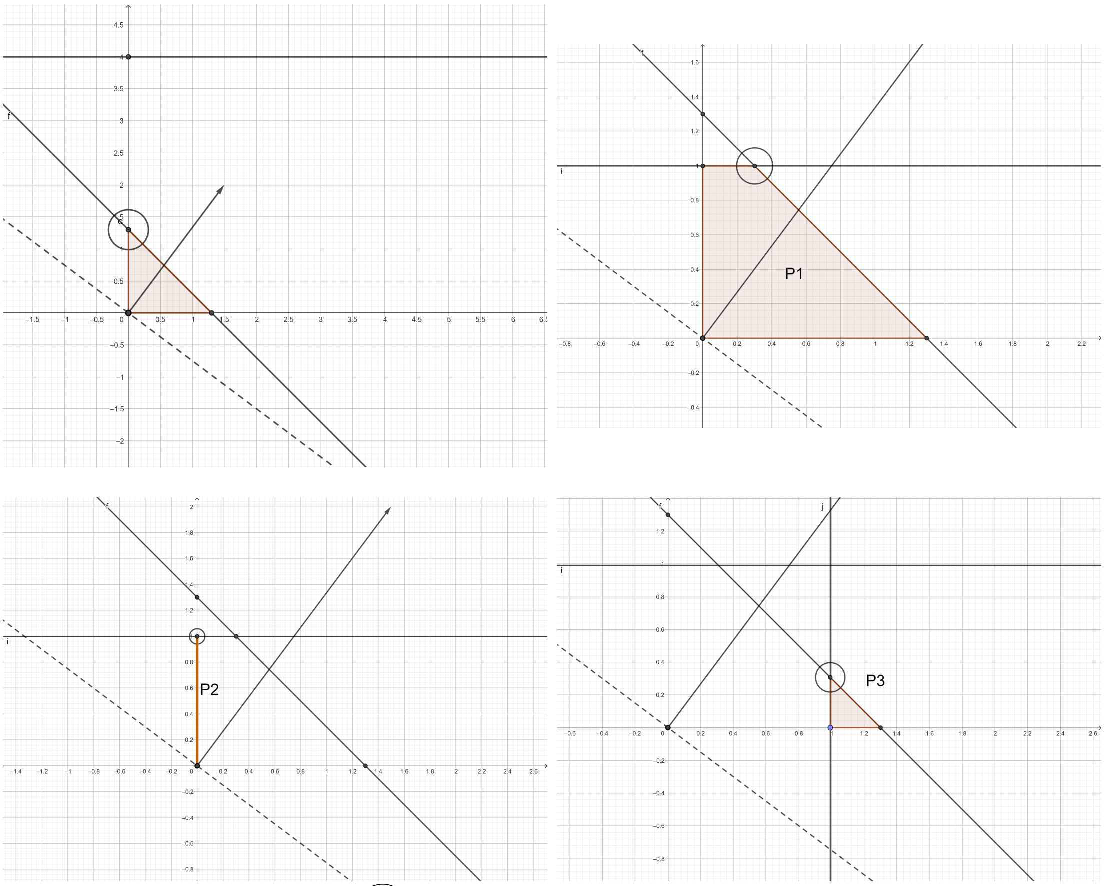
c. The dual problem is

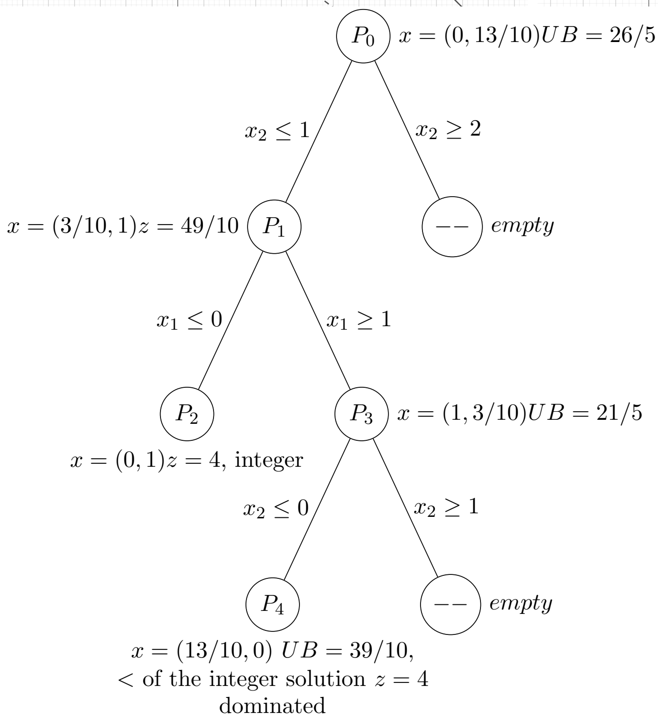
$$\begin{array}{lr}
\max & 10u_1 + 12u_2 \\
& u_1 + 3u_2 \leq 2 \\
& -2u_1 \leq 3 \\
& u_1 - u_2 \leq 0 \\
& 4u_1 + 3u_2 \leq 1 \\
& 2u_1 + 6u_2 \leq 4
\end{array}$$

The 3rd and 4th constraints are not satisfied, therefore the dual solution in unfeasible.

d. Since the dual is infeasible the primal solution cannot be optimal.

Exercise 3





The optimal solution is in problem P_2 with $x_1 = 0, x_2 = 1$ and value 4.