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AMATH 502 A
January 16, 2021
Homework #2

Problem #1

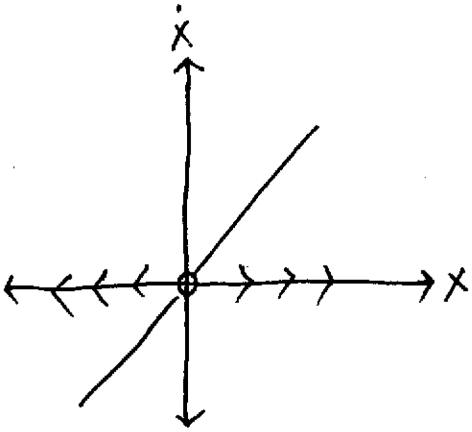
a) $\dot{x} = x - rx(1-x)$

First, we need to find the fixed points:

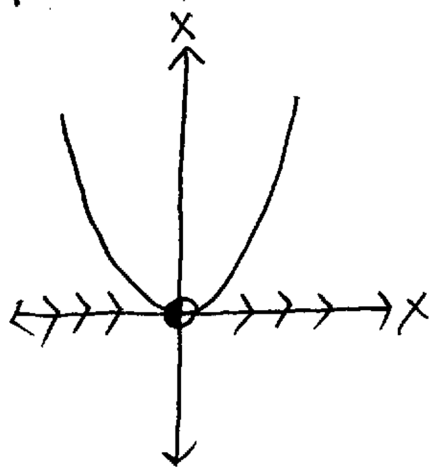
$x - rx(1-x) = 0 \Rightarrow$ fixed point at $x=0$ and

$$rx(1-x) = x$$
$$r(1-x) = 1$$
$$r - rx = 1$$
$$-rx = 1-r$$
$$x = \frac{r-1}{r} \Rightarrow \text{fixed point at } x = \frac{r-1}{r}$$

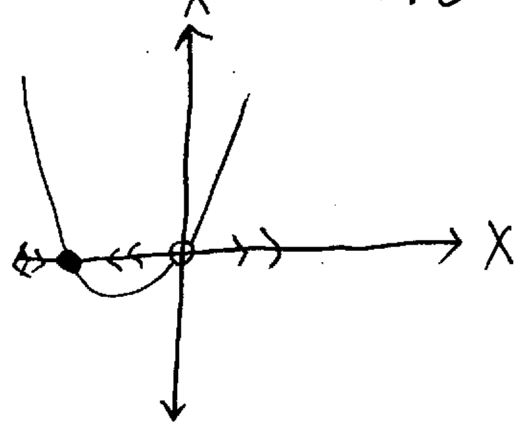
$r \neq 0$



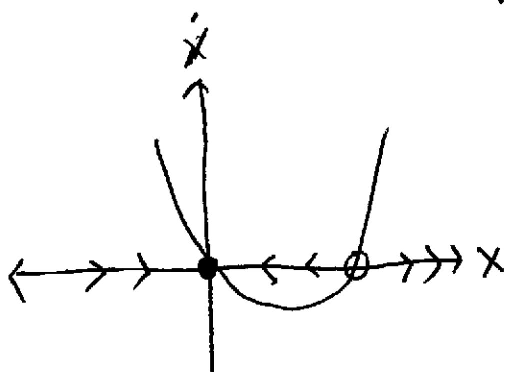
$r=0$



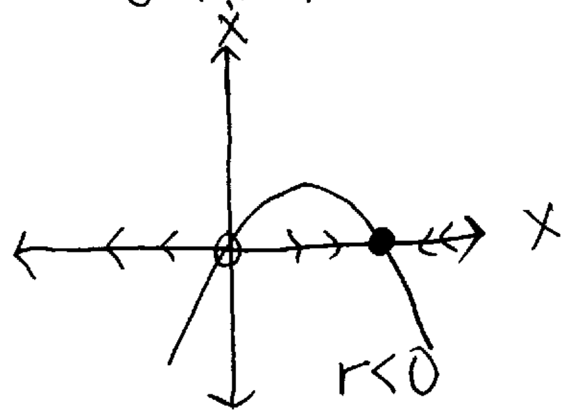
$r=1$



$0 < r < 1$



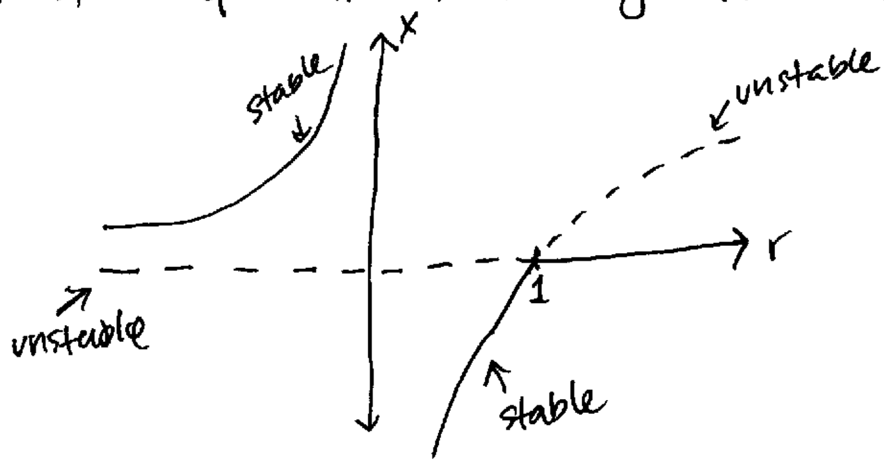
$r \cancel{2} > 1$



$r<0$

we have two fixed points at $x_1^* = 0$ and $x_2^* = \frac{r-1}{r}$.
The bifurcation point occurs when $r_c = 1$ and $x_c = 0$
because this is when $x_1^* = x_2^* \Rightarrow 0 = \frac{r-1}{1} \Rightarrow r_c = 1$
$x_c = 0$

Thus, we get the following bifurcation diagram:



$f(x) = x - rx(1-x) \Rightarrow f'(x) = 1 - r + 2rx$

we plug in two fixed points into $f'(x)$ to determine stability.

$$f'(0) = 1 - r \begin{cases} r<1 \rightarrow \text{point is unstable because } f'(0)>0 \\ r>1 \rightarrow \text{stable} \\ r=1 \rightarrow \text{semi-stable by looking @ the graph} \end{cases}$$

$f'\left(\frac{r-1}{r}\right) = 1 - r + 2r\left(\frac{r-1}{r}\right) = 1 - r + 2r - 2 = r - 1$

$$\text{when } x_2^* = \frac{r-1}{1} \begin{cases} r<1 \rightarrow \text{stable} \\ r=1 \rightarrow \text{semi stable} \\ r>1 \rightarrow \text{unstable} \end{cases}$$

Expanding RHS $f(x,r)$ about the bifurcation point to put in the ODE in normal form near the bifurcation point(s)

From lecture notes on 01/11/2021, we have the following Taylor expansion series of $f(x,r)$ near bifurcation point

$$f(x,r) = \underbrace{f(x_c, r_c)}_{\text{goes to zero (F.P.)}} + \overbrace{(x - x_c) \frac{df}{dx}\Big|_{\substack{x=x_c \\ r=r_c}}}^{\text{zero @ bif. point}} + \underbrace{(r-r_c)\frac{df}{dr}\Big|_{(x_c,r_c)}}_{1} +$$

$$+ \underbrace{\frac{1}{2}(x-x_c)^2 \frac{d^2f}{dx^2}\Big|_{(x_c,r_c)}}_{2} + \underbrace{\frac{1}{2}(r-r_c)^2 \frac{d^2 f}{dr^2}\Big|_{(x_c r_c)}}_{3} +$$

$$+ \underbrace{(x-x_c)(r-r_c)\frac{d^2 f}{dx dr}\Big|_{(x_c,r_c)}}_{4} + O(3)$$

$$f(x,r) = x - rx + rx^2$$

1 $$\frac{df}{dr} = -x + x^2$$
$$= 0 \text{ @ bif. point @ } x_c = 0, r_c = 1$$

2 $$\frac{d^2f}{dx^2} = 2r \Big|_{\substack{x=0 \\ r=1}} = $$
$$= 2$$

3 $$\frac{d^2f}{dr^2} = 0 \text{ @ bif. point } x=0, r=1$$

4 $$\frac{d^2f}{dxdr} = -1 + 2x \Big|_{\substack{x=0 \\ r=1}} = -1$$

$$\Rightarrow f(x,r) = \frac{1}{2}(x-0)^2 \cdot 2 + (x-0)(r-1)\cdot(-1) =$$
$$= x^2 - rx + x$$
$$= x^2 + x(1-r) \Rightarrow \text{we can set } v = -x$$
$$\dot{v} = -\dot{x}$$
$$\dot{v} = -v^2 + v\underbrace{(1-r)}_{R}$$
$$\dot{v} = -v^2 + Rv$$

Thus, the normal form of trans. bifurcation.

Problem 1 part b

$\dot{x} = rx - \dfrac{x}{1+x^2}$, Pitchfork bifurcation

$= x\left(r - \dfrac{1}{1+x^2}\right) \Rightarrow$ fixed point @ $x=0$ and

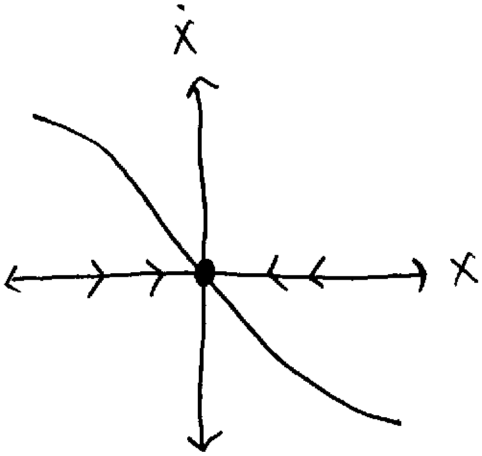
$r = \dfrac{1}{1+x^2} \Rightarrow 1+x^2 = \dfrac{1}{r} \Rightarrow x^2 = \dfrac{1}{r} - 1$

$\Rightarrow x = \pm\sqrt{\dfrac{1}{r} - 1}$

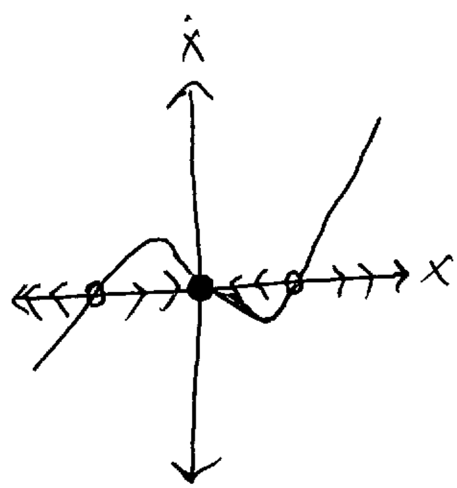
for $0 < r < 1$

Hence, we get three fixed points:

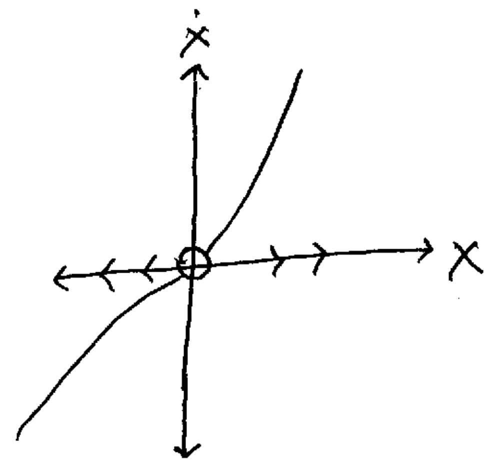
$x_1^* = 0$, $x_2^* = \sqrt{\frac{1}{r} - 1}$, $x_3^* = -\sqrt{\frac{1}{r} - 1}$



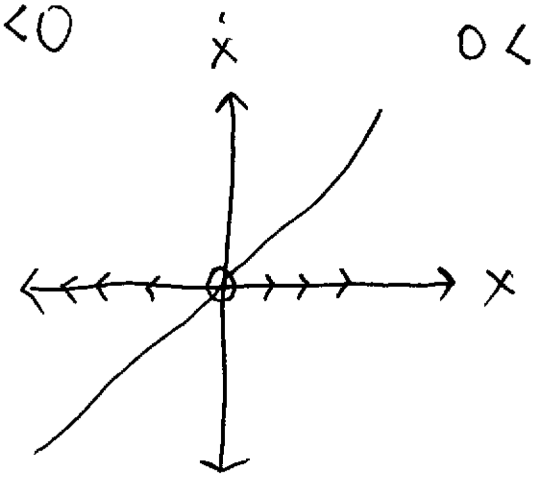
$r<0$



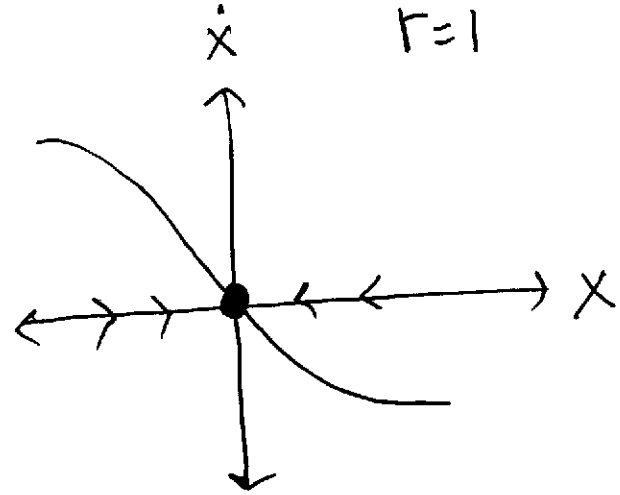
$0<r<1$



$r=1$



$r>1$



$r=0$

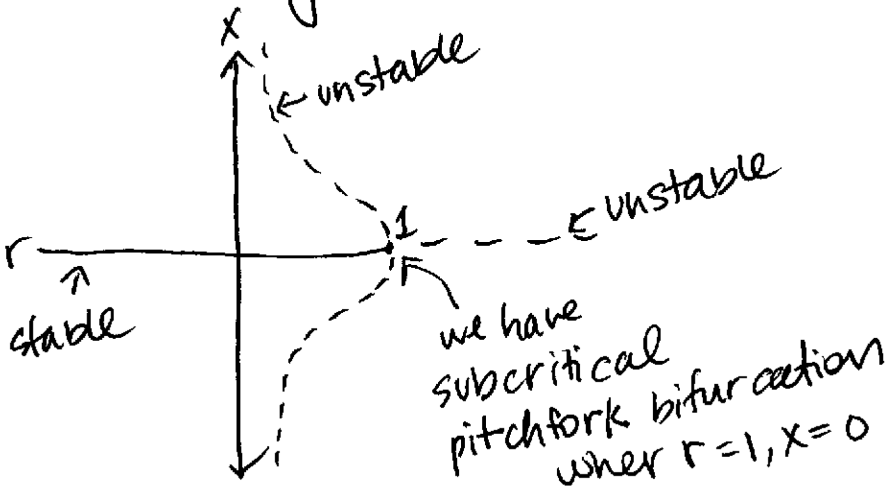
we have to make the fixed points equal in order to find the bifurcation point. It occurs when:

$$0 = \sqrt{\frac{1}{r} - 1} \quad \text{and/or} \quad 0 = -\sqrt{\frac{1}{r} - 1}$$

Thus, we get $1 = \frac{1}{r} \Rightarrow r = 1$.

The bifurcation point is at $r_c = 1$ and $x_c = 0$.

The following is the bifurcation diagram:



$$f(x) = rx - \frac{x}{1+x^2} \Rightarrow f'(x) = r - \frac{1-x^2}{(1+x^2)^2}$$

we plug in the fixed points into $f'(x)$ to determine stability.

$$f'(0) = r - 1$$

- $r < 1 \rightarrow$ stable
- $r = 1 \rightarrow$ unstable
- $r > 1 \rightarrow$ stable

$$f'\left(\sqrt{\tfrac{1}{r} - 1}\right) = r - \frac{1 - \frac{1}{r} + 1}{\left(1 + \frac{1}{r} - 1\right)^2} = r - \frac{2 - \frac{1}{r}}{\frac{1}{r^2}} = r - \left(2 - \frac{1}{r}\right) \cdot \frac{r^2}{1} =$$

$$= r - (2r^2 - r) = 2r - 2r^2 = 2r(1-r)$$

$\rightarrow$ unstable fixed point,

Expanding RHS $f(x,r)$ @ bifurcation point

$$f(x,r) = \underbrace{\cancel{f(x_c, r_c)}}_{\text{zero (F.P)}} + (x-x_c)\underbrace{\cancel{\frac{df}{dx}}\Big|_{x_c, r_c}}_{\text{zero @ bif. pt.}} + \underbrace{(r-r_c)\frac{df}{dr}}_{①}\Big|_{x_c, r_c} +$$

$$+ \underbrace{\frac{1}{2}(x-x_c)^2}_{②}\frac{d^2f}{dx^2}\Big|_{x_c, r_c} + \underbrace{\frac{1}{2}(r-r_c)^2\frac{d^2f}{dr^2}}_{③}\Big|_{x_c, r_c} +$$

$$+ \underbrace{(x-x_c)(r-r_c)}_{④}\frac{d^2f}{dx\,dr}\Big|_{x_c, r_c} + \underbrace{\frac{1}{6}(r-r_c)^3\frac{d^3f}{dr^3}}_{⑤}\Big|_{x_c, r_c} +$$

$$+ \underbrace{\frac{1}{2}(r-r_c)^2(x-x_c)}_{⑥}\frac{d^3f}{dx\,d^2r}\Big|_{x_c, r_c} + \underbrace{\frac{1}{2}(r-r_c)(x-x_c)^2}_{⑦}\frac{d^3f}{d^2x\,dr}$$

$$+ \underbrace{\frac{1}{6}(x-x_c)^3}_{⑧}\frac{d^3f}{dx^3}\Big|_{x_c, r_c} + O(4)$$

$$f(x,r) = rx - \frac{x}{1+x^2}$$

① $\frac{df}{dr} = x \Rightarrow (r-1)x$

② $\frac{d^2f}{dx^2} = \frac{4x}{(x^2+1)^2} - x\left(\frac{8x^2}{(x^2+1)^3} - \frac{2}{(x^2+1)^2}\right) = 0$

③ $\frac{d^2f}{dr^2} = 0$

④ $\frac{d^2f}{dx\,dr} = 0$

⑤ $\frac{d^3f}{dr^3} = 0$

⑥ $\frac{d^3f}{dx\,d^2r} = 0$

⑦ $\frac{d^3f}{d^2x\,dr} = 0$

⑧ $\frac{d^3f}{dx^3} = \frac{1}{6}(x-0)^3 \cdot 6 = x^3$

$$\Rightarrow f(x,r) = (r-1)x + x^3 = x\underbrace{(r-1)}_{R} + x^3$$

$$= xR + x^3$$

↳ Normal form

Problem 1 part c

Saddle-Node bifurcation: $\dot{x} = 1 + rx + x^2$

First, we need to find the fixed points

$$\dot{x} = 1 + rx + x^2$$

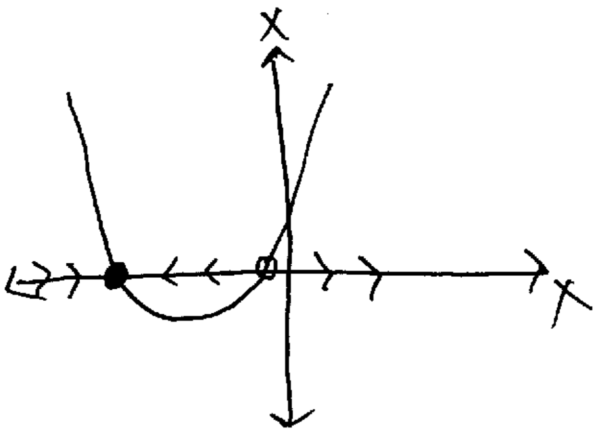
$$0 = 1 + rx + x^2 \Rightarrow x = \frac{-r \pm \sqrt{r^2-4}}{2} \quad \text{when } |r| \geq 2$$

so, there are two fixed points

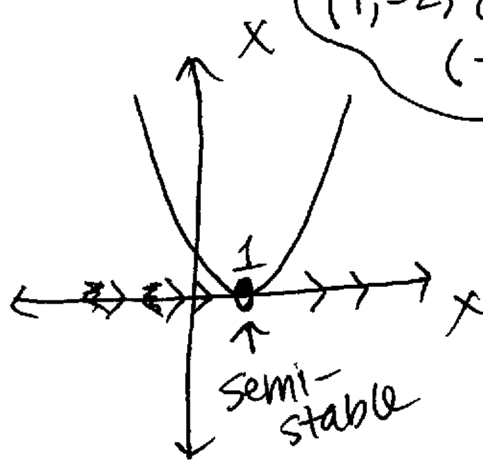
$$x_1^* = \frac{-r + \sqrt{r^2-4}}{2} \quad \text{and} \quad x_2^* = \frac{-r - \sqrt{r^2-4}}{2}$$

The bifurcation occurs when $r^2 - 4 = 0 \Rightarrow r = \pm 2$

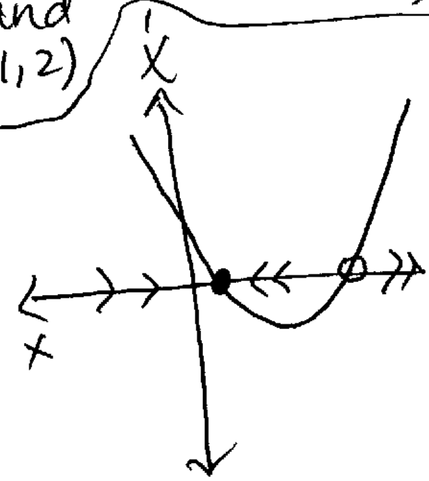
Thus, we have two bifurcation points at (1,-2) and (-1,2)



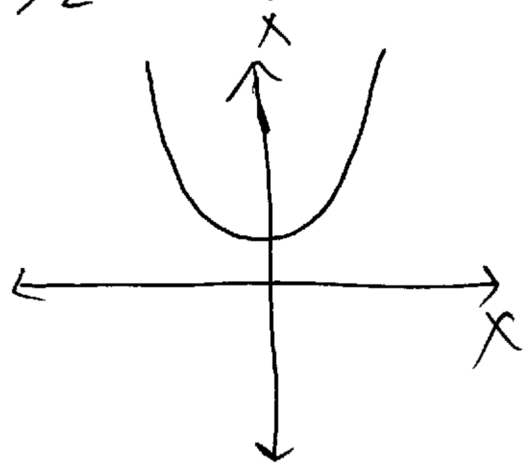
$r > 2$



$r = -2$

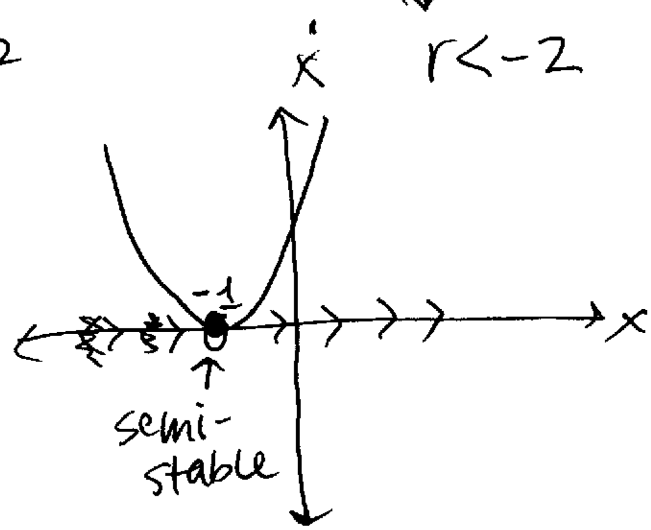


$r < -2$



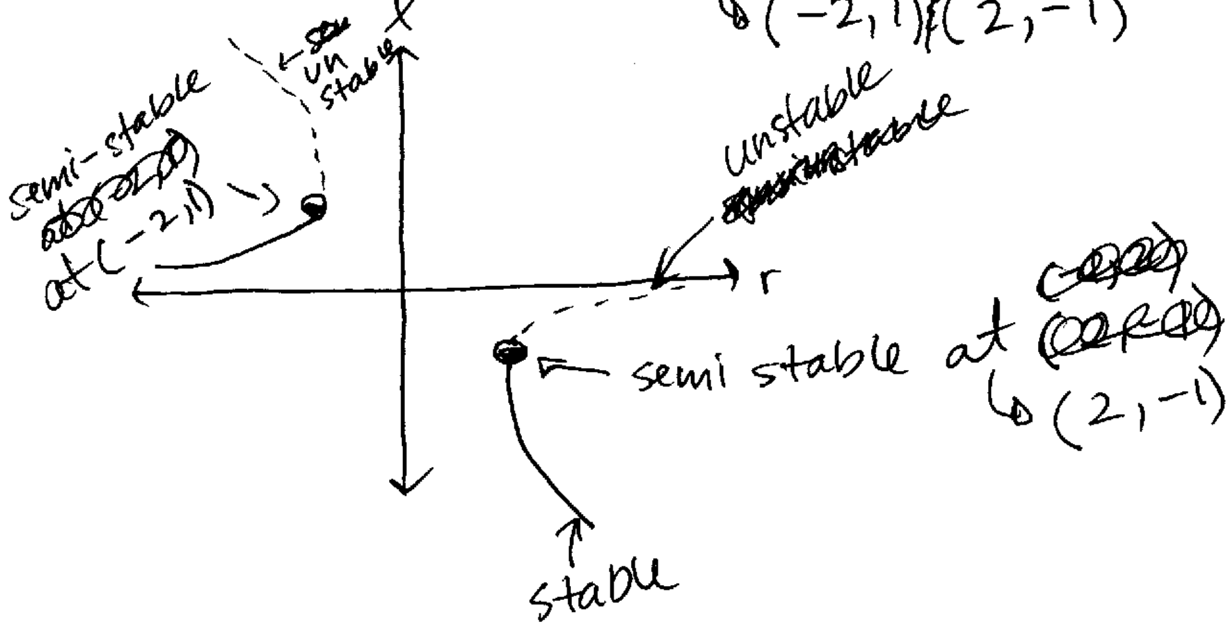
$|r| < 2 \Rightarrow$ no fixed points

$-2 < r < 2$



$r = 2$

The following is the bifurcation diagram.
Two bifurcation points at ~~(2,1) and (2,−1)~~ $(-2,1)$ & $(2,-1)$



$f(x) = 1 + rx + x^2 \Rightarrow f'(x) = r + 2x$

$f'\left(\frac{-r+\sqrt{r^2-4}}{2}\right) = r + 2\left(\frac{-r+\sqrt{r^2-4}}{2}\right) = \sqrt{r^2-4}$

This is unstable when $r=2$ or $r=-2$ because

$f\left(\frac{-r+\sqrt{r^2-4}}{2}\right) > 0$ when $r>2$ & $r>-2$

$f\left(\frac{-r-\sqrt{r^2-4}}{2}\right) = r + 2\left(\frac{-r-\sqrt{r^2-4}}{2}\right) = -\sqrt{r^2-4}$

This is stable when $r=2$ or $r=-2$ because

$f\left(\frac{-r-\sqrt{r^2-4}}{2}\right) < 0$ when $r>2$ & $r>-2$

Expanding RHS $f(x,r)$ about the bifurcation points to put in the ODE in normal form near the bifurcation points.

$$f(x,r) = \overbrace{f(x_c, r_c)}^{\text{goes to zero}} + (x - x_c)\,\overbrace{\cancel{\frac{df}{dx}}}^{\text{zero at bif. point}}\Big|_{\substack{x_c\\ r_c}} + \underbrace{(r - r_c)\frac{df}{dr}}_{①}\Big|_{\substack{x_c\\ r_c}} +$$

$$+ \underbrace{\frac{1}{2}(x - x_c)^2}_{②}\frac{d^2f}{dx^2}\Big|_{\substack{x_c\\ r_c}} + \underbrace{\frac{1}{2}(r - r_c)^2}_{③}\frac{d^2f}{dr^2}\Big|_{\substack{x_c\\ r_c}} +$$

$$+ \underbrace{(x - x_c)(r - r_c)}_{④}\frac{d^2f}{dxdr}\Big|_{\substack{x_c\\ r_c}} + \mathcal{O}(3)$$

$f(x,r) = 1 + rx + x^2$ bifurcation point at $(1, -2)$

① $\frac{df}{dr} = x = 1 \Rightarrow (r+2)$ ② $\frac{d^2f}{dx^2} = 2 \Rightarrow (x-1)^2$

③ $\frac{d^2f}{dr^2} = 0$ ④ $\frac{d^2f}{dxdr} = 1 \Rightarrow (x-1)(r+2)$

$$\Rightarrow f(x,r) = (r+2) + (x-1)^2 + (x-1)(r+2) =$$

$$= r + 2 + x^2 - 2x + 1 + xr + 2x - r - 2 =$$

$$= \underbrace{x^2 + xr + 1}_{\text{back to original function}}$$

$$f(x,r) = \underbrace{f(x_c, r_c)}_{\text{goes to zero}} + (x - x_c)\underbrace{\cancel{\frac{df}{dx}}}_{\text{zero @ bif. pt.}} + \underbrace{(r - r_c)\frac{df}{dr}}_{①}\Big|_{\substack{x_c\\r_c}} +$$

$$+ \frac{1}{2}(x - x_c)^2 \frac{d^2f}{dx^2}\Big|_{\substack{x_c\\r_c}} + \frac{1}{2}(r - r_c)^2 \frac{d^2f}{dr^2}\Big|_{\substack{x_c\\r_c}} +$$

$$+ (x - x_c)(r - r_c)\frac{d^2f}{dxdr}\Big|_{\substack{x_c\\r_c}} + O(3)$$

$f(x,r) = 1 + rx + x^2$ bifurcation point @ $(-1, 2)$

① $\frac{df}{dr} = x \Rightarrow -(r-2) = 2 - r$ ② $\frac{d^2f}{dx^2} = 2 \Rightarrow (x+1)^2$

③ $\frac{d^2f}{dr^2} = 0$ ④ $\frac{d^2f}{dxdr} = 1 \Rightarrow (r - 2)(x+1)$

$$\Rightarrow f(x,r) = (2-r) + (x+1)^2 + (r-2)(x+1) =$$

$$= 2 - r + x^2 + 2x + 1 + rx + r - 2x - 2 =$$

$$= \underbrace{x^2 + xr + 1}_{\text{back to original function}}$$

Problem #1. Part d.

Strogatz 3.1.5 $\dot{x} = r^2 - x^2$

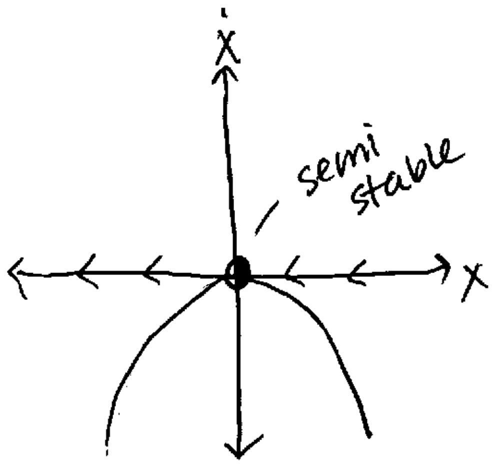
To find the fixed points $\Rightarrow$ $0 = r^2 - x^2$

$x^2 = r^2 \Rightarrow x = \pm\sqrt{r^2}$

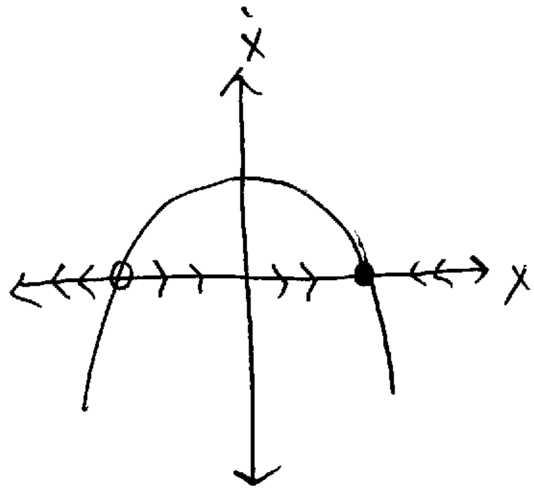
$x = \pm|r|$

Two fixed points $\Rightarrow$ $x_1^* = +|r|$

$x_2^* = -|r|$

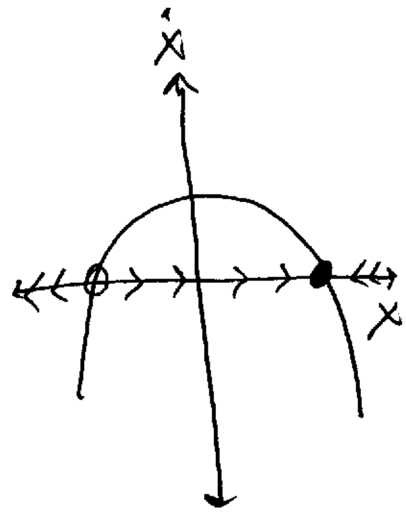


$r = 0$



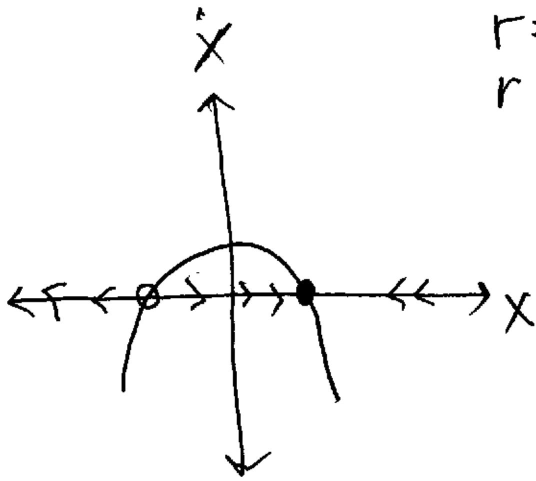
$r = -2$

$r < 0$



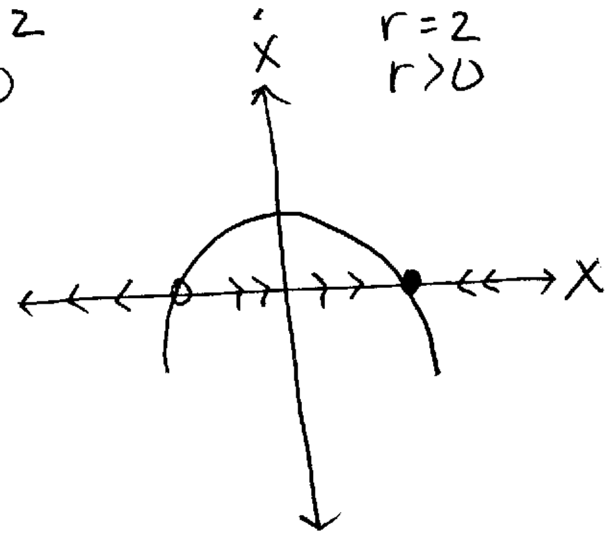
$r = 2$

$r > 0$



$r = -1$

$r < 0$

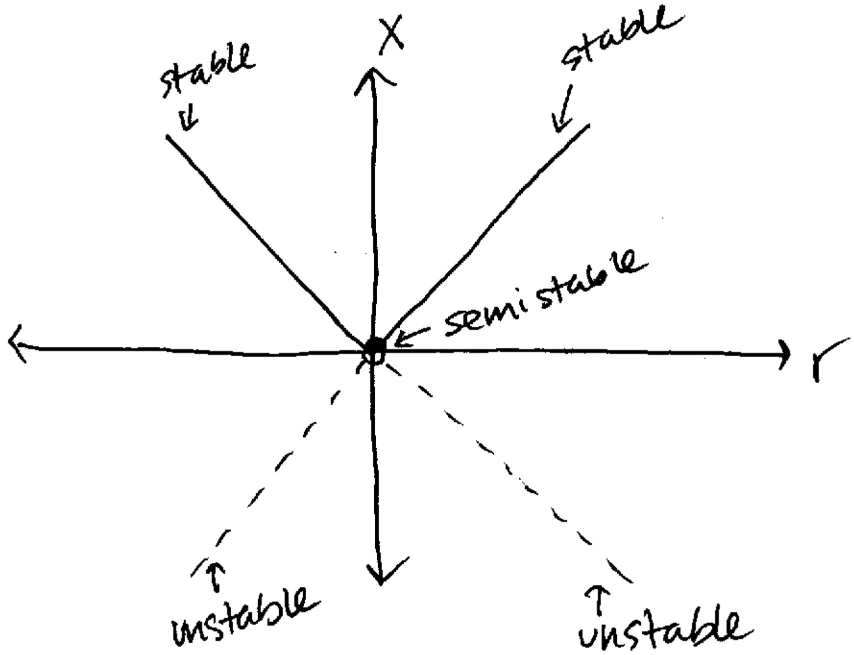


$r = 1$ $r > 0$

Bifurcation happens when $x_1^* = x_2^* \Rightarrow +|r| = -|r|$

Thus, it's zero. So, $x_c = 0$ and $r_c = 0$ $(0,0)$

The following is the bifurcation diagram. One bifurcation point at $(0,0)$



$f(x) = r^2 + x^2 \Rightarrow f'(x) = 2x$

$f'(|r|) = 2 \cdot |r| = 2|r|$

we have $f'(|r|) > 0$, thus it's unstable

$f'(-|r|) = 2 \cdot (-|r|)$

we have $f'(-|r|) < 0$, thus it's stable

Hence, the bifurcation point at $(0,0)$ is semi-stable

$$f(x,r) = \underbrace{f(x_c, r_c)}_{\text{goes to zero}} + (x - x_c)\underbrace{\frac{df}{dx}}_{\text{zero @ bif. pt.}} + \underbrace{(r - r_c)}_{①}\frac{df}{dr}\Big|_{\substack{x_c\\r_c}} +$$

$$+ \underbrace{\frac{1}{2}(x - x_c)^2}_{②}\frac{d^2f}{dx^2}\Big|_{\substack{x_c\\r_c}} + \underbrace{\frac{1}{2}(r - r_c)^2}_{③}\frac{d^2f}{dr^2}\Big|_{\substack{x_c\\r_c}} +$$

$$+ \underbrace{(x - x_c)(r - r_c)}_{④}\frac{d^2f}{dxdr}\Big|_{\substack{x_c\\r_c}} + O(3)$$

$$f(x,r) = r^2 - x^2 \text{ @ } (0,0)$$

① $\frac{df}{dr} = 2r \Big|_{\substack{r=0\\x=0}} = 0$

② $\frac{d^2f}{dx^2} = -2 \Rightarrow -x^2$

③ $\frac{d^2f}{dr^2} = 2 \Rightarrow r^2$

④ $\frac{d^2f}{dxdr} = 0$

$$\Rightarrow f(x,r) = \underbrace{r^2 - x^2}_{\text{back to original function}}$$

$\boxed{\text{Describe what we see}}$

Based on the bifurcation diagram, we see that this diagram does not represent a saddle-node bifurcation. Because as r changes, the positions of the fixed points change as well. This means that the bifurcation diagram appears to look like the transcritical bifurcation.

Problem 2

$$\dot{x} = (rx + x^3)(r + 2 - x^2)$$

Fixed points $rx + x^3 = 0 \Rightarrow x = 0,\ x = \pm\sqrt{-r}$

$r + 2 - x^2 = 0 \Rightarrow x = \pm\sqrt{r+2}$ $r > -2$

This gives us 5 F.P.s $x_1^* = 0$, $x_2^* = +\sqrt{-r}$ ($r<0$), $x_3^* = -\sqrt{-r}$ ($r<0$), $x_4^* = \sqrt{r+2}$ ($r>-2$), $x_5^* = -\sqrt{r+2}$ ($r>-2$)

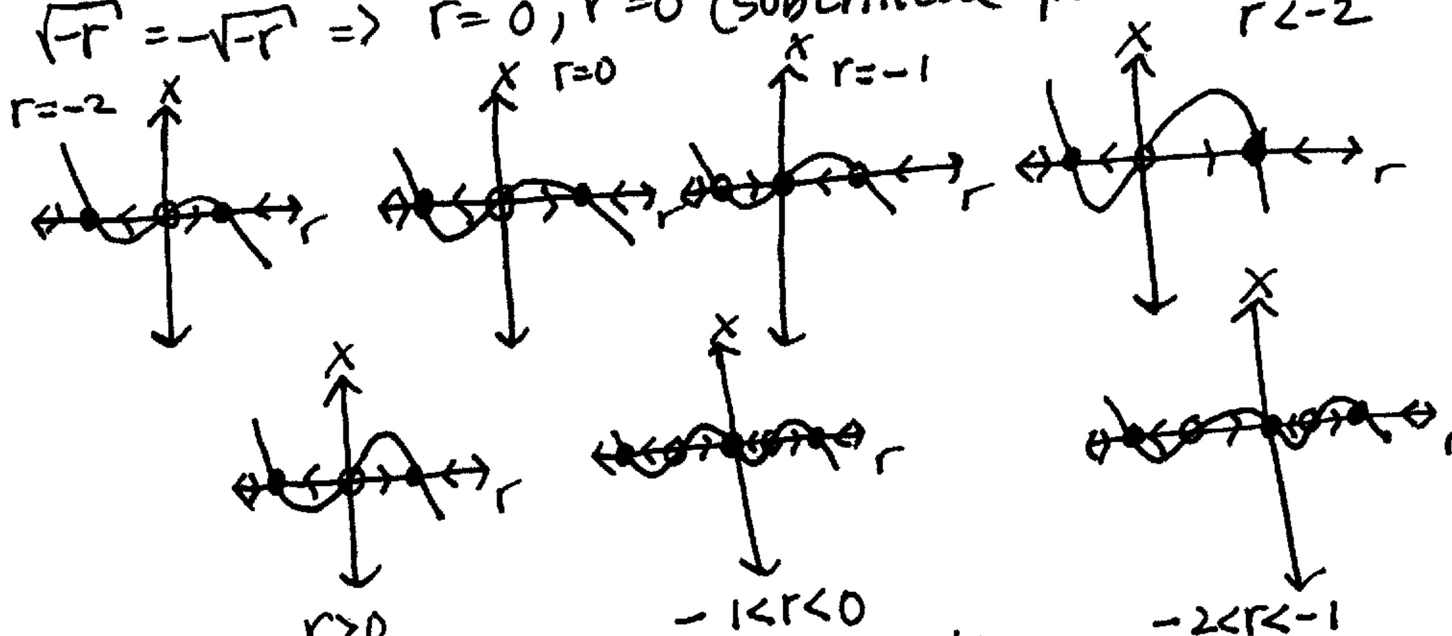
We get four bifurcation points:

① $-\sqrt{-r} = -\sqrt{r+2} \Rightarrow r = -1, x = -1$ (transcritical)

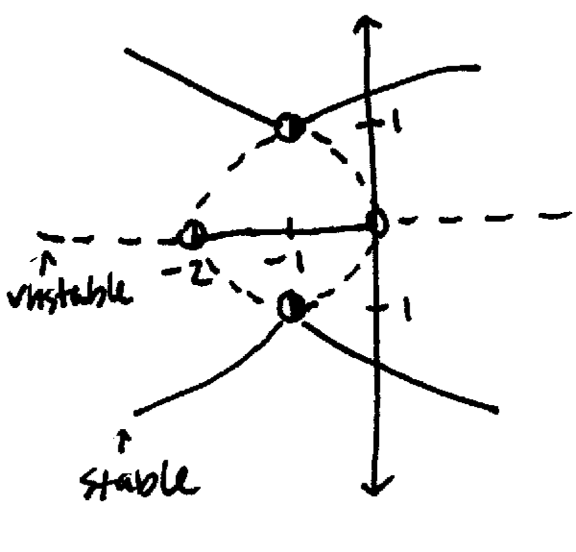
② $\sqrt{-r} = \sqrt{r+2} \Rightarrow r = -1, x = 1$ (transcritical)

③ $0 = \sqrt{r+2} \Rightarrow r = -2, x = 0$ (subcritical pitchfork)

④ $\sqrt{-r} = -\sqrt{-r} \Rightarrow r = 0, r = 0$ (subcritical pitchfork)



We get the following bifurcation diagram from the graphs



Transcritical bifurcation at (-1,1) (-1,1) because stability changes without the loss of two fixed points. Subcritical pitchfork bifurcation at (0,0) and (-2,0).

Problem 3

a) $\dot{N} = rN\left(1 - \frac{N}{K}\right) - H$

$\Rightarrow$ dividing by rK we get

$$\frac{\dot{N}}{rK} = \frac{N}{K}\left(1 - \frac{N}{K}\right) - \frac{H}{rK}$$

$\Rightarrow$ now substitute $x = \frac{N}{K}$, $\tau = rt$, $h = \frac{H}{rK}$

$$\frac{dx}{d\tau} = d\left(\frac{N}{K}\right) \cdot \frac{1}{d(rt)} \Rightarrow \left(\frac{1}{rK}\right) \cdot \dot{N} = \frac{\dot{N}}{rK}$$

Thus, we get by substitution the following system:

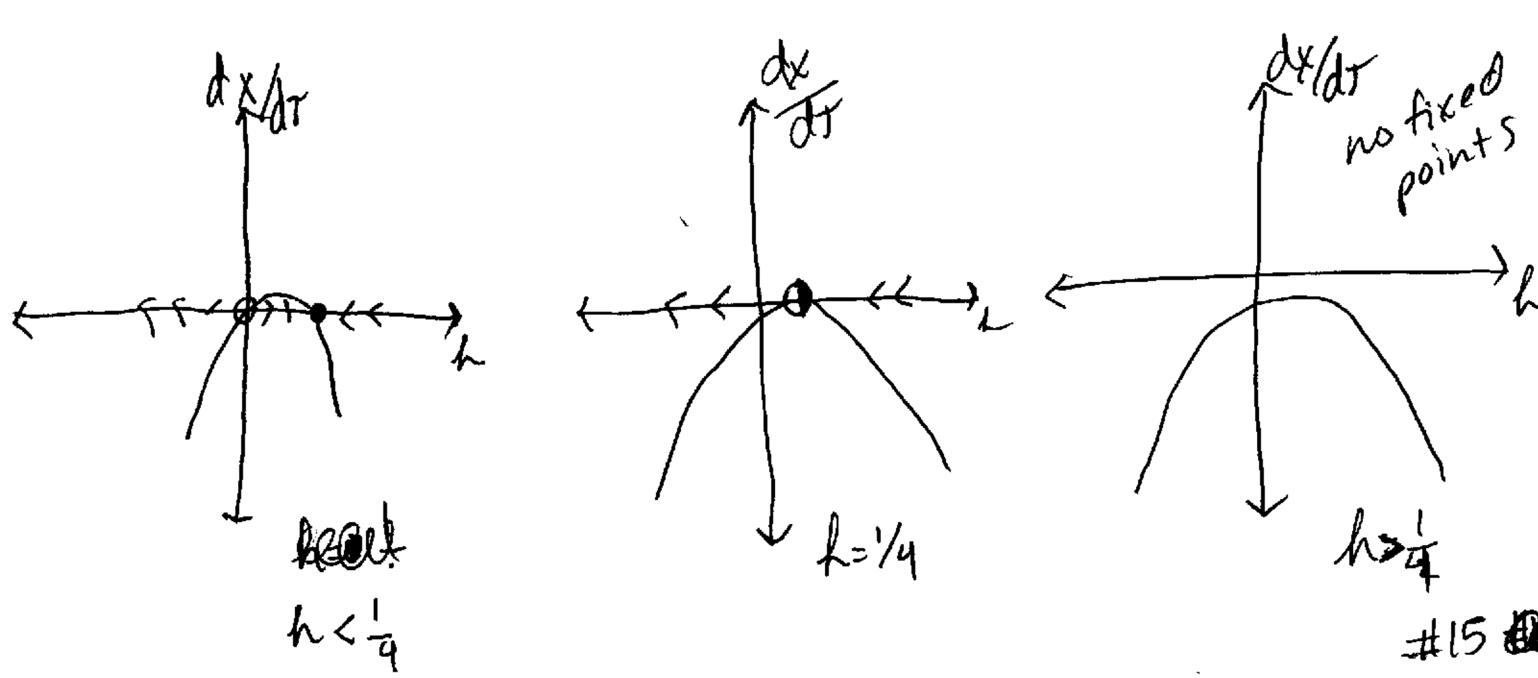
$$\frac{dx}{d\tau} = x(1-x) - h \text{ as desired}$$

b) In order to draw the qualitatively different phase portraits, we need to find the fixed points.

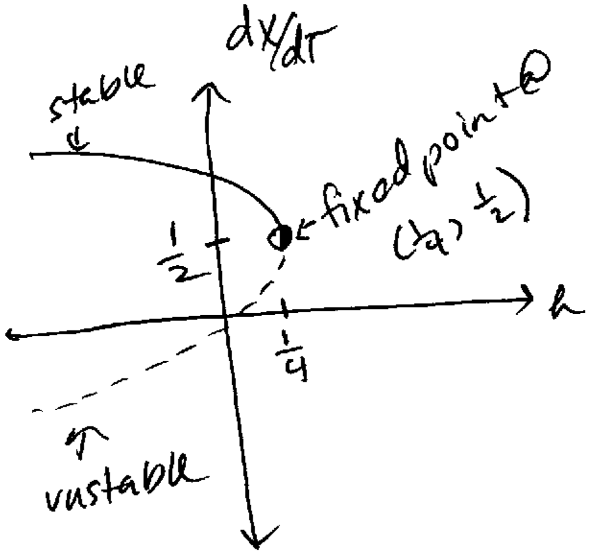
$0 = x - x^2 - h$ using quadratic equation, we get

$$x = \frac{1 \pm \sqrt{1-4h}}{2}$$

To get bifurcation point, we make $1-4h=0$ and get $h_c = \frac{1}{4}$ and $x_c = \frac{1}{2}$.



we draw the bifurcation diagram:



Based on the diagram we have a saddle node bifurcation because the stability changes from ~~unstable~~ stable to unstable and the fixed point becomes semi-stab

c) Long-term behavior

when $h < h_c$ ($h < \frac{1}{4}$) we have two fixed points. One point is stable and the other is unstable. If the fish population is close to the unstable point(0,0) then ~~the~~ fishermen will harvest all the fish and the population may die out. If the fishing population is close to the stable point, then the fishing population is increasing and there's always enough fish.

when $h > h_c$, then the fishing population is being overly demanded by fishermen and eventually they may harvest out all the fish.

Problem: when fishing population is zero (extinct) it can not go to negative infinity but this model allows it. Instead, the model ~~level~~ should remain at extinction level.

Problem 4

a) $\dot{g} = K_1 S_0 - K_2 g + \frac{K_3 g^2}{K_4^2 + g^2}$ let $g = K_4 X$

we get

$\Rightarrow K_4 \dot{X} = K_1 S_0 - K_2 K_4 X + \frac{K_3 \cdot K_4^2 X^2}{K_4^2 + (K_4^2)(X^2)}$

$K_4 \dot{X} = K_1 S_0 - K_2 K_4 X + K_3 \cdot \frac{X^2}{(1+X^2)}$

$\Rightarrow$ divide both sides by K_3

$$\frac{K_4}{K_3} \dot{X} = \frac{K_1 S_0}{K_3} - \frac{K_2 K_4}{K_3} \cdot X + \frac{X^2}{(1+X^2)}$$

$\Rightarrow$ Now substitute the following

$$s = \frac{K_1 S_0}{K_3} \qquad r = \frac{K_2 K_4}{K_3} \qquad \tau = \frac{K_3}{K_4} t$$

we get $\Rightarrow \frac{dx}{d\tau} = s - r \cdot X + \frac{X^2}{(1+X^2)}$ as desired.

b) If we ~~ste~~ set $s=0$, we get the following

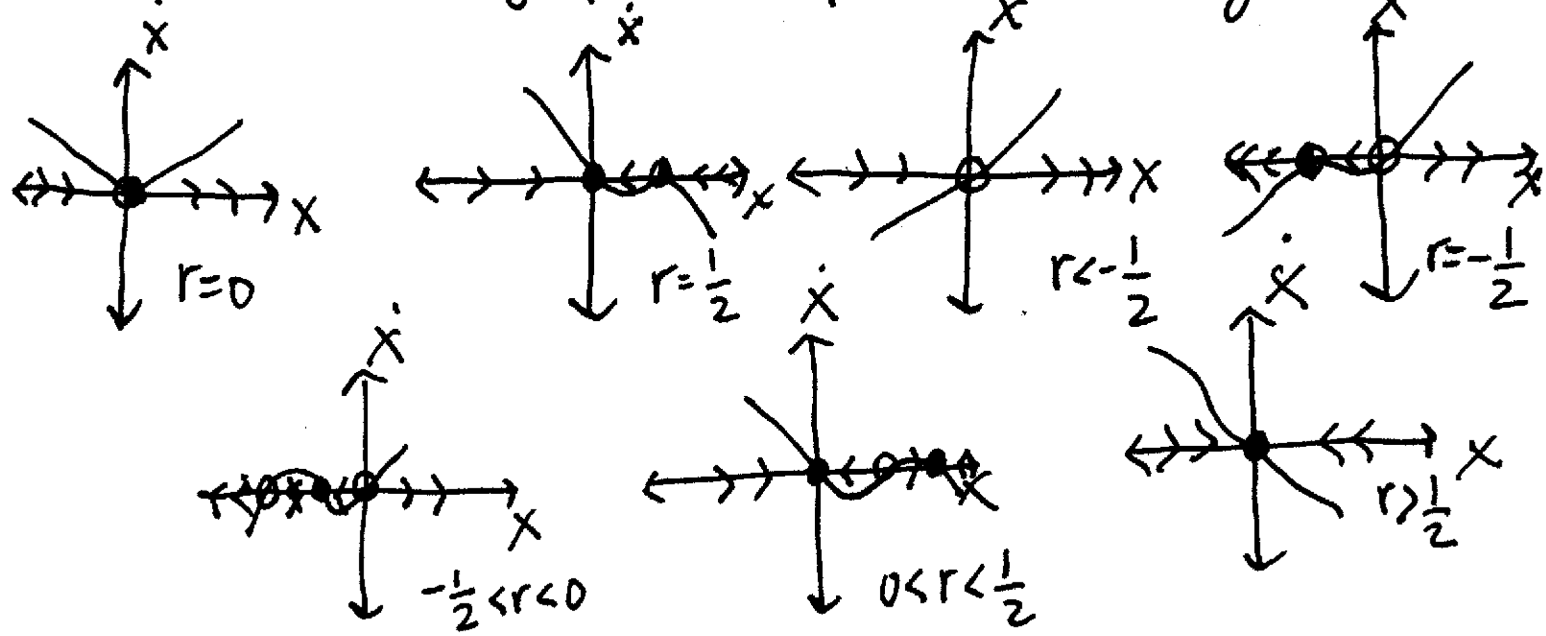
$\frac{dx}{d\tau} = 0 - rX + \frac{X^2}{(1+X^2)}$, we have three fixed points

$$X_1^* = 0, \; X_2^* = \frac{1 + \sqrt{1-4r^2}}{2r}, \; X_3^* = \frac{1 - \sqrt{1-4r^2}}{2r}$$

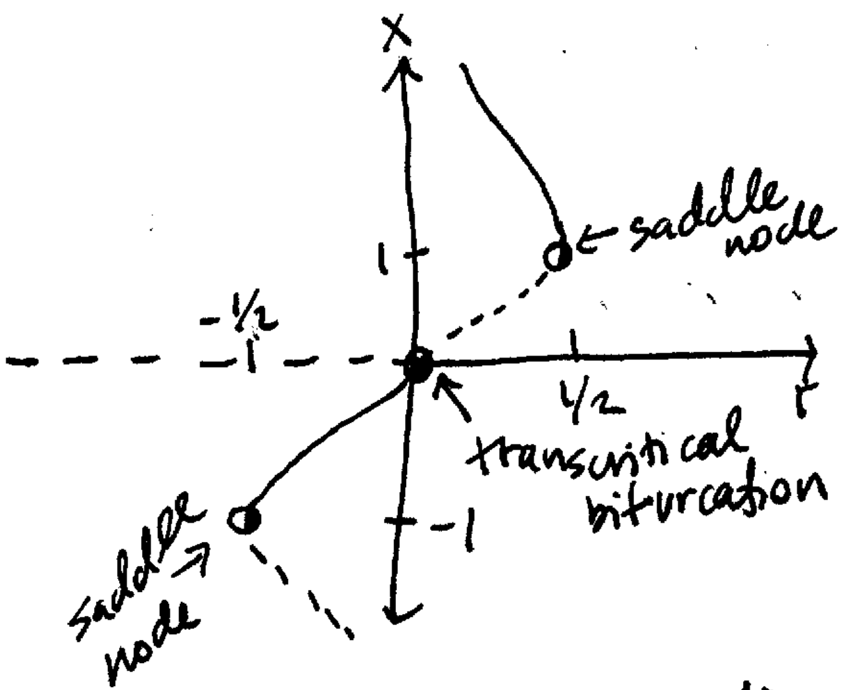
But only two are positive when $r_c = \frac{1}{2}$, $X_c = 1$

fixed points X_2^* and X_3^* are the two positive fixed points between $0 < r < 1/2$

c) First we draw 7 graphs to represent stabity



From these graph we can draw all bifurcations in a diagram



d) Based on the bifurcation diagram we have three bifurcations:

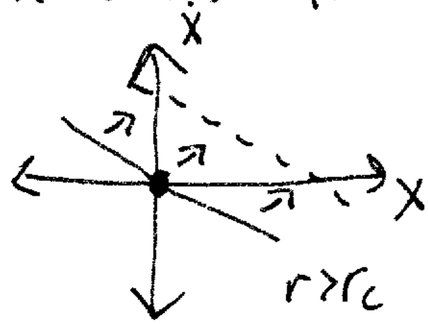
① at $r=\frac{1}{2}$ and $x=1$. This is a saddle node bifurcation

② at $r=0$ and $x=0$. This is a transcritical bifurcation

③ at $r=-\frac{1}{2}$ and $x=-1$. This is a saddle node bifurcation

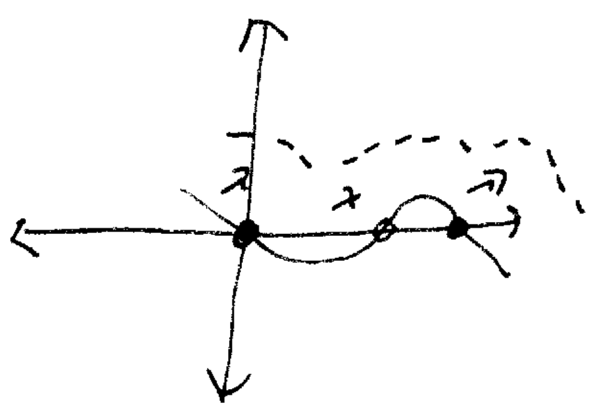
e. when we have $r > r_c$, we will have only one fixed stable point at (0,0). So, if the system increases g and t increase as well. The system will therefore go back to zero after bifurcation and will cause the gene to turn off.

Graph:



If $r < r_c$, then the system may not go back to it's to zero because it may be that the system went to far out and ended up coming back to the larger stable point instead of zero. So, if it does that the gene will not turn off. In order to make sure that the system goes back to zero, we need to make sure that it doesn't stay out for too long and comes back in time for the gene to turn off.

Graph:



Problem 5 (part a based on Professor's video)

a) $$\frac{dN}{dt} = RN\left(1-\frac{N}{K}\right) - \frac{BN}{A+N}; \quad N = N_0 x, \quad t = t_0 \tau$$

$$\frac{dN}{dt} = \frac{d(N_0 x)}{dt} = N_0 \frac{dx}{dt} = N_0 \frac{dx}{d\tau}\frac{d\tau}{dt} = N_0 \dot{x}\frac{1}{t_0} = \frac{N_0}{t_0}\dot{x}$$

$$\frac{N_0}{t_0}\dot{x} = RN_0 x\left(1 - \frac{N_0 x}{K}\right) - \frac{B(N_0 x)}{A + N_0 x}$$

$$\dot{x} = Rt_0 x\left(1 - \frac{N_0 x}{K}\right) - \frac{BN_0 t_0 x}{A + N_0 x}$$

$$= Rt_0 x\left(1 - \frac{N_0 x}{K}\right) - \frac{Bt_0 x}{N_0\left(\frac{A}{N_0} + x\right)}$$

① $\frac{A}{N_0} = 1 \Rightarrow A = N_0$ ② $\frac{Bt_0}{N_0} = 1 = \frac{B}{A}t_0 = 1$

$$\Rightarrow t_0 = \frac{A}{B}$$

If $N = Ax$, $t = \frac{A}{B}\tau$

$$\Rightarrow \dot{x} = \frac{RA}{B}x\left(1 - \frac{A}{K}x\right) - \frac{x}{1+x}$$

③ $\frac{RA}{B} = \alpha$, $\frac{K}{A} = \frac{1}{k}$

$$\Rightarrow \dot{x} = \alpha x\left(1 - \frac{x}{k}\right) - \frac{x}{1+x}$$ as desired

Problem 5 part b

$k=1 \qquad \dot{x} = \alpha x\left(1 - \frac{x}{k}\right) - \frac{x}{1+x}$ (from part a)

$\Rightarrow \dot{x} = \alpha x\left(1 - \frac{x}{1}\right) - \frac{x}{1+x}$

Need to find the fixed points by making the equation equal to zero:

$x\left(\alpha(1-x) - \frac{1}{1+x}\right) = 0 \Rightarrow$ first F.P. @ $x=0$

$\alpha(1-x) - \frac{1}{1+x} = 0 \Rightarrow$ F.P.s @ $x = \pm\sqrt{1-\frac{1}{\alpha}}$

Thus, we have three fixed points as follows:

$x_1^* = 0, \ \underbrace{x_2^* = \sqrt{1-\frac{1}{\alpha}}}_{a<1 \text{ or } a>1}, \ x_3^* = \underbrace{-\sqrt{1-\frac{1}{\alpha}}}_{a<0 \text{ or } a>1}$

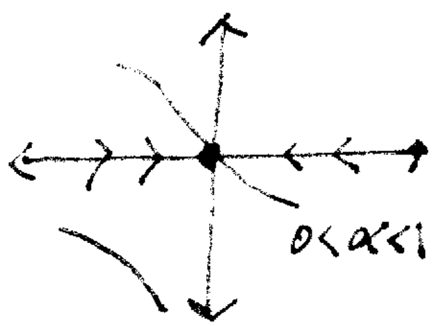
Now, we find the bifurcation points:

$0 = \sqrt{1-\frac{1}{\alpha}} \Rightarrow \alpha = 1$ and $x=0$

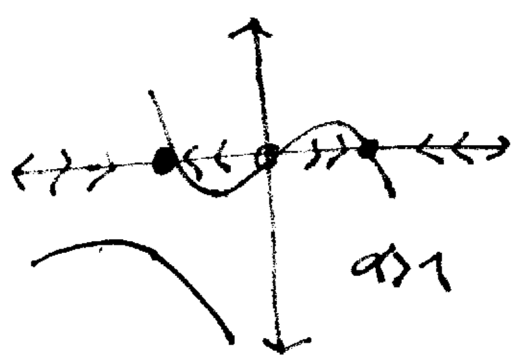
so, bifurcation point at $(1,0)$

Phase portraits of different a parameters:

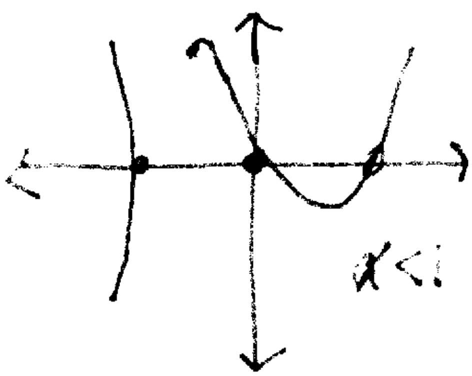
$0<\alpha<1$



$0<\alpha<1$

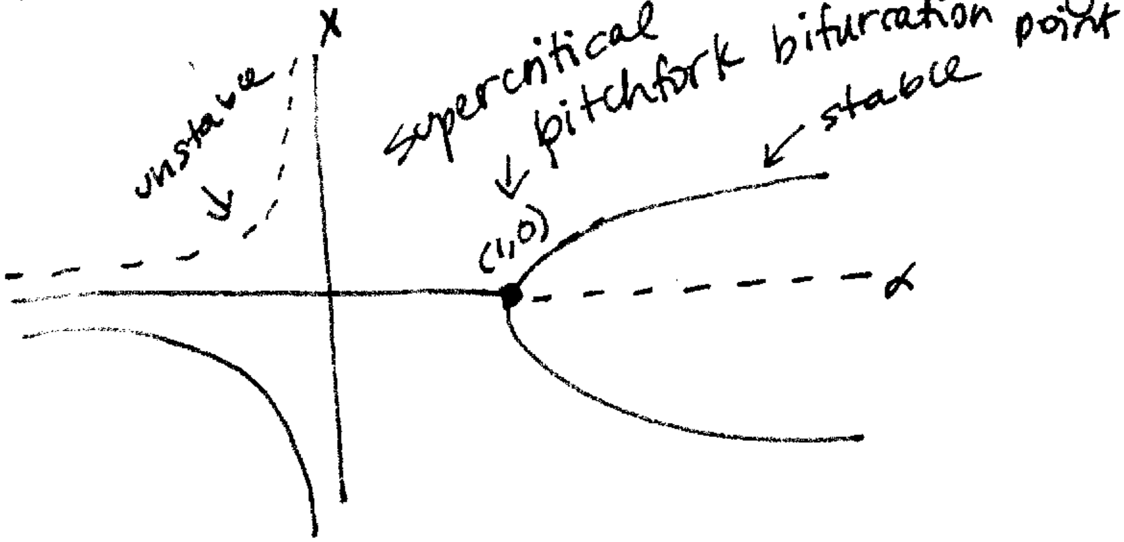


$\alpha>1$



$\alpha<1$

Now, we can draw the bifurcation diagram:



There is no hysteresis here because we have supercritical pitchfork bifurcation and the system will return to its' stable point after it shifts

c) $K=2$ $\dot{x} = \alpha x(1 - x/K) - x/(1+x)$ (from part a)

$$\Rightarrow \dot{x} = \alpha x\left(1 - \frac{x}{2}\right) - \frac{x}{(1+x)}$$

$$x\left(\alpha(1-x) - \frac{1}{(1+x)}\right) = 0 \Rightarrow \text{first F.P.} \Rightarrow x = 0$$

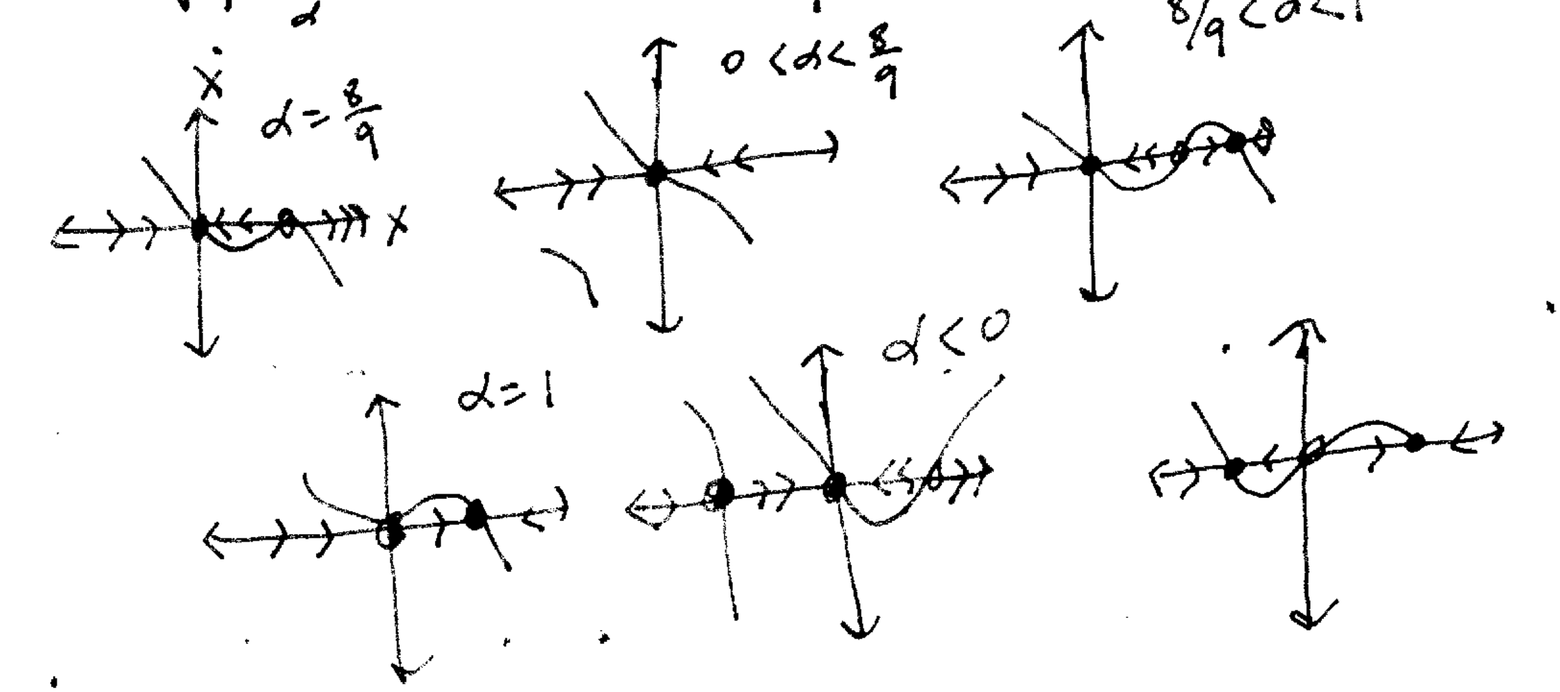
$$\alpha(1-x) - 1/(1+x) = 0 \Rightarrow \text{F.P.s @ } x = \frac{1 \pm \sqrt{9 - 8/\alpha}}{2}$$

Three F.Ps:

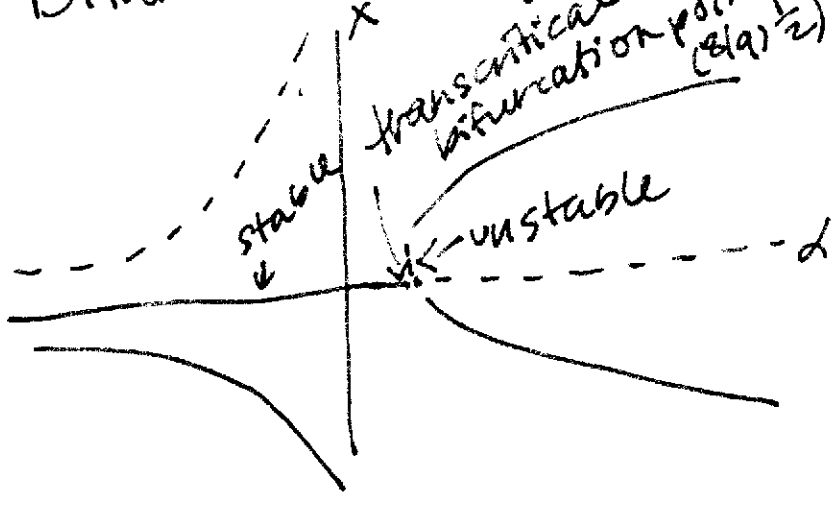
$$x_1^* = 0, \quad x_2^* = \frac{1 + \sqrt{9 - 8/\alpha}}{2}, \quad x_3^* = \frac{1 - \sqrt{9 - 8/\alpha}}{2}$$

Now, we find the bifurcation point:

$$\sqrt{9 - \frac{8}{\alpha}} = 0 \Rightarrow \alpha = \frac{8}{9} \quad x = \frac{1}{2} \text{ (Transcritical point)}$$



Bifurcation diagram



This system has hysteresis because the system will not return to the same stability point that it started with because we have a transcritical bifurcation point.