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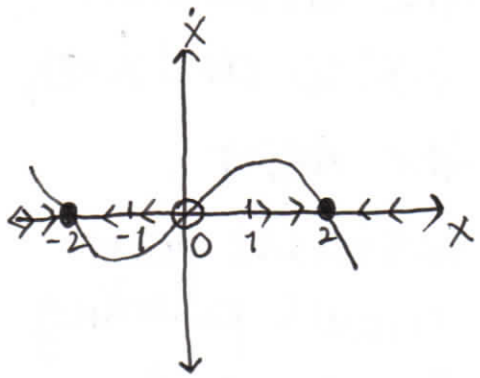
AMATH 502 A

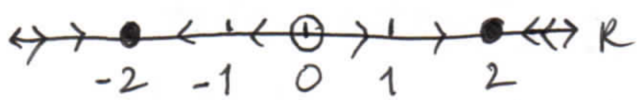
Problem # 1 Part a)

a) $\dot{x} = 4x - x^3$

First, let's draw a graph as follows:



Based on the graph, we can draw a phase portrait:



To solve for the equilibrium (fixed) points, we need to set the right hand side of the equation to zero:

$$4x - x^3 = 0$$

~~$x(4x -$~~

$x(4 - x^2) = 0 \Rightarrow$ we have first fixed point at zero

$$4 - x^2 = 0$$

$$x^2 = 4$$

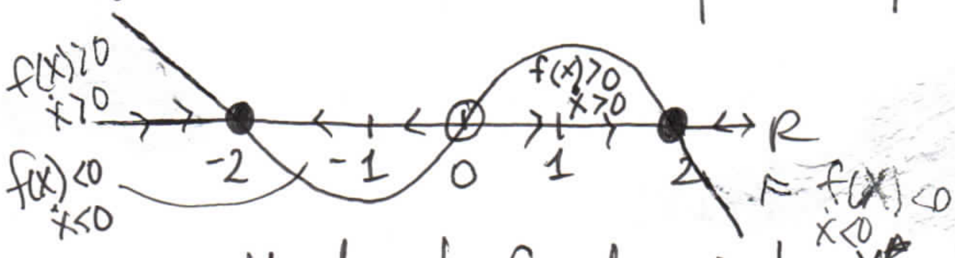
$$x = \sqrt{4}$$

$x = 2, -2 \Rightarrow$ Two more equilibrium points at $x = 2$ and $x = -2$

Therefore, there are three equilibrium points

$$x_1^* = 0, \; x_2^* = 2, \; x_3^* = -2$$

To indicate whether the points are stable or unstable, we need to look at the phase portrait.

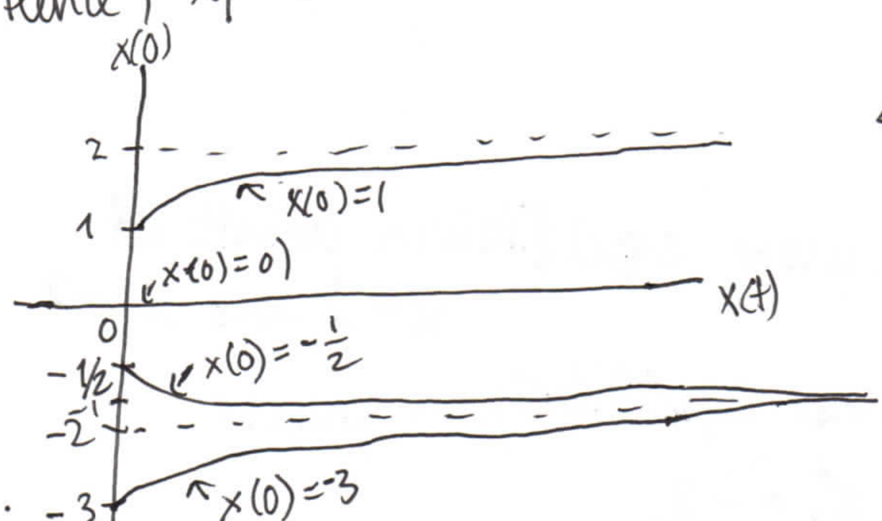


we see that at fixed point $x_1^*=0$, the arrows on left hand side point to the left because $f(x)<0$ and $\dot{x}<0$, the arrows to the right of $x_1^*=0$ point to the right because $f(x)>0$ and $\dot{x}>0$. Thus, $x_1^*=0$ is unstable point.

For fixed point $x_2^*=-2$, we have the arrows pointing to the right of the left hand side of $x_2^*=-2$, and the arrows to the right are pointing to the left because $f(x)<0$ and $\dot{x}<0$. Thus, $x_2^*=-2$ is a stable point.

For fixed point $x_3^*=2$, we have arrows to the left pointing in the right direction ($f(x)>0$ and $\dot{x}>0$) and the arrows to the right pointing ~~to~~ in the left direction ($f(x)<0$ & $\dot{x}<0$). Thus, $x_3^*=2$ is a stable point.

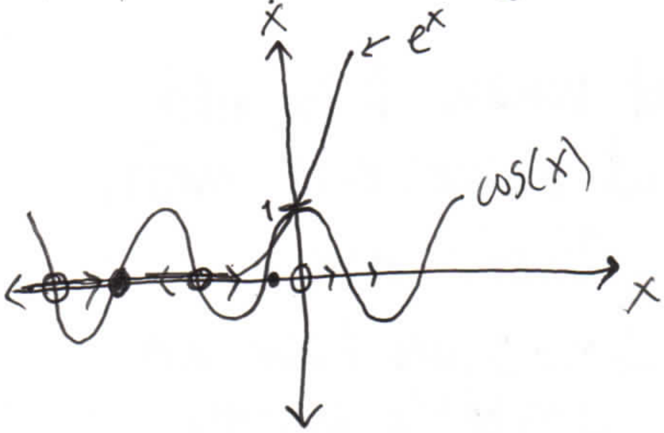
Hence, $x_1^*=0$ is unstable, $x_2^*=-2$ is stable, $x_3^*=2$ is stable.



← Graph of the solutions of four initial conditions $x(0)\in\{1,0,-\frac{1}{2},-3\}$

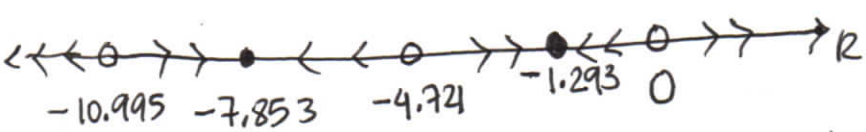
Problem #1 Part b. $\dot{x} = e^x - \cos(x)$

First, let's draw a graph as follows:



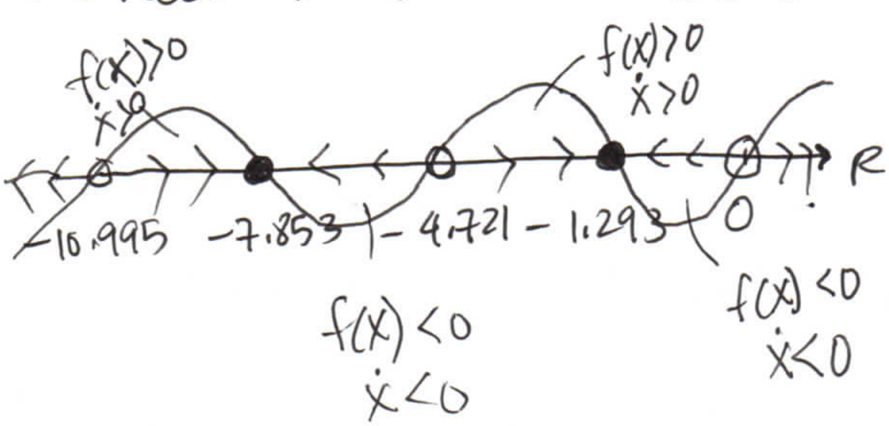
we draw the function $\dot{x} = e^x - \cos(x)$ as two separate functions to see the intersection points when $\dot{x} = 0$.

Based on the graph, we can draw a phase portrait:



To solve for the equilibrium points, we need to look at the intersection of both graphs when $\dot{x} = 0$. From the above graph, we see that e^x goes to zero as x goes to $-\infty$. Therefore, when $\cos(x) = 0$, we get equilibrium (fixed) points. We know that $\cos(x) = 0$ for $x = \frac{\pi}{2} - k\pi$ for $k = 1, 2, 3, 4 \ldots$ Thus, the first few equilibrium points are when $x = 0, -1.293, -4.721, -7.853, -10.995$.

To indicate whether the points are stable or unstable, we need to look at the directions of the arrows.

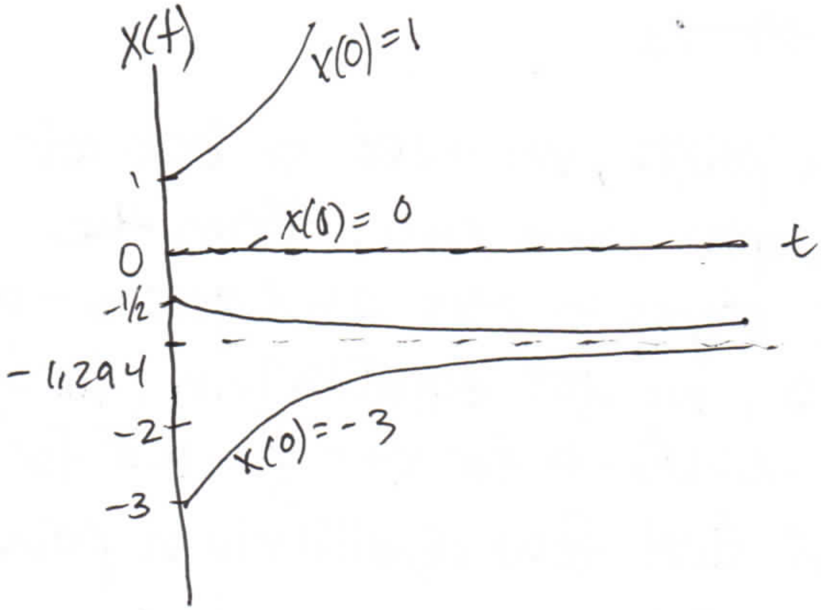


So, we get fixed points at $x \approx \frac{\pi}{2} - k\pi$ for $k \in \mathbb{N}$ and

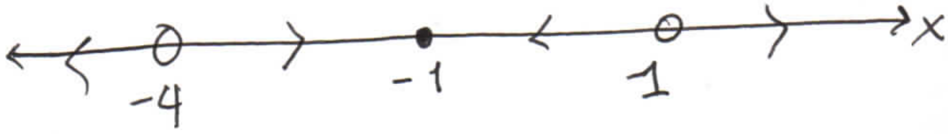
$x = 0$

Based on the graph, we see that when k is odd, the fixed points are stable, and when k is even, the fixed points are unstable. There are infinitely many fixed points. A $x=0$, we have an unstable point.

Sketh of the graph for each of the four initial conditions



Problem 2 Part a)



To find a dynamic system that has three fixed points at $x=-4$ unstable point, $x=-1$ stable point, and $x=1$ unstable point, we need to have a right hand side of the equation equal to zero, while all of these fixed points are present. One of the solutions for this given phase portrait is the following equation:

$$\dot{x} = (x+4)(x-1)(x+1)$$

Here, we see when the RHS is equal to zero, we get three equilibrium points:

$$(x+1)(x+4)(x-1) = 0$$

$$x_1^* = -4, \quad x_2^* = -1, \quad x_3^* = 1$$

To check the stability of these point for the given equation, we need to draw the graph of the equation.

Problem #2 Part b.

To find a dynamic system that has two equilibrium points at $x=-3$ semi stable point, and $x=1$ unstable point we need to have a right hand side equation that equals to zero, while all these fixed points are present. One of the solutions for this given phase portrait is the following equation:

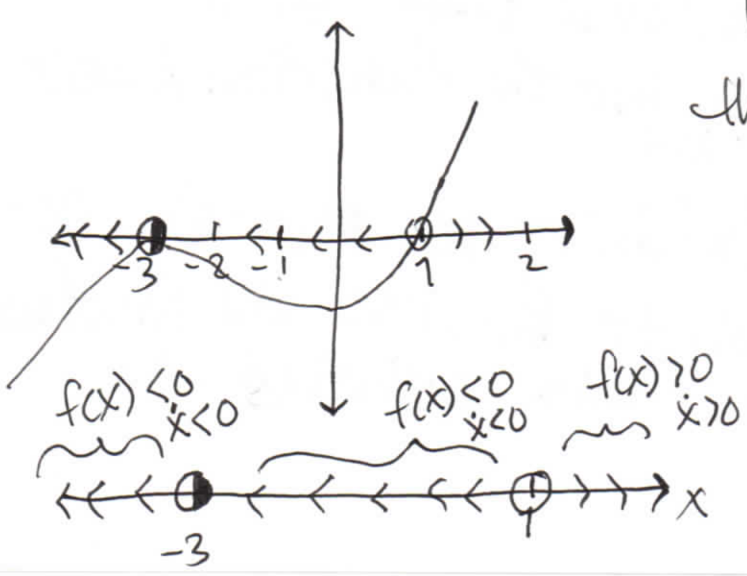
$$\dot{x} = (x-1)(x+3)^2$$

Here, we see when the RHS is equal to zero, we get two equilibrium points:

$$(x-1)(x+3)^2 = 0$$

$$x_1^* = 1, \quad x_2^* = -3$$

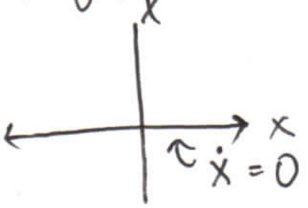
To check stability of these points for $\dot{x} = (x-1)(x+3)^2$ we need to draw the graph of the equation



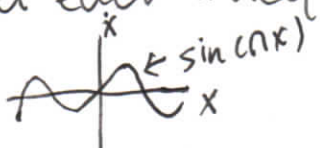
Thus, based on the graph and the phase portrait of the function $\dot{x} = (x-1)(x+3)^2$ we get the solution to the given problem, where there are two fixed points $x_1^* = 1$ unstable point and $x_2^* = -3$ semi stable point

Problem 3

a) Solution $= \dot{x} = 0$ because the flow of the function is zero for all values of x. Thus, it satisfies the requirement where every real number is a fixed point.



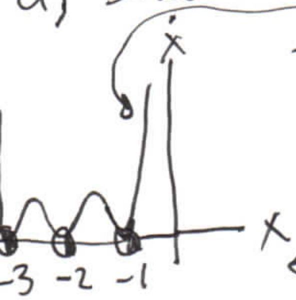
b) Solution: $\dot{x} = \sin(\pi x)$ because the flow of the function at each integer is a fixed point, and there are no others.



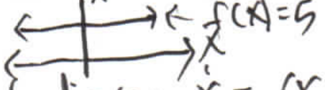
c). Solution: No solutions here because there has to be a point of opposite stability between two points of the same stability (because of the mean value theorem).

d) Solution: $\dot{x} = (x+3)^2 (x+1)^2 (x+2)^2$. $(x+3)^2 (x+1)^2 (x+2)^2 = 0$

There are three fixed points at $-3, -1, -2$ and they are all semi-stable, thus meeting the requirement of three unstable points.



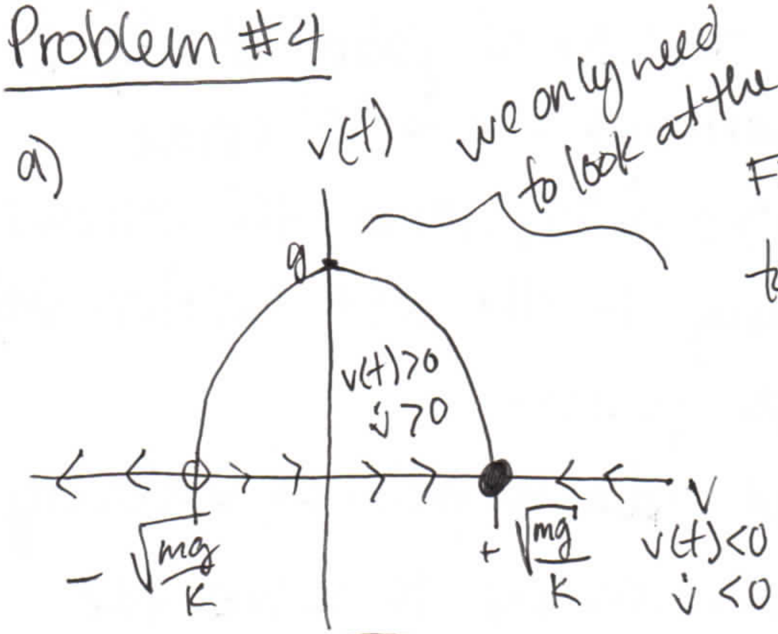
e) solution: $\dot{x} = 5$ satisfied the requirement where there are no fixed points because there are no intersection points when $x = 0$ and the function $\dot{x} = 5$ is just a constant.



f) Solution: $\dot{x} = (x-1)(x-2)(x-3)(x-4)(x-5) \ldots\ldots (x-100)$ has precisely 100 fixed points because the function intersects the x-axis at 100 points. This satisfies the requirement.

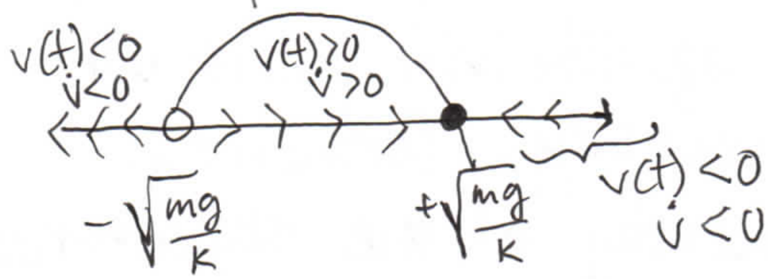
Problem #4

a)



First, let's draw a graph to vizualize the given function.

Based on the graph we can draw the phase portrait



In order to get the terminal velocity, $v(t)$ as $t \to \infty$, we need to find the equilibrivm point as follows

$$\dot{v} = \frac{mg - \text{sgn}(v)kv^2}{m}$$

$$0 = \frac{mg - \text{sgn}(v)kv^2}{m} = g - \frac{\text{sgn}(v)kv^2}{m}$$

$$g = \frac{\text{sgn}(v)kv^2}{m}$$

$$\frac{gm}{k} = \text{sgn}(v)v^2$$

$$\pm\sqrt{\frac{gm}{k}} = \text{sgn}(v)v$$

Because we have $v(t)>0$, we will only consider the positive $\sqrt{\frac{gm}{k}}$ as a fixed point.

Based on the phase portrait, the fixed point at $\sqrt{\frac{gm}{k}}$ is a stable point because the arrows on the left side are going to the right ($v(t)>0, \dot{v}>0$) and the arrows to the right side are going to the left ($v(t)<0, \dot{v}<0$). Hence, $\dot{v}=\sqrt{\frac{gm}{k}}$ is a stable point.

Furthermore, we can get the terminal velocity from phase portrait (without having to solve the equation) because the equilibrium point will show us the terminal velocity. Terminal velocity is the final velocity of the skydiver as it approaches the stable fixed point.

b) Used Mathematica to find the solution as follows:

$$v(t)=\sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{gk}{m}}\,t\right)$$

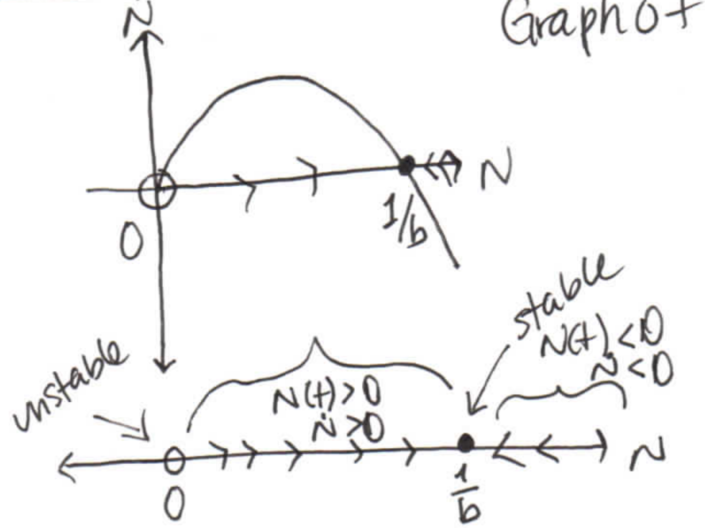
c)

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty}\sqrt{\frac{mg}{k}}\underbrace{\tanh\left(\sqrt{\frac{gk}{m}}\,t\right)}_{\text{goes to } 1}$$

Thus, we get $\lim_{t\to\infty} v(t)=\sqrt{\frac{mg}{k}}$

Problem 5

Graph of $\dot{N}$



Phase portrait

In order to find the equilibrium points, we need to set the right hand side of the equation equal to zero:

$$-aN\ln(bN)=0$$

$$-aN=0 \Rightarrow N=0$$

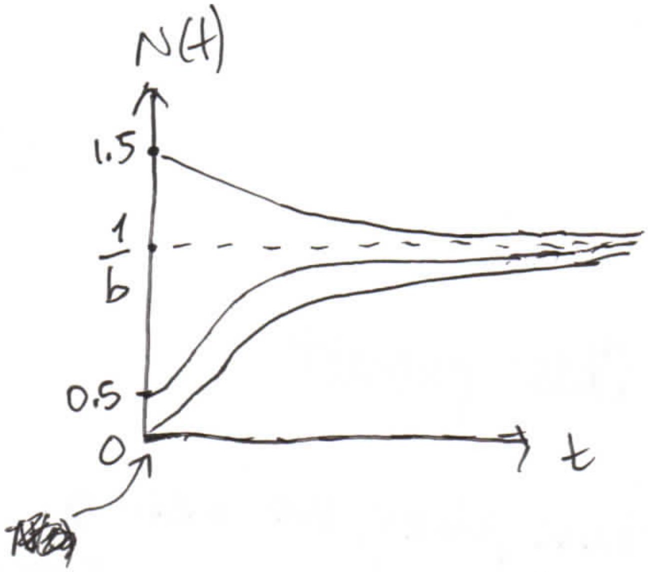
$$\ln(bN)=0 \Rightarrow N=\frac{1}{b}$$

Therefore, we have two equilibrium points $N=0$ and $N=\frac{1}{b}$.

Fixed point $N=\frac{1}{b}$ is a stable point because on the left hand side of the point the arrows are going to the right $(N(t)>0, \dot{N}>0)$ and on the left hand side the arrows are going to the left $(N(t)<0, \dot{N}<0)$

Fixed point $N=0$ is unstable because on the right hand side the arrows are going away from the point $(N(t)>0, \dot{N}>0)$, thus it's unstable.

Sketch of $N(t)$ for several initial conditions



Two initial conditions are chosen such as

$N(0) = \frac{1}{2}$ and $N(0) = 1.5$

Problem 6

a) We have the following equation:

$$\dot{x} = s(1-x)x^a - (1-s)x(1-x)^a$$

we can factor out $(1-x)$ and x to get the following:

$$\dot{x} = x(1-x)(sx^{a-1} - (1-s)(1-x)^{a-1})$$

Now, we can find the equilibrium points by having the right hand side equal to zero:

$$0 = x(1-x)(sx^{a-1} - (1-s)(1-x)^{a-1})$$

we see that we have three fixed points

$$x_1^* = 0$$

$$x_2^* = 1 \Leftarrow (1-x) = 0 \quad x = 1$$

$$x_3^* = (sx^{a-1} - (1-s)(1-x)^{a-1})$$

Let's define x_3^* as $h(x) = (sx^{a-1} - (1-s)(1-x)^{a-1})$

and when we let $h(0) = -(1-s)$ (0 is one of fixed points

when we let $h(1) = s$ (1 is the other fixed point)

So, we have $h(0) < 0$ and $h(1) > 0$, which means that x_3^* is a third fixed point in between the two fixed points of $x_1^* = 0$ and $x_2^* = 1$.

We can solve for x_3^* as follows.

$$sx^{a-1} - (1-s)(1-x)^{a-1} = 0$$

$$sx^{a-1} = (1-s)(1-x)^{a-1}$$

Problem 6 continued

part a)

$$x^{a-1} = \frac{(1-s)(1-x)^{a-1}}{s}$$

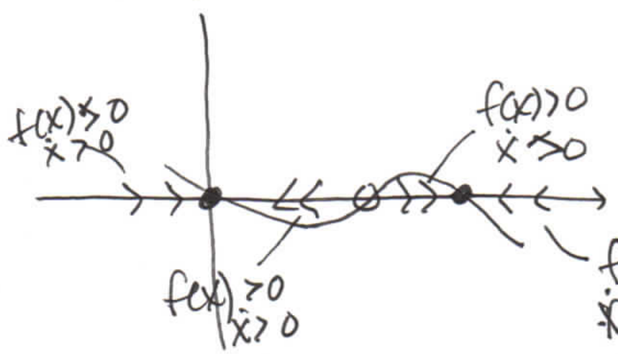
$$x = \left(\frac{1-s}{s}\right)^{1/a-1}(1-x)$$

$$x + \left(\frac{1-s}{s}\right)^{1/a-1} \cdot x = \left(\frac{1-s}{s}\right)^{1/a-1}$$

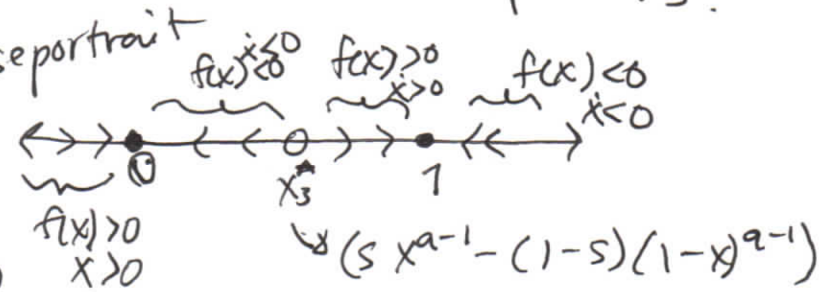
$$x = \frac{\left(\frac{1-s}{s}\right)^{1/a-1}}{1+\left(\frac{1-s}{s}\right)^{1/a-1}}$$

Hence, $x_3^* = \dfrac{\left(\frac{1-s}{s}\right)^{1/a-1}}{1+\left(\frac{1-s}{s}\right)^{1/a-1}}$

b) We graph the function to see the fixed points.



phaseportrait $\Rightarrow$



From the phase portrait, $x_2^*=1$ is a stable point ($f(x)>0, \dot{x}>0$) because the arrows to the left go in the right direction and the arrows to the right go in the left direction ($f(x)<0$ $\dot{x}<0$). Hence, $x_2^*=1$ is a stable point.

$x_1^*=0$ is a stable point because the arrows to the left of it go in the right direction ($f(x)>0, \dot{x}>0$) and the arrows to the right go in the left direction ($f(x)<0$, $\dot{x}<0$) making $x_1^*=0$ a stable point.

c) There are a few cases here for long-term behavior of the system.

If the model goes to either of the stable points at $x=0$ or $x=1$ then it will stay there meaning that everyone speaks either X or Y languages. Hence, both X and Y languages cannot stably coexist.

However, there is a third case when the model is equal to the unstable point

$$x = \frac{\left(\frac{1-s}{s}\right)^{1/a-1}}{1+\left(\frac{1-s}{s}\right)^{1/a-1}}$$

and this is when both languages can coexist.

Problem 7

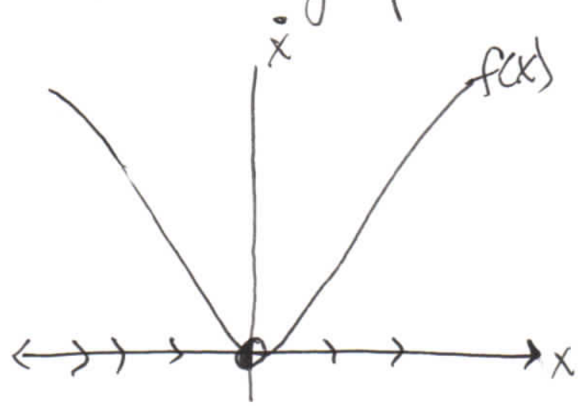
a) $\dot{x} = 1 - e^{-x^2}$

we have $f(x) = 1 - e^{-x^2}$

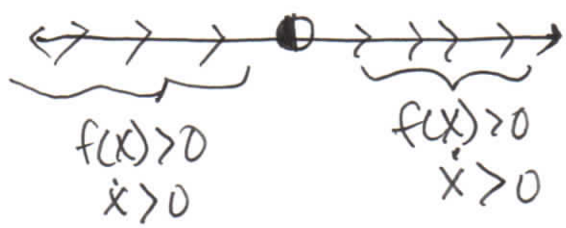
$$f'(x) = 2x \cdot e^{-x^2}$$

$f'(0) = 0$, thus linear stability analysis is inconclusive.

We use the graphic method



we see that $x_1^* = 0$ is semi-stable because the arrows to the left go in the right direction ($f(x) > 0$, $\dot{x} > 0$) and the ~~arrows~~ arrows to the right hand side go in the right direction ($f(x) > 0$, $\dot{x} > 0$)



b) $\dot{x} = ax - x^3$ $\quad f'(x) = a - 3x^2$

we can factor x out and get

$$\dot{x} = x(a - x^2)$$

we see the following fixed points

$$x_1^* = 0$$

$$(a - x^2) = 0$$

$$-x^2 = -a$$

$$x = \sqrt{a} \Rightarrow \quad x_2^* = \sqrt{a}$$

$$x_3^* = -\sqrt{a}$$

There are three cases to discuss here:

$a > 0$

$x_1^* = 0 = f'(x) = a - 3x^2$

$f'(0) = a > 0 \rightarrow$ unstable point

$x_2^* = \sqrt{a} = f'(x) = a - 3x^2$

$f'(\sqrt{a}) = a - 3\sqrt{a}^2 = -2a$

$-2a < 0 \rightarrow$ stable point

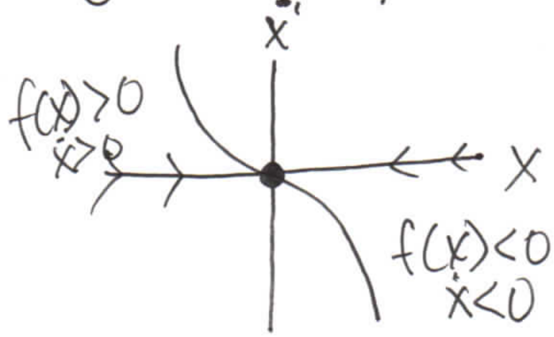
$x_3^* = -\sqrt{a} = f'(x) = a - 3x^2$

$f'(-\sqrt{a}) = a - 3 \cdot (-\sqrt{a})^2 = -2a$

$-2a < 0 \rightarrow$ stable point

$a = 0$ $\quad f'(x) = a - 3x^2 = -3x^2 = f'(a)$

$x^* = 0 = f'(0) = 0$ inconclusive so we graph



we see

$a = 0$ is a stable point based on the graph (arrows going towards eachother)

$a < 0$ $\quad f'(x) = a - 3x^2$

$x^* = 0 \quad f'(0) = a < 0 \rightarrow$ stable point