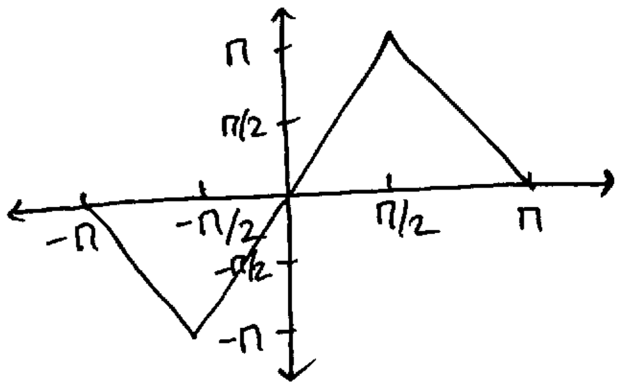
Solomiya Bilyk
AMATH 502
Homework #3
01/22/2021

Problem #1

Part a)



$$f(\varphi) = \begin{cases} 2\varphi & |\varphi| \le \pi/2 \\ 2\,\mathrm{sgn}(\varphi)\pi - 2\varphi & |\varphi| > \pi/2 \end{cases}$$

b) $\varphi = \theta_s - \theta_S$, $\tau = At$, $\mu = (\Omega - \omega)/A$

$\varphi = \theta_s - \theta_S \Rightarrow \dot{\varphi} = \theta_s - \theta_S = \Omega - \omega - Af(\theta_s - \theta_S)$

$\tau = At$, $d\tau = A\,dt$, $\frac{dt}{d\tau} = \frac{1}{A}$

$\Rightarrow \frac{d\varphi}{d\tau} = \frac{d\varphi}{dt} \cdot \frac{dt}{d\tau} = (\Omega - \omega - Af(\varphi)) \cdot \frac{1}{A} = \frac{\Omega - \omega}{A} - f(\varphi)$

$\mu = \frac{\Omega - \omega}{A}$, $\frac{d\varphi}{dt} = \varphi'$ $\Rightarrow$ $\varphi' = \mu - f(\varphi)$ as desired

c) Phase-locked to the stimulus when $-\pi \le \mu \le \pi$, this is the maximum and minimum of $f(\varphi)$.

$\dot{\varphi} = 0 \Rightarrow \dot{\varphi} = \mu - f(\varphi) = 0$

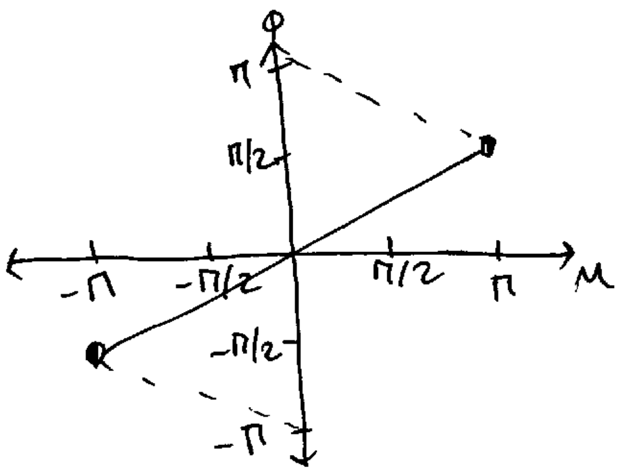
d) $\mu = \dfrac{\Omega - \omega}{A}$, $-\pi \leq \mu \leq \pi$

$$-\pi \leq \frac{\Omega - \omega}{A} \leq \pi$$

$$-\pi A \leq \Omega - \omega \leq \pi A$$

$\omega - \pi A \leq \Omega \leq \pi A + \omega$ => range of entrainment

e) Saddle Node bifurcation when $\mu = \pm\pi$ based on the following bifurcation diagram:



f) Formula of phase difference when phase-locked to the stimulus.
we know the range for μ is $-\pi \leq \mu \leq \pi$
we need to find the formula for μ, which is the slope of the line

Formula $= \frac{1}{2} \cdot \mu$ for $-\pi < \mu < \pi$

Problem 1

~~Parg~~ Part g

(c) for arbitrary smooth 2π-periodic function, we have a single maximum M, a single minimum m on $-\pi \le \varphi \le \pi$

Thus, values of μ where firefly will be phased locked is

$\dot{\varphi} = \mu - f(\varphi) = 0 \Rightarrow \boxed{m \le \mu \le M}$

d) $m \le \mu \le M$, $\mu = \dfrac{\Omega - \omega}{A}$

$m \le \dfrac{\Omega - \omega}{A} \le M \Rightarrow Am \le \Omega - \omega \le AM$

$\boxed{\omega - mA \le \Omega \le AM + \omega}$

f) We can't always find a formula for the stable fixed point because the function may be discontinuous and then we won't be able to find a formula. If a function is continuous, then we can find a formula.

Problem #2

a) $\dot{x} = 3x - 2y$, $\dot{y} = 2y - x$

Matrix form $\Rightarrow$ $A = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}$

Mathematica $\Rightarrow$ $\lambda_1 = 4$, $\lambda_2 = 1 \Rightarrow$ eigenvalues $\underbrace{\begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\text{eigenvectors}}$

λ_1 & λ_2 are positive, thus the origin is unstable and not attracting.

λ_1 & λ_2 are positive, it is an unstable node.
Attaching phase portrait in Gradescope.

b) $\dot{x} = x$, $\dot{y} = 5x - y$

Matrix form $\Rightarrow$ $A = \begin{bmatrix} 1 & 0 \\ 5 & -1 \end{bmatrix}$

Mathematica $\Rightarrow$ $\lambda_1 = -1$ $\lambda_2 = 1$ $\Rightarrow$ eigenvalues

$\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix} \Rightarrow$ eigenvectors

This is unstable and not attracting. This is a saddle point. Saddle points have a stable and an unstable manifold.

stable manifold when $\lambda_1 = -1 = \text{span}\left\{\begin{matrix} 0 \\ 1 \end{matrix}\right\}$

unstable manifold when $\lambda_2 = 1 = \text{span}\left\{\begin{matrix} 2 \\ 5 \end{matrix}\right\}$

Attaching phase portrait in Gradescope.

c) $\ddot{x} + \dot{x} + x = 0$

$$x_1 = x \qquad \dot{x}_1 = \dot{x} = x_2$$
$$x_2 = \dot{x} \quad \Rightarrow \quad \dot{x}_2 = \ddot{x} = -\dot{x} - x = -x_2 - x_1$$

Matrix form $\Rightarrow$ $A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$

Mathematica $\Rightarrow$ $\lambda_1 = \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2} \right)$ $\lambda_2 = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2} \right)$

There are two complex eigenvalues with real parts that are negative, therefore the origin is stable and is attracting. This makes the origin asymptotically stable

~~Eigenvalue~~: Eigenvectors (mathematica): $\begin{pmatrix} -\frac{1}{2} + \frac{i\sqrt{3}}{2} \\ 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} - \frac{i\sqrt{3}}{2} \\ 1 \end{pmatrix}$

The origin is a stable spiral because the eigenvalues are complex ~~and~~ with real parts.

Attaching phase portrait in Gradescope.

#5

Problem #3

a) $\dot{R} = aR + J$
$\dot{J} = -R - aJ$ $\Rightarrow$ Matrix form

$$\begin{pmatrix} \dot{R} \\ \dot{J} \end{pmatrix} = \begin{pmatrix} a & 1 \\ -1 & -a \end{pmatrix} \begin{pmatrix} R \\ J \end{pmatrix}$$

$$a > 0$$

There are four parameters in the matrix $\Rightarrow a, 1, -1, -a$.
Ray's response to his own feelings depends if R is positive or negative. If R is positive, Ray has more positive feelings. If R is negative, than he has more negative feelings. Ray's feelings are self reinforcing.

Jun's response to her own feelings depend on $-a$, so she is more cautious. If she is mad at Ray, then her feelings tell her to forgive Ray. However, if she if she is in love with Ray, then she remains in that stage.

Jun's response to Ray's feelings (this is when -1).
Jun reacts in an opposite manner to Ray's feelings. This is when Ray has more positive feelings, then Jun has negative feelings.
When Ray has more negative feelings, then Jun has positive feelings.
Ray's response to Jun's feelings (this is when 1)
Ray feels the same ~~whe~~ way as Jun.
If Jun feels positive, Ray does also.
If ~~Ray~~ Jun feels negative, Ray does also.

Role the a parameter play in the relationship.
Parameter a decides how Ray and Jun feel for their own feelings or how they respond to their own feelings.

b) $a>1$

$$\begin{pmatrix} \dot{R} \\ \dot{J} \end{pmatrix} = \begin{pmatrix} a & 1 \\ -1 & -a \end{pmatrix} \begin{pmatrix} R \\ J \end{pmatrix} \Rightarrow \text{matrix form}$$

Mathematica $\Rightarrow$ $\lambda_1 = -\sqrt{-1+a^2}$, $\lambda_2 = \sqrt{-1+a^2}$ $\Rightarrow$ eigenvalue

Eigenvectors $\Rightarrow$ $\begin{pmatrix} -a+\sqrt{-1+a^2} \\ 1 \end{pmatrix}$, $\begin{pmatrix} -a-\sqrt{-1+a^2} \\ 1 \end{pmatrix}$

This is unstable and not attracting. This is a saddle point. Saddle points have a stable and an unstable manifold.

stable manifold when $\lambda_1 = -\sqrt{-1+a^2}$, so the span is $\left\{ \begin{matrix} -a+\sqrt{-1+a^2} \\ 1 \end{matrix} \right\}$

Unstable manifold when $\lambda_2 = \sqrt{-1+a^2}$, so the span is $\left\{ \begin{matrix} -a-\sqrt{-1+a^2} \\ 1 \end{matrix} \right\}$

i. Attaching phase portrait in Gradescope.
Long-term behavior $R(t) \to +\infty$, $J(t) \to -\infty$

Ray will have more positive feelings for Jun. Jun will have more negative feelings for Ray.

#8

$R(t) \to -\infty, J(t) \to \infty$

Ray has more negative feelings for Jun. Jun will have more positive feeling for Jun

Part b continued

i. $0<a<1$

$$\begin{pmatrix} \dot{R} \\ \dot{J} \end{pmatrix} = \begin{pmatrix} a & 1 \\ -1 & -a \end{pmatrix} \begin{pmatrix} R \\ J \end{pmatrix} \Rightarrow \text{matrix form}$$

Mathematica $\Rightarrow$ $\underbrace{\lambda_1 = -\sqrt{-1+a^2}, \lambda_2 = \sqrt{-1+a^2}}_{\text{eigenvalues}}$

eigenvectors $\Rightarrow$ $\begin{pmatrix} -a+\sqrt{-1+a^2} \\ 1 \end{pmatrix}, \begin{pmatrix} -a-\sqrt{-1+a^2} \\ 1 \end{pmatrix}$

since $a<1$, there will be two complex eigenvalues. Thus, the origin is a center. The origin of centers are Lyapunov stable, but not asymptotically stable. Thus, the origin is neutrally stable.

ii. Attaching Gradescope the phase portraits.

Based on the graph, Jun's and Ray's feelings go in circles and alternate between hate and love. First and fourth Quadrant show us that their feelings are opposite of each other. When Ray loves Jun, Jun hates him (1st Q) and when Jun loves Ray, Ray hates her (4th Q) so, their feelings alternate between positive and negative and the feelings are opposite of each other.

Problem 3 continued.

b. i. $a = 1$

$$\begin{pmatrix} \dot{R} \\ \dot{J} \end{pmatrix} = \begin{pmatrix} a & 1 \\ -1 & -a \end{pmatrix}\begin{pmatrix} R \\ J \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} R \\ J \end{pmatrix}$$

Mathematica $\lambda_1 = 0$, $\lambda_2 = 0 \Rightarrow$ eigenvalues

Eigenvectors $\Rightarrow \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

The origin is unstable and not attracting. It is sheer flow because $\lambda_1 = \lambda_2 = 0$

ii. Attaching phase portrait to Gradescope

Long-term behavior

↳ The origin is unstable and not attracting therefore when $R(t) \to +\infty$ and $J(t) \to \infty$, the feelings of Ray and Jun are opposite. This is shown in (Quadrant 4) Ray has positive feelings for Jun and Jun has negative feelings for Ray

when $R(t) \to -\infty$ and $J(t) \to \infty$ again the feelings of Ray and Jun are opposite. This is shown in Quadrant 2. Ray has negative feelings for Jun, while Jun has positive feelings for Ray

Problem 3

Part c

Parameter a decides how Ray and Jun feel for their own feelings or how they respond to their own feelings.

For Ray, his parameter a is positive and his feelings ~~are~~ show exactly how he feels. If he has positive feelings, then he has more positive feeling. If he doesn't feel happy, he keeps feeling sad.

For Jun, her parameter is $-a$ and this shows that she is more cautious. This depends on how her feeling are towards Ray. If she is mad at him, then her cautious level tells her to forgive him. If she is in love with Ray, her feelings tell her to be careful.

when

Jun and Ray have opposite feelings. So, when

$a=0 \rightarrow$ Jun loves Ray, Ray hates Jun and vice versa

$a<1 \rightarrow$ Jun and Ray go in cycles. They are in love in one cycle and hate each other in next

$a>1 \rightarrow$ Jun and Ray have feelings for each other but the speed of their feelings is different. Ray loves Jun, but Jun first goes into a neutral state before loving Ray.

Problem 4

Part a

$$L\ddot{I} + \dot{I}R + \frac{I}{C} = 0$$

$$L\ddot{I} = -\frac{I}{C} - \dot{I}R$$

$$\ddot{I} = -\frac{I}{CL} - \frac{\dot{I}R}{L} = -\frac{1}{CL}\cdot I - \frac{R}{L}\dot{I}$$

Now, let $x_1 = I \quad x_2 = \dot{I} \quad \dot{x}_2 = \ddot{I}$

$$\dot{x}_1 = \dot{I} \Rightarrow \dot{x}_1 = x_2 = I$$

$$\Rightarrow \dot{x}_2 = -\frac{1}{CL}\cdot x_1 - \frac{R}{L}\cdot x_2$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{1}{CL}\cdot x_1 - \frac{R}{L}\cdot x_2$$

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1/CL & -R/L\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} \Rightarrow \text{Matrix form}$$

Problem 4

Part b Using Mathematica:

$$\lambda_1, \lambda_2 = \frac{-R \pm \sqrt{-\frac{4L}{C} + R^2}}{2L} \quad R>0$$

① We have two complex eigenvalues when $R>1$, so $\pm\sqrt{-\frac{4L}{C}+R^2}$ becomes imaginary and complex. The eigenvalues are $\lambda_1, \lambda_2 = -\frac{R}{2L} \pm$ complex number. Thus, the origin is asymptotically stable because the real parts are negative of the eigenvalues ($\lambda_1, \lambda_2 < 0$)

②. When $0<R<1$, we will get eigenvalues:

$$\lambda_1, \lambda_2 = -\frac{R}{2L} \pm \left(\text{number} < \frac{R}{2L}\right)$$

Thus, we ~~a~~ have negative eigenvalues and the origin is asymptotically stable as $\lambda_1, \lambda_2 < 0$

Hence, we have shown that when $R>0$, the origin is asymptotically stable.

Part b continued

$R=0$

$$\lambda_1,\lambda_2 = \frac{-R \pm \sqrt{-\frac{4L}{C}+R^2}}{2L} = \frac{\pm\sqrt{-\frac{4L}{C}}}{2L} = \frac{\pm\sqrt{-\frac{L}{C}}}{2L}$$

$\lambda_1,\lambda_2 = \pm i\beta$

Now, we have two purely imaginary eigenvalues, so the origin is a center and neutrally stable.

Problem 4 continued

Part c) $\dfrac{-CR-\sqrt{C}\sqrt{-4L+CR^2}}{2CL}$ (per Mathematica)

$R^2C-4L>0$, we get $R^2>\dfrac{4L}{C}$, thus

we get $\lambda_1,\lambda_2 = -\dfrac{R}{2L} \pm \dfrac{\text{number }(R>\sqrt{\frac{4L}{C}})}{2L}$ This will make it smaller than R

Thus, both λ_1,λ_2 are negative, and real and the origin is a stable node.

$R^2C-4L<0 \Rightarrow 4L>CR^2$, thus we get

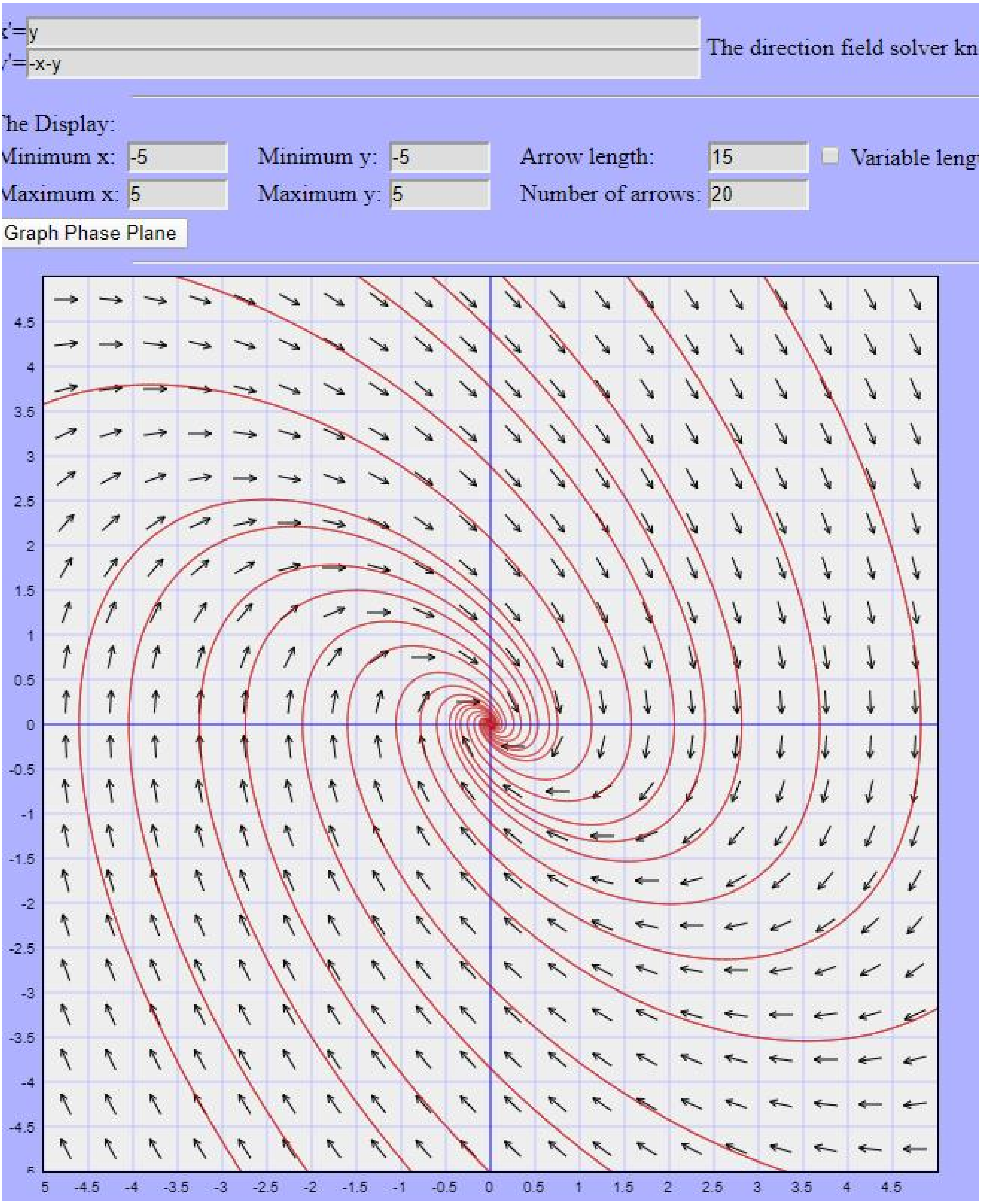
$\lambda_1,\lambda_2 = -\dfrac{R}{2L} \pm$ complex number

Real part of complex number is negative thus $\lambda_1,\lambda_2<0$ and we have a ~~spi~~ stable spiral and it's stable.

$R^2C-4L=0$, then we get

$\lambda_1,\lambda_2 = -\dfrac{R}{2L} \pm 0$

Both $\lambda_1,\lambda_2<0$ and λ_1,λ_2 are equal to each other. Hence, it's a degenerate node and it's stable.



y
-x-y
The direction field solver kn
The Display:
Minimum x: -5
Minimum y: -5
Arrow length: 15
Variable leng
Maximum x: 5
Maximum y: 5
Number of arrows: 20
Graph Phase Plane
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
-4
-4.5
-4.5
-4
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5

For a much more sophisticated phase plane plotter, see the [MATLAB plotter] written by John C. Polking of Rice University.

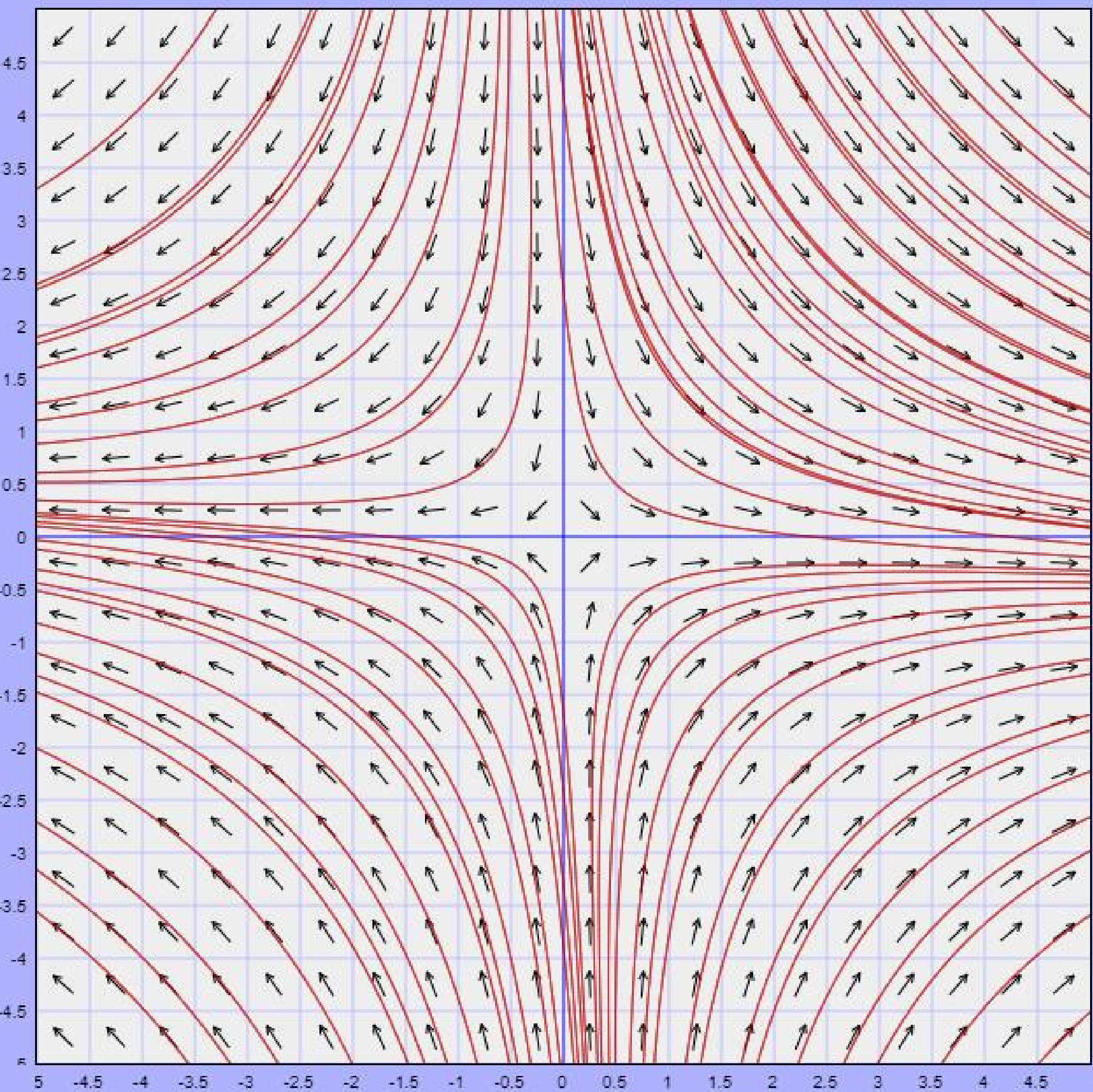
x'= 10*x+y
y'= -x-10*y

The direction field solver knows about trigonometric,

The Display:

Minimum x:	-5	Minimum y:	-5	Arrow length:	15	Variable length arrows
Maximum x:	5	Maximum y:	5	Number of arrows:	20	

Graph Phase Plane



For a much more sophisticated phase plane plotter, see the [MATLAB plotter] written by John C. Polking of Rice Univers

x'=x
y'=5*x-y

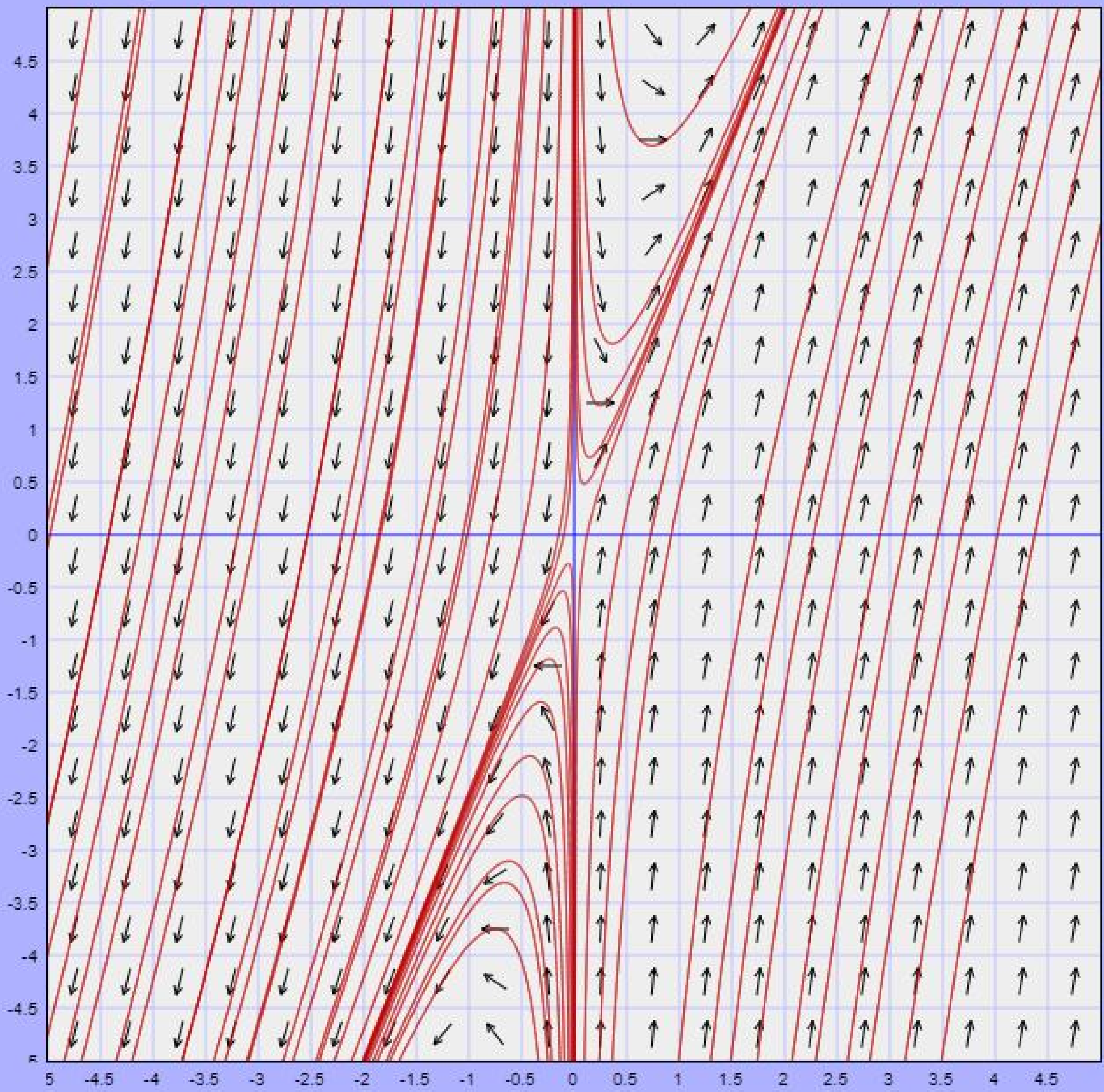
The direction field solver knows about trigon

The Display:

Minimum x: -5 Minimum y: -5 Arrow length: 15 Variable length arrows

Maximum x: 5 Maximum y: 5 Number of arrows: 20

Graph Phase Plane



x'= x+y

y'= -x-y

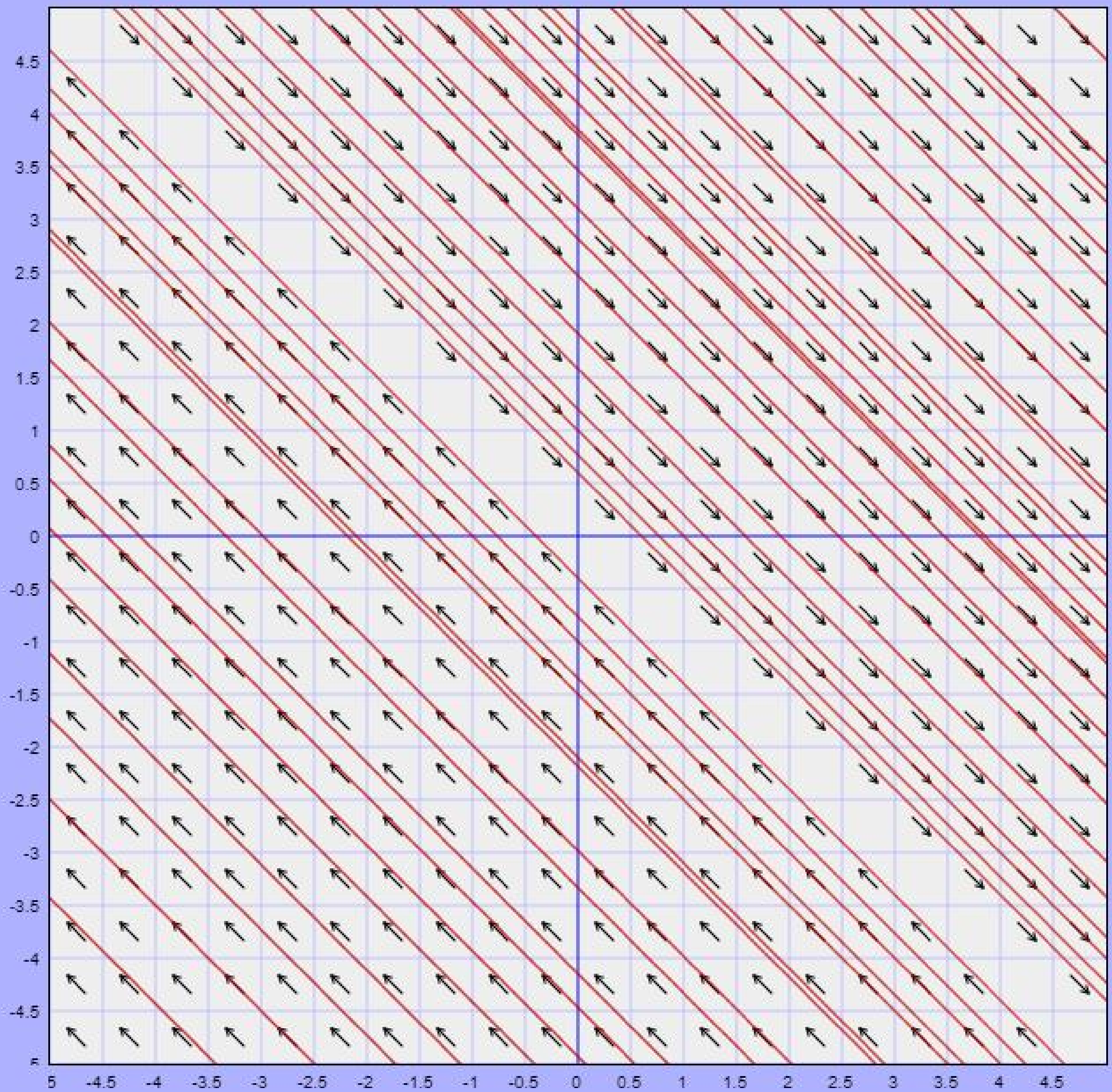
The direction field solver knows about trigo

The Display:

Minimum x: -5 Minimum y: -5 Arrow length: 15 Variable length arrows

Maximum x: 5 Maximum y: 5 Number of arrows: 20

Graph Phase Plane



x'=3*x-2*y

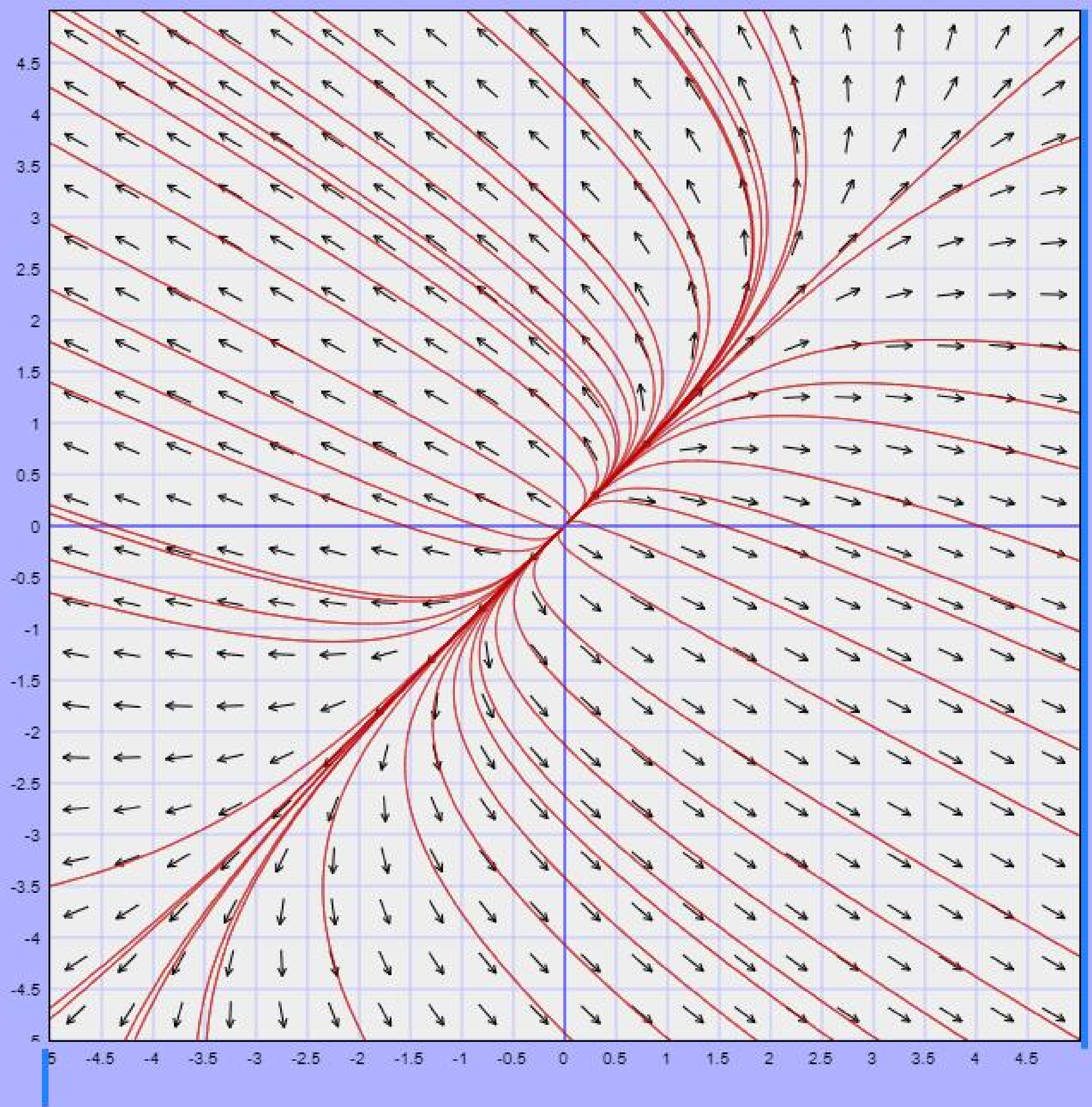
y'=2*y-x

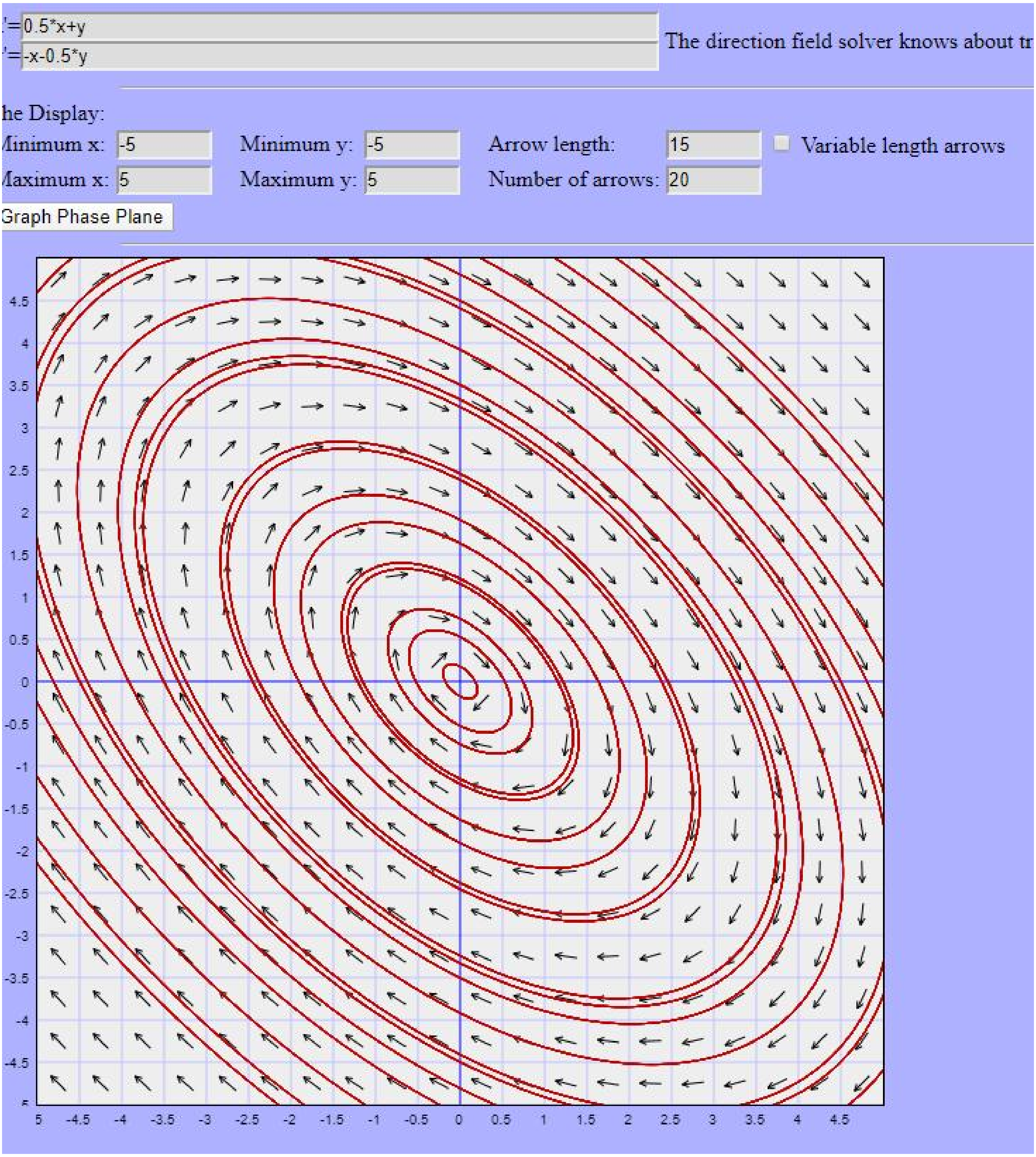
The direction field solver knows about trigor

The Display:

Minimum x:	-5	Minimum y:	-5	Arrow length:	15	Variable length arrows
Maximum x:	5	Maximum y:	5	Number of arrows:	20	

Graph Phase Plane




'=0.5*x+y
'=-x-0.5*y
The direction field solver knows about tr
he Display:
linimum x: -5
Minimum y: -5
Arrow length: 15
Variable length arrows
laximum x: 5
Maximum y: 5
Number of arrows: 20
Graph Phase Plane